Uniqueness of classical solution to a two-phase Stefan problem with supercooling

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Abstract

For the two-phase Stefan problem with undercooling, it is proved by Götz and Zaltzman [4] that the free boundary may have a jump. While they prove that the solution becomes classical after this jump, the uniqueness of a classical solution is unknown. By comparison principle type arguments, we prove the uniqueness of classical solution for this problem under some mild restriction on the initial data.

1 Introduction

The Stefan problem models the evolution of temperature in a material with a solid phase and liquid phase, which we shall refer to as "ice region" and "water region" respectively. We define the curve $B(t)$ as the interface that separates the ice region $[-1, B(t))$ and the water region $(B(t), 1]$. The Stefan problem is given by the following system of equations

$$\begin{cases} u_t = u_{xx} \quad \text{in} \quad (-1, B(t)) \cap (B(t), 1) \\ \dot{B}(t) = -u_x(B(t) - 0, t) + u_x(B(t) + 0, t) \\ u(x, 0) = u_0(x) \\ u(-1, 0) = f_1(t) \\ u(1, 0) = f_2(t). \end{cases} \tag{1.1}$$

Götz and Zaltzman [4] used kinetic approximation to show the free boundary of the solution to (1.1) is smooth everywhere, except at the time when it may have a jump. After the jump, they proved the existence of a classical solution, however the uniqueness was unclear. A similar one-phase Stefan problem of this type is studied by Fasano and Primicerio [3], however their method does not apply to the two-phase problem. In this paper we study the Stefan system where the initial data is taken from the density right after a jump, and our goal is to show that the classical solution is unique.

The main tool we use is comparison principle between solutions to (1.1) with different initial data. We mention that although all arguments in this note can be written in a PDE argument, the motivation indeed comes from the particle system method used by Chayes and Swindle [1], where they constructed a particle system to show the hydrodynamic limiting behavior corresponds to the solution of the one-phase Stefan problem.

The organization of this paper is as follows. In Section 1.1, we state the assumptions on the initial data u_0, and state our main theorem. In Section 2, we present three technical lemmas regarding the flux of the heat equation with Dirichlet boundary data. In Section 3, we will compare the Stefan problem with another Stefan problem with a modified initial data, whose solution is already known to be unique. We will show for all $\delta > 0$, the free boundary of the original problem will be within distance δ of the free boundary of the new problem, therefore it is unique since δ can be arbitrarily small.

1.1 Summary of Results

Let $u_1(x,t)$ and $B_1(t)$ be the solution of the two-phase Stefan problem (1.1) with initial condition $\lim_{x\to 0-} u_1(x,0) = 0, \lim_{x\to 0+} u_1(x,0) = 1, B_1(0) = 0$.

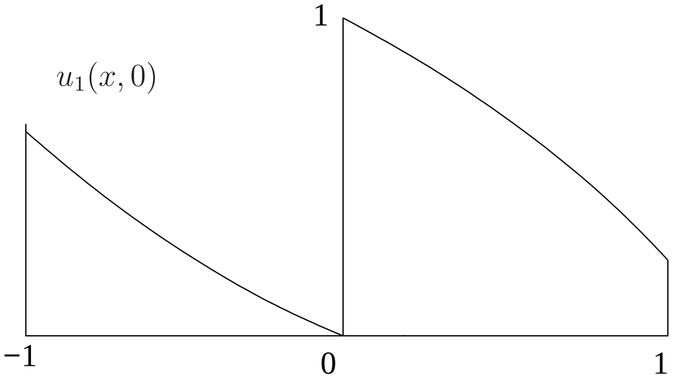

Figure 1: The initial data $u_1(x,0)$

We make the following assumptions on the initial data:

(H1) $u_1(x,0) \in C^1((-1,0)\cup(0,1))$, and the derivative is bounded by M. Moreover, $u_1(x,0) \geq 0$ for all $x \in [-1,1]$.

