# Definability theory

Once we know that there are functions on $\mathbb{N}$ which are not computable, the subject splits into two:

- The theory of computable functions, comprising
  the theory of algorithms, and complexity theory
- definability theory, which can be viewed as
  complexity theory for non-computable functions (and relations)

• In definability theory we study the special (regularity) properties of functions and relations which can be defined in various ways, and structure properties of sets of definable functions.

**Prim** = the set of all primitive recursive functions and relations (on $\mathbb{N}$)
**Rec** = the set of all recursive functions

• **Prim** and **Rec** are closed under substitutions, negations, conjunctions and bounded quantification $(\exists i \le t)$

…but **Prim** $\subsetneq$ **Rec** (the Ackermann function)

# More notations

Variations on the notation: with $\vec{x} = (x_1, \ldots, x_n)$,

$$\{e\}(\vec{x}) = \varphi_e(\vec{x}) = \varphi_e^n(\vec{x}) = U(\mu y T_n(e, \vec{x}, y))$$

- The notation $\{e\}(\vec{x})$ puts the *program* $e$ on the same level as the *data* $\vec{x}$ and gives in which the $S_n^m$-Theorem takes the form

$$\{e\}(\vec{z}, \vec{x}) = \{S_n^m(e, \vec{z})\}(\vec{x}) \qquad [\vec{z} = (z_1, \ldots, z_m), \vec{x} = (x_1, \ldots, x_n)]$$

$$W_e = \{x \mid \{e\}(x)\downarrow\} = \text{ the domain of convergence of } \varphi_e^1$$

# Operations on total functions and relations

| | | |
|---|---|---|
| $(\neg)$ | $P(\vec{x}) \iff \neg Q(\vec{x})$ | (negation) |
| $(\&)$ | $P(\vec{x}) \iff Q(\vec{x}) \,\&\, R(\vec{x})$ | (conjunction) |
| $(\vee)$ | $P(\vec{x}) \iff Q(\vec{x}) \vee R(\vec{x})$ | (disjunction) |
| $(\Rightarrow)$ | $P(\vec{x}) \iff Q(\vec{x}) \implies R(\vec{x})$ | (implication) |
| $(\exists)$ | $P(\vec{x}) \iff (\exists y)Q(\vec{x}, y)$ | (existential quantification) |
| $(\exists_{\leq})$ | $P(z, \vec{x}) \iff (\exists i \leq z)Q(\vec{x}, i)$ | (bounded ex. quant.) |
| $(\forall)$ | $P(\vec{x}) \iff (\forall y)Q(\vec{x}, y)$ | (universal quantification) |
| $(\forall_{\leq})$ | $P(z, \vec{x}) \iff (\forall i \leq z)Q(\vec{x}, i)$ | (bounded univ. quant.) |
| (substitution) | $P(\vec{x}) \iff Q(f_1(\vec{x}), \dots, f_m(\vec{x}))$ | |

- **Prim** *and* **Rec** *are closed under all these operations except for* $(\exists)$ *and* $(\forall)$

(with total *prim substitution* for **Prim** and *recursive substitution* for **Rec**)

- *Neither of these sets of functions and relations is closed under* $(\exists)$ *or* $(\forall)$

# Semirecursive and $\Sigma^0_1$

(1) A relation $P(\vec{x})$ is **semirecursive** if for some recursive partial function $f(\vec{x})$,

$$P(\vec{x}) \iff f(\vec{x})\downarrow \iff \{e\}(\vec{x})\downarrow \text{ for some } e$$

(2) A relation $P(\vec{x})$ is $\Sigma^0_1$ if for some recursive relation $Q(\vec{x}, y)$

$$P(\vec{x}) \iff (\exists y) Q(\vec{x}, y).$$

- *The following are equivalent, for any relation $P(\vec{x})$:*

(1) $P(\vec{x})$ *is semirecursive.*
(2) $P(\vec{x})$ *is $\Sigma^0_1$.*
(3) $P(\vec{x})$ *satisfies an equivalence*

$$P(\vec{x}) \iff (\exists y) Q(\vec{x}, y)$$

*with some primitive recursive $Q(\vec{x}, y)$*

**Proof**. (1) $\Longrightarrow$ (3) by the normal form theorem; (3) $\Longrightarrow$ (2) trivially; and (2) $\Longrightarrow$ (1) setting $f(\vec{x}) = \mu y Q(\vec{x}, y)$, so that

$$(\exists y) Q(\vec{x}, y) \iff f(\vec{x})\downarrow \quad \square$$

# $\Sigma_1^0 \cap \neg\Sigma_1^0 = \mathbf{Rec}$

- The Halting problem $H(e, x)$ is semirecursive but not recursive.

