Homework 4 Solutions

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Problem 1: Let $P_3(x)$ be the interpolating polynomial for the data $(0,0)$, $(0.5,y)$, $(1,3)$ and $(2,2)$. Find y if the coefficient of x^3 in $P_3(x)$ is 6.

Solution: We have $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, $x_3 = 2$, and
$f(x_0) = 0$, $f(x_1) = y$, $f(x_2) = 3$, $f(x_3) = 2$.

The Lagrange polynomial of order 3, connecting the four points, is given by

$$P_3(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3),$$

where

$$\begin{aligned}
L_0(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)},\\
L_1(x) &= \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)},\\
L_2(x) &= \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)},\\
L_3(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}.
\end{aligned}$$

Thus, for this problem,

$$\begin{aligned}
L_0(x) &= \frac{(x-0.5)(x-1)(x-2)}{(0-0.5)(0-1)(0-2)} = \frac{x^3 - \frac{7}{2}x^2 + \frac{7}{2}x - 1}{-1} = -x^3 + \frac{7}{2}x^2 - \frac{7}{2}x + 1,\\
L_1(x) &= \frac{(x-0)(x-1)(x-2)}{(0.5-0)(0.5-1)(0.5-2)} = \frac{x^3 - 3x^2 + 2x}{\frac{3}{8}} = \frac{8}{3}x^3 - 8x^2 + \frac{16}{3}x,\\
L_2(x) &= \frac{(x-0)(x-0.5)(x-2)}{(1-0)(1-0.5)(1-2)} = \frac{x^3 - \frac{5}{2}x^2 + x}{-\frac{1}{2}} = -2x^3 + 5x^2 - 2x,\\
L_3(x) &= \frac{(x-0)(x-0.5)(x-1)}{(2-0)(2-0.5)(2-1)} = \frac{x^3 - \frac{3}{2}x^2 + \frac{1}{2}x}{3} = \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{6}x.
\end{aligned}$$

Thus,

$$\begin{aligned}
P_3(x) &= L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3)\\
&= L_0(x)\cdot 0 + L_1(x)\cdot y + L_2(x)\cdot 3 + L_3(x)\cdot 2\\
&= \Big(\frac{8}{3}x^3 - 8x^2 + \frac{16}{3}x\Big)y \quad - 6x^3 + 15x^2 - 6x \quad + \frac{2}{3}x^3 - x^2 + \frac{1}{3}x\\
&= \Big(\frac{8}{3}y - 6 + \frac{2}{3}\Big)x^3 + \Big(-8y + 15 - 1\Big)x^2 + \Big(\frac{16}{3}y - 6 + \frac{1}{3}\Big)x\\
&= \Big(\frac{8y-16}{3}\Big)x^3 + \Big(-8y + 14\Big)x^2 + \Big(\frac{16y-17}{3}\Big)x.
\end{aligned}$$

Since we want the coefficient of x^3 to be equal to 6, we need:

$$\frac{8y - 16}{3} = 6,$$

or $y = \frac{17}{4} = 4.25.$ ✓

With such y, the polynomial becomes

$$P_3(x) = 6x^3 - 20x^2 + 17x.$$

We can check whether this polynomial interpolates function f, that is, whether we got the correct answer. Note that

$$\begin{aligned} P_3(0) &= 0, \\ P_3(0.5) &= 4.25, \\ P_3(1) &= 3, \\ P_3(2) &= 2. \end{aligned}$$

Problem 2: Let $f(x) = e^x$ for $0 \le x \le 2$. Approximate $f(0.25)$ using linear interpolation with $x_0 = 0$ and $x_1 = 0.5$.

Solution: Linear interpolation is achieved by constructing the Lagrange polynomial P_1 of order 1, connecting the two points. We have:

$$P_1(x) = L_0(x)f(x_0) + L_1(x)f(x_1),$$

where

$$\begin{aligned} L_0(x) &= \frac{x - x_1}{x_0 - x_1} = \frac{x - 0.5}{-0.5}, \\ L_1(x) &= \frac{x - x_0}{x_1 - x_0} = \frac{x}{0.5}. \end{aligned}$$

Thus,

$$P_1(x) = -\frac{x - 0.5}{0.5} \cdot e^0 + \frac{x}{0.5} \cdot e^{0.5} = -2x + 1 + 3.2974x = 1.2974x + 1.$$

We can check whether this polynomial interpolates function f, that is, whether we got the correct answer. Note that

$$\begin{aligned} P_1(0) &= 1 = e^0, \\ P_1(0.5) &= 1.6487 = e^{0.5}. \end{aligned}$$

Now we can evaluate

$$P_1(0.25) = 1.2974 \cdot 0.25 + 1 = 1.3243, \quad ✓$$

which is an approximation of $f(0.25)$. The true value of f at $x = 0.25$ is $f(0.25) = e^{0.25} = 1.2840$. Thus, we obtained a reasonable approximation.

Problem 3: For a function f, the forward divided differences are given by

$$\begin{array}{lllll}
x_0 & = & 0.0 & f[x_0] & \\
 & & & & f[x_0, x_1] \\
x_1 & = & 0.4 & f[x_1] & \qquad\qquad f[x_0, x_1, x_2] = \dfrac{50}{7} \\
 & & & & f[x_1, x_2] = 10 \\
x_2 & = & 0.7 & f[x_2] = 6 &
\end{array}$$

Determine the missing entries.

Solution: This problem is on Newton's divided differences.