(H2) At the free boundary, the initial density of ice is 0, and the initial density of supercooled water is 1:

$$\lim_{x\to 0-} u(x,0) = 0, \lim_{x\to 0+} u(x,0) = 1.$$

(H3) On the other hand, near the free boundary, we assume the density of supercooled water is not too close to 1 in the sense that

$$-\lim_{x\to 0-} u_x(x,0) \leq -\lim_{x\to 0+} u_x(x,0) := C,$$

Which is true in Zaltzman's paper, since $u_1(x,0)$ is the density after a jump.

(H4) For the boundary condition, we assume $u_1(-1,t) \geq 0$, $u_1(1,t) \geq 0$, which are both Lipschitz continuous in time.

For the Stefan equation (1.1) with initial data u_0 satisfying **(H1)-(H4)**, Götz and Zaltzman [4] prove the existence of a classical solution (u_1, B_1), while the uniqueness of such solution is unknown. Note that the main difficulty here is caused by the assumption **(H2)**. Here the initial density of supercooled water is 1 at the free boundary, which is exactly the borderline case: if the density is greater than 1, then it is proved in [4] that the free boundary would have a jump immediately; and if the density is strictly less than on, then it is proved in [2] that there exists a unique classical solution.

Our main result is as follows:

Theorem 1.1. *Consider the Stefan equation* (1.1) *with initial data* u_0 *satisfying the assumptions* ***(H1)-(H4)****. Then there exists a unique pair of classical solution* (u_1, B_1) *to equation* (1.1).

2 Lemmas on the comparison and estimation of flux

In this section, we present three technical lemmas. We consider the heat equation with Dirichlet boundary condition, where the boundary is some function of time, and estimate the flux across the boundary.

Lemma 2.1. *Let* $v_1(x,t)$ *be the solution of the heat equation with initial data* $v_1(x,0) > 0$*. Assume* v_1 *satisfies the Dirichlet boundary condition* $v_1(L(t),t) = f(t)$ *and* $v_1(R_1(t),t) = 0$*, where* $f(t) \geq 0$ *for all* t*. Let* $v_2(x,t)$ *be the solution of the heat equation with the same initial data and same value on left boundary, however the right boundary condition is* $v_2(R_2(t),t) = 0$*, where* $R_1(t) \leq R_2(t)$ *for* $0 < t < T$*, then*

$$\int_0^T -\frac{d}{dx} v_1(R_1(t),t)dt \geq \int_0^T -\frac{d}{dx} v_2(R_2(t),t)dt \tag{2.1}$$

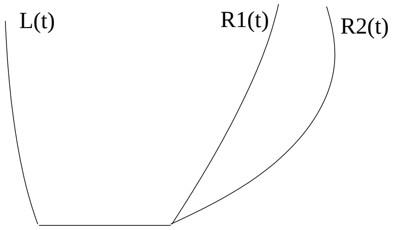

Figure 2: Two solutions to heat equation with same left boundary $L(t)$ and different right boundaries $R_1(t)$ and $R_2(t)$.

Proof. We compare $v_1(x,t)$ with $v_2(x,t)$ in the region $Q_T = \{(x,t) : 0 \leq t \leq T, L(t) \leq x \leq R_1(t)\}$.

Note that v_1 and v_2 agree on the left boundary and at $t = 0$, while on the right boundary $x = R_1(t)$, we have $v_1(R_1(t),t) = 0, v_2(R_1(t),t) \geq 0$, hence $v_1 \geq v_2$ on the parabolic boundary of Q_T. Therefore, by maximum principle, we have

$$v_2(x,t) \geq v_1(x,t) \text{ in } Q_T. \tag{2.2}$$

Intergrate it in x at $t = T$, and from the fact that $R_2(T) \geq R_1(T)$, we have

$$\int_{L(T)}^{R_1(T)} v_1(x,T)dx \leq \int_{L(T)}^{R_2(T)} v_2(x,T)dx \tag{2.3}$$

Since $v_2(L(t),t) = v_1(L(t),t),\ v_2(x,t) \geq v_1(x,t)$ for $x > L(t)$, we have

$$\int_0^T \frac{d}{dx} v_1(L(t),t)dt \leq \int_0^T \frac{d}{dx} v_2(L(t),t)dt \tag{2.4}$$

