Math 273b: Calculus of Variations

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Homework #3

[1] Consider the 1D length functional minimization problem

\[ \min_u F(u) = \int_0^1 L(u'(x)) \, dx, \text{ or } \min_u \int_0^1 \sqrt{1 + (u'(x))^2} \, dx, \]

for twice differentiable functions \( u : [0, 1] \to \mathbb{R} \) with boundary conditions \( u(0) = 0, \)
\( u(1) = 1. \)

(a) Show that the functional \( u \mapsto F(u) \) is convex.

(b) Formally compute the Gateaux-differential and then obtain the Euler-Lagrange equation associated with the minimization.

(c) Find the exact solution of the problem.

**Solution:** (a) To prove convexity of \( F \), first prove it for \( L \). Start by differentiating \( L(x) \) twice. We get

\[
\frac{d^2}{dx^2} \sqrt{1+x^2} = \frac{d}{dx} \frac{1}{2} (1+x^2)^{-\frac{3}{2}} 2x \\
= \frac{d}{dx} (1+x^2)^{-\frac{3}{2}} x \\
= (1+x^2)^{-\frac{1}{2}} + x \left(-\frac{1}{2} (1+x^2)^{-\frac{3}{2}} 2x \right) \\
= (1+x^2)^{-\frac{1}{2}} - x^2 (1+x^2)^{-\frac{3}{2}} \\
= (1+x^2)^{-\frac{1}{2}} \left(1 - x^2 (1+x^2)^{-1} \right) \\
= (1+x^2)^{-\frac{1}{2}} \left(1 - \frac{x^2}{(1+x^2)} \right) \\
= (1+x^2)^{-\frac{1}{2}} \left(\frac{1}{(1+x^2)} \right) > 0.
\]
This proves that $L$ is convex and so, $F(\lambda u + (1 - \lambda)v) = \int_0^1 L(\lambda u'(x) + (1 - \lambda)v'(x))dx \leq \int_0^1 \lambda L(u'(x)) + (1 - \lambda)L(v'(x))dx = \lambda F(u) + (1 - \lambda)F(v)$. So $F$ is convex.

(b) To compute the Gateaux derivative, consider an arbitrary smooth $v$ such that $v(0) = v(1) = 0$ and an arbitrary $\epsilon > 0$. We see we get

$$\lim_{\epsilon \to 0} \frac{F(u + \epsilon v) - F(u)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\int_0^1 \sqrt{1 + (u' + \epsilon v')^2} - \int_0^1 \sqrt{1 + (u')^2}}{\epsilon}$$

$$= \lim_{\epsilon \to 0} \int_0^1 \left(\sqrt{1 + (u' + \epsilon v')^2} - \sqrt{1 + (u')^2}\right) \frac{v'}{\epsilon v'}$$

$$= \int_0^1 \frac{u'v'}{\sqrt{1 + (u')^2}}$$

$$= u'v|_0^1 - \int_0^1 \frac{d}{dx} \left(\frac{u'}{\sqrt{1 + (u')^2}}\right) v.$$

As $v$ vanishes at boundaries, we are left with $-\int_0^1 \frac{d}{dx} \left(\frac{u'}{\sqrt{1 + (u')^2}}\right) v$. Since we are minimizing, the Gateaux derivative must vanish for all $v$, that is $\int_0^1 \frac{d}{dx} \left(\frac{u'}{\sqrt{1 + (u')^2}}\right) v = 0$ for all functions $v$ which implies $\frac{d}{dx} \left(\frac{u'}{\sqrt{1 + (u')^2}}\right) = 0$. Differentiating leads to $\frac{u''}{(1+(u')^2)^{3/2}}$ so we get $u'' = 0$. This is the Euler Lagrange Equation for this problem.

(c) Since the second derivative vanishes, the function must be linear, and since only one line passes through (0,0) and (1,1), we get that the unique solution to the 1D length minimization is $u(x) = x$.

[2] Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear self-adjoint operator, $b \in \mathbb{R}^n$, and consider the quadratic function for $x \in \mathbb{R}^n$

$$x \mapsto q(x) := \langle Ax, x \rangle - 2\langle b, x \rangle.$$

Show that the three statements

(i) $\inf \{ q(x) : x \in \mathbb{R}^n \} > -\infty$

(ii) $A \geq O$ and $b \in \text{Im}A$. 

(iii) the problem \( \inf \{ q(x) : x \in \mathbb{R}^n \} > -\infty \) has a solution are equivalent. When they hold, characterize the set of minimum points of \( q \), in terms of the pseudo-inverse of \( A \).

