Mathematics 114C, Winter 2019
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Final examination, Thursday, March 21, 2017

Name (last name first): ______________________________

Signature: ______________________________

There are 17 problems or parts of problems and each of them is worth 7 points, for a total of 119 points, 19 more than the 100 points which will count as a "perfect" test. The problems vary widely in difficulty; *do first those which are easy*, there are enough of them to give you an A.

Most of the problems test your knowledge of what we learned and do not require any great ingenuity. You may—and in some places you will need to—use some of the basic results we established, e.g., the Normal Form and S^m_n Theorems, the Σ^0_1-Selection Lemma, the Second Recursion Theorem and the closure properties of the several collections of relations we have studied.

There is a blank page at the end which you can use for scratch.

A solved copy will be posted after the test.

Solved!

Problem 1. ________

Problem 2. ________

Problem 3. ________

Problem 4. ________

Problem 5. ________

Problem 6. ________

Problem 7. ________

Problem 8. ________

Total: ____________

Problem 1. Suppose $g(y)$ is a primitive recursive function and $f(n, x)$ is defined by the equations

$$f(0, x) = x, \quad f(n + 1, x) = g(f(n, x)).$$

(1a) Prove that $f(n, x)$ is primitive recursive, using only the definition of primitive recursive functions.

Solution. $f(n, x)$ is defined by the primitive recursion
$f(0, x) = x = P^1_1(x), \quad f(n+1, x) = h(f(n, x), n, x)$ where $h(u, n, x) = g(P^3_1(u, n, x))$, and $h(u, n, x)$ is the composition of two primitive recursive functions and hence primitive recursive.

(1b) Assume further that for all y, $g(y) \geq 2y + 1$ and prove that

$$\text{for all } n \geq 1,\ f(n, x) \geq 2^n(x + 1) - 1.$$

Solution. The proof is by induction on $n \geq 1$.

Basis, $n = 1$: $f(1, x) = g(x) \geq 2x + 1 = 2^1(x + 1) - 1$.

Induction Step. Assume the inductive hypothesis, that $f(n, x) \geq 2^n(x + 1) - 1$ and compute:

$$\begin{aligned} f(n + 1, x) = g(f(n, x)) \geq 2f(n, x) + 1 &\geq 2[2^n(x + 1) - 1] + 1 \text{ (ind. hyp)} \\ &= 2^{n+1}(x + 1) - 2 + 1 = 2^{n+1}(x + 1) - 1. \end{aligned}$$

Problem 2. Suppose $g : \mathbb{N} \to \mathbb{N}$ is a total recursive function and consider the recursive equation

$$(*) \qquad p(x) = \mathsf{if}\ (\mathrm{Prime}(x))\ \mathsf{then}\ g(x)\ \mathsf{else}\ p(x+1).$$

Let

$$\overline{p}(x) = g(\mu y \geq x\ \mathrm{Prime}(y)).$$

(2a) Prove that $\overline{p}$ satisfies $(*)$.

Solution. We plug $\overline{p}$ for p in the equation to see if it makes both sides equal, taking cases on x.

If x is prime, then

$$\begin{aligned} \overline{p}(x) &= g(x) && \text{by the def of } \overline{p}(x), \\ &= \mathsf{if}\ (\mathrm{Prime}(x))\ \mathsf{then}\ g(x)\ \mathsf{else}\ \overline{p}(x+1). \end{aligned}$$

If x is not prime, then

$$\begin{aligned} \overline{p}(x) &= \overline{p}(x+1) && \text{by the def of } \overline{p}(x), \\ &= \mathsf{if}\ (\mathrm{Prime}(x))\ \mathsf{then}\ g(x)\ \mathsf{else}\ \overline{p}(x+1). \end{aligned}$$

(2b) Prove that $\overline{p}$ is the unique solution of $(*)$, i.e., if $p : \mathbb{N} \rightharpoonup \mathbb{N}$ is any partial function satisfying $(*)$, then for every x, $p(x) = \overline{p}(x)$.