Moreover, by intergrating the heat equation, we have

$$\int_{L(T)}^{R_i(T)} v_i(x,T)dx - \int_{L(0)}^{R_i(0)} v_i(x,0)dx = \int_0^T \frac{d}{dx} v_i(R_i(t),t)dt - \int_0^T \frac{d}{dx} v_i(L(t),t)dt \tag{2.5}$$

for $i = 1, 2$. From Equation (2.3), (2.4), (2.5) and the fact that v_1 and v_2 have the same initial data, we have

$$\int_0^T -\frac{d}{dx} v_1(R_1(t),t)dt \geq \int_0^T -\frac{d}{dx} v_2(R_2(t),t)dt. \tag{2.6}$$

□

Lemma 2.2. *Let $v(x,t)$ be the solution of the heat equation with initial data $v(x,t) \equiv 1$ for $x < 0$, and right boundary condition $v(0,t) \equiv 0$, (there is no left boundary). Let F(T) be the flux across the right boundary within time T, i.e.*

$$F(T) = \int_0^T -\frac{d}{dx}v(0,t)dt \tag{2.7}$$

Then $F(T) = 2\sqrt{T/\pi}$.

Proof. The solution of heat equation is given by

$$v(x,t) = \int_{\mathbb{R}} \Phi(x-y,t)g(y)dy, \tag{2.8}$$

where $\Phi(x,t)$ is the fundamental solution to heat equation, i.e.

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}, \tag{2.9}$$

and

$$g(y) = 1_{(-\infty,0)}(y) - 1_{(0,\infty)}(y). \tag{2.10}$$

Therefore

$$\begin{aligned}
v_x(0,t) &= \int_{\mathbb{R}} \frac{\partial \Phi}{\partial x}(-y,t)g(y)dy \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}}\frac{y}{2t}e^{-\frac{y^2}{4t}}(1_{(-\infty,0)}(y) - 1_{(0,\infty)}(y))dy \\
&= \frac{2}{\sqrt{4\pi t}}\int_{-\infty}^0 \frac{y}{2t}e^{-\frac{y^2}{4t}}dy \\
&= \frac{1}{\sqrt{\pi t}}\int_{-\infty}^0 -\frac{\partial}{\partial y}(e^{-\frac{y^2}{4t}})dy \\
&= \frac{1}{\sqrt{\pi t}},
\end{aligned}$$

□

hence the flux is given by

$$\begin{aligned}
F(T) &= \int_0^T -v_x(0,t)dt \\
&= \int_0^T \frac{1}{\sqrt{\pi t}}dt \\
&= 2\sqrt{\frac{t}{\pi}}
\end{aligned}$$

Lemma 2.3. *Let $v(x,t)$ be the solution of the heat equation with initial data $v(x,t) \equiv 1$ for $x > 0$, and v vanishes at the left boundary $B(t)$, where $B(t) \geq \sqrt{8t/\pi}$, (and there is no right boundary). Let $F(T)$ be the flux across the left boundary within time T, i.e.*

$$F(T) = \int_0^T \frac{d}{dx} v(B(t),t) dt \tag{2.11}$$

Then $F(t) \geq 1.1\sqrt{8t/\pi}$ as $t \to 0$.

Proof. If $B(t) \equiv 0$, then it is the same situation with lemma 2. However, since now $B(t)$ is moving towards inside, we expect a bigger flux.

For a particle starting at y, where $y > B(t)$, if use $Y(t)$ to denote the position of the particle at time t, since $Y(t) \sim N(y,t)$, we have

$$\begin{aligned} P(\max_{s\in[0,t]} Y(s) < B(t)) &\geq P(Y(t) < B(t)) \\ &= \int_{\frac{y-B(t)}{\sqrt{t}}}^{\infty} e^{-\frac{x^2}{2}} dx \end{aligned}$$

So the flux generated by the particles in $[B(t), \infty)$ is:

$$\begin{aligned} F_1(t) &= \int_{B(t)}^{\infty} \int_{\frac{y-B(t)}{\sqrt{t}}}^{\infty} e^{-\frac{x^2}{2}} dx\ dy \\ &= \int_0^{\infty} \int_{\frac{y}{\sqrt{t}}}^{\infty} e^{-\frac{x^2}{2}} dx\ dy \\ &\sim \sqrt{\frac{2t}{\pi}} \end{aligned}$$

