Journal of Number Theory 129 (2009) 2557–2568

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Distribution of powers of the partition function modulo ℓ^j

Jim L. Brown [a,*], Yingkun Li [b,1]

[a] Department of Mathematical Sciences, Clemson University, Clemson, SC 29634, United States
[b] Department of Mathematics, California Institute of Technology, Pasadena, CA 91125, United States

ARTICLE INFO

Article history:
Received 27 October 2008
Revised 17 December 2008
Available online 13 February 2009
Communicated by James W. Cogdell

MSC:
11F33
11P83

Keywords:
Newman's conjecture
Partitions
Modular forms

ABSTRACT

In this paper we study Newman's conjecture for powers of the partition function. While this conjecture is known for powers of primes ℓ that are not exceptional for the power under consideration, it is an open problem for exceptional primes. We settle this conjecture in many cases for small powers of the partition function by generalizing results of Ono and Ahlgren. It should be noted our method requires a case by case examination of each power and does not yield a general method for dealing with different powers simultaneously.

Published by Elsevier Inc.

1. Introduction and statement of results

A *partition* of a positive integer n is a non-increasing sequence of positive integers whose sum is n. The partition function $p(n)$ is defined to be the number of partitions of n. By convention, $p(0) = 1$ and $p(n) = 0$ for $n < 0$.

Euler showed that the partition function satisfies the following generating function relationship:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

* Corresponding author.
E-mail addresses: jimlb@clemson.edu (J.L. Brown), yingkun@caltech.edu (Y. Li).

[1] Supported by a Summer Undergraduate Research Fellowship at California Institute of Technology and the John and Maria Laffin Trust.

0022-314X/$ – see front matter Published by Elsevier Inc.
doi:10.1016/j.jnt.2008.12.007

An r-colored partition of a positive integer n is a partition of n, where one of r colors is assigned to each integer in the sequence. The rth power of the partition function, $p_r(n)$, counts the number of r-colored partitions of n. It satisfies a generating function relationship similar to that of $p(n)$:

$$\sum_{n=0}^{\infty} p_r(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^r}. \tag{1}$$

In his 1921 paper [13], Ramanujan proved the following beautiful and intriguing congruences, which became known as the Ramanujan congruences

$$p(\ell n - \delta_\ell) \equiv 0 \pmod{\ell} \tag{2}$$

where $\ell = 5, 7, 11$, $\delta_\ell = \frac{\ell^2-1}{24}$, and n is any positive integer.

Since then, congruences similar to (2) have been extensively studied. Although Ahlgren and Boylan showed in [2] that the Ramanujan congruences are the only ones of the form (2), Kiming and Olsson [9] have shown that congruences like (2) exist for $p_r(n)$. They define these congruences as follows.

Definition 1.1. Let $\ell \geqslant 5$ be a prime number, r a positive integer with $1 \leqslant r < \ell$ and $r \neq \ell - 1, \ell - 3$. We say that ℓ is *exceptional* for r if there exists an integer a such that $0 \leqslant a \leqslant \ell - 1$ with $p_r(\ell n + a) \equiv 0 \pmod{\ell}$ for all positive integers n.

In the same paper, Kiming and Olsson also proved the following theorem, which makes the Ramanujan congruences special cases of exceptional congruences

Theorem 1.2. *(See [9, Thm. 1].) If $\ell \geqslant 5$ is prime and exceptional for r, then $24a \equiv r \pmod{\ell}$.*

In 1960, Newman made the following conjecture about the distribution of the partition function modulo a positive integer M.

Conjecture 1.3. *(See [11].) If M is an integer, then for every integer s there are infinitely many non-negative integers n such that $p(n) \equiv s \pmod{M}$.*

Ahlgren and Boylan proved Conjecture 1.3 for $M = \ell^j$, with $\ell \geqslant 5$ a prime number and j a positive integer [3]. For certain M with multiple prime factors, conditions to check the validity of the conjecture were obtained in [1].

Since $p(n)$ is a special case of $p_r(n)$, we will consider the following generalization of Newman's conjecture to $p_r(n)$.

Conjecture 1.4. *If M and r are positive integers, then for every integer s there are infinitely many non-negative integers n such that $p_r(n) \equiv s \pmod{M}$.*

Notice that for $r = 1$, we have $p_r(n) = p(n)$ and recover Newman's original conjecture. Using a procedure similar to the one in [2], Kilbourn [8] proved Conjecture 1.4 for $M = \ell^j$ when $\ell > r + 4$ is not exceptional for r and outlined a method to check the conjecture when ℓ is exceptional for $r < 24$.

In this paper, we will follow the procedure outlined in [8, §5] to verify Conjecture 1.4 for certain $M = \ell^j$ with ℓ a prime number and exceptional for $r < 48$. All the pairs (r, ℓ) with ℓ exceptional for $r < 48$ are listed in [6]. We will look at the pairs in the following set

$$S = \{(r,\ell) \mid \ell \text{ exceptional for } r,\ r \leqslant 24\} \\ \cup \left\{(r,\ell) \,\middle|\, \ell \text{ exceptional for } r,\ 24 < r < 48 \text{ and } \left(\frac{24-r}{\ell}\right) = \left(\frac{-r}{\ell}\right)\right\}.$$

Throughout this paper we will use $(\frac{\bullet}{\ell})$ to denote the Legendre symbol and set $\delta_\ell = \frac{\ell^2-1}{24}$ and $q = e^{2\pi i z}$.

The main result of this paper is the following theorem.

Theorem 1.5. *Let $(r, \ell) \in S$ and j a positive integer. Then for every integer s there are infinitely many non-negative integers n such that $p_r(n) \equiv s \pmod{\ell^j}$.*

The proof depends on the action of Hecke operators on certain modular forms whose coefficients are congruent to $p_r(n)$ modulo a prime. First, we will construct a half-integral weight modular form for each pair in S following a similar method in [4] using eta-quotients and twists of modular forms by characters. We then compute the action of certain Hecke operators on those modular forms. Note that the proof of Theorem 5 in [3] has demonstrated the cases for $(r, \ell) = (1, 5), (1, 7), (1, 11)$, and [8, §5] has a sketch for the case $(r, \ell) = (3, 11)$.