- **Kleene's Theorem**. *A relation $P(\vec{x})$ is recursive if and only if both $P(\vec{x})$ and its negation $\neg P(\vec{x})$ are semirecursive.*

**Proof**. $P(\vec{x}) \iff (\exists y)P(\vec{x})$, $\neg P(\vec{x}) \iff (\exists y)\neg P(\vec{x})$ by "vacuous quantification", so if $P(\vec{x})$ is recursive, then both $P(\vec{x})$ and $\neg P(\vec{x})$ are semirecursive.

In the other direction, if

$$P(\vec{x}) \iff (\exists y)Q(\vec{x}, y), \quad \neg P(\vec{x}) \iff (\exists y)R(\vec{x}, y)$$

with recursive relations $Q$ and $R$, then the function

$$f(\vec{x}) = \mu y[R(\vec{x}, y) \vee Q(\vec{x}, y)]$$

is recursive, total, and $P(\vec{x}) \iff Q(\vec{x}, f(\vec{x}))$ $\square$

# Closure properties of $\Sigma^0_1$

• $\Sigma^0_1$ *is closed under recursive substitutions, under the "positive" propositional operators* $\&, \vee$, *under the bounded quantifiers* $\exists_\leq$, $\forall_\leq$, *and under the existential quantifier* $\exists$

**Proof**. For closure under recursive substitution notice that if $P(\vec{x}) \iff Q(f_1(\vec{x}), f_2(\vec{x}))$, then

$$\chi_P(\vec{x}) = \chi_Q(f_1(\vec{x}), f_2(\vec{x})).$$

For the rest use the equivalences

$$\begin{aligned}
(\exists y)Q(\vec{x}, y) \vee (\exists y)R(\vec{x}, y) &\iff (\exists u)[Q(\vec{x}, u) \vee R(\vec{x}, u)] \\
(\exists y)Q(\vec{x}, y) \ \&\ (\exists y)R(\vec{x}, y) &\iff (\exists u)[Q(\vec{x}, (u)_0) \ \&\ R(\vec{x}, (u)_1)] \\
(\exists z)(\exists y)Q(\vec{x}, y, z) &\iff (\exists u)R(\vec{x}, (u)_0, (u)_1) \\
(\exists i \leq z)(\exists y)Q(\vec{x}, y, i) &\iff (\exists u)[(u)_0 \leq z \ \&\ Q(\vec{x}, (u)_1, (u)_0) \\
(\forall i \leq z)(\exists y)Q(\vec{x}, y, i) &\iff (\exists u)(\forall i \leq z)Q(\vec{x}, (u)_i, i). \quad \square
\end{aligned}$$

• $\Sigma^0_1$ *is not closed under negation or the universal quantifier*, since otherwise the basic halting relation

$$H(e,x) \iff (\exists y)T_1(e,x,y)$$

would have a semirecursive negation and so it would be recursive by Kleene's Theorem.

