Homework 5 Solutions

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Problem 1: Using Taylor expansion, show that

$$f'(x_0) \quad = \quad \frac{f(x_0+h)-f(x_0)}{h} \;-\; \frac{h}{2}f''(\xi),$$

for some ξ lying in between x_0 and x_0+h.

Solution: We expand the function f in a first order Taylor polynomial around x_0:

$$f(x) \quad = \quad f(x_0)+(x-x_0)f'(x_0)+(x-x_0)^2\frac{f''(\xi)}{2},$$

where ξ is between x and x_0. Let $x = x_0+h$:

$$f(x_0+h) \quad = \quad f(x_0)+hf'(x_0)+\frac{h^2}{2}f''(\xi).$$

Solving for $f'(x_0)$, we obtain:

$$f'(x_0) \quad = \quad \frac{f(x_0+h)-f(x_0)}{h} \;-\; \frac{h}{2}f''(\xi). \quad \checkmark$$

Problem 2: Derive an $O(h^4)$ five-point formula to approximate $f'(x_0)$ using nodes $x_0 - h,\quad x_0,\quad x_0 + h,\quad x_0 + 2h,\quad x_0 + 3h.$

Solution: Consider the expression

$$f'(x_0) = af(x_0 - h) + bf(x_0) + cf(x_0 + h) + df(x_0 + 2h) + ef(x_0 + 3h). \tag{1}$$

We will expand the right hand side in fourth order Taylor polynomial. Then, we will equate coefficients to obtain a, b, c, d, e.

$$\begin{aligned}
f'(x_0) &= a\Big[f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(x_0) - \frac{h^3}{6}f'''(x_0) + \frac{h^4}{24}f^{(4)}(x_0) - \frac{h^5}{120}f^{(5)}(\xi_1)\Big] \\
&+ b\Big[f(x_0)\Big] \\
&+ c\Big[f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{6}f'''(x_0) + \frac{h^4}{24}f^{(4)}(x_0) + \frac{h^5}{120}f^{(5)}(\xi_2)\Big] \\
&+ d\Big[f(x_0) + 2hf'(x_0) + \frac{(2h)^2}{2}f''(x_0) + \frac{(2h)^3}{6}f'''(x_0) + \frac{(2h)^4}{24}f^{(4)}(x_0) + \frac{(2h)^5}{120}f^{(5)}(\xi_3)\Big] \\
&+ e\Big[f(x_0) + 3hf'(x_0) + \frac{(3h)^2}{2}f''(x_0) + \frac{(3h)^3}{6}f'''(x_0) + \frac{(3h)^4}{24}f^{(4)}(x_0) + \frac{(3h)^5}{120}f^{(5)}(\xi_4)\Big] \\
&= (a + b + c + d + e)\, f(x_0) \\
&+ (-a + c + 2d + 3e)\, h\, f'(x_0) \\
&+ (a + c + 4d + 9e)\, \frac{h^2}{2}\, f''(x_0) \\
&+ (-a + c + 8d + 27e)\, \frac{h^3}{6}\, f'''(x_0) \\
&+ (a + c + 16d + 81e)\, \frac{h^4}{24}\, f^{(4)}(x_0) \\
&+ \big(-af^{(5)}(\xi_1) + cf^{(5)}(\xi_2) + 32df^{(5)}(\xi_3) + 243ef^{(5)}(\xi_4)\big)\, \frac{h^5}{120}.
\end{aligned}$$

Thus, we have the following 5 equations in 5 unknowns:

$$\begin{aligned}
a + b + c + d + e &= 0, \\
(-a + c + 2d + 3e)h &= 1, \\
(a + c + 4d + 9e)\frac{h^2}{2} &= 0, \\
(-a + c + 8d + 27e)\frac{h^3}{6} &= 0, \\
(a + c + 16d + 81e)\frac{h^4}{24} &= 0.
\end{aligned}$$

Solving this linear system, we obtain coefficients $a = -\frac{3}{12h}$, $b = -\frac{10}{12h}$, $c = \frac{18}{12h}$, $d = -\frac{6}{12h}$, $e = \frac{1}{12h}$, and we plug these into (1) to get:

$$f'(x_0) = -\frac{3}{12h}f(x_0 - h) - \frac{10}{12h}f(x_0) + \frac{18}{12h}f(x_0 + h) - \frac{6}{12h}f(x_0 + 2h) + \frac{1}{12h}ef(x_0 + 3h) + O(h^4),$$

or

$$f'(x_0) = \frac{-3f(x_0 - h) - 10f(x_0) + 18f(x_0 + h) - 6f(x_0 + 2h) + f(x_0 + 3h)}{12h} + O(h^4). \quad \checkmark$$