In Section 2, we will give some facts about modular forms modulo ℓ and eta-quotients as in [10,14,15] and [12]. We will describe the construction of the modular form in Section 3 and prove Theorem 1.5 in Section 4.

2. Preliminaries

Let $M_k(\Gamma)$ and $S_k(\Gamma)$ denote the space of modular forms and cusp forms of weight k and level Γ respectively for $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ a congruence subgroup. If $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\mathrm{SL}_2(\mathbb{Z})) \cap \mathbb{Z}[\![q]\!]$, let $\widetilde{f(z)} := \sum_{n=0}^{\infty} \overline{a(n)}q^n$ be the coefficient-wise reduction of $f(z)$ modulo ℓ. Note that we fix ℓ in this section so the reductions are always assumed modulo ℓ unless otherwise noted. Define

$$\widetilde{M}_k\big(\mathrm{SL}_2(\mathbb{Z})\big) := \big\{\widetilde{f(z)} \bigm| f(z) \in M_k\big(\mathrm{SL}_2(\mathbb{Z})\big)\big\}$$

as the space of weight k modular forms reduced modulo ℓ.

Let $f(z) \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ have nonzero reduction modulo ℓ. If $g(z) \in \mathbb{Z}[\![q]\!]$ has the property that $g(z) \equiv f(z) \pmod{\ell}$, then define the *filtration* $\omega_\ell(g)$ of $g(z)$ modulo ℓ by

$$\omega_\ell(g) := \min\big\{k' \bigm| \text{there exists } \widetilde{f} \in \widetilde{M}_{k'}\big(\mathrm{SL}_2(\mathbb{Z})\big) \text{ s.t. } \widetilde{f} = \widetilde{g}\big\}.$$

Note that one clearly has $\omega_\ell(g) \leqslant k$. If $g(z) \equiv 0 \pmod{\ell}$, then we set $\omega_\ell(g) = -\infty$.

Recall the Ramanujan operator for $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\mathrm{SL}_2(\mathbb{Z})) \cap \mathbb{Z}[\![q]\!]$ is defined as

$$\Theta(f) := \sum_{n=1}^{\infty} na(n)q^n.$$

From [12, Prop. 2.44] we know that $\widetilde{\Theta(f)} \in \widetilde{M}_{k+\ell+1}(\mathrm{SL}_2(\mathbb{Z}))$. In addition, we have the following facts about filtrations and the effect of the theta operator on filtrations from [14] and [15].

Lemma 2.1. *Let $\ell \geqslant 5$ be a prime number and $f(z) \in M_k(\mathrm{SL}_2(\mathbb{Z})) \cap \mathbb{Z}[\![q]\!]$ with $\widetilde{f} \neq 0$. Then*

1. $\omega_\ell(f) \equiv k \pmod{\ell - 1}$;
2. $\omega_\ell(f^i) = i\omega_\ell(f)$ *for all integers i*;
3. $\omega_\ell(\Theta(f)) \leqslant \omega_\ell(f) + \ell + 1$ *with equality if and only if $\omega_\ell(f) \not\equiv 0 \pmod{\ell}$.*

Let d be a positive integer. We define the U-operator and V-operator by

$$\left(\sum_{n\geqslant n_0} c(n)q^n\right)\Big|U(d) := \sum_{n\geqslant n_0} c(dn)q^n, \tag{3}$$

$$\left(\sum_{n\geqslant n_0} c(n)q^n\right)\Big|V(d) := \sum_{n\geqslant n_0} c(n)q^{dn}. \tag{4}$$

Unlike the Ramanujan operator, both $U(d)$ and $V(d)$ transform modular forms to modular forms in the following manner.

Lemma 2.2. *(See [12, Prop. 2.22].) Suppose that $f(z) \in M_k(\Gamma_0(N), \chi)$ and d is a positive integer dividing N. Then*

$$f(z)|U(d) \in M_k\big(\Gamma_0(N), \chi\big),$$
$$f(z)|V(d) \in M_k\big(\Gamma_0(dN), \chi\big).$$

Moreover, if $f(z)$ is a cusp form, so are $f(z)|U(d)$ and $f(z)|V(d)$.

Recall Dedekind's eta function and its q-expansion:

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}\big(1 - q^n\big). \tag{5}$$

We have that $\eta(24z) \in S_{1/2}(\Gamma_0(576), \chi_{12})$ where $\chi_{12} = (\frac{12}{\bullet})$. One also has that every integral weight modular form on $\mathrm{SL}_2(\mathbb{Z})$ and every half-integral weight modular form on $\Gamma_0(4)$ can be expressed as a rational function in $\eta(z), \eta(2z)$, and $\eta(4z)$ (see [12, Thm. 1.67] or [10, §4.2]). In particular, $\eta^{24}(z) = \Delta(z) \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ is the cusp form with the smallest integral weight. Thus it is easy to see that $\eta(z)$ is an important building block for both integral and half-integral weight modular forms. Here are some facts about eta-quotients.

Theorem 2.3. *(See [12, Thms. 1.64, 1.65].)*

1. *If $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ is an eta-quotient with $k = \sum_{\delta|N} r_\delta \in \mathbb{Z}$ and with the additional properties that*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then $f(z)$ satisfies

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for every $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(N)$. Here the character χ is defined by $\chi(d) := (\frac{(-1)^k s}{d})$, where $s := \prod_{\delta|N} \delta^{r_\delta}$.

2. *Let c, d and N be positive integers with $d|N$ and $\gcd(c, d) = 1$. If $f(z)$ is an eta-quotient satisfying the conditions above for N, then the order of vanishing of $f(z)$ at cusp $\frac{c}{d}$ is*

$$\frac{N}{24}\sum_{\delta|N}\frac{\gcd(d,\delta)^2 r_\delta}{\gcd(d,\frac{N}{d})d\delta}.$$

Proposition 2.4. *Let t be a positive integer. Define the eta-quotient $E_{\ell,t}(z)$ as*

$$E_{\ell,t}(z) = \frac{\eta^{\ell^t}(z)}{\eta(\ell^t z)}.$$

One has $E_{\ell,t}(z) \in M_{\frac{\ell^t-1}{2}}(\Gamma_0(\ell^t), \chi_{\ell,t})$ where $\chi_{\ell,t} = (\frac{(-1)^{(\ell^t-1)/2}\ell^t}{\bullet})$. Furthermore, $E_{\ell,t}(z)$ vanishes at every cusp not equivalent to ∞ under $\Gamma_0(\ell^t)$. Also, $E_{\ell,t}^{\ell^{m-1}}(z) \equiv 1 \pmod{\ell^m}$ for every positive integer m.

