Math 114C, Winter 2019, Solutions to HW #8

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x4D.3. Are they true or not—and you must prove your answers:

(1) There is a recursive, partial function $f(e)$, such that for every e,

$$(*) \qquad \text{if } W_e \text{ is infinite, then } f(e)\!\downarrow\ \&\ f(e) \in W_e\ \&\ f(e) > e.$$

(2) There exists a *total* recursive function $f(e)$ which satisfies $(**)$.

Solution. (1) The relation

$$R(e, y) \iff y \in W_e\ \&\ y > e$$

is Σ^0_1, and therefore, by the Σ^0_1-Selection Lemma there exists a recursive partial function $f(e)$ such that for every e,

$$(**) \qquad (\exists y)[y \in W_e\ \&\ y > e] \Longrightarrow f(e)\!\downarrow\ \&\ f(e) \in W_e\ \&\ e < f(e);$$

this $f(e)$ is the required one, since the hypothesis of the implication $(**)$ is satisfied by any e such that W_e is infinite.

(2) We show that this does not hold, by contradiction. Assuming that it holds with some total $f(e)$, we choose by the 2nd Recursion Theorem some e such that

$$\varphi_e(x) = \begin{cases} 1, & \text{if } f(e) < x, \\ \uparrow, & \text{otherwise,} \end{cases}$$

and we notice that the set $W_e = \{x \mid x > f(e)\}$ is infinite, so that from the properties of f which we have assumed it follows that $f(e) \in W_e$, which implies together with the characteristic property of φ_e that $f(e) < f(e)$, which is absurd.

x4D.8. Let $g(e)$ be a recursive, partial function such that for all e,

$$\text{if } W_e = \mathbb{N}, \text{ then } g(e)\!\downarrow;$$

prove that there exist numbers m and k, such that

$$W_m = \{0, 1, \ldots, k\} \text{ and } g(m)\!\downarrow.$$

Hint: Apply the 2nd Recursion Theorem to the partial function

$$f(m, x) = \begin{cases} 1, & \text{if } (\forall y \le x)\neg T_1(\widehat{g}, m, y), \\ \uparrow, & \text{otherwise,} \end{cases}$$

where $g(e) = \{\widehat{g}\}(e)$.

Solution. Following the hint, let m from the 2nd Recursion Theorem such that

$$\varphi_m(x) = \begin{cases} 1, & \text{if } (\forall y \le x)\neg T_1(\widehat{g}, m, y), \\ \uparrow, & \text{otherwise,} \end{cases}$$

from which follows that

$$W_m = \{x \mid (\forall y \le x)\neg T_1(\widehat{g}, m, y)\}.$$

If $g(m)\uparrow$, then for every x, we have that $(\forall y \le x)\neg T_1(\widehat{g}, m, y)$ holds, so that $W_m = \mathbb{N}$ and $g(m)\!\downarrow$; it follows that $g(m)\!\downarrow$, and the required conclusion holds with $k = \mu y T_1(\widehat{g}, m, y) \dot{-} 1$. (Here we use the property $\neg T_1(\widehat{g}, m, 0)$ of the basic relation of Kleene, which could be avoided with some more careful formulation of the problem.)

x5A.4. Classify in the arithmetical hierarchy the relation

$$\begin{aligned} Q(e,m) &\iff \varphi_e \sqsubseteq \varphi_m \\ &\iff (\forall x)\Big(\varphi_e(x)\!\downarrow\ \Longrightarrow [\varphi_m(x)\!\downarrow\ \&\ \varphi_e(x) = \varphi_m(x)]\Big). \end{aligned}$$

Solution. $Q(e,m)$ is in the class $\Pi^0_2 \setminus \Sigma^0_2$. To show that it is Π^0_2, we compute:

$$Q(e,m) \iff (\forall x)(\forall w)[\varphi_e(x) = w \Longrightarrow \varphi_m(x) = w],$$

which shows that $Q(e,m)$ is Π^0_2, since the relation

$$\varphi_z(x) = w \iff (\exists y)[T_1(z,x,y)\ \&\ U(y) = w]$$

is Σ^0_1. The basic idea for the computation of the lower bound is the reduction

$$\varphi_e \text{ is total} \iff (\lambda t)0 \sqsubseteq (\lambda t)[0 \cdot \varphi_m(t)],$$

where $(\lambda t)[\dots t \dots]$ is the partial function with value $\dots t \dots$ for each t. In detail, let e_0 be a code of the constant function C^1_0,

$$\varphi_{e_0}(t) = 0,$$

and let $g(e)$ be a recursive function such that for every e,

$$\varphi_{g(e)}(t) = 0 \cdot \varphi_e(t);$$

it follows that

$$\varphi_e \text{ is total} \iff Q(e_0, g(e)),$$

and so the set

$$F = \{e \mid \varphi_e \text{ is total}\}$$

would be Σ^0_2, if Q were Σ^0_2, while F is Π^0_2-complete, 5A.6.