Now consider the flux generated by the particles in $[B(t)/2, B(t))$. For a particle starting at y, where $B(t)/2 < y < B(t)$, since $B(t) > y$, we have

$$P(\max_{s\in[0,t]} Y(s) < B(t)) \geq P(Y(t) < B(t)) > P(Y(t) < y) = \frac{1}{2} \tag{2.12}$$

So the flux generated by the particles in $[B(t)/2, B(t))$ is:

$$F_2(t) > \frac{1}{2}\,\frac{B(t)}{2} \geq \sqrt{\frac{t}{2\pi}} \tag{2.13}$$

At last we consider the flux generated by the particles in $[0, B(t)/2)$. For a particle starting at y, where $0 < y < B(t)/2$, we have

$$\begin{aligned}
P(\max_{s\in[0,t]} Y(s) < B(t)) &\geq P(Y(t) < B(t)) \\
&> P(Y(t) < y + B(t)/2) \\
&= \int_{-\infty}^{B(t)/2\sqrt{t}} e^{-\frac{x^2}{2}} dx \\
&\geq \int_{-\infty}^{\sqrt{2/\pi}} e^{-\frac{x^2}{2}} dx \geq 0.7
\end{aligned}$$

So the flux generated by the particles in $[0, B(t)/2)$ is:

$$F_3(t) > 0.7\,\frac{B(t)}{2} \geq 0.7\sqrt{\frac{2t}{\pi}} \tag{2.14}$$

Add up F_1, F_2, F_3, we have

$$F(t) = F_1(t) + F_2(t) + F_3(t) \geq 1.1\sqrt{\frac{8t}{\pi}} \tag{2.15}$$

$\square$

3 Comparison with a new Stefan problem

Next we will show that the Stefan problem with initial data u_0 satisfying assumptions (u, B) has a unique classical solution. Let (u_1, B_1) be a pair of classical solution to (1.1) with initial data u_0. For any $\delta > 0$, we will construct a more regular initial data $u_2(\cdot, 0)$ to the Stefan problem, where the solution (u_2, B_2) is known to be unique, and constructed such that $B_2(t) < B_1(t) < B_2(t) + \delta$. Since δ is arbitrary, we have B_1 is unique.

For each δ, the initial data of u_2 is constructed as following:

- We let the free boundary starts at $x = \delta$, i.e. $B_2(0) = \delta$.
- In the ice region $\{x < B_2(0)\}$, we shift the density of u_1 to the right by δ, i.e. $u_2(x, 0) = u_1(x - \delta, 0)$ for $-1 + \delta < x < \delta$. And in this modified Stefan problem, we let the left boundary be at $x = -1 + \delta$.
- In the water region, for every fixed δ, we can find some slope k, such that the area between $x = 0$, $y = u_1(x, 0)$, $y = 0$ and $y = k(x - \delta)$ is δ. (See Figure 3.)

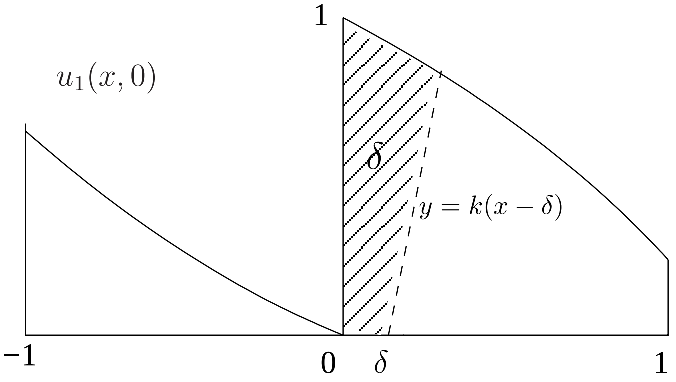

Figure 3: Find a cutoff function for $u_1(x,0)$

We will only keep the density of u_1 below the line $y = k(x-\delta)$ to be the initial density of u_2 in the water region. That is,

$$u_2(x,0) = \begin{cases} k(x-\delta) & \text{for } \delta < x < x' \\ u_1(x,0) & \text{for } x' < x < 1 \end{cases}$$

where x' is the intersection of $y = u_1(x,0)$ and $y = k(x-\delta)$.