# The Graph Lemma

The **graph** of a partial function $f(\vec{x})$ is the relation

$$G_f(\vec{x}, w) \iff f(\vec{x}) = w,$$

• **Graph Lemma** *A partial function $f(\vec{x})$ is recursive if and only if its graph $G_f(\vec{x}, w)$ is a semirecursive relation.*

**Proof** If $f(\vec{x})$ is recursive with code $\widehat{f}$, then

$$G_f(\vec{x}, w) \iff (\exists y)[T_n(\widehat{f}, \vec{x}, y) \ \& \ U(y) = w],$$

so that $G_f(\vec{x}, w)$ is semirecursive; and if

$$f(\vec{x}) = w \iff (\exists u) R(\vec{x}, w, u)$$

with some recursive $R(\vec{x}, w, u)$, then

$$f(\vec{x}) = \Big(\mu u R(\vec{x}, (u)_0, (u)_1)\Big)_0,$$

so that $f(\vec{x})$ is recursive. $\square$

# The $\Sigma^0_1$-Selection Lemma

• **$\Sigma^0_1$-Selection Lemma** *For every semirecursive relation $R(\vec{x}, w)$, there is a recursive partial function $f(\vec{x})$ such that for all $\vec{x}$,*

$$\begin{aligned} (\exists w)R(\vec{x}, w) &\iff f(\vec{x})\downarrow \\ (\exists w)R(\vec{x}, w) &\implies R(\vec{x}, f(\vec{x})). \end{aligned}$$

**Proof**. By the hypothesis, there exists a recursive relation $P(\vec{x}, w, y)$ such that

$$R(\vec{x}, w) \iff (\exists y)P(\vec{x}, w, y),$$

and the conclusion of the Lemma follows easily if we set

$$f(\vec{x}) = \Big(\mu u P(\vec{x}, (u)_0, (u)_1)\Big)_0. \quad \square$$

# The aim to study and classify definable relations by the complexity of their definitions

- *The following are equivalent, for any relation $P(\vec{x})$:*

(1) *$P(\vec{x})$ is semirecursive, i.e.,*

$$P(x) \iff f(x)\downarrow$$

*with some recursive $f : \mathbb{N}^n \rightharpoonup \mathbb{N}$.*

(2) *$P(\vec{x})$ is $\Sigma^0_1$, i.e.,*

$$P(\vec{x}) \iff (\exists y)[R(\vec{x}, y)]$$

*with sone recursive $R(\vec{x}, y)$*

We also proved several *closure properties* of these sets of relations

The proofs in this lecture (and later) depend heavily on these closure properties and illustrate how closure properties of sets of relations can be used

# Recursively enumerable sets

A set $A \subseteq \mathbb{N}$ is **recursively enumerable** (r.e.), if $A = \emptyset$ or some recursive total function $f : \mathbb{N} \to \mathbb{N}$ enumerates $A$,

$$A = f[\mathbb{N}] = \{f(0), f(1), \ldots\}.$$

- *The following are equivalent for any $A \subseteq \mathbb{N}$:*

(a) *$A$ is r.e.*

(b) *The relation $x \in A$ is semirecursive (equivalently $\Sigma^0_1$)*

(c) *$A$ is finite, or there is a recursive injection $f : \mathbb{N} \rightarrowtail \mathbb{N}$ which enumerates $A = f[\mathbb{N}]$ without repetitions.*

**Proof**. (a), (b), (c) are all true if $A$ is empty, so assume $A$ is not empty.
(a) $\Longrightarrow$ (b): If $A = f[\mathbb{N}]$, then $x \in A \iff (\exists i)[f(i) = x]$, so $A$ is $\Sigma^0_1$
(b) $\Longrightarrow$ (a): if $x_0 \in A$ and $x \in A \iff (\exists y)R(x, y)$, then $A$ is enumerated by the recursive total function

$$f(u) = \text{if } (R(u)_0, (u)_1) \text{ then } (u)_0 \text{ else } x_0$$

(c) $\Longrightarrow$ (a) is trivial. This leaves (a) $\Longrightarrow$ (c) for infinite $A$.