x5A.5. Classify in the arithmetical hierarchy the set

$$A = \{e \mid W_e \text{ has at least } e \text{ members}\}.$$

Solution. A is recursively enumerable, i.e., Σ^0_1, with the computation

$$\begin{aligned} e \in A \iff (\exists u)[(\forall i < e)(\forall j < e)[i \neq j \Longrightarrow (u)_i \neq (u)_j] \\ \&\ (\forall i < e)][(u)_i \in W_e]]. \end{aligned}$$

It is not Π^0_1, because it is Σ^0_1-complete, as follows.

Let

$$P(x) \iff (\exists y)R(x,y)$$

be a Σ^0_1 relation, with $R(x,y)$ recursive. We set

$$g(x,t) = \mu y R(x,y),$$

so that g is a recursive partial function; if $\hat{g}$ is a code of $g(x)$, then

$$\begin{aligned} P(x) &\Longrightarrow (\exists y)R(x,y) \\ &\Longrightarrow (\forall t)[\{\hat{g}\}(x,t)\!\downarrow] \\ &\Longrightarrow (\forall t)[\{S^1_1(\hat{g},x)\}(t)\!\downarrow], \end{aligned}$$

so that if we define

$$f(x) = S^1_1(\hat{g}, x),$$

then

$$P(x) \Longrightarrow W_{f(x)} = \mathbb{N} \Longrightarrow f(e) \in A.$$

On the other hand,

$$f(x) \in A \Longrightarrow (\exists t) g(x,t)\downarrow \ \Longrightarrow (\exists y) R(x,y) \Longrightarrow P(x),$$

so that $P(x) \iff f(x) \in A$.

x5A.6. Classify in the arithmetical hierarchy the set of codes of bounded recursive partial functions,

$$B = \{e \mid \text{for some } w \text{ and all } x,\ \varphi_e(x)\downarrow \ \Longrightarrow \varphi_e(x) \le w\}.$$

Solution. For the easy direction, we compute:

$$\begin{aligned}(\exists w)(\forall x)[\varphi_e\downarrow \ \Longrightarrow \varphi_e(x) \le w] &\iff (\exists w)(\forall x)[(\exists y)T_1(e,x,y) \Longrightarrow U(y) \le w] \\ &\iff (\exists w)(\forall x)[(\forall y)\neg T_1(e,x,y) \vee U(y) \le w],\end{aligned}$$

and the relation on the right is obviously Σ^0_2.

To show that this set is Σ^0_2-complete, let

$$P(x) \iff (\exists w)(\forall z) R(x,w,z)$$

where the relation $R(x,w,z)$ is recursive, we set

$$g(x,t) = \begin{cases} (\mu w \le t)(\forall z \le t) R(x,w,z) & \text{if } (\exists w \le t)(\forall z \le t) R(x,w,z), \\ t & \text{otherwise,}\end{cases}$$

and finally

$$f(x) = S^1_1(\widehat{g}, x), \quad \text{where } g(x,t) = \varphi_{\widehat{g}}(x,t).$$

It is enough to show that

$$(\exists w)(\forall z) R(x,w,z) \iff \varphi_{f(x)} \text{ is bounded},$$

and to this end we will use the obvious (from the definitions) facts that *for every x, $\varphi_{f(x)}$ is total*, and

$$t < s \Longrightarrow g(x,t) \le g(x,s);$$

from these it follows that what we need to prove is the simple equivalence

$$(\exists w)(\forall z) R(x,w,z) \iff (\exists w)(\exists t)(\forall s \ge t)[g(x,s) = w],$$

which is now (almost) obvious, by the definition of $g(x,t)$.

x5A.7. Let A be any recursive set such that $A \subsetneq \mathbb{N}$; classify in the arithmetical hierarchy the set

$$B = \{e \mid W_e \subseteq A\}.$$

Solution. For the upper bound, we compute:

$$\begin{aligned} e \in B &\iff (\forall x)[x \in W_e \Longrightarrow x \in A] \\ &\iff (\forall x)[x \notin W_e \vee x \in A],\end{aligned}$$

so that B is Π^0_1.