Proof. From Theorem 2.3, we know that $E_{\ell,t}(z)$ transforms correctly under $\Gamma_0(\ell^t)$ with weight $\frac{\ell^t-1}{2}$. Also, one easily checks that $E_{\ell,t}(z)$ is holomorphic at every cusp and vanishes at every cusp not equivalent to ∞ under $\Gamma_0(\ell^t)$.

To show that $E_{\ell,t}^{\ell^{m-1}}(z) \equiv 1 \pmod{\ell^m}$, notice $E_{\ell,t}(z)$ has the following expansion

$$E_{\ell,t}(z) = \frac{\prod_{n=1}^{\infty}(1-q^n)^{\ell^t}}{\prod_{n=1}^{\infty}(1-q^{\ell^t n})}. \tag{6}$$

By the binomial theorem, we have $(1-q)^\ell \equiv (1-q^\ell) \pmod{\ell}$ and Eq. (6) implies that $E_{\ell,t}(z) \equiv 1 \pmod{\ell}$. Now with an induction on m, we see that for every positive integer m

$$E_{\ell,t}^{\ell^{m-1}}(z) \equiv 1 \pmod{\ell^m}. \qquad \square \tag{7}$$

3. Construction

We assume throughout this section that $\ell \geqslant 5$ is a prime number that is exceptional for r. This allows us to assume that r is odd by work of Kiming and Olsson [9]. We define

$$\epsilon_{r,\ell} = \left(\frac{-6r}{\ell}\right).$$

We begin this section by proving the existence of a half-integral weight modular form $F_{r,\ell,j}(z)$ whose coefficients are congruent to values of p_r modulo ℓ^j. One can see [4, Thm. 2.1] for the statement in terms of the partition function.

Lemma 3.1. *Let $\ell \geqslant 5$ be prime and j a positive integer. If $(r,\ell) \in S$, then there is a modular form $F_{r,\ell,j}(z) \in S_{\lambda_{r,\ell,j}-\frac{1}{2}}(\Gamma_0(576\ell^3), \chi_{\ell,3}\chi_\ell\chi_{12}) \cap \mathbb{Z}[\![q]\!]$ such that*

$$F_{r,\ell,j}(z) \equiv \sum_{(\frac{n}{\ell})=-(\frac{-r}{\ell})} p_r\left(\frac{n+r}{24}\right)q^n \pmod{\ell^j},$$

where $\lambda_{r,\ell,j}$ is an integer, $\chi_\ell = (\frac{\bullet}{\ell})$, $\chi_{\ell,t} = (\frac{(-1)^{(\ell^t-1)/2}\ell^t}{\bullet})$, and $\chi_{12} = (\frac{12}{\bullet})$.

Proof. Consider the following eta-quotient

$$f_{r,\ell}(z) := \left(\frac{\eta^\ell(\ell z)}{\eta(z)}\right)^r.$$

From Theorem 2.3 we know that $f_{r,\ell}(z) \in M_{\frac{r(\ell-1)}{2}}(\Gamma_0(\ell), \chi_\ell)$ where we have used that $\chi_\ell^r = \chi_\ell$ since r is necessarily odd. Using Eqs. (1), (5) and the fact that $f_{r,\ell}(z)$ vanishes at ∞, we can write

$$f_{r,\ell}(z) = \left(\sum_{n=0}^{\infty} p_r(n) q^{n+r\delta_\ell}\right) \prod_{n=1}^{\infty} \left(1-q^{\ell n}\right)^{r\ell} = \sum_{n=1}^{\infty} a_{r,\ell}(n) q^n. \tag{8}$$

Given a modular form f and a character χ, we denote the twist of f by χ as $f \otimes \chi$. Consider the function

$$g_{r,\ell}(z) := f_{r,\ell}(z) - \epsilon_{r,\ell} f_{r,\ell}(z) \otimes \chi_\ell = \sum_{n=1}^{\infty} \big(1 - \epsilon_{r,\ell}\chi_\ell(n)\big) a_{r,\ell}(n) q^n.$$

Standard facts (cf. [10, §3.3]) imply that $g_{r,\ell}(z) \in M_{\frac{r(\ell-1)}{2}}(\Gamma_0(\ell^3), \chi_\ell)$. Since all of the exponents of the q's in the product $\prod(1-q^{\ell n})^{r\ell}$ are divisible by ℓ, we can conclude that

$$g_{r,\ell}(z) = \left(\sum_{\ell \mid n+r\delta_\ell} p_r(n) q^{n+r\delta_\ell} + 2 \sum_{(\frac{n+r\delta_\ell}{\ell}) = -\epsilon_{r,\ell}} p_r(n) q^{n+r\delta_\ell}\right) \prod_{n=1}^{\infty} \left(1-q^{\ell n}\right)^{r\ell}. \tag{9}$$

Consider $f_{r,\ell}(z) | U(\ell) | V(\ell) = \sum_{n=1}^{\infty} a_{r,\ell}(\ell n) q^{\ell n}$. By Lemma 2.2, we know that $f_{r,\ell}(z) | U(\ell) | V(\ell) \in M_{\frac{r(\ell-1)}{2}}(\Gamma_0(\ell^2), \chi_\ell)$. Since ℓ divides the exponent of every term in the q-expansion of $f_{r,\ell} | U(\ell) | V(\ell)$ in Eq. (8), it has the following q-expansion:

$$f_{r,\ell}(z) | U(\ell) | V(\ell) = \left(\sum_{\ell \mid n+r\delta_\ell} p_r(n) q^{n+r\delta_\ell}\right) \prod_{n=1}^{\infty} \left(1-q^{\ell n}\right)^{r\ell}. \tag{10}$$