Since $\lim_{x\to 0-} u_2(x,0) = 0, \lim_{x\to 0+} u_2(x,0) = 0$, by the following theorem, (u_2, B_2) is unique:

Theorem 3.1 (Kim-Chayes [2])**.** *Consider the 1D Stefan problem (1.1), where the initial data satisfies that $u_0 < 1$ in a neighborhood of 0. Then there is a unique classical solution to this system.*

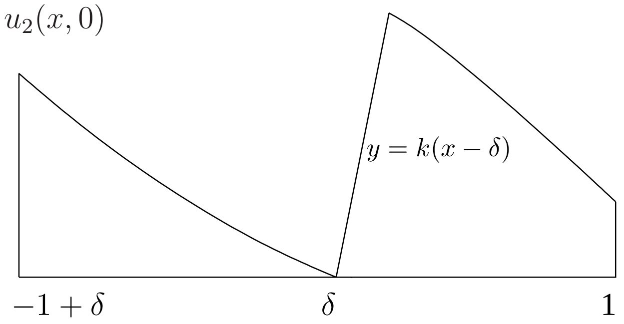

Figure 4: The initial data $u_2(x,0)$

Our goal is to prove that

$$B_2(t) < B_1(t) < B_2(t) + \delta \text{ for } 0 < t < t_0, \tag{3.1}$$

where t_0 only depends on $u_1(x,0)$ and does not depend on δ. Since δ can be chosen to be arbitrarily small, (3.1) directly implies the uniqueness of $B_1(t)$, and the uniqueness of $u_1(x,t)$ follows immediately. Hence to prove Theorem 1.1, it suffices to prove (3.1).

The following proposition gives one direction of (3.1).

Proposition 3.2. *For any $t_1 > 0$, if $B_1(t) > B_2(t) - \delta$ for $0 < t < t_1$, then $B_1(t) < B_2(t)$ for $0 < t < t_1$.*

Proof. Prove by contradiction. Suppose s is the first time such that $B_1(s) = B_2(s)$, where $s < t_1$.

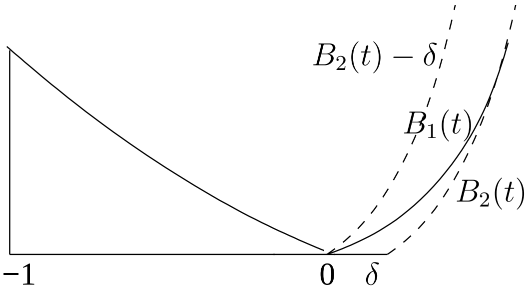

Figure 5: Until $t = s$, $B_1(t)$ is bounded by $B_2(t) - \delta$ and $B_2(t)$

Then since $B_1(s) - B_1(0) = B_2(s) - B_2(0) + \delta$, we have

$$\int_0^s \dot{B}_1(t)dt = \int_0^s \dot{B}_2(t)dt + \delta \tag{3.2}$$

Therefore,

$$F_1^-(s) + F_1^+(s) = F_2^-(s) + F_2^+(s) + \delta \tag{3.3}$$

Here we use $F_1^-(s)$ to denote the flux through the boundary B_1 from the ice side in time $[0,s]$, and $F_1^+(s)$ is the flux from the water side, i.e.

$$F_1^-(s) = \int_0^s -\frac{d}{dx}u_1(B_1(t) - 0, t)dt$$
$$F_1^+(s) = \int_0^s \frac{d}{dx}u_1(B_1(t) + 0, t)dt$$

First, we compare the flux from the ice side. Since $B_2(t) - \delta < B_1(t)$ for $0 < t < s$, by Lemma 2.1, we have

$$F_1^-(s) \leq F_2^-(s) \tag{3.4}$$

Now we will show u_2 also has bigger flux on the water side.