`Let me know of errors or better solutions.`

For the lower bound, let

$$x \in C \iff (\forall u)R(x,u)$$

be any Π^0_1 set (with the relation $R(x,u)$ recursive), let $x_0 \notin A$ (that exists from the hypothesis), and set

$$g(x,v) = \mu u[v = x_0 \;\&\; \neg R(x,u)].$$

It follows that for all x, v,

$$\begin{aligned} g(x,v)\downarrow &\iff v = x_0 \;\&\; (\exists u)\neg R(x,u) \\ &\iff v = x_0 \;\&\; x \notin C, \end{aligned}$$

so that with $\hat{g}$ a code of g and $f(x) = S^1_1(\hat{g}, x)$,

$$\begin{aligned} x \in C &\Longrightarrow W_{f(x)} = \emptyset \Longrightarrow W_{f(x)} \subseteq A, \\ x \notin C &\Longrightarrow W_{f(x)} = \{x_0\} \not\subseteq A, \end{aligned}$$

in other words,

$$x \in C \iff W_{f(x)} \subseteq A.$$

x5A.9. Prove that the graph

$$G_f(\vec{x}, w) \iff f(\vec{x}) = w$$

of a *total* function $f(\vec{x})$ is Σ^0_k if and only if it is Δ^0_k.

Solution. Since f is total, we have the equivalence

$$\neg[f(\vec{x}) = w] \iff (\exists v)[w \neq v \;\&\; f(\vec{x}) = v],$$

which shows that if the relation G_f is Σ^0_k, then its negation is also Σ^0_k.

x5A.10. A total function $f(\vec{x})$ is *limit recursive* if there exists a recursive, total function $g(m, \vec{x})$ such that

$$f(\vec{x}) = \lim_{m\to\infty} g(m, \vec{x}),$$

where the limit of a sequence of natural numbers is defined as usually,

$$\lim_{m\to\infty} a_m = w \iff (\exists k)(\forall m \geq k)[a_m = w].$$

Prove that a total $f(\vec{x})$ is limit recursive if and only if the graph G_f of $f(\vec{x})$ is Δ^0_2.

Solution. If f is limit recursive, then

$$f(\vec{x}) = w \iff (\exists n)(\forall m)[m \geq n \Longrightarrow g(m, \vec{x}) = w],$$

so that the graph G_f is Σ^0_2, so also Δ^0_2, by the preceding problem. For the converse direction, let

$$f(\vec{x}) = w \iff (\exists u)(\forall v)P(\vec{x}, u, v, w)$$

with some recursive relation $P(\vec{x}, u, v, w)$. We set

$$h(m, \vec{x}) = (\mu z \leq m)[z = \langle (z)_0, (z)_1 \rangle \;\&\; (\forall v \leq m)P(\vec{x}, (z)_0, v, (z)_1)]$$

and we Prove that

(*) $\quad [f(\vec{x}) = w \;\&\; z = \mu t[t = \langle (t)_0, w \rangle \;\&\; (\forall v)P(\vec{x}, (t)_0, v, w)]$

$$\Longrightarrow \lim_{n\to\infty} h(n,\vec{x}) = z;$$

this will complete the proof, as it implies that

$$f(\vec{x}) = \lim_{n\to\infty} (h(n,\vec{x}))_1.$$

Let w and z be as in the hypothesis of (*), so that

$$y = \langle (y)_0, (y)_1 \rangle < z \Longrightarrow (\exists v)\neg P(\vec{x}, (y)_0, v, (y)_1),$$

and for every $y < z$, let v_y be such that

$$\neg P(\vec{x}, (y)_0, v_y, (y)_1);$$

and, finally, let

$$n > z, v_0, \ldots, v_{z-1}.$$

Since $z < n$, if $h(n,\vec{x}) = y$ for some $y \neq z$, we must have $y < z$; but then $v_y < n$, so $(\exists v \leq n)\neg P(\vec{x}, (y)_0, v, (y)_1)$ which is not compatible with $h(n,\vec{x}) = y$, and so completes the proof.

`Let me know of errors or better solutions.`