

1. Consider the Hamilton-Jacobi equation $u_t + H(\nabla_x u) = 0$, where H depends only on $\nabla_x u = (u_{x_1}, u_{x_2}, \dots, u_{x_n})$.

- (a) Obtain the characteristic equations;
- (b) Verify that $u(x, t) = \mathbf{a} \cdot x - tH(\mathbf{a}) + b$ is a complete integral.
- (c) Use envelopes to generate a solution of $u_t + |\nabla u|^2 = 0$ which is not linear in x and t .

Solution: Let

$$\tilde{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ t \end{pmatrix} = \begin{pmatrix} x \\ t \end{pmatrix}, \quad \tilde{p} = \begin{pmatrix} u_{x_1} \\ u_{x_2} \\ \vdots \\ u_{x_n} \\ u_t \end{pmatrix} = \begin{pmatrix} p \\ p_{n+1} \end{pmatrix}, \quad \tilde{z} = u$$

then the original equation is just $F(\tilde{x}, \tilde{z}, \tilde{p}) = p_{n+1} + H(p)$ and the characteristic equations are

$$\frac{d\tilde{x}}{d\tau} = D_{\tilde{p}}F = \begin{pmatrix} \nabla H(p) \\ 1 \end{pmatrix}, \quad (1)$$

$$\frac{d\tilde{p}}{d\tau} = -D_{\tilde{x}}F - \tilde{p}D_{\tilde{z}}F = \mathbf{0}, \quad (2)$$

$$\frac{d\tilde{z}}{d\tau} = \tilde{p} \cdot D_{\tilde{p}}F = p \cdot \nabla H(p) + p_{n+1}. \quad (3)$$

If we identify τ as t , then the characteristic equation is

$$\frac{dx}{dt} = \nabla H(p), \quad \frac{dp}{dt} = \mathbf{0}, \quad \frac{dz}{dt} = p \cdot \nabla H(p) - H(p). \quad (4)$$

(b) Since for $u(x, t; \mathbf{a}, b) = \mathbf{a} \cdot x - tH(\mathbf{a}) + b$,

$$u_t = -H(\mathbf{a}), \quad \nabla_x u = \mathbf{a}.$$

Substituting it into the equation, we have

$$u_t + H(\nabla_x u) = -H(\mathbf{a}) + H(\mathbf{a}) = 0.$$

Hence $u(x, t; a, b)$ solves the PDE.

Moreover, we have

$$\begin{pmatrix} D_a u & D_{ax} u & D_{at} u \\ D_b u & D_{bx} u & D_{bt} u \end{pmatrix} = \begin{pmatrix} x - t\nabla H(\mathbf{a}) & \mathbf{I} & -\nabla H(\mathbf{a}) \\ 1 & 0 & 0 \end{pmatrix}$$

The rank of above matrix is obviously $n + 1$. Therefore, $u(x, t; a, b)$ is a complete integral.

(c) Taking the derivative of $u(x, t; \mathbf{a}, b)$ with respect to \mathbf{a} , we find that

$$0 = x - t\nabla H(\mathbf{a}) = x - 2t\mathbf{a}$$

This gives $\mathbf{a} = x/(2t)$. Substituting it back to $u(x, t; \mathbf{a}, b)$ (we can choose any b , say $b = 0$) then we have the solution

$$u(x, t; x/(2t), 0) = \frac{|x|^2}{2t} - t \frac{|x|^2}{4t^2} = \frac{|x|^2}{4t}$$

2. (a) Solve the following equation, using characteristics,

$$u_t + uu_x = 0, \quad u(x, 0) = a \sin x,$$

where $a > 0$ is a constant. Determine the first time when a shock forms.

(b) Now consider

$$u_t + uu_x + cu = 0$$

with the same initial data and a positive constant c . How large does c need to be in order to prevent shock formation?

