Homework 1 Solutions

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Theorem 5.4: Suppose that $D = \{(t, y) \mid a \le t \le b, -\infty < y < \infty\}$ and that $f(t, y)$ is continuous on D. If f satisfies a Lipschitz condition on D in the variable y, then the initial-value problem

$$\begin{aligned} &y'(t) = f(t, y), \qquad a \le t \le b, \\ &y(a) = \alpha, \end{aligned}$$

has a unique solution $y(t)$ for $a \le t \le b$.

Section 5.1, Problem 1(d): Use Theorem 5.4 to show that

$$\begin{aligned} &y' = \frac{4t^3 y}{1 + t^4}, \qquad 0 \le t \le 1, \\ &y(0) = 1. \end{aligned}$$

has a unique solution, and find the solution.

Solution: Note that

$$f(t, y) = \frac{4t^3 y}{1 + t^4}$$

is continuous on $D = \{(t, y) \mid 0 \le t \le 1, -\infty < y < \infty\}$.
Also, f satisfies a Lipschitz condition on D in the variable y:

$$\left|\frac{\partial f}{\partial y}(t, y)\right| = \left|\frac{4t^3}{1 + t^4}\right| \le 2, \qquad 0 \le t \le 1.$$

Thus, the initial-value problem has a unique solution for $a \le t \le b$.

We now solve the initial-value problem.

$$\begin{aligned} &\frac{dy}{dt} = \frac{4t^3 y}{1 + t^4}, \\ &\int \frac{dy}{y} = \int \frac{4t^3}{1 + t^4} dt, \\ &\log y = \log(1 + t^4) + C_1, \\ &y = C(1 + t^4). \end{aligned}$$

Thus, $y(t) = C(1 + t^4)$, and using initial condition, we obtain $y(0) = C = 1$. Hence, the solution to the initial value problem is $y(t) = 1 + t^4$. ✓

It is always recommended to check if your solution ($y(t) = 1 + t^4$) is correct, i.e. whether it satisfies the initial value problem. Note that,

$$\begin{aligned} \frac{dy}{dt} \quad &=^? \quad \frac{4t^3 y}{1 + t^4}, \\ 4t^3 \quad &= \quad \frac{4t^3(1 + t^4)}{(1 + t^4)}, \quad \checkmark \end{aligned}$$

and also, $y(0) = 1$. ✓

Section 5.2, Problem 1(b): Use Euler's method to approximate the solution for the following initial-value problem:

$$y' = 1 + (t-y)^2, \qquad 2 \le t \le 3,$$
$$y(2) = 1,$$

with $h = 0.5$.

Solution: We have $f(t,y) = 1 + (t-y)^2$.
Since $h = 0.5$, $t_i = 2 + 0.5i$. Given the initial condition $w_0 = 1$, Euler's method calculates w_i, $i = 0, 1, 2, \ldots$:

$$\begin{aligned} w_{i+1} &= w_i + hf(t_i, w_i) \\ &= w_i + h(1 + (t_i - w_i)^2) \\ &= w_i + 0.5(1 + (2 + 0.5i - w_i)^2). \end{aligned}$$

So,

$$\begin{aligned} w_1 &= w_0 + 0.5(1 + (2 - w_0)^2) = 1 + 0.5(1 + (2-1)^2) = 2.0, \\ w_2 &= w_1 + 0.5(1 + (2 + 0.5 - w_1)^2) = 2 + 0.5(1 + (2 + 0.5 - 2)^2) = 2.625. \end{aligned}$$

Section 5.2, Problem 1(c): Use Euler's method to approximate the solution for the following initial-value problem:

$$y' = 1 + y/t, \qquad 1 \le t \le 2,$$
$$y(1) = 2,$$

with $h = 0.25$.

Solution: We have $f(t,y) = 1 + y/t$.
Since $h = 0.25$, $t_i = 1 + 0.25i$, $w_0 = 2$. We have

$$\begin{aligned} w_{i+1} &= w_i + hf(t_i, w_i) \\ &= w_i + 0.25(1 + w_i/t_i) \\ &= w_i + 0.25(1 + w_i/(1 + 0.25i)). \end{aligned}$$

So,

$$\begin{aligned} w_1 &= w_0 + 0.25(1 + w_0) = 2 + 0.25(3) = 2.75, \\ w_2 &= w_1 + 0.25(1 + w_1/(1 + 0.25)) = 2.75 + 0.25(1 + 2.75/1.25) = 3.55, \end{aligned}$$

and, similarly, calculate w_3 and w_4.