Solution. Using the fact that there are infinitely many prime numbers, we set

$$h(x) = \mu t\ \mathrm{Prime}(x+t),$$

and we prove that if p satisfies $(*)$ then

$$\overline{p}(x) = w \Longrightarrow p(x) = w$$

by induction on $h(x)$.

Basis. If $h(x) = 0$, then x is a prime and so

$$\overline{p}(x) = g(x) \text{ and also } p(x) = \mathsf{if}\ (\mathrm{Prime}(x))\ \mathsf{then}\ g(x)\ \mathsf{else}\ p(x+1) = g(x).$$

Induction Step. If $h(x) > 0$, then x is not a prime, $h(x) = h(x+1) + 1$,

$$\overline{p}(x) = g(\mu y \geq (x+1)\ \mathrm{Prime}(y)) = \overline{p}(x+1),$$

and also

$$p(x) = \mathsf{if}\ (\mathrm{Prime}(x))\ \mathsf{then}\ g(x)\ \mathsf{else}\ p(x+1) = p(x+1),$$

so the Induction Hypothesis insures that $\overline{p}(x) = \overline{p}(x+1) = p(x+1) = p(x)$.

Problem 3. The first two parts are about a collection $\mathcal{E}$ of relations (of all arities) on $\mathbb{N}$, e.g., the collection of all primitive recursive relations, Σ^0_k, Π^0_k, etc.

(3a) Define what it means for $\mathcal{E}$ to be closed under primitive recursive substitutions.

Solution. If $R(u_1, \dots, u_m)$ is in $\mathcal{E}$, $f_1(\vec{x}), \dots, f_m(\vec{x})$ are primitive recursive functions and

$$P(\vec{x}) \iff R(f_1(\vec{x}), \dots, f_m(\vec{x})),$$

then $P(\vec{x})$ is also in $\mathcal{E}$.

(3b) Define what it means for $\mathcal{E}$ to be closed under bounded universal quantification $\forall_\leq$.

Solution. If $R(i, \vec{x})$ is in $\mathcal{E}$, then so is the relation

$$P(n, \vec{x}) \iff (\forall i \leq n) R(i, \vec{x}).$$

(3c) For each of the collections of relations below determine whether it is closed or not under each of the indicated operations by marking Y (for YES) or N (for NO) the relevant block in the table; for example, the square to the right of PR and below PrimSubs should be marked Y if you think that the collection of primitive recursive relations is closed under primitive recursive substitutions. (No proofs are required.)

	PrimSubs	&	$\vee$	$\neg$	$\exists_\leq$	$\forall_\leq$	$\exists$	$\forall$
PR								
Δ^0_1								
Σ^0_k								
Π^0_k								

Solution. Omitted, sorry; too messy to type and the problem is quite easy.

Problem 4. Recall that $W_e = \{x \mid \varphi_e(x)\downarrow\} = \{x \mid (\exists y)T_1(e, x, y)\}$.

(4a) Prove that there is a recursive partial function $f(e, n)$ such that for all e, n,

$(*)$ *if W_e has at least n members,*

$$\textit{then } \Big(f(e,n)\downarrow \ \ \& \ W_e \textit{ has at least } n \textit{ members which are } < f(e,n)\Big).$$

Solution. Put

$$P(e, n, u) \iff \mathrm{Seq}(u) \ \& \ (\forall i < n)[(u)_i \in W_e] \ \& \ (\forall i, j < n)[(i \neq j) \Longrightarrow (u)_i \neq (u)_j].$$

This is a Σ^0_1 relation, and the Σ^0_1-Selection Lemma gives us a recursive partial $g(e, n)$ such that for all e, n,

$$W_e \text{ has at least } n \text{ members} \Longrightarrow (\exists u)P(e, n, u) \Longrightarrow [g(e, n)\downarrow \ \ \& \ P(e, n, g(e, n))].$$

The required result follows by setting

$$f(e, n) = \max\{g(e, n)_i \mid i < n\} + 1.$$

(4b) Prove that there is no total, recursive function $f(e, n)$ which satisfies $(*)$.