Now we are able to show that for sufficiently large m the following function has the desired property

$$F_{r,\ell,j,m}(z) := \frac{\ell^j + 1}{2} \cdot \frac{(E_{\ell,3}^{\ell^{m-1}}(z)(g_{r,\ell}(z) - f_{r,\ell}(z) | U(\ell) | V(\ell))) | V(24)}{\eta^{r\ell}(24\ell z)}. \tag{11}$$

(In particular, we require that $m \geqslant j$.) Using Eqs. (7), (9) and (10), we can compute the q-expansion of $F_{r,\ell,j,m}(z)$ modulo ℓ^j (using $m \geqslant j$)

$$\begin{aligned} F_{r,\ell,j,m}(z) &\equiv \frac{\ell^j+1}{2}\left(2 \sum_{(\frac{n+r\delta_\ell}{\ell}) = -\epsilon_{r,\ell}} p_r(n) q^{24n-r}\right) \equiv \frac{\ell^j+1}{2}\left(2 \sum_{(\frac{n}{\ell}) = -(\frac{-r}{\ell})} p_r\left(\frac{n+r}{24}\right) q^n\right) \\ &\equiv \sum_{(\frac{n}{\ell}) = -(\frac{-r}{\ell})} p_r\left(\frac{n+r}{24}\right) q^n \pmod{\ell^j}. \end{aligned}$$

The fact that $\eta^{r\ell}(24\ell z) \in M_{\frac{r\ell}{2}}(\Gamma_0(576\ell), \chi_{12})$ with $\chi_{12} := (\frac{12}{\bullet})$ gives that $F_{r,\ell,j,m}(z)$ transforms correctly under $\Gamma_0(576\ell^3)$ with weight $\frac{(\ell^3-1)\ell^{m-1}}{2} - \frac{1}{2}$ and character $\chi_{\ell,3}\chi_\ell\chi_{12}$. To check that it vanishes at the cusps, consider the function

$$h(z) := \frac{E_{\ell,3}^{24\ell^{m-1}}(z)(g_{r,\ell}(z) - f_{r,\ell}(z)|U(\ell)|V(\ell))^{24}}{\Delta^{r\ell}(\ell z)}, \tag{12}$$

where $\Delta(z)$ is the unique normalized cusp form on $\mathrm{SL}_2(\mathbb{Z})$ of weight 12.

Clearly, $h(z)$ transforms correctly under $\Gamma_0(\ell^3)$. At ∞, we can use (9) and (10) to obtain the q-expansion for $g_{r,\ell}(z) - f_{r,\ell}(z)|U(\ell)|V(\ell)$:

$$g_{r,\ell}(z) - f_{r,\ell}(z)\big|U(\ell)\big|V(\ell) = \left(2 \sum_{(\frac{n+r\delta_\ell}{\ell}) = -\epsilon_{r,\ell}} p_r(n) q^{n+r\delta_\ell}\right) \prod_{n=1}^{\infty} \big(1-q^{\ell n}\big)^{r\ell}. \tag{13}$$

When $r < 24$, the first term in the summation of Eq. (13) has order at least $1 + r\delta_\ell$ since $(\frac{0+r\delta_\ell}{\ell}) = \epsilon_{r,\ell} \neq -\epsilon_{r,\ell}$. When $24 < r < 48$ and $r \in S$, the first two terms, with $n = 0$ and 1, have the property that

$$\left(\frac{0+r\delta_\ell}{n}\right) = \left(\frac{1+r\delta_\ell}{n}\right) = \epsilon_{r,\ell} \neq -\epsilon_{r,\ell}.$$

by the definition of S. So it has order at least $2 + r\delta_\ell$. Since $E_{\ell,3}(z)$ is holomorphic at ∞, we see from Eq. (12) that $\mathrm{ord}_\infty(h(z)) \geqslant 24(1 + r\delta_\ell) - r\ell^2$ and $\mathrm{ord}_\infty(h(z)) \geqslant 24(2 + r\delta_\ell) - r\ell^2$ for $r < 24$ and $24 < r < 48$ respectively. In both cases, we have $\mathrm{ord}_\infty(h(z)) > 0$. Since $E_{\ell,3}^{\ell^{m-1}}(z)$ will vanish at cusps not equivalent to ∞ under $\Gamma_0(\ell^3)$, we can choose m large enough so that $h(z)$ also vanishes at those cusps.

By Lemma 2.2, $h(24z) = h(z)|V(24)$ is a cusp form, hence vanishes at all cusps. Since $F_{r,\ell,j}(z) = \frac{\ell^j+1}{2}(h(24z))^{\frac{1}{24}}$, it also vanishes at all the cusps. Thus, $F_{r,\ell,j}(z) \in S_{\lambda_{r,\ell,j}-\frac{1}{2}}(\Gamma_0(576\ell^3), \chi_{\ell,3}\chi_\ell\chi_{12})$, where $\lambda_{r,\ell,j} = \frac{(\ell^3-1)\ell^{m-1}}{2}$ is an integer. (Note that it may look odd to have a dependence on j on the left-hand side but the right-hand side in terms of m. We write it this way to emphasize that the m can be arbitrary as long as it is large enough to ensure cuspidality and $m \geqslant j$.) $\square$

We now construct a modular form $P_{r,\ell}(z)$ such that its level is not divisible by ℓ and $P_{r,\ell}(z) \equiv F_{r,\ell,j}(z) \pmod{\ell}$. The construction for $r = 1$ is carried out in [5] and an analogous result is sketched for $r < 24$ in [8, §5].

