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AN EXAMPLE OF A SMOOTH LINEAR PARTIAL DIFFERENTIAL EQUATION WITHOUT SOLUTION

BY HANS LEWY

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Introduction

In dealing with the existence of solutions of partial differential equations it was customary during the nineteenth century and it still is today in many applications, to appeal to the theorem of Cauchy-Kowalewski which guarantees the existence of analytic solutions for analytic partial differential equations. On the other hand a deeper understanding of the nature of solutions requires the admission of non-analytic functions in equations and solutions. For large classes of equations this extension of the range of equation and solution has been carried out since the beginning of this century. In particular much attention has been given to linear partial differential equations and systems of such. Uniformly the experience of the investigated types has shown that—speaking of existence in the local sense—there always were solutions, indeed, smooth solutions, provided the equations were smooth enough. It was therefore a matter of considerable surprise to this author, to discover that this inference is in general erroneous. More precisely, there exist linear partial differential equations with coefficients in C^∞ which possess not a single smooth solution in any neighborhood. The example to be presented in this paper is an equation of first order in three independent variables with complex-valued coefficients and unknown function, or, what amounts to the same, a system of two equations of first order for two functions of three variables, all real.

A theorem

We begin the discussion of this example by first deriving the following

THEOREM. *Let x_1, x_2, y_1 be independent real variables, u a dependent complex-valued variable and $\psi(y_1)$ a real function of C^1. Consider the linear equation*

$$[-(\partial/\partial x_1) - i(\partial/\partial x_2) + 2i(x_1 + ix_2)(\partial/\partial y_1)]u = \psi'(y_1). \tag{1}$$

Assume that there exists a solution $u(x_1, x_2, y_1)$ of (1) in a neighborhood $N = N(0, 0, y_1^0)$ of the point $(0, 0, y_1^0)$, with u in C^1. Then $\psi(y_1)$ is analytic at $y_1 = y_1^0$.

PROOF. Integrate (1) over a circle in N of equations

$$x_1^2 + x_2^2 = \text{const.} = y_2, \qquad y_1 = \text{const.}$$

On introducing the angle θ by $x_1 + ix_2 = y_2^{\frac{1}{2}}e^{i\theta}$ we find

$$\partial/\partial x_1 + i\partial/\partial x_2 = \alpha(\partial/\partial \log y_2^{\frac{1}{2}} + i\partial/\partial\theta) \tag{2}$$

with α determined by applying (2) on $\log y_2^{\frac{1}{2}}$. Therefore $\alpha = (x_1 + ix_2)y_2^{-1}$ and

$$\begin{aligned}\int_0^{2\pi} (\partial/\partial x_1 + i\partial/\partial x_2)u \, d\theta &= \int_0^{2\pi} e^{i\theta} y_2^{-\frac{1}{2}}[(\partial/\partial \log y_2^{\frac{1}{2}})u + i(\partial/\partial\theta)u] \, d\theta \\ &= \int_0^{2\pi} e^{i\theta} y_2^{-\frac{1}{2}}[(\partial/\partial \log y_2^{\frac{1}{2}})u + u] \, d\theta \\ &= (\partial/\partial y_2)2 \int_0^{2\pi} e^{i\theta} y_2^{\frac{1}{2}} u \, d\theta.\end{aligned}$$

Setting

$$U(y_1, y_2) = i \int_0^{2\pi} e^{i\theta} y_2^{\frac{1}{2}} u \, d\theta \tag{3}$$

we obtain from (1)

$$\partial U/\partial y_1 + i\partial U/\partial y_2 = \pi\psi'(y_1). \tag{4}$$

Note that (3) implies $U(y_1, 0) = 0$. Furthermore,

$$V(y_1, y_2) = V(y) = U(y_1, y_2) - \pi\psi(y_1)$$

is in C^1 and satisfies

$$\partial V/\partial y_1 + i\partial V/\partial y_2 = 0$$

which states that $V(y)$ is an analytic function of y whose domain of existence certainly includes all those points (y_1, y_2) for which $y_2 > 0$ is sufficiently small and y_1 is such that (x_1, x_2, y_1) lies in N for $x_1^2 + x_2^2 \leqq y_2$. As $V = -\pi\psi$ on $y_2 = 0$ with ψ real by hypothesis, V can be continued across $y_2 = 0$ as analytic function of y. Therefore V is analytic on $y_2 = 0$. Hence $\psi(y_1)$ is analytic at and near $y_1 = \overset{0}{y_1}$.

We apply the Theorem in a negative sense. Suppose we take an equation (1) in which $\psi(y_1)$ is real and in C^∞, but not analytic at $y_1 = \overset{0}{y_1}$, then the equation (1) can have no solution u which is in C^1 in any $N = N(0, 0, \overset{0}{y_1})$.

It becomes desirable to remove the restriction to special neighborhoods which occurs in this example.

Example of an equation without solution

With the aid of a **periodic** real function $\psi(y_1)$ of C^∞, which is analytic in no y_1-interval, we can construct a function $F(x_1, x_2, y_1)$ of C^∞ such that

$$[-\partial/\partial x_1 - i\partial/\partial x_2 + 2i(x_1 + ix_2)\partial/\partial y_1]u = F(x_1, x_2, y_1) \tag{5}$$

has no H^1-solution, no matter what open (x_1, x_2, y_1)-set is taken as domain of existence. A function is said to be in H^1 if its first partial derivatives satisfy a Hölder condition with positive exponent, provided the distance of the points involved does not exceed 1.