Solution: (a) The characteristic equations are

$$\frac{dx}{dt} = u, \quad \frac{du}{dt} = 0$$

with the initial condition

$$x(y, 0) = y, \quad u(y, 0) = a \sin y.$$

The solution is simply given by

$$x(y, t) = y + at \sin y, \quad u(y, t) = a \sin y$$

Then the solution is given implicitly by

$$u = a \sin(x - ut)$$

The shock forms when $x_y(y, t) = 0$, i.e. $1 + at \cos y = 0$. Then first time it forms is

$$T^* = \min_{\cos y < 0} \frac{1}{-a \cos y} = \frac{1}{a}$$

(b) Similarly, the characteristic equation is

$$\frac{dx}{dt} = u, \quad \frac{du}{dt} = -cu.$$

with initial condition

$$x(y, 0) = y, \quad u(y, 0) = a \sin y.$$

The solution to the equation for u is

$$u(y, t) = ae^{-ct} \sin y,$$

and therefore the equation for x is

$$\frac{dx}{dt} = ae^{-ct} \sin y.$$

The solution for x is

$$x(y, t) = y + \frac{1 - e^{-ct}}{c} a \sin y.$$

There is no shock only if

$$x_y(y, t) = 1 + \frac{1 - e^{-ct}}{c} a \cos y \neq 0$$

for any y and any $t > 0$. Since $1 - e^{-ct} \geq 0$, the possible shock forms only at $\cos y = -1$ and thus only for t at which

$$1 = \frac{1 - e^{-ct}}{c} a$$

The above equation has a solution for t if and only if $c < a$. In another word, c has to be at least a to prevent shock formation.

3. Determine the type of the equation and then find its general solution

$$x^2 u_{xx} + 2xy u_{xy} + u_{yy} = u_y.$$

Solution: Let characteristics is the solution of the equation

$$x^2 \left(\frac{dy}{dx} \right)^2 - 2xy \frac{dy}{dx} + 1 = 0, \quad (\text{Notice the minus sign})$$

or

$$\frac{dy}{dx} = \frac{y \pm \sqrt{y^2 - 1}}{x}.$$

Therefore, the equation is elliptic when $|y| < 1$, parabolic when $|y| = 1$ and hyperbolic when $|y| > 1$. We are going to find the general solution only for hyperbolic equations. Without loss of generality, we assume $y > 1$. Then we have to use the following change of variable

$$\xi = \ln x - \int \frac{dy}{y + \sqrt{y^2 - 1}}, \quad \eta = \ln x - \int \frac{dy}{y - \sqrt{y^2 - 1}}.$$

where the two variables are chosen to be the integration constant of the characteristic equation.

Since

$$\begin{aligned} \int \frac{dy}{y + \sqrt{y^2 - 1}} &= \int (y - \sqrt{y^2 - 1}) dy \\ &= \frac{y^2}{2} - \frac{y\sqrt{y^2 - 1}}{2} + \frac{1}{2} \ln(y + \sqrt{y^2 - 1}) \end{aligned} \quad (5)$$

ξ can be written explicitly as

$$\xi = \ln x - \frac{y^2}{2} - \frac{y\sqrt{y^2 - 1}}{2} + \frac{1}{2} \ln(y + \sqrt{y^2 - 1}).$$

Similarly

$$\eta = \ln x - \frac{y^2}{2} + \frac{y\sqrt{y^2 - 1}}{2} - \frac{1}{2} \ln(y + \sqrt{y^2 - 1}).$$

It is easy to see that

$$\xi_x = \frac{1}{x}, \quad \xi_y = -y - \sqrt{y^2 - 1}, \quad \xi_{xx} = -\frac{1}{x^2}, \quad \xi_{xy} = 0, \quad \xi_{yy} = -1 - \frac{y}{\sqrt{y^2 - 1}}.$$

$$\eta_x = \frac{1}{x}, \quad \eta_y = -y + \sqrt{y^2 - 1}, \quad \eta_{xx} = -\frac{1}{x^2}, \quad \eta_{xy} = 0, \quad \eta_{yy} = -1 + \frac{y}{\sqrt{y^2 - 1}}.$$