Lemma 3.2. *Suppose $(r, \ell) \in S$. Then there is a cusp form $P_{r,\ell}(z) \in S_{\frac{\ell^2-r-1}{2}}(\Gamma_0(576), \chi_{12}) \cap \mathbb{Z}[\![q]\!]$ such that*

$$P_{r,\ell}(z) \equiv F_{r,\ell,j}(z) \pmod{\ell}.$$

Proof. Let $f_{r,\ell}(z) = \sum_{n=1}^{\infty} a_{r,\ell}(n) q^n$ be defined as in Lemma 3.1. By considering its q-expansion in Eq. (8) and applying the binomial theorem, we have $f_{r,\ell}(z) \equiv \Delta^{r\delta_\ell}(z) \pmod{\ell}$. Now Lemma 2.1 implies that

$$\omega_\ell(f_{r,\ell}) = \omega_\ell\big(\Delta^{r\delta_\ell}\big) = r\delta_\ell\omega_\ell(\Delta).$$

The fact that $M_k(\mathrm{SL}_2(\mathbb{Z}))$ is either 0 or a one-dimensional vector space for $k < 12$ allows one to easily check that $\omega_\ell(\Delta) = 12$. Thus $\omega_\ell(f_{r,\ell}) = \frac{r(\ell^2-1)}{2}$.

Since $\frac{r(\ell^2-1)}{2}+k(\ell+1)\not\equiv 0 \pmod{\ell}$ for $k=0,1,\dots,\frac{\ell-1}{2}$, we can apply the Ramanujan operator $\frac{\ell-1}{2}$ times and conclude from Lemma 2.1(3) that

$$\omega_\ell\bigl(\Theta^{\frac{\ell-1}{2}}(f_{r,\ell})\bigr)=\frac{(r+1)(\ell^2-1)}{2}. \tag{14}$$

By considering the q-expansion of $\Theta^{\frac{\ell-1}{2}}(f_{r,\ell}(z))$, we have

$$\Theta^{\frac{\ell-1}{2}}\bigl(f_{r,\ell}(z)\bigr)\equiv\sum_{n=1}^{\infty}\left(\frac{n}{\ell}\right)a_{r,\ell}(n)q^n \pmod{\ell}. \tag{15}$$

Combining Eqs. (14) and (15), we know that there is a cusp form $Q_{r,\ell}(z)\in S_{\frac{(r+1)(\ell^2-1)}{2}}(\mathrm{SL}_2(\mathbb{Z}))$ such that

$$Q_{r,\ell}(z)\equiv\sum_{n=1}^{\infty}\left(\frac{n}{\ell}\right)a_{r,\ell}(n)q^n\equiv f_{r,\ell}(z)\otimes\left(\frac{\bullet}{\ell}\right) \pmod{\ell}. \tag{16}$$

Let $E_k(z)$ be the normalized Eisenstein series of weight k. Using the fact that $E_{\ell-1}(z)\equiv 1 \pmod{\ell}$, we can define the cusp form $R_{r,\ell}$ in $S_{\frac{(r+1)(\ell^2-1)}{2}}(\mathrm{SL}_2(\mathbb{Z}))\cap\mathbb{Z}[\![q]\!]$ as

$$R_{r,\ell}(z):=\Delta^{r\delta_\ell}(z)E_{\ell-1}^{\frac{\ell+1}{2}}(z)-\epsilon_{r,\ell}Q_{r,\ell}(z).$$

Using Eq. (16), we can calculate the q-expansion of $R_{r,\ell}(z)$ as

$$R_{r,\ell}(z)\equiv\Biggl(\sum_{n\equiv 0 \pmod{\ell}}p_r(n-r\delta_\ell)q^n+2\sum_{(\frac{n}{\ell})=-\epsilon_{r,\ell}}p_r(n-r\delta_\ell)q^n\Biggr)\prod_{n=1}^{\infty}\bigl(1-q^n\bigr)^{r\ell^2} \pmod{\ell}.$$

When ℓ is exceptional for r, we can find an integer a such that $p_r(\ell n+a)\equiv 0 \pmod{\ell}$ for every integer n. If $\ell|n$, then we can deduce from Theorem 1.2 that $n-r\delta_\ell\equiv a \pmod{\ell}$. So we can write $n-r\delta_\ell=\ell m+a$ for some integer m. Since ℓ is exceptional for r, $p_r(n-r\delta_\ell)\equiv 0 \pmod{\ell}$ whenever ℓ divides n. Thus the q-expansion of $R_{r,\ell}(z)$ modulo ℓ is as follows:

$$R_{r,\ell}(z)\equiv 2\Biggl(\sum_{(\frac{24n-r}{\ell})=-(\frac{r}{\ell})}p_r(n)q^{n+r\delta_\ell}\Biggr)\prod_{n=1}^{\infty}\bigl(1-q^n\bigr)^{r\ell^2} \pmod{\ell}. \tag{17}$$

Define

$$P_{r,\ell}(z):=\frac{\ell+1}{2}\cdot\frac{R_{r,\ell}(24z)}{\eta^{r\ell^2}(24z)}.$$

From Eq. (17), we see that $P_{r,\ell}(z)\equiv F_{r,\ell,j}(z) \pmod{\ell}$. To check that it is a cusp form, consider the space $S_{\frac{(r+1)(\ell^2-1)}{2}}(\mathrm{SL}_2(\mathbb{Z}))$. It is generated by $E_4^i(z)\Delta^j(z)$ with $4i+12j=\frac{(r+1)(\ell^2-1)}{2}$. Also, the first nonzero term of $R_{r,\ell}(z)$ has exponent at least $r\delta_\ell+1$ for $r<24$ and $r\delta_\ell+2$ for $24<r<48$ by construction. So we can write

$$R_{r,\ell}(z)=\Delta^{r\delta_\ell+1}(z)C_1(z)$$

with $C_1(z) \in M_{\frac{\ell^2-25}{2}}(\mathrm{SL}_2(\mathbb{Z}))$ for $r < 24$ and

$$R_{r,\ell}(z) = \Delta^{r\delta_\ell+2}(z)C_2(z)$$

with $C_2(z) \in M_{\frac{\ell^2-49}{2}}(\mathrm{SL}_2(\mathbb{Z}))$ for $24 < r < 48$. In either case, $P_{r,\ell}(z)$ is the product of some power of $\eta(24z)$ and another modular form. (See Appendix A for some examples of expressing $P_{r,\ell}(z)$ explicitly in terms of $\eta(24z)$ and $E_4(24z)$.) Hence we have $P_{r,\ell}(z) \in S_{\frac{\ell^2-r-1}{2}}(\Gamma_0(576), \chi_{12}) \cap \mathbb{Z}[\![q]\!]$. $\square$

4. Proof of main theorem

The proof of Theorem 1.5 is similar to that of [3, Thm. 5] and the sketch in [8, §5]. First, we need the following definition from [3].