The zeroth divided difference of f with respect to x_i is

$$f[x_i] = f(x_i).$$

The first divided difference of f with respect to x_i and x_{i+1} is

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}.$$

The second divided difference is

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}.$$

① Thus, first we find $f[x_1]$:

$$\begin{aligned}
f[x_1, x_2] &= \frac{f[x_2] - f[x_1]}{x_2 - x_1}, \\
10 &= \frac{6 - f[x_1]}{0.7 - 0.4}, \\
f[x_1] &= \mathbf{3}. \quad \checkmark
\end{aligned}$$

② We find $f[x_0, x_1]$:

$$\begin{aligned}
f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}, \\
\frac{50}{7} &= \frac{10 - f[x_0, x_1]}{0.7 - 0.0}, \\
f[x_0, x_1] &= \mathbf{5}. \quad \checkmark
\end{aligned}$$

③ We now find $f[x_0]$:

$$\begin{aligned}
f[x_0, x_1] &= \frac{f[x_1] - f[x_0]}{x_1 - x_0}, \\
5 &= \frac{3 - f[x_0]}{0.4 - 0.0}, \\
f[x_0] &= \mathbf{1}. \quad \checkmark
\end{aligned}$$

Note that steps ① and ② could be interchanged. However, step ③ could only be done last.

Problem 4: Let $i_0, i_1, \ldots, i_n$ be a rearrangement of the integers $0, 1, \ldots, n$.
Show that $f[x_{i_0}, x_{i_1}, \ldots, x_{i_n}] \;=\; f[x_0, x_1, \ldots, x_n]$.

Solution: Let P_c and P_d be two polynomials, such that P_c interpolates f at $x_0, x_1, \ldots, x_n$ and P_d interpolates f at $x_{i_0}, x_{i_1}, \ldots, x_{i_n}$:

$$\begin{aligned} P_c &= c_0 + c_1(x - x_0) \;+\; \ldots \;+\; c_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}), \\ P_d &= d_0 + d_1(x - x_{i_0}) \;+\; \ldots \;+\; d_n(x - x_{i_0})(x - x_{i_1})\cdots(x - x_{i_{n-1}}), \end{aligned}$$

We can rewrite the polynomials above as

$$\begin{aligned} P_c &= c_n x^n \;+\; \text{lower order terms}, \\ P_d &= d_n x^n \;+\; \text{lower order terms}. \end{aligned}$$

Since P_c and P_d were defined to be in the form of Newton's polynomials, we know that c_n and d_n are nth divided differences, $c_n = f[x_0, x_1, \ldots, x_n]$ and $d_n = f[x_{i_0}, x_{i_1}, \ldots, x_{i_n}]$:

$$\begin{aligned} P_c &= f[x_0, x_1, \ldots, x_n]x^n \;+\; \text{lower order terms}, \\ P_d &= f[x_{i_0}, x_{i_1}, \ldots, x_{i_n}]x^n \;+\; \text{lower order terms}. \end{aligned}$$

We also know that the polynomial interpolating the same nodes is unique, that is, $P_c = P_d$. Thus,

$$f[x_0, x_1, \ldots, x_n] \;=\; f[x_{i_0}, x_{i_1}, \ldots, x_{i_n}]. \quad \checkmark$$

Problem 5: Give explicit formulas for $f[a]$, $f[a,b]$, $f[a,b,c]$ in terms of $f(a)$, $f(b)$, $f(c)$. *Optional:* Give an explicit formula for $f[x, x+h, x+2h, \ldots, x+nh]$.

Solution:
Similar to problem 3, we can write the Newton's divided difference formulas as:

$$\begin{aligned} f[a] &= f(a), \quad \checkmark \\ f[a,b] &= \frac{f[b]-f[a]}{b-a} \\ &= \frac{f(b)-f(a)}{b-a}, \quad \checkmark \\ f[a,b,c] &= \frac{f[b,c]-f[a,b]}{c-a} \\ &= \frac{\frac{f(c)-f(b)}{c-b} - \frac{f(b)-f(a)}{b-a}}{c-a} \\ &= \frac{(f(c)-f(b))(b-a)-(f(b)-f(a))(c-b)}{(c-b)(b-a)(c-a)}. \quad \checkmark \end{aligned}$$

Optional: Note that

$$\begin{aligned} f[x] &= f(x), \\ f[x,x+h] &= \frac{f(x+h)-f(x)}{h}, \\ f[x,x+h,x+2h] &= \frac{f[x+h,x+2h]-f[x,x+h]}{2h} \\ &= \frac{\frac{f(x+2h)-f(x+h)}{h} - \frac{f(x+h)-f(x)}{h}}{2h} \\ &= \frac{f(x+2h)-2f(x+h)+f(x)}{2h^2}, \\ f[x,x+h,x+2h,x+3h] &= \frac{f[x+h,x+2h,x+3h]-f[x,x+h,x+2h]}{3h} \\ &= \frac{\frac{f(x+3h)-2f(x+2h)+f(x+h)}{2h^2} - \frac{f(x+2h)-2f(x+h)+f(x)}{2h^2}}{3h} \\ &= \frac{f(x+3h)-3f(x+2h)+3f(x+h)-f(x)}{6h^3}, \end{aligned}$$

By observation, we have

$$f[x,x+h,x+2h,\ldots,x+nh] = \frac{f(x+nh)-nf(x+(n-1)h)+\ldots \pm nf(x+h) \mp f(x)}{n!\,h^n}.$$

The exact signs of $\pm$ and $\mp$ depend on whether n is even or odd.