Section 5.2, Problem 11: Given the initial-value problem:

$$y' = -y + t + 1, \qquad 0 \le t \le 5,$$
$$y(0) = 1,$$

with exact solution $y(t) = e^{-t} + t$.
a) Approximate $y(5)$ using Euler's method with $h = 0.2$, $h = 0.1$, and $h = 0.05$.
b) Determine the optimal value of h to use in computing $y(5)$, assuming $\delta = 10^{-6}$ and that the following equation

$$h = \sqrt{\frac{2\delta}{M}}$$

is valid.

Solution:
a) Note how small the time-step h is compared to the length of the time interval $t \in [0, 5]$. The book, hence, wants you to use the computer (e.g. Matlab) to solve this problem.

$y(5) = 5.00673795$

$N = 25, h = 0.20, w = 5.00377789, E = 0.00296005;$
$N = 50, h = 0.10, w = 5.00515378, E = 0.00158417;$
$N = 100, h = 0.05, w = 5.00592053, E = 0.00081742.$

b) Since the exact solution is $y(t) = e^{-t} + t$, we have $y''(t) = e^{-t}$. Hence, $|y''(t)| \le 1 = M$.

$$h = \sqrt{\frac{2\delta}{M}} = \sqrt{\frac{2 \cdot 10^{-6}}{1}} = 0.00141.$$

Section 5.2, Problem 12: Consider the initial-value problem:

$$y' = -10y \quad 0 \le t \le 2,$$
$$y(0) = 1,$$

which has solution $y(t) = e^{-10t}$. What happens when Euler's method is applied to this problem with $h = 0.1$? Does this behavior violate Theorem 5.9?

Solution: Using Euler's method, we get:

$$\begin{aligned} w_{i+1} &= w_i + hf(t_i, w_i) \\ &= w_i + 0.1 \cdot (-10 w_i) \\ &= w_i - w_i = 0, \quad \text{for all } i. \end{aligned}$$

We can also run the program to get the following results:

after the first step ($t = 0.1$):
$N = 1, h = 0.10, t = 0.10, w = 0.0000000000e + 000, y = 3.6787944117e - 001, E = 3.6787944117e - 001$;

after 20 steps ($t = 2$):
$N = 20, h = 0.10, t = 2.00, w = 0.0000000000e + 000, y = 2.0611536224e - 009, E = 2.0611536224e - 009$.

Theorem 5.9 gives the following estimate:

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right].$$

For our problem, $h = 0.1$, $|y''(t)| = |100e^{-10t}| \le 100 = M$, $\left|\frac{\partial f}{\partial y}(t, y)\right| = |-10| = 10 = L$, $a = 0$.
We now see if the estimate holds. After the first step, we have

$$|0.369 - 0| \le \frac{10}{20}\left[e^{10\cdot(0.1-0)} - 1\right] = 0.859. \qquad \checkmark$$

After the final step, we have

$$|2.06 \cdot 10^{-9} - 0| \le \frac{10}{20}\left[e^{10\cdot(2-0)} - 1\right] = 2.43 \cdot 10^{8}. \qquad \checkmark$$

Thus, even though we obtain an incorrect solution, this behavior does not violate the theorem.

Section 5.2, Problem 15: Let

$$E(h) = \frac{hM}{2} + \frac{\delta}{h}.$$

a) For the initial-value problem

$$\begin{aligned} &y' = -y + 1, \qquad 0 \leq t \leq 1, \\ &y(0) = 0, \end{aligned} \tag{1}$$

compute the value of h to minimize $E(h)$. Assume $\delta = 5 \cdot 10^{-(n+1)}$ if you will be using n-digit arithmetic in part (c).

b) For the optimal h computed in part (a), use the following equation

$$|y(t_i) - u_i| \leq \frac{1}{L}\Big(\frac{hM}{2} + \frac{\delta}{h}\Big)\big[e^{L(t_i - a)} - 1\big] + |\delta_0| e^{L(t_i - a)} \tag{2}$$

to compute the minimal error obtainable.

c) Compare the actual error obtained using $h = 0.1$ and $h = 0.01$ to the minimal error in part (b).