Definition 4.1. Let M be a positive integer and $F(z)$ a half-integral weight modular form with

$$F(z) = \sum_{n=1}^{\infty} a(n)q^n.$$

The coefficients of $F(z)$ are said to be *well-distributed* modulo M if for every integer s, we have

$$\#\{1 \leqslant n \leqslant X \mid a(n) \equiv s \pmod{M}\} \gg_{s,M} \begin{cases} \frac{\sqrt{X}}{\log X} & \text{if } r \not\equiv 0 \pmod{M}, \\ X & \text{if } r \equiv 0 \pmod{M}. \end{cases}$$

Clearly, if the form $F_{r,\ell,j}(z)$ constructed in the previous section is well-distributed modulo $M = \ell^j$, then Conjecture 1.4 is true for $p_r(n)$ modulo $M = \ell^j$. Furthermore, we would have a lower bound for how often $p_r(n)$ falls into each congruence class of $M = \ell^j$. The following lemma, which is a direct consequence of [3, Thm. 1], gives a condition on when $F_{r,\ell,j}(z)$ is well-distributed modulo $M = \ell^j$.

Lemma 4.2. *Let $F_{r,\ell,j}(z) = \sum_{n=1}^{\infty} a_{r,\ell,j}(n)q^n$ be defined as in Lemma* 3.1. *If $M = \ell^j$ is the power of a prime number and r is a positive integer with $(r,\ell) \in S$, then at least one of the following is true*:

(1) *$F_{r,\ell,j}(z)$ is well-distributed modulo $M = \ell^j$*;
(2) *There are finitely many square-free integers $n_1, n_2, \ldots, n_t$ for which*

$$F_{r,\ell,j}(z) \equiv \sum_{i=1}^{t}\sum_{m=1}^{\infty} a_{r,\ell,j}(n_i m^2)q^{n_i m^2} \pmod{\ell}. \tag{18}$$

One should note here that conditions (1) and (2) can be simultaneously satisfied. The point here is that if condition (2) fails, then we must have the validity of condition (1).

The following proposition gives us a way to check the validity of condition (2) in Lemma 4.2 for each pair $(r,\ell) \in S$. It combines the results [3, Lem. 4.1] and [7, Thm. 1].

Proposition 4.3. *Suppose $P_{r,\ell}(z) = \sum_{n=1}^{\infty} a_{r,\ell}(n)q^n \in S_{\lambda_{r,\ell}+\frac{1}{2}}(\Gamma_0(576), \chi_{12})$ can be written in the form of Eq.* (18) *and $a_{r,\ell}(n_1 m_1^2) \not\equiv 0 \pmod{\ell}$ for some positive integers m_1 and $n_i \in \{n_1, \ldots, n_t\}$. Without loss of generality we assume $i = 1$. Then the following condition is true*

$$a_{r,\ell}(p^2 n_1 m_1^2) - \left(\frac{n_1}{p}\right)\left(\frac{(-1)^{\lambda_{r,\ell}}}{p}\right)\chi_{12}(p)p^{\lambda_{r,\ell}}a_{r,\ell}(n_1 m_1^2) \equiv 0 \pmod{\ell}, \tag{19}$$

where p is a prime number with $p \nmid 576\ell n_1 m_1$ and $p \not\equiv 1 \pmod{\ell}$.

Table 1
Table for values of p, n_1, and $(r,\ell) \in S$.

$(r,\ell)\in S$	p	n_1	Eq. (19)	$(r,\ell)\in S$	p	n_1	Eq. (19)
(1, 5)	7	23	$1 \not\equiv 0 \bmod 5$	(15, 29)	5	33	$27 \not\equiv 0 \bmod 29$
(1, 7)	5	23	$4 \not\equiv 0 \bmod 7$	(17, 23)	5	7	$11 \not\equiv 0 \bmod 23$
(1, 11)	5	23	$4 \not\equiv 0 \bmod 11$	(19, 23)	7	5	$2 \not\equiv 0 \bmod 23$
(3, 11)	5	93	$4 \not\equiv 0 \bmod 11$	(21, 29)	5	123	$6 \not\equiv 0 \bmod 29$
(3, 17)	5	21	$11 \not\equiv 0 \bmod 17$	(21, 31)	5	3	$7 \not\equiv 0 \bmod 31$
(5, 11)	5	67	$10 \not\equiv 0 \bmod 11$	(21, 47)	5	3	$4 \not\equiv 0 \bmod 47$
(5, 23)	5	19	$12 \not\equiv 0 \bmod 23$	(25, 29)	5	47	$6 \not\equiv 0 \bmod 29$
(7, 11)	5	17	$8 \not\equiv 0 \bmod 11$	(25, 31)	5	47	$29 \not\equiv 0 \bmod 31$
(7, 19)	7	17	$18 \not\equiv 0 \bmod 19$	(27, 31)	5	21	$25 \not\equiv 0 \bmod 31$
(9, 17)	7	39	$16 \not\equiv 0 \bmod 17$	(27, 41)	5	21	$12 \not\equiv 0 \bmod 41$
(9, 19)	5	39	$16 \not\equiv 0 \bmod 19$	(33, 41)	7	15	$36 \not\equiv 0 \bmod 41$
(9, 23)	7	39	$9 \not\equiv 0 \bmod 23$	(39, 47)	7	57	$37 \not\equiv 0 \bmod 47$
(13, 17)	5	11	$10 \not\equiv 0 \bmod 17$	(39, 61)	7	33	$46 \not\equiv 0 \bmod 61$
(13, 19)	5	59	$4 \not\equiv 0 \bmod 19$	(43, 47)	5	29	$9 \not\equiv 0 \bmod 47$
(13, 23)	5	59	$18 \not\equiv 0 \bmod 23$	(45, 53)	7	195	$33 \not\equiv 0 \bmod 53$
(15, 23)	5	33	$11 \not\equiv 0 \bmod 23$	(45, 59)	5	3	$42 \not\equiv 0 \bmod 59$

