Math 114C, Winter 2019, Solutions to HW #6

---

x4B.8. Prove that the set

$$K_0 = \{e \mid (\exists y)[T_1((e)_0, e, y) \ \& \ (\forall z \leq y)\neg T_1((e)_1, e, z)]\}$$

is r.e.-complete.

Solution. If we define the partial function

$$g(x,t) = (\mu y)T_1(x, x, y),$$

with code $\widehat{g}$ so that

$$\{\widehat{g}\}(x,t) = \{S^1_1(\widehat{g}, x)\}(t) = g(x,t),$$

then, by the definitions, with $f(x) = S^1_1(\widehat{g}, x)$,

$$\begin{aligned} x \in K &\implies W_{f(x)} = \mathbb{N}, \\ x \notin K &\implies W_{f(x)} = \emptyset. \end{aligned}$$

We set

$$h(x) = \langle f(x), e^* \rangle, \qquad \text{where } W_{e^*} = \emptyset,$$

and we compute,

$$\begin{aligned} h(x) \in K_0 &\implies (\exists y)[T_1(f(x), h(x), y) \ \& \ (\forall z \leq y)\neg T_1(e^*, h(x), z)] \\ &\implies h(x) \in W_{f(x)} \\ &\implies W_{f(x)} \neq \emptyset \\ &\implies x \in K, \end{aligned}$$

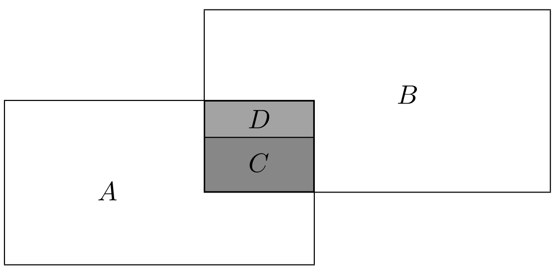
and, in the opposite case,

$$\begin{aligned} h(x) \notin K_0 &\implies (\forall y)[T_1(f(x), h(x), y) \implies (\exists z \leq y)T_1(e^*, h(x), z)] \\ &\implies h(x) \notin W_{f(x)} \qquad \text{because } W_{e^*} = \emptyset \\ &\implies W_{f(x)} \neq \mathbb{N} \\ &\implies x \notin K; \end{aligned}$$

so $K \leq_1 K_0$ and K_0 is complete.

x4B.9. (The **reduction property** for the class of r.e. sets). Prove that if A and B are r.e., then there exist r.e. sets A_1, B_1, such that

$$A_1 \subseteq A, \quad B_1 \subseteq B, \quad A_1 \cup B_1 = A \cup B, \quad A_1 \cap B_1 = \emptyset.$$



$$A_1 = (A \setminus B) \cup C, \quad B_1 = (B \setminus A) \cup D$$

Solution. If $x \in A \iff (\exists y)P(x,y)$ and $x \in B \iff (\exists z)Q(x,z)$ for suitable recursive relations $P(x,y)$ and $Q(x,z)$, we set

$$x \in A_1 \iff (\exists y)[P(x,y) \ \&\ (\forall z \le y)\neg Q(x,z)],$$
$$x \in B_1 \iff (\exists z)[Q(x,z) \ \&\ (\forall y < z)\neg P(x,y)].$$

x4B.10. Prove that if A and B are complements of some r.e. sets A^c and B^c and $A \cap B = \emptyset$, then there exists a recursive C which separates A from B, i.e.,

$$A \subseteq C, \quad C \cap B = \emptyset.$$

Hint. Apply the preceding Problem to the complements A^c and B^c.

Solution. The complements A^c and B^c are r.e., and we notice that their union is $\mathbb{N}$, since

$$A^c \cup B^c = (A \cap B)^c = \emptyset^c = \mathbb{N}.$$

By the preceding problem there exist r.e. A_1 and B_1 such that (at first)

$$A_1 \cup B_1 = A^c \cup B^c = \mathbb{N}, \quad A_1 \cap B_1 = \emptyset,$$

which implies that $B_1 = A_1^c$, and so A_1 is recursive; also

$$A_1 \subseteq A^c, \quad B_1 \subseteq B^c$$

i.e., $A \subseteq A_1^c$, $B \subseteq B_1^c = A_1$, so that A_1 separates A from B.

x4B.11. One of the two following propositions is true while the other is not. Give a proof of the true one and a counterexample of the one which is not true.

(1) If $A \subseteq B$ and A, B^c are r.e., then there exists a recursive set C such that $A \subseteq C \subseteq B$.

(2) If $A \subseteq B$ and A^c, B are r.e., then there exists a recursive set C such that $A \subseteq C \subseteq B$.

Solution. The picture becomes clear if we observe that for arbitrary $A, B, C \subseteq \mathbb{N}$,

$$A \subseteq C \subseteq B \iff [A \subseteq C \ \&\ C \cap B^c = \emptyset],$$

namely, in both parts, the question is if there exists a recursive C which separates A from B^c. So we can formulate in another way the two parts of the problem:

(1) *For any r.e. A, B^c, if $A \cap B^c = \emptyset$, then there exists a recursive set C which separates A from B^c.* This does not always hold, by Theorem 4B.12.

(2) *For any r.e. complements A, B^c, if $A \cap B^c = \emptyset$, then there exists a recursive set C which separates A from B^c.* This holds by Problem x4B.10.

x2D.3. Prove that the only total solution of the equation $(*)$ in Problem x1C.13* is recursive, but the equation $(*)$ also has solutions which are not recursive partial functions.

Solution. The total, constant function $\mathsf{p}(0,y) = 1$ obviously satisfies the equation $(*)$, and (by an easy induction on x), for every total function $\mathsf{p}'(x,y)$ which satisfies the equation $(*)$, $\mathsf{p}'(x,y) = 1$, for all x.

In order to find non-recursive, partial solutions of the equation $(*)$, let $g(y)$ be any partial function such that

$$(**) \qquad g(y)\!\downarrow \implies g(y) = 1;$$

We set

$$\mathsf{p}_g(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ g(y), & \text{if } x = 1, \\ \uparrow, & \text{if } x \geq 2, \end{cases}$$

and we verify easily, by considering cases for x that the function $\mathsf{p}_g(x, y)$ satisfies the equation $(*)$. To complete the proof, we have to use the fact that *there exist partial functions $g(y)$ which satisfy* (**) *and which are not recursive*; we will prove this fact in the next chapter, but we can see it also with some set theory: because there exist uncountably many functions which satisfy (**) (as many as the sets of natural numbers), while the set of recursive partial functions is countable.