Solution: **a)** In order to find the minimum of $E(h)$, we find $E'(h)$ and set it to equal 0:

$$E'(h) = \frac{M}{2} - \frac{\delta}{h^2} = 0.$$

Therefore,

$$h = \sqrt{\frac{2\delta}{M}}.$$

In order to find M such that $|y''(t)| \leq M$, $t \in [0, 1]$, we need to find the analytic solution of the initial-value problem (2). We have

$$\begin{aligned} &\frac{dy}{dt} = -y + 1, \\ &\int \frac{dy}{-y+1} = \int dt, \\ &-\log(-y + 1) = t + C_1, \\ &\log\left(\frac{1}{-y+1}\right) = t + C_1, \\ &\frac{1}{-y+1} = Ce^t, \\ &y(t) = 1 - \frac{1}{Ce^t}. \end{aligned}$$

We now employ initial condition in order to find constant C:

$$y(0) = 1 - \frac{1}{C} = 0.$$

Thus, $C = 1$, which gives $y(t) = 1 - \frac{1}{e^t}$. We also have: $y'(t) = e^{-t}$ and $y''(t) = -e^{-t}$. Hence, $|y''(t)| = |-e^{-t}| \leq 1$ if $0 \leq t \leq 1$, which gives $M = 1$. We can calculate h now:

$$h = \sqrt{\frac{2\delta}{M}} = \sqrt{\frac{2 \cdot 5 \cdot 10^{-(n+1)}}{1}} = \sqrt{2 \cdot 5 \cdot 10^{-(n+1)}} = \sqrt{10^{-n}} = 10^{-n/2}. \quad \checkmark$$

b) We have $\left|\frac{\partial f}{\partial y}\right| = 1 = L$. $\delta_0 = \epsilon/2$, where $\epsilon = 10^{-n}$ is the machine epsilon. So,

$$\begin{aligned}
|y(t_i) - u_i| &\leq \frac{1}{L}\Big(\frac{hM}{2} + \frac{\delta}{h}\Big)\big[e^{L(t_i-a)} - 1\big] + |\delta_0|e^{L(t_i-a)} \\
&= \frac{1}{1}\Big(\frac{10^{-n/2}\cdot 1}{2} + \frac{5\cdot 10^{-(n+1)}}{10^{-n/2}}\Big)\big[e^{1\cdot(1-0)} - 1\big] + 5\cdot 10^{-n-1}\cdot e^{1\cdot(1-0)} \\
&= \Big(\frac{10^{-n/2}}{2} + \frac{5\cdot 10^{-(n+1)}}{10^{-n/2}}\Big)(e-1) + 5\cdot 10^{-n-1}\cdot e \\
&= \Big(5\cdot 10^{-n/2-1} + 5\cdot 10^{-n/2-1}\Big)(e-1) + 5\cdot 10^{-n-1}\cdot e \\
&= 10^{-n/2}(e-1) + 5\cdot 10^{-n-1}\cdot e. \quad \checkmark
\end{aligned}$$

c) In Matlab, $n = 15$, that is, your arithmetic is exact to 15 digits.

$$\begin{aligned}
|y(t_i) - u_i| &\leq 10^{-n/2}(e-1) + 5\cdot 10^{-n-1}\cdot e = 10^{-7.5}(e-1) + 5\cdot 10^{-16} \\
&= 5.4337\cdot 10^{-8}.
\end{aligned}$$

The solution in Matlab with $h = 0.1$, and $h = 0.01$ will give:
$N = 10, h = 0.10, t = 1.00, w = 6.5132155990e-001, y = 6.3212055883e-001, error = 1.9201001071e-002$;
$N = 100, h = 0.01, t = 1.00, w = 6.3396765873e-001, y = 6.3212055883e-001, error = 1.8470998982e-003$.

That is, if you take a smaller timestep (as long as it is not smaller than $h = 10^{-n/2} = 10^{-7.5}$), the error will get smaller.