Choose a countable set of points $P_1, P_2, \cdots$, which is dense in the (x_1, x_2, y_1)-space, and positive radii $\rho_1, \rho_2, \cdots$ with $\lim_{n\to\infty} \rho_n = 0$ and denote by N_j the sphere of radius ρ_j about P_j. An arbitrary open set always contains some

N_j. Designate by p_j and q_j the x_1 and x_2-coordinates of P_j and put

$$c_j = \max [j, | p_j |, | q_j |].$$

Consider the sequences ε of real numbers $\varepsilon_1, \varepsilon_2, \cdots$ with

$$\text{l.u.b.}_{j=1,2,\cdots} | \varepsilon_j | < \infty \tag{6}$$

and set

$$F_\varepsilon(x_1, x_2, y_1) = \textstyle\sum_1^\infty \varepsilon_j c_j^{-c_j} \psi'(y_1 + 2q_j x_1 - 2p_j x_2).$$

F_ε is itself in C^∞, for if we formally construct a ν^{th} derivative D^ν by termwise differentiation

$$D^\nu F_\varepsilon(x_1, x_2, y_1) = \textstyle\sum_{j=1}^\infty \varepsilon_j c_j^{-c_j} \psi^{(\nu+1)}(y_1 + 2q_j x_1 - 2p_j x_2) q_j^{\nu_1}(-p_j)^{\nu_2} 2^{\nu_1+\nu_2},$$
$$\nu_1 + \nu_2 \leqq \nu$$

the series of the right member converges absolutely and uniformly as $\psi^{(\nu+1)}$ is bounded on account of the periodicity of ψ and $\sum_{j=1}^\infty c_j^{-c_j+\nu} < \infty$.

Recall that the sequences ε with norm (6) form a complete metric (Banach) space and consequently this space is not exhausted by a countable sum of nondense sets. This fact enables us to demonstrate the existence of a sequence ε^* in this space such that for $F = F_{\varepsilon^*}$ there is no open set of points (x_1, x_2, y_1) on which (5) has an H^1-solution.

PROOF. Denote by H_{nm}^1 the property of a function of having first partial derivatives satisfying a Hölder condition of exponent $1/n$ and constant m, where $n, m = 1, 2, 3, \cdots$. Evidently $H^1 = \sum_{n,m} H_{nm}^1$ and all functions which vanish at P_j and satisfy H_{nm}^1 are compact so that the functions of H^1 vanishing at P_j and existing in N_j are all contained in a countable sum of compact sets. Let E_{jnm} be the set of sequences ε such that (5) with $F = F_\varepsilon$ has a solution existing in N_j and belonging there to H_{nm}^1. E_{jnm} is closed for the following reason: we can always suppose that the solutions vanish at P_j and out of any sequence of solutions belonging to various ε of E_{jnm} tending to a limit sequence we can select a suitable subsequence of sequences ε and corresponding solutions u of (5) with $F = F_\varepsilon$ such that the solutions converge in N_j together with their first partials. The limit function then satisfies the limit equation in N_j and lies in H_{nm}^1, proving that E_{jnm} is closed.

But E_{jnm} is nondense. In view of the closure property of E_{jnm} it suffices to ascertain that every sequence of E_{jnm} is limit of sequences of the Banach space none of which is in E_{jnm}. Observe that if α and β are elements of E_{jnm} so is $(\alpha - \beta)/2$. Consider the particular sequence $\delta^j = (\delta_1^j, \delta_2^j, \cdots)$ where δ_k^j is the Kronecker symbol. It will be seen in the last § that (5) has no C^1-solution in N_j for $F = F_{\delta^j}$. Hence since (5) is linear, none of the sequences $\varepsilon + \lambda\delta^j$, $\lambda \neq 0$, ε in E_{jnm}, can be E_{jnm}. Taking $| \lambda |$ arbitrarily small, we obtain the sequence ε of E_{jnm} as limit of sequences not in E_{jnm}.

It follows that $\sum_{j,n,m} E_{jnm}$ does not exhaust the space of all sequences ε

with (6). Let ε^* be a sequence not in $\sum E_{jnm}$. Then (5) with $F = F_{\varepsilon^*}$ has no H^1-solution in any N_j, $j = 1, 2, \cdots$ or any other open set.

The only thing remaining is the proof that

$$(7)\qquad \begin{aligned} Lu \equiv [-\partial/\partial x_1 - i\partial/\partial x_2 + 2i(x_1 + ix_2)\partial/\partial y_1]u \\ = \psi'(y_1 + 2q_jx_1 - 2p_jx_2) \end{aligned}$$

has no C^1-solution in N_j. Consider the transformation

$$X_1 = x_1 - p_j, \qquad X_2 = x_2 - q_j, \qquad Y_1 = y_1 + 2q_jx_1 - 2p_jx_2$$

whose inverse is

$$x_1 = X_1 + p_j, \qquad x_2 = X_2 + q_j, \qquad y_1 = Y_1 - 2q_jX_1 + 2p_jX_2$$

and for which

$$\partial/\partial x_1 = (\partial/\partial X_1) + 2q_j(\partial/\partial Y_1), \qquad \partial/\partial x_2 = (\partial/\partial X_2) - 2p_j(\partial/\partial Y_1),$$
$$\partial/\partial y_1 = \partial/\partial Y_1.$$

We verify that (7) is transcribed into

$$(8)\qquad Lu \equiv [-\partial/\partial X_1 - i\partial/\partial X_2 + 2i(X_1 + iX_2)\partial/\partial Y_1]u = \psi'(Y_1).$$

If (7) had a C^1-solution in N_j, (8) would have a C^1-solution in the neighborhood of the transform of the center of N_j, whose new coordinates are

$$X_1 = X_2 = 0, \qquad Y_1 = Y_1^0 = y_1(P_j).$$

But here our Theorem applies that for the solution to exist $\psi(Y_1)$ would have to be analytic at this Y_1^0, contrary to the hypothesis about ψ.

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