**Solution:** (i) \( \Rightarrow \) (ii) We do this by contrapositive, suppose that \( A \) is not positive semi-definite, then \( A \) has a negative eigenvalue \( c < 0 \) with some fixed eigenvector \( x' \), i.e. \( Ax = cx \). Consider \( x_\alpha = \alpha x' \). Now look at \( q(x_\alpha) = \langle Ax_\alpha, x_\alpha \rangle - 2\langle b, x_\alpha \rangle = x'_TAx_\alpha - 2b^T x_\alpha = c\|x_\alpha\|^2 - 2b^Tx_\alpha = ca^2\|x'\|^2 - \alpha b^T x' = \). As \( c\|x'\|^2 \) and \( -b^T x' \) are fixed, call these \( d_1 < 0 \) and \( d_2 \), we can make \( q(x_\alpha) = d_1 \alpha^2 + d_2 \alpha \) as negative as we want as it is quadratic in \( \alpha \) with negative leading coefficient, thus \( \inf \{ q(x) \} \not> -\infty \).

This proves \( \inf \{ q(x) \} > -\infty \) implies \( A \geq 0 \). Next assume \( b \not\in \text{Im} A \). Consider an orthonormal basis for \( A \) \( \{ v_i \} \) with eigenvalues \( \{ \lambda_i \} \). We have that there does not exist a solution to \( \sum b_i v_i = b = Ax = A \sum x_i v_i = \sum x_i \lambda_i v_i \) which means \( \exists j \) such that \( b_j \neq 0 \) but \( \lambda_j = 0 \). Let \( x_\alpha = \alpha v_j \). Now we have \( q(x_\alpha) = \langle Ax_\alpha, x_\alpha \rangle - 2\langle b, x_\alpha \rangle = \langle 0, x_\alpha \rangle - 2\langle \sum b_i v_i, x_\alpha \rangle = -2\alpha b_j \), this can be made as positive or negative as we want as it is linear in \( \alpha \). Thus again \( \inf \{ q(x) \} \not> -\infty \). This proves \( \inf \{ q(x) \} > -\infty \) implies \( b \in \text{Im} A \).

(ii) \( \Rightarrow \) (iii) As \( A \geq 0 \) we have that there exists an orthonormal eigenbasis \( \{ v_i \} \) whose eigenvalues are all non-negative. Let’s arrange the eigenvalues in decreasing order (repeated eigenvalues are fine, any order is okay) and so we have that the first \( k \) eigenvalues are positive and the last \( n-k \) are 0, where \( n-k \) is the dimension of the kernel. We also have that \( b = \sum b_i v_i \in \text{Im} A \) and so we know that \( b_i = 0 \) for \( i > k \).

Evaluating \( q(x) \) we get \( \langle Ax, x \rangle - 2\langle b, x \rangle = \langle A \sum x_i v_i, \sum x_i v_i \rangle - 2\langle \sum b_i v_i, \sum x_i v_i \rangle = \sum \lambda_i(x_i)^2 - 2\sum b_i x_i = \sum \lambda_i(x_i)^2 - 2\sum b_i x_i \). Notice that we only need to sum up to \( k \) as for \( i > k \), \( \lambda_i = b_i = 0 \). Thus we have that \( q(x) = \sum \lambda_i(x_i)^2 - 2b_i x_i \). This is just a sum of quadratic polynomial in \( k \) variables with positive leading coefficients, and it can be minimized by minimizing each separately as can be seen partial differentiation techniques. We find that setting \( x_i = \frac{b_i}{\lambda_i} \) for \( i \leq k \) is the minimizer, with \( x_i \) free in the other coordinates as they will vanish. Thus \( \inf \{ q(x) \} > -\infty \) has a solution.

(iii) \( \Rightarrow \) (i) Trivial.

To construct the pseudo-inverse of \( A \), let \( U \) be the \( n \times k \) matrix whose columns are the eigenvectors with nonzero eigenvalues and \( D \) the diagonal \( k \times k \) matrix consisting of those nonzero eigenvalues. The pseudo-inverse is then \( UD^{-1}U^T \). Our solution set is \( UD^{-1}U^T b + \text{Ker} A \). We note that \( U^T b \) projects \( b \) onto to basis of the first \( k \) eigenvectors, which is just a basis transformation since \( b \) is in the image of \( A \) so the projection does not lose any information, we just get the \( b_i \) terms from the expansion \( b = \sum b_i v_i \). Then multiplication by \( D^{-1} \) precisely gives \( \frac{b_i}{\lambda_i} \) and multiplication by \( U \) gives us \( x = \sum \frac{b_i}{\lambda_i} v_i \). The pseudo-inverse gives the solution from the solution set that minimizes \( \| x \| \), because it does not add anything from the kernel.

(a) Consider the minimization problem
\[
\inf_u F(u) = \int_{x_0}^{x_1} L(x, u(x), u'(x)) dx,
\]
with \(u(x_0) = u_0, u(x_1) = u_1\) given constants, and \(L\) a sufficiently smooth function. Obtain formally the Euler-Lagrange equation of the minimization problem that is satisfied by a smooth optimal \(u\).

