Homework 6 Solutions

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Problem 1: Consider the numerical quadrature rule to approximate $\int_0^1 f(x)\,dx$ given by

$$\int_0^1 f(x)\,dx \approx w_1 f(0) + w_2 f(x_1).$$

Find the maximum possible degree of precision you can attain by appropriate choices of w_1, w_2 and x_1. With such choices of w_1 and w_2, approximate $\int_0^1 x^3\,dx$ and compare with the exact value.

Solution: We want the formula

$$\int_0^1 f(x)\,dx = w_1 f(0) + w_2 f(x_1)$$

to hold for polynomials $1, x, x^2, \ldots$. Plugging these into the formula, we obtain:

$$f(x) = x^0 \qquad \int_0^1 1\,dx = x\big|_0^1 = 1 = w_1 \cdot 1 + w_2 \cdot 1,$$

$$f(x) = x^1 \qquad \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2} = w_1 \cdot 0 + w_2 \cdot x_1,$$

$$f(x) = x^2 \qquad \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} = w_1 \cdot 0 + w_2 \cdot x_1^2.$$

We have 3 equations in 3 unknowns:

$$\begin{aligned} w_1 + w_2 &= 1, \\ w_2 x_1 &= \frac{1}{2}, \\ w_2 x_1^2 &= \frac{1}{3}, \end{aligned}$$

or

$$\begin{aligned} w_2 &= 1 - w_1, \\ x_1(1 - w_1) &= \frac{1}{2}, \\ x_1^2(1 - w_1) &= \frac{1}{3}. \end{aligned}$$

Multiplying the second equation by x_1 and subtracting the third equation, we obtain $x_1 = \frac{2}{3}$. Then, $w_2 = \frac{3}{4}$ and $w_1 = \frac{1}{4}$.
Thus, the quadrature formula is

$$\int_0^1 f(x)\,dx = \frac{1}{4} f(0) + \frac{3}{4} f\Big(\frac{2}{3}\Big). \quad \checkmark$$

The accuracy of this quadrature formula is $n = 2$, since this formula holds for polynomials $1, x, x^2$.
We can check how well this formula approximates $\int_0^1 x^3\,dx$:

$$\int_0^1 x^3\,dx \quad = \quad \frac{1}{4}\cdot 0 + \frac{3}{4}\cdot\frac{8}{27} \quad = \quad \frac{2}{9} \quad = \quad 0.2222. \quad \checkmark$$

The exact value of this integral is

$$\int_0^1 x^3\,dx \quad = \quad \frac{x^4}{4}\Big|_0^1 \quad = \quad \frac{1}{4} \quad = \quad 0.2500. \quad \checkmark$$

Problem 2: Determine constants a, b, c, d that will produce a quadrature formula

$$\int_{-1}^1 f(x)\,dx \quad \approx \quad af(-1) + bf(1) + cf'(-1) + df'(1)$$

that has degree of precision 3.

Solution: We want the formula

$$\int_{-1}^1 f(x)\,dx \quad = \quad af(-1) + bf(1) + cf'(-1) + df'(1)$$

to hold for polynomials $1, x, x^2, \ldots$. Plugging these into the formula, we obtain:

$$\begin{array}{lllll}
f(x) = x^0 & \displaystyle\int_{-1}^1 1\,dx = x\Big|_{-1}^1 & = 2 & = & a\cdot 1 + b\cdot 1 + c\cdot 0 + d\cdot 0, \\
f(x) = x^1 & \displaystyle\int_{-1}^1 x\,dx = \frac{x^2}{2}\Big|_{-1}^1 & = 0 & = & a\cdot(-1) + b\cdot 1 + c\cdot 1 + d\cdot 1, \\
f(x) = x^2 & \displaystyle\int_{-1}^1 x^2\,dx = \frac{x^3}{3}\Big|_{-1}^1 & = \frac{2}{3} & = & a\cdot 1 + b\cdot 1 + c\cdot(-2) + d\cdot 2, \\
f(x) = x^3 & \displaystyle\int_{-1}^1 x^3\,dx = \frac{x^4}{4}\Big|_{-1}^1 & = 0 & = & a\cdot(-1) + b\cdot 1 + c\cdot 3 + d\cdot 3.
\end{array}$$

We have 4 equations in 4 unknowns:

$$\begin{aligned}
a + b &= 2, \\
-a + b + c + d &= 0, \\
a + b - 2c + 2d &= \frac{2}{3}, \\
-a + b + 3c + 3d &= 0.
\end{aligned}$$

Solving this system, we obtain:

$$a = 1,\ b = 1,\ c = \frac{1}{3},\ d = -\frac{1}{3}.$$

Thus, the quadrature formula with accuracy $n = 3$ is:

$$\int_{-1}^1 f(x)\,dx \quad = \quad f(-1) + f(1) + \frac{1}{3}f'(-1) - \frac{1}{3}f'(1). \quad \checkmark$$

Computational Problem: Approximate $\int_0^2 x^2 \sin(-x)\, dx \approx -2.4694834$ by the following quadrature rules to 10^{-6} accuracy and also find the size of h required for each rule.

Solution: See the code for the implementation of the composite numerical integration of the rules listed below.

The number of intervals specified below was sufficient to get an answer within 10^{-6} accuracy. The corresponding subinterval size is also specified.

(a) **Composite left point rule**.
Number of intervals: $n = 4,000,000$; interval size: $h = 5 \cdot 10^{-7}$.

(b) **Composite right point rule**.
Number of intervals: $n = 4,000,000$; interval size: $h = 5 \cdot 10^{-7}$.

(c) **Composite midpoint rule**.
Number of intervals: $n = 600$; interval size: $h = 0.0033$.

(d) **Composite trapezoidal rule**.
Number of intervals: $n = 850$; interval size: $h = 0.0024$.

(e) **Composite Simpson's rule**.
Number of intervals: $n = 18$; interval size: $h = 0.11$.