# Infinite r.e. sets can be enumerated without repetitions

- *If $A$ is infinite and $A = f[\mathbb{N}]$ with a recursive $f : \mathbb{N} \to \mathbb{N}$, then $A = g[\mathbb{N}]$ with a recursive injection $g : \mathbb{N} \rightarrowtail \mathbb{N}$*

**Proof**. The hypothesis gives a total, recursive $f$ such that

$$A = \{f(0), f(1), \dots\},$$

and the obvious idea is to *delete all the repetitions* in this enumeration of $A$, which will leave an enumeration of $A$ without repetitions. In full detail, let

$$B = \{n \mid (\forall i < n)[f(n) \neq f(i)]\},$$

the (infinite) set of positions where $f(n)$ puts a new element in $A$. Now $n \in B \iff (\forall i < n)[f(n) \neq f(i)]$, so $B$ is recursive and it is enumerated by $h : \mathbb{N} \rightarrowtail \mathbb{N}$ defined by the primitive recursion

$$h(0) = f(0), \quad h(n+1) = \mu i[i > n \;\&\; i \in B]$$

Finally, the composition $g(n) = f(h(n))$ is a recursive injection and enumerates $A$ without repetitions $\square$

# Enumerations of recursive sets

- *An infinite set $A$ is recursive if and only if it can be enumerated by an increasing recursive function $f$, i.e.,*

$$A = \{f(0) < f(1) < f(2) < \cdots\}$$

**Proof**. Suppose $A$ is infinite and recursive and define $f$ by

$$f(0) = \mu i[i \in A], \quad f(n+1) = \mu i[i > f(n) \;\&\; i \in A].$$

This is total (because $A$ is infinite); recursive (because $A$ is recursive); increasing (obviously); and it clearly enumerates $A$.

For the converse we need the simple fact that

$$\text{if } f : \mathbb{N} \rightarrowtail \mathbb{N} \text{ is increasing, then } (\forall n)[n \leq f(n)],$$

easily proved by induction; it implies that if $A$ is enumerated by an increasing $f$, then

$$n \in A \iff (\exists i)[n = f(i)] \iff (\exists i \leq n)[n = f(i)],$$

so $A$ is recursive. $\square$

# Comparing the complexity of arbitrary subsets of $\mathbb{N}$

A **reduction** of $A$ to $B$ is any total function $f : \mathbb{N} \to \mathbb{N}$ such that

$$x \in A \iff f(x) \in B \quad (x \in \mathbb{N})$$

$$\begin{aligned}
A \leq_m B &\iff A \text{ is } \textit{many-one reducible to } B \\
&\iff \text{ there is a recursive reduction of } A \text{ to } B \\
A \leq_1 B &\iff A \text{ is } \textit{one-one reducible} \text{ to } B \\
&\iff \text{ there is a recursive, injective reduction of } A \text{ to } B \\
A \equiv B &\iff A \text{ is } \textit{recursively equivalent to } B \\
&\iff \text{ there is a recursive permutation } f : \mathbb{N} \rightarrowtail\!\!\!\!\rightarrow \mathbb{N} \\
&\qquad\qquad\qquad\qquad \text{which reduces } A \text{ to } B
\end{aligned}$$

- $A \equiv B \implies A \leq_1 B \implies A \leq_m B$
- $A \leq_m A$, $[A \leq_m B \;\&\; B \leq_m C] \implies A \leq_m C]$ *and similarly with* $\leq_1$
- $\equiv$ *is an equivalence relation on the subsets of* $\mathbb{N}$

# Complete r.e. sets

The structure of r.e. sets has been studied intensively since the 30's—and is still an active area of research today

Notation: $W_e = \{x \mid \{e\}(x) = \varphi_e(x)\downarrow\}$,
so $W_0, W_1, \dots$ enumerates all r.e. sets

**Def**. A set $B$ is **r.e. complete** if it is r.e. and every r.e. set $A$ is one-one reducible to it, $A \leq_1 B$

- *The set* $H' = \{x \mid \{(x)_0\}((x)_1))\downarrow\}\}$ *is r.e. complete*

**Proof**. $H'$ is r.e., because

$$x \in H' \iff \mu y T_1((x)_0, (x)_1, y)\downarrow,$$

so $H'$ is the domain of convergence of a recursive partial function
It is r.e. complete, because for any, fixed $e$,

$$x \in W_e \iff \{e\}(x)\downarrow \iff \langle e, x\rangle \in H'$$

and the map $x \mapsto \langle e, x\rangle$ is recursive and one-to-one

# Emil Post's r.e. complete set $K$

**Def**. $K = \{x \mid x \in W_x\} = \{x \mid \{x\}(x)\downarrow\} = \{x \mid (\exists y) T_1(x, x, y)\}$

- $K$ *r.e.* (because the relation $x \in K$ is $\Sigma^0_1$)