Hint: Consider smooth test functions \(v\), such that \(v(x_0) = v(x_1) = 0\). Since \(u\) is a minimizer, we must have \(F(u) \leq F(u + \epsilon v)\) for all such sufficiently smooth functions \(v\) and every real \(\epsilon\). Apply integration by parts to obtain the desired result. You should obtain a second-order differential equation.

(b) Let now \(u(x, t)\) be a smooth solution of the time-dependent partial differential equation (PDE)
\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} L_u(x, u, u') - L_{uu}(x, u, u'),
\]
with \(u(x, 0) = u_0(x)\) on \((x_0, x_1)\) and \(u(x_0, t) = U_0, u(x_1, t) = U_1\) for \(t \geq 0\). Show that the function \(E(t) = F(u(\cdot, t))\) is decreasing in time, where \(F(u) = \int_{x_0}^{x_1} L(x, u, u') dx\).

Solution: (a) We want to minimize \(F(u)\), that is, if \(u\) is the minimizer \(F(u) \leq F(u + w)\) for all functions \(w\) such that \(w\) vanishes at the boundary. So let’s assume \(u\) is such a minimizer of \(F\), now consider a smooth test function \(v\) such that \(v(x_0) = v(x_1) = 0\) and an \(\epsilon > 0\). Let’s examine \(G(\epsilon) = \int_{x_0}^{x_1} L(x, u(x) + \epsilon v(x), u'(x) + \epsilon v'(x)) dx\). We should have that this \(G(\epsilon)\) attains a minimum at \(\epsilon = 0\), which we prove by taking derivative.
\[
\frac{dG(\epsilon)}{d\epsilon} = \int_{x_0}^{x_1} \frac{d}{d\epsilon} L(x, u(x) + \epsilon v(x), u'(x) + \epsilon v'(x)) dx
\]
where we have a total derivative. Let \(L(\epsilon) = L(x, u(x) + \epsilon v(x), u'(x) + \epsilon v'(x))\) and \(L = L(x, u(x), u'(x))\) and we get
\[
\int_{x_0}^{x_1} \frac{d}{d\epsilon} L(\epsilon) dx = \int_{x_0}^{x_1} \left( \frac{d}{d\epsilon} \frac{d\epsilon}{dx} L(\epsilon) + \frac{d}{d(\epsilon v)} \frac{d\epsilon}{dx} L(\epsilon) + \frac{d}{d\epsilon} \frac{d\epsilon}{d(\epsilon v)} L(\epsilon) \right) dx
\]
\[
= \int_{x_0}^{x_1} \left( v \frac{dL(\epsilon)}{d(\epsilon v)} + v' \frac{dL(\epsilon)}{d(\epsilon v)} \right) dx.
\]

Now we can set \(\epsilon = 0\) and this integral should vanish. That is, we get
\[
\int_{x_0}^{x_1} \left( v \frac{dL}{du} + v' \frac{dL}{dw} \right) dx = 0.
\]
An integration by parts gives
\[
\int_{x_0}^{x_1} v \left( \frac{dL}{du} - \frac{d}{dx} \frac{dL}{du'} \right) dx - v \frac{dL}{du'} \bigg|_{x_0}^{x_1} = 0.
\]

However, we know that \( v \) vanishes at the boundary, and since \( v \) is an arbitrary smooth function,
\[
\frac{dL}{du} - \frac{d}{dx} \frac{dL}{du'} = 0.
\]
That is to say, the minimizer of \( u \) satisfies that above second order differential equation.

(b) We want to show \( E \) is decreasing in time. This is equivalent to show \( \frac{dE}{dt} \leq 0 \). We have that
\[
\frac{d}{dt} \int_{x_0}^{x_1} L(x, u(x, t), u'(x, t)) dx = \int_{x_0}^{x_1} \left( \frac{dx}{dt} \frac{dL}{du} + \frac{du}{dt} \frac{dL}{du} + \frac{d}{dt} \frac{dL}{du} \right) dx
\]
\[
= \int_{x_0}^{x_1} \left( \frac{du}{dt} \frac{dL}{du} + \frac{d}{dt} \frac{dL}{du} \right) dx
\]
\[
= \int_{x_0}^{x_1} \frac{du}{dt} \left( \frac{dL}{du} - \frac{d}{dx} \frac{dL}{du'} \right) dx + \frac{d}{dt} \frac{dL}{du} \bigg|_{x_0}^{x_1} \tag{4}
\]
\[
= \int_{x_0}^{x_1} \left( \frac{du}{dt} \right)^2 dx \leq 0.
\]

Where we used integration by parts and the fact that \( u \) is constant in time at \( x_0 \) and \( x_1 \), which meant the boundary term vanished. Thus the energy \( E \) decreases in time.