Problem 3: Derive an $O(h^4)$ five-point formula to approximate $f'(x_0)$ using nodes $x_0 - 2h,\quad x_0 - h,\quad x_0,\quad x_0 + h,\quad x_0 + 2h.$

Solution: Consider the expression

$$f'(x_0) \;=\; af(x_0-2h)+bf(x_0-h)+cf(x_0)+df(x_0+h)+ef(x_0+2h). \tag{2}$$

We will expand the right hand side in fourth order Taylor polynomial. Then, we will equate coefficients to obtain a, b, c, d, e.

$$\begin{aligned}
f'(x_0) &= a\Big[f(x_0) - 2hf'(x_0) + \frac{(2h)^2}{2}f''(x_0) - \frac{(2h)^3}{6}f'''(x_0) + \frac{(2h)^4}{24}f^{(4)}(x_0) - \frac{(2h)^5}{120}f^{(5)}(\xi_1)\Big] \\
&+ b\Big[f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(x_0) - \frac{h^3}{6}f'''(x_0) + \frac{h^4}{24}f^{(4)}(x_0) - \frac{h^5}{120}f^{(5)}(\xi_2)\Big] \\
&+ c\Big[f(x_0)\Big] \\
&+ d\Big[f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{6}f'''(x_0) + \frac{h^4}{24}f^{(4)}(x_0) + \frac{h^5}{120}f^{(5)}(\xi_3)\Big] \\
&+ e\Big[f(x_0) + 2hf'(x_0) + \frac{(2h)^2}{2}f''(x_0) + \frac{(2h)^3}{6}f'''(x_0) + \frac{(2h)^4}{24}f^{(4)}(x_0) + \frac{(2h)^5}{120}f^{(5)}(\xi_4)\Big] \\
&= (a+b+c+d+e)\,f(x_0) \\
&+ (-2a-b+d+2e)\,h\,f'(x_0) \\
&+ (4a+b+d+4e)\,\frac{h^2}{2}\,f''(x_0) \\
&+ (-8a-b+d+8e)\,\frac{h^3}{6}\,f'''(x_0) \\
&+ (16a+b+d+16e)\,\frac{h^4}{24}\,f^{(4)}(x_0) \\
&+ \big(-32af^{(5)}(\xi_1) - bf^{(5)}(\xi_2) + df^{(5)}(\xi_3) + 32ef^{(5)}(\xi_4)\big)\,\frac{h^5}{120}.
\end{aligned}$$

Thus, we have the following 5 equations in 5 unknowns:

$$\begin{aligned}
a+b+c+d+e &= 0, \\
(-2a-b+d+2e)h &= 1, \\
(4a+b+d+4e)\frac{h^2}{2} &= 0, \\
(-8a-b+d+8e)\frac{h^3}{6} &= 0, \\
(16a+b+d+16e)\frac{h^4}{24} &= 0.
\end{aligned}$$

Solving this linear system, we obtain coefficients $a = \frac{1}{12h}$, $b = -\frac{8}{12h}$, $c = 0$, $d = \frac{8}{12h}$, $e = -\frac{1}{12h}$, and we plug these into (2) to get:

$$f'(x_0) \;=\; \frac{1}{12h}f(x_0-2h) - \frac{8}{12h}f(x_0-h) + \frac{8}{12h}f(x_0+h) - \frac{1}{12h}ef(x_0+2h) + O(h^4),$$

or

$$f'(x_0) \;=\; \frac{f(x_0-2h) - 8f(x_0-h) + 8f(x_0+h) - f(x_0+2h)}{12h} + O(h^4). \quad \checkmark$$

Problem 4: Compare two error terms obtained in the above two problems (2) and (3) and decide which is better.

Solution: The truncation error obtained in Problem 2 is:

$$
\begin{aligned}
\tau_1(h) &= \left(\frac{3}{12h}f^{(5)}(\xi_1) + \frac{18}{12h}f^{(5)}(\xi_2) - 32\cdot\frac{6}{12h}f^{(5)}(\xi_3) + 243\cdot\frac{1}{12h}f^{(5)}(\xi_4)\right)\frac{h^5}{120} \\
&= \left(\frac{3}{12}f^{(5)}(\xi_1) + \frac{18}{12}f^{(5)}(\xi_2) - 32\cdot\frac{6}{12}f^{(5)}(\xi_3) + 243\cdot\frac{1}{12}f^{(5)}(\xi_4)\right)\frac{h^4}{120} \\
&= 6f^{(5)}(\xi)\,\frac{h^4}{120} \\
&= \frac{h^4}{20}f^{(5)}(\xi).
\end{aligned}
$$