In system 1, u_1 can be seen as the sum of two parts u_{11} and u_{12}, with initial condition

$$\begin{aligned} u_{11}(x,0) &= u_2(x,0) \\ u_{12}(x,0) &= u_1(x,0) - u_{11}(x,0) \end{aligned}$$

$u_{11}(x,t)$ and $u_{12}(x,t)$ both satisfy the heat equation in $[0,s] \times [B_1(t),1]$, and they both vanish at $x = B_1(t)$. As for the right boundary, let $u_{11}(1,t) = u_1(1,t), u_{12}(1,t) \equiv 0$. Then we have $u_1(x,t) = u_{11}(x,t) + u_{12}(x,t)$.

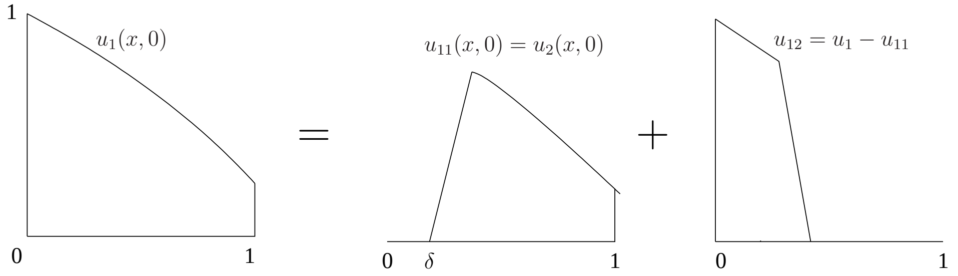

Figure 6: Divide the water side $u_1(x,0)$ into two parts

Since u_{11} and u_2 have the same initial data and the same value at right boundary, and $B_1(t) < B_2(t)$ for $0 < t < s$, Lemma 2.1 yields that $F_2^+(s) \geq F_{11}^+(s)$.

And since $F_{12}^+(s) < \int_0^1 u_{12}(x,0)dx = \delta$, we obtain

$$F_1^+(s) < F_2^+(s) + \delta \tag{3.5}$$

Adding equation (3.4) and (3.5), we obtain a contradiction with equation (3.3). ☐

Proposition 3.3. *There exists a t_0 only depending on $u_1(x,0)$, s.t. for any δ small enough, $B_1(t) > B_2(t) - \delta$ for $0 < t < t_0$.*

Proof. For any $\delta > 0$, since $\dot{B}_1(0) = \infty, \dot{B}_2(0) < \infty$, we know $B_1(t)$ is ahead of $B_2(t) - \delta$ for a short time. Let $t_1 > 0$ be the first time such that $B_1(t_1) = B_2(t_1) - \delta$.

Since $B_1(t_1) - B_1(0) = B_2(t_1) - B_2(0)$, we have

$$F_1^-(t_1) + F_1^+(t_1) = F_2^-(t_1) + F_2^+(t_1) \tag{3.6}$$

We want to show that t_1 cannot be too close to 0, for any δ sufficiently small.

First, by applying Proposition 3.2, we obtain $B_1(t) < B_2(t)$ for $0 < t < t_1$. Therefore $B_2(t) - \delta < B_1(t) < B_2(t)$ for $0 < t < t_1$.

We want to show that if t_1 is smaller than some constant, then

$$F_1^+(t_1) - F_2^+(t_1) > F_2^-(t_1) - F_1^-(t_1) \tag{3.7}$$

which would contradict with equation (3.6).

We first estimate the difference of flux in the ice part. Compare $u_1(x,t)$ to $u_3(x,t)$, where $u_3(x,0) = u_1(x,0)$ for $x < 0$, and u_3 vanishes at $B_3(t)$, which is equal to $B_2(t)$.

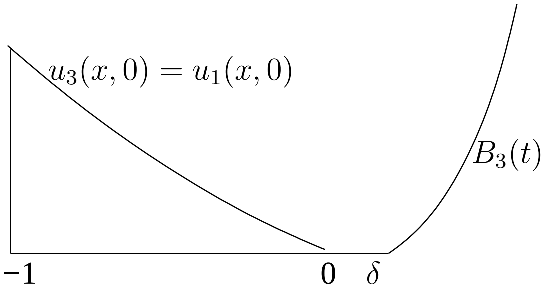

Figure 7: The initial data and right boundary for $u_3(x,t)$

$$\begin{cases} \frac{d}{dt}u_3(x,t) = \frac{d^2}{dx^2}u_3(x,t) \quad \text{for } 0 < t < t_1, -1 < x < B_2(t) \\ u_3(x,0) = u_1(x,0) \text{ for } x < 0 \\ u_3(-1,t) = u_1(-1,t) \\ u_3(B_3(t),t) = 0 \\ B_3(t) = B_2(t) \end{cases}$$