Solution. Assume that some total $f(e, n)$ satisfies $(*)$, choose a total recursive $g(e, x)$ such that

$$W_{g(e,x)} = \{y \mid T_1(e, x, y)\}$$

and compute:

$$(\exists y)T_1(e, x, y) \iff W_{g(e,x)} \neq \emptyset \iff (\exists y < f(g(e, x), 1))T_1(e, x, y);$$

but this equivalence implies that the Halting Relation

$$H(e, x) \iff (\exists y)T_1(e, x, y)$$

is recursive, which it is not.

Problem 5. Recall that a r.e. set A is *simple* if A has infinite complement and every infinite r.e. set intersects A.

(5a) Prove that if A is simple and $X = \{x_0, \dots, x_n\}$ is a finite set, then the sets $A \cup X$ and $A \setminus X$ are both simple.

SOLUTION. For $A \cup X$ first: its complement is $A^c \setminus X$, so infinite, since A^c is infinite and X is finite; and every infinite r.e. set intersects A, so it intersects the larger set $A \cup X$.

For $A \setminus X$, similarly: its complement is $A^c \cup X$, so it contains A^c which is infinite; and if B is any infinite r.e. set. then $B \setminus X$ is also infinite r.e., so there is some x such that $x \in A$ and $x \in B \setminus X$, i.e., $x \in A \setminus X$ and $x \in B$, so B intersects $A \setminus X$.

(5b) For the next two claims, decide whether they are true or not, and justify your answer by a proof or a counterexample:

(1) For every infinite r.e. set A, there is a total, recursive function f, such that for every x,

$$f(x) > x \text{ and } f(x) \in A.$$

(2) For every r.e. set B with infinite complement, there is a total recursive function f, such that for every x,

$$f(x) > x \text{ and } f(x) \notin B.$$

SOLUTION. (1) This holds: because the relation

$$R(x, y) \iff y > x \ \&\ y \in A$$

is semirecursive and for every x there exists a $y > x$ such that $y \in A$ by the hypothesis, so by the Σ_1^0-Selection Lemma, there exists a total, recursive function $f(x)$ as the claim requires.

(2) This is not true when B is simple, because if such an f existed, then $f[\mathbb{N}] = \{f(x) \mid x \in \mathbb{N}\}$ would be an r.e. subset of B^c which is unbounded, so infinite.

Problem 6. This problem is about the 2nd Recursion Theorem.

(6a) Suppose $f(x)$ is a total recursive function; prove that there is a number z such that

$$W_z = \{f(z)\}.$$

SOLUTION. Choose z by the 2nd Recursion Theorem such that

$$\varphi_z(x)\downarrow \iff x = f(z),$$

so that

$$x \in W_z \iff \varphi_z(x)\downarrow \iff x = f(z).$$

(6b) Let $g(e)$ be a recursive partial function such that for all e,

$$\text{if } W_e = \emptyset, \text{ then } g(e)\downarrow;$$

Prove that there is some m such that $W_m = \{m\}$ and $g(m)\downarrow$.

SOLUTION. By the 2nd Recursion Theorem, there exists some number m such that

$$\varphi_m(x) = \begin{cases} 1, & \text{if } g(m)\downarrow \ \& \ x = m, \\ \uparrow, & \text{otherwise.} \end{cases}$$

If $g(m)\uparrow$, then $W_m = \emptyset$ and therefore $g(m)\downarrow$; so $g(m)\downarrow$, and by the definition $\varphi_m(x)\downarrow$ only if $x = m$, i.e., $W_m = \{m\}$, which is what we wanted.

Problem 7. Let G be the set of all codes of total recursive functions which are increasing, i.e.,

$$e \in G \iff \varphi_e \text{ is total and } (\forall x, y)[x < y \Longrightarrow \varphi_e(x) < \varphi_e(y)].$$

(7a) Prove that G is Π^0_2.

SOLUTION.

$$\begin{aligned} e \in G \iff & (\forall x)(\exists s)T_1(e, x, s) \\ & \& \ (\forall x)(\forall y)(\forall s)(\forall t)\Big([x < y \ \& \ T_1(e, x, s) \ \& \ T_1(e, y, t)] \Longrightarrow U(s) < U(t)\Big) \end{aligned}$$

(7b) Prove that G is not Σ^0_2.