- *$K$ is r.e. complete*

**Proof**. If $A$ is r.e., then there is a recursive $g : \mathbb{N} \rightharpoonup \mathbb{N}$ such that

$$x \in A \iff g(x)\downarrow \iff h(x, y)\downarrow$$

where $h(x, y) = g(x)$ is also recursive. Choose a code $\widehat{h}$ of $h$, and compu

$$x \in A \iff h(x, y)\downarrow \iff \{\widehat{h}\}(x, y)\downarrow \iff \{S^1_1(\widehat{h}, x)\}(y)\downarrow$$

This equivalence holds for every $y$, so in particular

$$x \in A \iff \{S^1_1(\widehat{h}, x)\}(S^1_1(\widehat{h}, x))\downarrow \iff S^1_1(\widehat{h}, x) \in K$$

and the function $x \mapsto S^1_1(\widehat{h}, x)$ is (primitive) recursive and injective □

This is the first of many important applications of the $S^m_n$ functions

# Proofs of undecidability using $K$

- *If $K \leq_m A$, then $A$ is not recursive* — because $K \leq_m A$ with a recursive $A$ implies that every r.e. set is recursive, which is not true
- *The set $A = \{e \mid W_e \neq \emptyset\}$ is r.e. but not recursive.*

**Proof**. $e \in A \iff (\exists x)[x \in W_e]$, so $A$ is r.e.

To show that $K \leq_1 A$, we set

$$g(e, x) = \mu y T_1(e, e, y)$$

so that the value $g(e, x)$ is independent of $x$ and

$$e \in K \iff g(e, x)\downarrow \qquad (\text{any } x);$$

so if $\widehat{g}$ is a code of $g(x, y)$, then

$$e \in K \iff (\exists x)[\{\widehat{g}\}(e, x)\downarrow] \iff (\exists x)[\{S^1_1(\widehat{g}, e)\}(x)\downarrow]$$
$$\iff W_{S^1_1(\widehat{g}, e)} \neq \emptyset \iff S^1_1(\widehat{g}, e) \in A \quad \square$$

# Myhill's Theorem

- *For any two* (arbitrary) *sets* $A, B \subseteq \mathbb{N}$

$$\Big(A \le_1 B \ \& \ B \le_1 A\Big) \implies A \equiv B.$$

**Proof**. We are given two injections $f, g : \mathbb{N} \rightarrowtail \mathbb{N}$ and we must produce a permutation $h : \mathbb{N} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathbb{N}$ such that for any two $A, B$,

$$(*) \quad \Big((\forall x)[x \in A \iff f(x) \in B] \ \& \ (\forall y)[y \in B \iff g(y) \in A]\Big)$$
$$\implies (\forall x)[x \in A \iff h(x) \in B]$$

The difficulty in the proof stems from the fact **we must construct** $h$ **using only** $f$ **and** $g$, since we know nothing about the arbitrary sets $A$ and $B$ other than $(*)$

# Proof of Myhill's Theorem

We are given injections $f : \mathbb{N} \rightarrowtail \mathbb{N}$ and $g : \mathbb{N} \rightarrowtail \mathbb{N}$

**Def** A finite sequence of pairs $\boxed{W = (x_0, y_0), \ldots, (x_n, y_n)}$ is **injective** if

$$i \neq j \implies [x_i \neq x_j \ \& \ y_i \neq y_j] \quad (i, j \leq n)$$

$W$ is **good for** $A, B$ if in addition $\boxed{x_i \in A \iff y_i \in B \quad (i \leq n)}$

We think of an injective sequence which is good for $A, B$ as a finite approximation of a permutation $h : \mathbb{N} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathbb{N}$ which reduces $A$ to $B$

and the construction of the required $h$ will follows from two Lemmas which allow us to extend injective sequences so that "goodness" for any $A, B$ is preserved