Proof. By [3, Lem. 4.1] there exist primes $p_1, \ldots, p_s$ distinct from ℓ and p, and a modular form $G_{r,\ell}(z) \in S_{\lambda_{r,\ell}+\frac{1}{2}}(\Gamma_0(576p_1^2 \cdots p_s^2), \chi_{12}) \cap \mathbb{Z}[\![q]\!]$ with

$$G_{r,\ell}(z) \equiv \sum_{\substack{m=1 \\ \gcd(m,\prod p_i)=1}}^{\infty} a_{r,\ell}\big(n_1 m^2\big) q^{n_1 m^2} \not\equiv 0 \pmod{\ell}. \tag{20}$$

Now we can study the action of the Hecke operator $T(p^2, \lambda_{r,\ell}, \chi_{12})$ on $G_{r,\ell}(z)$. Recall the Hecke operator $T(p^2, \lambda, \chi)$ acts on a half-integral weight modular form $F(z) = \sum_{n=1}^{\infty} a(n)q^n \in M_{\lambda+\frac{1}{2}}(\Gamma_0(N), \chi)$ by

$$F(z) | T\big(p^2, \lambda, \chi\big) = \sum_{n=0}^{\infty} \left(a\big(p^2 n\big) + \left(\frac{n}{p}\right) \chi^*(p) p^{\lambda-1} a(n) + \chi^*(p) p^{2\lambda-1} a\big(n/p^2\big) \right) q^n, \tag{21}$$

where $\chi^*(p) = (\frac{(-1)^\lambda}{p})\chi(p)$ and $a(n/p^2) = 0$ if $p^2 \nmid n$.

By the choices of $p_1, \ldots, p_s$, we have $\gcd(\ell, 576p_1^2 \cdots p_s^2) = 1$ and $p \nmid 576p_1^2 \cdots p_s^2 \ell$. Using Eq. (20), we can write $G_{r,\ell}(z)$ in the following form

$$G_{r,\ell}(z) \equiv \sum_{(\frac{n}{p}) \in \{0, (\frac{n_1}{p})\}} a_{r,\ell}(n) q^n \pmod{\ell}.$$

Applying [7, Thm. 1], we have

$$(p-1) G_{r,\ell}(z) | T\big(p^2, \lambda_{r,\ell}, \chi_{12}\big) \equiv \left(\frac{n_1}{p}\right) \chi_{12}^*(p) \big(p^{\lambda_{r,\ell}} + p^{\lambda_{r,\ell}-1}\big)(p-1) G_{r,\ell}(z) \pmod{\ell}. \tag{22}$$

Since $p^2 \nmid n_1 m_1^2$ and $p \not\equiv 1 \pmod{\ell}$, we can combine Eqs. (21) and (22) to obtain Eq. (19). $\square$

Since $P_{r,\ell}(z) \equiv F_{r,\ell,j}(z) \pmod{\ell}$, we can use Proposition 4.3 to check that condition (2) in Lemma 4.2 does not hold by finding p, n_1 and m_1 that do not satisfy Eq. (19). In Table 1, we list the choices of p, n_1 for each pair $(r,\ell) \in S$. For simplicity, we choose $m_1 = 1$ in all cases. Since Eq. (19) does not hold, condition (1) of Lemma 4.2 is true, and Theorem 1.5 is proved.

Appendix A

Here are some examples of expressing $P_{r,\ell}(z)$ in terms of $\eta(24z)$ and $E_4(24z)$ for some $(r,\ell) \in S$. These examples have less than 20 terms in their summation. Note: $P_{1,5}(z)$, $P_{1,7}(z)$ and $P_{1,11}(z)$ appeared as Eqs. (5.4), (5.5) and (5.6) in [3].

$$P_{1,5}(z) = \eta^{23}(24z),$$
$$P_{1,7}(z) = \eta^{23}(24z)E_4^3(24z) + 3\eta^{47}(24z),$$
$$P_{1,11}(z) = \eta^{23}(24z)E_4^{12}(24z) + 5\eta^{47}(24z)E_4^9(24z) + 4\eta^{71}(24z)E_4^6(24z) + \eta^{95}(24z)E_4^3(24z) + 8\eta^{119}(24z),$$

$$P_{3,11}(z) = 9\eta^{45}(24z)E_4^9(24z) + 6\eta^{69}(24z)E_4^6(24z) + 7\eta^{93}(24z)E_4^3(24z) + 6\eta^{117}(24z),$$

$$\begin{aligned} P_{3,17}(z) = {} & 3\eta^{21}(24z)E_4^{33}(24z) + \eta^{45}(24z)E_4^{30}(24z) + 2\eta^{69}(24z)E_4^{27}(24z) \\ & + 15\eta^{93}(24z)E_4^{24}(24z) + 5\eta^{117}(24z)E_4^{21}(24z) + 7\eta^{141}(24z)E_4^{18}(24z) \\ & + 10\eta^{165}(24z)E_4^{15}(24z) + 9\eta^{189}(24z)E_4^{12}(24z) + 8\eta^{213}(24z)E_4^9(24z) \\ & + 14\eta^{237}(24z)E_4^6(24z) + 14\eta^{261}(24z)E_4^3(24z) + 14\eta^{285}(24z), \end{aligned}$$

$$P_{5,11}(z) = 10\eta^{67}(24z)E_4^6(24z) + \eta^{91}(24z)E_4^3(24z) + 5\eta^{115}(24z),$$

$$\begin{aligned} P_{7,11}(z) = {} & 7\eta^{17}(24z)E_4^{12}(24z) + 3\eta^{41}(24z)E_4^9(24z) + 10\eta^{65}(24z)E_4^6(24z) \\ & + 2\eta^{89}(24z)E_4^3(24z) + 8\eta^{113}(24z), \end{aligned}$$