SOLUTION. Let $F = \{e \mid (\forall x)[\varphi_e(x)\downarrow]\}$. We know that this is Π^0_2 but not Σ^0_2, so it is enough to reduce F to G, i.e., to define a total, recursive $f(e)$ such that

$$e \in F \iff f(e) \in G.$$

For that it is sufficient to choose $f(e)$ so that

$$\varphi_{f(e)}(x) = \max\{\varphi_e(i) \mid i \leq x)\} + 1;$$

and we can define such an $f(e)$ using the Graph Lemma and the S^m_n-Theorem, as follows:

- Set

$$R(e, x, w) \iff w > 0 \ \& \ (\forall i \leq x)(\exists u < w)[\varphi_e(i) = u] \ \& \ (\exists i \leq x)[\varphi_e(i) = w - 1].$$

- Check that that $R(e, x, w)$ is semirecursive and the graph of a partial function $g(e, x)$, which is therefore recursive.

- Choose $\overline{g}$ such that $g(e, x) = \{\overline{g}\}(e, x) = \{S^1_1(\overline{g}, e)\}(x)$ and set $f(e) = S^1_1(\overline{g}, e)$,

- And finally check that

$$\varphi_e \text{ is total} \iff \varphi_{f(e)} \text{ is total and increasing.}$$

Problem 8. A collection $\mathcal{A}$ of r.e. sets is *semieffective* if there is a Σ^0_1 set A such that

$$W_e \in \mathcal{A} \iff e \in A.$$

For example, $\{W_e \mid W_e \neq \emptyset\}$ and $\{W_e \mid (\exists m)[2m+3 \in W_e \cap W_m]\}$ are semieffective. Notice that the definition implies the following *invariance property* of A,

$$(*) \qquad W_e = W_m \Longrightarrow \Big(e \in A \iff m \in A\Big).$$

Hint: For both parts, use the Second Recursion Theorem.

(8a) Prove that semieffective collections of r.e. sets are $\subseteq$-*monotone*, i.e.,

$$(**) \qquad \Big(W_e \in \mathcal{A} \;\&\; W_e \subseteq W_m\Big) \Longrightarrow W_m \in \mathcal{A}.$$

Solution. Fix e and m which satisfy the hypothesis of $(**)$ and use the Second Recursion Theorem to get a z such that

$$(***) \qquad x \in W_z \iff x \in W_e \vee \Big(z \in A \;\&\; x \in W_m\Big).$$

Now check:

(1) $W_e \subseteq W_z$; directly from $(***)$.

(2) $z \in A$; because if $z \notin A$, then $(***)$ gives $W_z = W_e$ and then the invariance property $(*)$ gives $z \in A$.

(3) $W_z = W_m$, easily from (2) using the hypothesis $W_e \subseteq W_m$.

Now $m \in A$ by (3) and the invariance property of A again, which completes the proof.

(8b) Prove that if $W_e \in \mathcal{A}$, then there is a finite $W_z \subseteq W_e$ such that $W_z \in \mathcal{A}$.

Solution. Choose $\widehat{f}$ such that

$$m \in A \iff (\exists y)T_1(\widehat{f}, m, y),$$

fix e such that $e \in A$ and then use the 2nd Recursion Theorem to get a z such that

$$(****) \qquad x \in W_z \iff x \in W_e \;\&\; (\forall u \leq x)\neg T_1(\widehat{f}, z, u).$$

Now check:

(1) $W_z \subseteq W_e$, directly from $(****)$.

(2) $z \in A$; because if not, then $(\forall u)\neg T_1(\widehat{f}, z, u)$; hence for every x, $(\forall u \leq x)\neg T_1(\widehat{f}, z, u)$; hence $W_z = W_e$ by $(****)$ and so $z \in A$ by the invariance property.

(3) If $y = \mu u T_1(\widehat{f}, z, u)$, then $x \in W_z \Longrightarrow x < y$, directly from $(****)$ again, so W_z is finite.