## Lemma X: Given $f : \mathbb{N} \rightarrowtail \mathbb{N}$, $x \in A \iff f(x) \in B$

- *From any injective sequence* $\boxed{W = (x_0, y_0), \dots, (x_n, y_n)}$ *and any* $x \notin \{x_0, \dots, x_n\}$, *we can compute a number* $y$ *such that*:

(X1) *The extension* $\boxed{W' = (x_0, y_0), \dots, (x_n, y_n), (x, y)}$ *is injective*, and

(X2) *For any* $A, B$, *if* $W$ *is good for* $A, B$, *then* $W'$ *is also good for* $A, B$

**Proof**. Follow the following procedure with

$$X = \{x_0, \dots, x_n\}, \quad Y = \{y_0, \dots, y_n\}$$

(1) If $f(x) \notin Y$, set $y := f(x)$ and check (X1), (X2)

(2) If for some $i$, $f(x) = y_i$ and $f(x_i) \notin Y$, set $y := f(x_i)$ and check (X1), (X2). For (X2):

$$\boxed{x \in A \iff f(x) \in B \iff y_i \in B \iff x_i \in A \iff f(x_i) = y \in B}$$

(3) If for some $i \neq j$, $f(x) = y_i$, $f(x_i) = y_j$ and $f(x_j) \notin Y$, set $y := f(x_j)$

$\dots$

The process eventually stops and yields a $y$ which works because $X \cup \{x\}$ has one more element than $Y$ (Pigeonhole Principle) $\square$

## Given $f, g : \mathbb{N} \rightarrowtail \mathbb{N}$, Good for $A, B$:

$$x \in A \iff f(x) \in B, \quad x \in B \iff g(x) \in A$$

- (Lemma X) *From any injective sequence*

$\boxed{W = (x_0, y_0), \ldots, (x_n, y_n)}$ *and any $x \notin \{x_0, \ldots, x_n\}$, compute $y$ s.t.*

(X1) *The extension* $\boxed{W' = (x_0, y_0), \ldots, (x_n, y_n), (x, y)}$ *is injective*, and

(X2) *For any $A, B$, if $W$ is good for $A, B$, then $W'$ is also good for $A, B$*

- (Lemma Y) *From any injective sequence*

$\boxed{W = (x_0, y_0), \ldots, (x_n, y_n)}$ *and any $y \notin \{y_0, \ldots, y_n\}$, compute $x$ s.t.*

(Y1) *The extension* $\boxed{W' = (x_0, y_0), \ldots, (x_n, y_n), (x, y)}$ *is injective*, and

(Y2) *For any $A, B$, if $W$ is good for $A, B$, then $x \in A \iff y \in B$*

**To finish the proof of Myhill's Theorem**: Start with $(x_0, y_0) = (0, f(0))$; use Lemma Y on $y_1 = \mu i \notin \{y_0\}$ to get $(x_0, y_0), (x_1, y_1)$; use Lemma $X$ on $x_2 = \mu i \notin \{x_0, x_1\}$ to get $(x_0, y_0), (x_1, y_1), (x_2, y_2)$; etc, to construct $h : \mathbb{N} \rightarrowtail\!\!\!\!\rightarrow \mathbb{N}$ $\square$

# Turing reducibility

**Def** $A \leq_T B \iff \chi_A$ is recursive in $(\mathbf{N}_0, \chi_B) = (\mathbb{N}, 0, 1, S, Pd, \chi_B)$

- $A \leq_m B \implies A \leq_T B$
- $A \leq_T (\mathbb{N} \setminus A)$ but $\mathbb{N} \not\leq_m \emptyset$

so $\leq_T$ is the weakest of the four reducibilities introduced by Post

Post proved (and we will show) that *there exist r.e. sets which are neither recursive nor r.e. complete*, but he could not answer the corresponding question of *intermediate Turing complexity for r.e. sets* he posed this problem in the following, stronger form:

**Post's Problem** (1944) *Are there r.e. sets $A$ and $B$ such that $A \not\leq_T B$ and $B \not\leq_T A$?*

Post's Problem was not solved until 1956 by Richard Friedberg and Albert Muchnik who proved (independently) that such Turing-incomparable r.e. sets exist using the same *priority method*