Moreover, by chain rule, we have

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x \\ u_y &= u_\xi \xi_y + u_\eta \eta_y \\ u_{xx} &= u_{\xi\xi} \xi_x^2 + 2u_{\xi\eta} \xi_x \eta_x + u_{\eta\eta} \eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx} \\ u_{xy} &= u_{\xi\xi} \xi_x \xi_y + u_{\xi\eta} (\xi_x \eta_y + \xi_y \eta_x) + u_{\eta\eta} \eta_x \eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy} \\ u_{yy} &= u_{\xi\xi} \xi_y^2 + 2u_{\xi\eta} \xi_y \eta_y + u_{\eta\eta} \eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy} \end{aligned} \quad (6)$$

Then the original equation can be transformed into an equation in ξ and η , i.e

$$\begin{aligned}
& x^2(u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_{\xi\xi\xi x} + u_{\eta\eta x x}) \\
& + 2xy(u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_{\xi\xi\xi y} + u_{\eta\eta x y}) \\
& + u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_{\xi\xi\xi y} + u_{\eta\eta y y} \\
& - u_{\xi\xi}\xi_y - u_{\eta\eta}\eta_y = 0
\end{aligned} \tag{7}$$

Above equation can be simplified as

$$-4y^2u_{\xi\eta} + (-2 - \frac{y}{\sqrt{y^2-1}} + y + \sqrt{y^2-1})u_{\xi} + (-2 + \frac{y}{\sqrt{y^2-1}} + y - \sqrt{y^2-1})u_{\eta} = 0$$

We need to transform the coefficients which is a function of y , into function of ξ and η . But it seems there is no explicit formula for this.

Remark: There is a typo ($2x$ instead of $2xy$ for the coefficient of u_{xy} in the problem. You can the solution from the book at Page 53, *Partial differential equations: methods and applications*, 2nd edition.

4. Consider the one-dimensional wave equation with dissipation

$$u_{tt} - c^2\Delta_x u + \alpha u_t = 0, \tag{8}$$

$$u(x, 0) = g(x), \quad u_t(x, 0) = h(x) \tag{9}$$

where g and h have compact support, $c > 0$ and $\alpha \geq 0$ are constant. (a) Show that the energy $E(t) = \frac{1}{2} \int_{R^n} (u_t^2 + c^2|\nabla_x u|^2) dx$ is non-increasing in $t > 0$; (b) Use the energy method to prove that solutions are uniquely determined by their Cauchy data.

Solution: (a) We can take the time derivative of the energy,

$$\begin{aligned}
\frac{d}{dt}E(t) &= \int_{R^n} (u_t u_{tt} + c^2 \nabla_x u \nabla_x u_t) dx \\
&= \int_{R^n} (u_t (c^2 \Delta u - \alpha u_t) + c^2 \nabla_x u \nabla_x u_t) dx \\
&= -\alpha \int_{R^n} u_t^2 dx + \int_{R^n} \nabla \cdot (u_t \nabla u) dx \\
&= -\alpha \int_{R^n} u_t^2 dx \leq 0
\end{aligned} \tag{10}$$

Therefore the energy is non-increasing in time.

(b) If u_1 and u_2 are two solutions of the same equation with the initial condition, then let $w = u_2 - u_1$, then w satisfies the equation

$$w_{tt} - c^2\Delta w + \alpha w_t = 0$$

and the zero initial condition, $w(x, 0) = 0$ and $w_t(x, 0) = 0$. If we define the same kind of energy

$$E(t) = \frac{1}{2} \int_{R^n} (w_t^2 + c^2|\nabla_x w|^2) dx$$

then $E(0) = 0$ and $E(t) \leq E(0) = 0$. This implies that $w_t \equiv 0$ and $\nabla w \equiv 0$. From the initial condition, we have $w \equiv 0$ and the solution is unique.