Note that we used the Intermideate Value Theorem above, similar to:

$$
f^{(5)}(\xi) = \frac{1}{2}(f^{(5)}(\xi_1) + f^{(5)}(\xi_2)).
$$

The truncation error obtained in Problem 3 is:

$$
\begin{aligned}
\tau_2(h) &= \left(-32\cdot\frac{1}{12h}f^{(5)}(\xi_1) + \frac{8}{12h}f^{(5)}(\xi_2) + \frac{8}{12h}f^{(5)}(\xi_3) - 32\cdot\frac{1}{12h}f^{(5)}(\xi_4)\right)\frac{h^5}{120} \\
&= \left(-32\cdot\frac{1}{12}f^{(5)}(\xi_1) + \frac{8}{12}f^{(5)}(\xi_2) + \frac{8}{12}f^{(5)}(\xi_3) - 32\cdot\frac{1}{12}f^{(5)}(\xi_4)\right)\frac{h^4}{120} \\
&= -4f^{(5)}(\xi)\,\frac{h^4}{120} \\
&= -\frac{h^4}{30}f^{(5)}(\xi).
\end{aligned}
$$

Thus, the error obtained in Problem 3 is smaller in magnitude than the error obtained in Problem 2, i.e. $|\tau_2(h)| < |\tau_1(h)|$. That shows that the formula that uses data that is evenly distributed around x_0 would give a better approximation than the one that uses data that is biased toward one of sides of x_0.

Problem 5: The forward-difference formula can be expressed as

$$f'(x_0) \quad = \quad \frac{1}{h}(f(x_0+h) - f(x_0)) - \frac{h}{2}f''(x_0) - \frac{h^2}{6}f'''(x_0) + O(h^3). \tag{3}$$

Use extrapolation to derive an $O(h^3)$ formula for $f'(x_0)$.

Solution: In general, Richardson's extrapolation is used to generate high-accuracy approximations while using low-order formulas.
Replacing h in (3) with $2h$ gives the new formula

$$f'(x_0) \quad = \quad \frac{1}{2h}(f(x_0+2h) - f(x_0)) - hf''(x_0) - \frac{4h^2}{6}f'''(x_0) + O(h^3). \tag{4}$$

Multiplying equation (3) by 2 and subtracting equation (4), we obtain:

$$\begin{aligned} f'(x_0) &= \frac{2}{h}(f(x_0+h) - f(x_0)) - \frac{1}{2h}(f(x_0+2h) - f(x_0)) - \frac{h^2}{3}f'''(x_0) + \frac{2h^2}{3}f'''(x_0) + O(h^3) \\ &= \frac{2}{h}(f(x_0+h) - f(x_0)) - \frac{1}{2h}(f(x_0+2h) - f(x_0)) + \frac{h^2}{3}f'''(x_0) + O(h^3). \end{aligned}$$

Rewriting this equation, we get an $O(h^2)$ formula for $f'(x_0)$:

$$f'(x_0) \quad = \quad \frac{-f(x_0+2h) + 4f(x_0+h) - 3f(x_0)}{2h} + \frac{h^2}{3}f'''(x_0) + O(h^3). \tag{5}$$

Replacing h in (5) with $2h$ gives:

$$f'(x_0) \quad = \quad \frac{-f(x_0+4h) + 4f(x_0+2h) - 3f(x_0)}{4h} + \frac{4h^2}{3}f'''(x_0) + O(h^3). \tag{6}$$

Multiplying equation (5) by 4 and subtracting equation (6), we obtain:

$$\begin{aligned} 3f'(x_0) &= \frac{-4f(x_0+2h) + 16f(x_0+h) - 12f(x_0)}{2h} - \frac{-f(x_0+4h) + 4f(x_0+2h) - 3f(x_0)}{4h} + O(h^3) \\ &= \frac{-8f(x_0+2h) + 32f(x_0+h) - 24f(x_0)}{4h} - \frac{-f(x_0+4h) + 4f(x_0+2h) - 3f(x_0)}{4h} + O(h^3) \\ &= \frac{f(x_0+4h) - 12f(x_0+2h) + 32f(x_0+h) - 21f(x_0)}{4h} + O(h^3), \end{aligned}$$

or

$$f'(x_0) \quad = \quad \frac{f(x_0+4h) - 12f(x_0+2h) + 32f(x_0+h) - 21f(x_0)}{12h} + O(h^3). \quad \checkmark$$