The study or Turing reducibility (on r.e. and larger classes of sets) is still a very active research area—which, however, we will not pursue

# Productive and creative sets

**Def**. A **productive function** for a set $B \subset \mathbb{N}$ is a recursive injection $p : \mathbb{N} \rightarrowtail \mathbb{N}$ such that

$$W_e \subseteq B \implies p(e) \in (B \setminus W_e)$$

and $B$ is **productive** if it has a productive function

**Def.** A set $A$ is **creative** if it is r.e. and its complement $A^c = \{x \in \mathbb{N} \mid x \notin A\}$ is productive

– Intuitively, an r.e. set is creative if it is *effectively not recursive*

- (Post) *The set $K$ is creative*

**Proof**. The defs give $t \in W_e \iff \{e\}(t)\downarrow$ and $e \in K \iff \{e\}(e)\downarrow$; and so to prove that the identity $p(e) = e$ is a productive function for $K^c$ we must check the trivial

$$\Big((\forall t)[\{e\}(t))\downarrow \implies \{t\}(t)\uparrow]\Big) \implies \{e\}(e)\uparrow \quad \square$$

- *Every r.e. complete set is creative* (Will be assigned)

- *Every productive $B$ set has an infinite r.e. subset*

**Lemma**. *There is a recursive injection $u(e, y)$ such that*

$$W_{u(e,y)} = W_e \cup \{y\} \quad (e, y \in \mathbb{N})$$

**Proof**. The relation $x \in W_u \vee x = y$ is $\Sigma^0_1$, so there is a $\widehat{g}$ such that

$$\begin{aligned} x \in W_e \cup \{y\} &\iff \{\widehat{g}\}(e, y, x)\downarrow \\ &\iff \{S^2_1(\widehat{g}, e, y)\}(x)\downarrow \iff x \in W_{u(e,y)} \end{aligned}$$

with $u(e, y) = S^2_1(\widehat{g}, e, y)$. $\square$(Lemma)

To prove the result in the caption, fix $e_0$ such that $W_{e_0} = \emptyset$, let $p(e)$ be a productive function for $B$, and define by primitive recursion

$$f(0) = e_0, \quad f(x+1) = u(f(x), p(f(x)));$$

Now $W_{f(0)} \subsetneq W_{f(1)} \subsetneq W_{f(1)} \cdots B$ and the set we need is $\bigcup_x W_{f(x)}$ $\square$

# Simple sets

**Def** (Post). A set $A$ is **simple** if it is r.e., its complement $A^c$ is infinite and $A^c$ has no infinite r.e. subsets, i.e.,

$$A \text{ is r.e., } A^c \text{ is infinite, and } (\forall e)[W_e \cap A = \emptyset \implies W_e \text{ is finite}].$$

- (Post) *There exists a simple set*

**Proof**. Let $\boxed{R(x,y) \iff y \in W_x \ \&\ y > 2x}$

This is $\Sigma^0_1$, so by $\Sigma^0_1$-Selection there is a recursive $f : \mathbb{N} \rightharpoonup \mathbb{N}$ s.t.

$$(*) \quad (\exists y)[y \in W_x \ \&\ y > 2x] \iff f(x)\downarrow,$$
$$\text{and } f(x)\downarrow \implies f(x) \in W_x \ \&\ f(x) > 2x$$

We set $\boxed{A = \{f(x) \mid f(x)\downarrow\}}$ and we need to prove that $A$ is simple

(1) *$A$ is r.e.*, because $y \in A \iff (\exists x)[f(x) = y]$

# Continuing the proof that there exists a simple set

We have proved that there is a recursive $f : \mathbb{N} \rightharpoonup \mathbb{N}$ such that

$$(*) \quad (\exists y)[y \in W_x \ \& \ y > 2x] \iff f(x)\downarrow, \\ \text{and } f(x)\downarrow \implies f(x) \in W_x \ \& \ f(x) > 2x,$$

set $\boxed{A = \{f(x) \mid f(x)\downarrow\}}$ and checked that $A$ is r.e.