$$\begin{aligned} P_{7,19}(z) = {} & 7\eta^{17}(24z)E_4^{42}(24z) + 11\eta^{41}(24z)E_4^{39}(24z) + 2\eta^{65}(24z)E_4^{36}(24z) + 5\eta^{89}(24z)E_4^{33}(24z) \\ & + 5\eta^{137}(24z)E_4^{27}(24z) + 16\eta^{161}(24z)E_4^{24}(24z) + 17\eta^{185}(24z)E_4^{21}(24z) \\ & + 6\eta^{209}(24z)E_4^{18}(24z) + 18\eta^{233}(24z)E_4^{15}(24z) + 4\eta^{257}(24z)E_4^{12}(24z) \\ & + 9\eta^{281}(24z)E_4^9(24z) + 17\eta^{305}(24z)E_4^6(24z) + 14\eta^{329}(24z)E_4^3(24z) + 14\eta^{353}(24z), \end{aligned}$$

$$\begin{aligned} P_{9,17}(z) = {} & 3\eta^{39}(24z)E_4^{30}(24z) + 5\eta^{63}(24z)E_4^{27}(24z) + 6\eta^{87}(24z)E_4^{24}(24z) \\ & + 6\eta^{111}(24z)E_4^{21}(24z) + 5\eta^{135}(24z)E_4^{18}(24z) + 3\eta^{159}(24z)E_4^{15}(24z) \\ & 16\eta^{207}(24z)E_4^9(24z) + 3\eta^{231}(24z)E_4^6(24z) + 8\eta^{255}(24z)E_4^3(24z) \\ & + 11\eta^{279}(24z), \end{aligned}$$

$$\begin{aligned} P_{9,19}(z) = {} & 16\eta^{39}(24z)E_4^{39}(24z) + 3\eta^{63}(24z)E_4^{36}(24z) + 4\eta^{87}(24z)E_4^{33}(24z) \\ & + 6\eta^{111}(24z)E_4^{30}(24z) + 3\eta^{135}(24z)E_4^{27}(24z) + 13\eta^{159}(24z)E_4^{24}(24z) \\ & + 4\eta^{183}(24z)E_4^{21}(24z) + 8\eta^{207}(24z)E_4^{18}(24z) + 5\eta^{231}(24z)E_4^{15}(24z) \\ & + 5\eta^{255}(24z)E_4^{12}(24z) + 5\eta^{279}(24z)E_4^9(24z) + 2\eta^{303}(24z)E_4^6(24z) \\ & + 4\eta^{327}(24z)E_4^3(24z) + 17\eta^{351}(24z), \end{aligned}$$

$$\begin{aligned}P_{13,17}(z) = {} & 13\eta^{11}(24z)E_4^{33}(24z) + 16\eta^{35}(24z)E_4^{30}(24z) + 12\eta^{59}(24z)E_4^{27}(24z) \\ & + 9\eta^{83}(24z)E_4^{24}(24z) + 3\eta^{131}(24z)E_4^{18}(24z) + 11\eta^{155}(24z)E_4^{15}(24z) \\ & + 3\eta^{179}(24z)E_4^{12}(24z) + 7\eta^{203}(24z)E_4^{9}(24z) + \eta^{227}(24z)E_4^{6}(24z) \\ & + 13\eta^{251}(24z)E_4^{3}(24z) + 4\eta^{275}(24z),\end{aligned}$$

$$\begin{aligned}P_{13,19}(z) = {} & 10\eta^{59}(24z)E_4^{36}(24z) + 13\eta^{83}(24z)E_4^{33}(24z) + 8\eta^{107}(24z)E_4^{30}(24z) \\ & + 14\eta^{131}(24z)E_4^{27}(24z) + 9\eta^{155}(24z)E_4^{24}(24z) + 4\eta^{179}(24z)E_4^{21}(24z) \\ & + 3\eta^{227}(24z)E_4^{15}(24z) + 13\eta^{275}(24z)E_4^{9}(24z) + 15\eta^{299}(24z)E_4^{6}(24z) \\ & + 10\eta^{323}(24z)E_4^{3}(24z) + 14\eta^{347}(24z).\end{aligned}$$

References

[1] S. Ahlgren, Distribution of the partition function modulo composite integers M, Math. Ann. 318 (2000) 795–803.
[2] S. Ahlgren, M. Boylan, Arithmetic properties of the partition function, Invent. Math. 153 (2003) 487–502.
[3] S. Ahlgren, M. Boylan, Coefficients of half-integral weight modular forms modulo ℓ^j, Math. Ann. 331 (2005) 219–239.
[4] S. Ahlgren, K. Ono, Congruences and conjectures for the partition function, Proc. Natl. Acad. Sci. USA 98 (23) (2001) 12882–12884.
[5] S. Ahlgren, K. Ono, Congruences and conjectures for the partition function, in: Contemp. Math., vol. 291, 2001, pp. 1–10.
[6] M. Boylan, Exceptional congruences for powers of the partition function, Acta Arith. 111 (2) (2004) 187–203.
[7] J. Bruinier, K. Ono, Coefficients of half-integral weight modular forms, J. Number Theory 99 (2003) 164–179.
[8] T. Kilbourn, The distribution of the powers of the partition function modulo ℓ^j, preprint, 2007.
[9] I. Kiming, B. Olsson, Congruences like Ramanujan's for powers of the partition function, Arch. Math. 59 (1992) 348–360.
[10] N. Koblitz, Introduction to Elliptic Curves and Modular Forms, Springer-Verlag, New York, 1993.
[11] M. Newman, Periodicity modulo m and divisibility properties of the partition function, Trans. Amer. Math. Soc. 97 (1960) 225–236.
[12] K. Ono, The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-Series, CBMS Reg. Conf. Ser. Math., vol. 102, Conference Board of the Mathematical Sciences, Washington, DC, 2004.
[13] S. Ramanujan, Congruence properties of partitions, Math. Z. 9 (1921) 147–153.
[14] J.-P. Serre, Formes modulaires et fonctions zêta p-adiques, in: Modular Functions of One Variable III, Proc. Internet. Summer School, Univ. Antwerp, 1972, in: Lecture Notes in Math., vol. 350, Springer, Berlin, 1973, pp. 191–268.
[15] H.P.F. Swinnerton-Dyer, On ℓ-adic representations and congruences for coefficients of modular forms, in: Modular Functions of One Variable III, Proc. Internet. Summer School, Univ. Antwerp, 1972, in: Lecture Notes in Math., vol. 350, Springer, Berlin, 1973, pp. 1–55.