Since $B_3(t) = B_2(t) > B_1(t)$ for $0 < t < t_1$, by Lemma 2.1, we have $F_3^-(t_1) \leq F_1^-(t_1)$. So $F_2^-(t_1) - F_1^-(t_1) < F_2^-(t_1) - F_3^-(t_1)$. Now we will find an upper bound for $F_2^-(t_1) - F_3^-(t_1)$. Notice $B_3(t) = B_2(t)$, and $u_2(x,0)$ is just a shift of $u_3(x,0)$ to the right by distance δ, we can consider the heat equation with Dirichlet boundary $B_2(t)$ and initial data

$$u_2(x,0) - u_3(x,0) = u_1(x-\delta,0) - u_1(x,0),$$

which is less than $C_1\delta$ by assumption (H1). Moreover, in assumption (H3), since we assume

$$-\lim_{x\to 0-}\frac{d}{dx}u_1(x,0) < C,$$

We can find $x_1 < 0$, s.t. $-\frac{d}{dx}u_1(x,0) < 1.01C$ for $x_1 < x < 0$, i.e.

$$u_2(x,0) - u_3(x,0) < 1.01C\delta \text{ for } x_1 < x < 0.$$

Hence Lemma 2.2 yields

$$F_2^-(t_1) - F_3^-(t_1) < 1.01C\delta\sqrt{\frac{8t}{\pi}} \text{ for small } t. \tag{3.8}$$

Now we estimate the difference of flux in the water part. Consider another heat equation $(u_4)_t = (u_4)_{xx}$, with initial data $u_4(x,0) = u_2(x,0)$ and let $u_4(x,t)$ vanishes at $B_4(t) = B_1(t) + \delta$, while the right boundary value is the same as u_2. Let the flux across the left boundary $B_4(t)$ to be $F_4^+(t)$. since $B_4(t) = B_1(t) + \delta > B_2(t)$ for $0 < t < t_1$, by lemma 1, we have $F_4^+(t_1) > F_2^+(t_1)$. Therefore $F_1^+(t_1) - F_2^+(t_1) > F_1^+(t_1) - F_4^+(t_1)$.

Next we will find a lower bound for $F_1^+(t_1) - F_4^+(t_1)$. Notice that B_4 is just B_1 shifted to the right by δ, and as for the initial value, in assumption (H3) we assume

$$-\lim_{x\to 0+}\frac{d}{dx}u_1(x,0) = C,$$

we can find a $x_2 > 0$, s.t. $-\frac{d}{dx}u_1(x,t) \geq 0.99C$ for $0 < x < x_2$, i.e.

$$\begin{aligned} u_1(x,0) - u_4(x+\delta,0) &= u_1(x,0) - u_2(x+\delta,0) \\ &\geq u_1(x,0) - u_1(x+\delta,0) \\ &\geq 0.99C\delta \text{ for } 0 < x < x_2 \quad \text{(by assumption (H3))} \end{aligned}$$

Finally we estimate the flux across the boundary B_1. Since $u_1(x,0) \to 1$ as $x \to 0+$, Lemma 2.2 yields that $B_1(t) \geq \sqrt{8t/\pi}$. Hence we can apply Lemma 2.3 and obtain

$$F_1^+(t_1) - F_4^+(t_1) \geq 1.1 \cdot 0.99C\delta\sqrt{\frac{8t}{\pi}} \text{ for } t_1 \text{ small.} \tag{3.9}$$

Note that by combining inequality (3.8) and (3.9), we obtain the following inequality

$$F_1^+(t_1) - F_4^+(t_1) > F_2^-(t_1) - F_3^-(t_1), \tag{3.10}$$

which implies $F_1^+(t_1) - F_2^+(t_1) > F_2^-(t_1) - F_1^-(t_1)$, hence contradicts with (3.6). $\square$

References

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