(2) $A^c$ *is infinite*, because for any $z$,

$$y \in A \ \& \ y \leq z \implies (\exists x)[y = f(x) \ \& \ 2x < y \leq 2z] \\ \implies (\exists x)[y = f(x) \ \& \ x < z];$$

which means that from the $2z+1$ numbers $\leq 2z$ at most $z$ belong to $A$; which means that for every $z$, sone $y \geq z$ belongs to $A^c$ so $A^c$ is infinite.

# Finishing the proof that there exists a simple set

We have proved that there is a recursive $f : \mathbb{N} \rightharpoonup \mathbb{N}$ such that

$$(*) \quad (\exists y)[y \in W_x \ \&\ y > 2x] \iff f(x)\downarrow,$$
$$\text{and } f(x)\downarrow \implies f(x) \in W_x \ \&\ f(x) > 2x,$$

set $\boxed{A = \{f(x) \mid f(x)\downarrow\}}$ and proved that $A$ is r.e. and its complement $A^c$ is infinite. It remains to prove

(3) *For any r.e. set $W_e$,*

$$W_e \textit{ is infinite} \implies (\exists y)[y \in W_e \ \&\ y > 2e]$$
$$\implies f(e)\downarrow \ \&\ f(e) \in W_e \implies f(e) \in W_e \cap A,$$

*so there is no infinite, r.e. subset of $A^c$* $\square$

• (Post) A simple set $A$ is r.e., but neither recursive nor r.e. complete

Recap: we have defined creative sets and proved

(1) *Every r.e. complete set is creative*

(2) *If $B$ is creative, then its complement $B^c$ has an infinite, r.e. subset*

By def, a simple set $A$ is r.e. and its complement $A^c$ is infinite but has no infinite r.e. subsets

• *$A$ is not recursive*; because if it were, then its infinite complement $A^c$ would be recursive and hence an r.e. infinite subset of itself

• *$A$ is not r.e complete*, by (1) and (2)

# The Second Recursion Theorem

- (Kleene) *For every recursive partial function $f(z, \vec{x})$, there is a number $z^*$ such that for all $\vec{x}$*

$$\text{(RT2)} \qquad \phi_{z^*}(\vec{x}) = \{z^*\}(\vec{x}) = f(z^*, \vec{x})$$

- *There is a number $z^*$ such that*

$$\phi_{z^*}(x) = z^* \quad (x \in \mathbb{N})$$

The function $\phi_{z^*} : \mathbb{N} \to \mathbb{N}$ is total, constant, and assigns to every number $x$ *a code of itself*.

Using this, you can produce a recursive program $E$ on an algebra of strings of symbols which on the empty input outputs itself, i.e.,

$$(E) \qquad p_0 \;:\; \to \cdots \to : \; E$$

It is by no means obvious how todo this!

- *There exist $z_1, z_2$ such that $W_{z_1} = \{z_1\}, \quad W_{z_2} = \{0, 1, \ldots, z_2\}$*

# Proof of the Second Recursion Theorem

- (Kleene) *For every recursive partial function $f(z, \vec{x})$, there is a number $z^*$ such that for all $\vec{x}$*

$$\text{(RT2)} \qquad f(z^*, \vec{x}) = \{z^*\}(\vec{x}) \quad (= \phi_{z^*}(\vec{x})).$$

**Proof**. Given $f(z, \vec{x})$ with $\vec{x} = (x_1, \dots, x_n)$, define the recursive partial function

$$g(z, \vec{x}) = f(S^1_n(z, z), \vec{x})$$

and let $\widehat{g}$ be a code of it, so that

$$f(S^1_n(z, z), \vec{x}) = \{\widehat{g}\}(z, \vec{x}) = \{S^1_n(\widehat{g}, z)\}(\vec{x}) \quad \text{(for all } z, \vec{x});$$

and if we set $z := \widehat{g}$ in this equation, we get

$$f(S^1_n(\widehat{g}, \widehat{g}), \vec{x}) = \{S^1_n(\widehat{g}, \widehat{g})\}(\vec{x})$$

which is the desired result with $z^* = S^1_n(\widehat{g}, \widehat{g})$ $\square$