Eulerian Vortex Motion in Two and Three Dimensions

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1 Objective

An Eulerian approach is used to solve the motion of an incompressible fluid, in two and three dimensions, in which the vorticity is concentrated on a lower dimensional set. This is a level set approach described in a paper by Harabetian, Osher, and Shu. Numerical examples include vortex sheets, vortex dipole sheets, and point vortices in two spatial dimensions, and vortex sheets in three spatial dimensions.

2 The General Formulation

The three-dimensional incompressible Navier-Stokes equations written in the velocity-pressure formulation are

$$\begin{aligned} \vec{v}_t + \vec{v} \cdot \nabla \vec{v} &= -\nabla p + \nu \triangle \vec{v}, \\ \nabla \cdot v &= 0. \end{aligned}$$

Taking the curl of the above equation gives

$$\begin{aligned} \vec{\omega}_t + \vec{v} \cdot \nabla \vec{\omega} &= \nabla \vec{v} \vec{\omega} + \nu \triangle \vec{\omega}, \\ \nabla \times \vec{v} &= \vec{\omega}, \\ \nabla \cdot \vec{v} &= 0, \end{aligned}$$

where $\nabla \vec{v} \vec{\omega}$ is the vorticity stretching.
In this application, we consider the Euler equations, i.e. $\nu = 0$.
We decompose $\vec{\omega}$ into a product of

$$\vec{\omega} = P(\phi)\, \eta,$$

where P is a scalar function, ϕ is a scalar function whose zero level set represents the points where the vorticity concentrates, and $\vec{\eta}$ is the vorticity strength vector.
Once the decomposition is found, the system of equations becomes:

$$\begin{aligned} \phi_t + \vec{v} \cdot \nabla \phi &= 0, \\ \vec{\eta}_t + \vec{v} \cdot \nabla \vec{\eta} - \nabla \vec{v} \vec{\eta} &= 0, \\ \nabla \times \vec{v} &= P(\phi) \vec{\eta}, \\ \nabla \cdot \vec{v} &= 0. \end{aligned}$$

For the discretization of the equations in two and three spatial dimensions, we use the fifth-order WENO scheme and a third-order TVD Runge-Kutta time-stepping, coupled with a second order Poisson solver.

3 The 2D Equations

We are going to solve the 2D vortex sheet problem with global level-set approach.
The 2D incompressible Euler Equations in Vorticity-Stream formulation are:

$$\omega_t + u\omega_x + v\omega_y = 0, \quad (1)$$

$$\nabla \times (u, v) = \omega, \quad (2)$$

$$\nabla \cdot (u, v) = 0. \quad (3)$$

Suppose ω concentrates only on a thin curve called vortex sheet. Let

$$\omega = P(\phi)\eta, \quad (4)$$

where ϕ is the level set function, which is 0 on the sheet, and nonzero elsewhere. η is the strength of vorticity.
Inserting (4) into (1), we get:

$$P'(\phi)\phi_t\eta + P(\phi)\eta_t + uP'(\phi)\phi_x\eta + uP(\phi)\eta_x + vP'(\phi)\phi_y\eta + vP(\phi)\eta_y = 0.$$

Thus, we get the following equations:

$$\begin{aligned} \phi_t + u\phi_x + v\phi_y &= 0, \\ \eta_t + u\eta_x + v\eta_y &= 0, \\ \nabla \times (u, v) &= P(\phi)\eta, \\ \nabla \cdot (u, v) &= 0. \end{aligned}$$

If the vortex sheet strength η does not change sign along the curve, it can be normalized to $\eta \equiv 1$ and the equations become

$$\phi_t + \vec{v}(\phi)\nabla\phi = 0,$$

where the velocity $v(\phi)$ is given by

$$\begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} -\partial_y \\ \partial_x \end{pmatrix} \triangle^{-1} P(\phi).$$

In this case, the vortex sheet strength along the curve is given by $1/|\nabla\phi|$.

3.1 Algorithm

Initialize the level set function ϕ and do the following steps iteratively:

1. Find stream function Ψ, the solution to the Poisson equation:

$$\triangle\Psi = -P(\phi),$$

with appropriate boundary conditions for Ψ.

2. Find the velocity vector (u, v):

$$u = -\frac{\partial\Psi}{\partial y}, \qquad v = \frac{\partial\Psi}{\partial x}.$$

3. Evolve the level set function ϕ by:

$$\phi_t + (u, v) \cdot \nabla\phi = 0.$$

3.2 Examples

The figures of the results of the following examples may be found at the end of this report. The movies of these and other examples may be found at the following address:

http://www.math.ucla.edu/∼yanovsky/VortexSheets.htm

Vortex Sheets in 2D
We consider the periodic vortex sheet in two dimensions:

$$P(\phi) = \delta(\phi) = \begin{cases} \frac{1}{2\varepsilon}\left(1+\cos\left(\frac{\pi\phi}{\varepsilon}\right)\right) & \text{if } |\phi| \le \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

The boundary conditions for the stream function are:

$$\Psi(x, \pm 1) = 0,$$
$$\Psi \text{ periodic in } x.$$

The boundary conditions for ϕ are periodic in x and Neumann in y.

Vortex Sheet Dipole
In this example, we define P as:

$$P(\phi) = \alpha \cdot \begin{cases} -\frac{1}{2\varepsilon^2}\left(1+\cos\left(\frac{\pi(\phi+\varepsilon)}{\varepsilon}\right)\right) & \text{if } -2\varepsilon \le \phi \le 0, \\ +\frac{1}{2\varepsilon^2}\left(1+\cos\left(\frac{\pi(\phi-\varepsilon)}{\varepsilon}\right)\right) & \text{if } 0 \le \phi \le 2\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

The boundary conditions for the stream function Ψ and for ϕ are the same as above.

Point Vortices
In this example, P is a δ function and ϕ is supported at finite number of points in the plane, x_1, x_2, $\cdots$, x_n. For example, if the vorticity is positive initially, we may choose

$$\phi_+(x) = \min_i \left(\frac{|x - x_i|}{a_i}\right),$$

where $a_i > 0$ are the vortex point strengths.
For point vortices with negative strength, we introduce a second level set function:

$$\phi_-(x) = \min_i \left(\frac{|x - x_i|}{a_i}\right),$$

where $a_i < 0$ are the vortex point strengths.
Function P is given by:

$$P(\phi_1, \phi_2) = \delta(\phi_+) + \delta(\phi_-).$$

The boundary condition for the stream function is $\Psi = 0$ at all four boundaries.
The boundary condition for ϕ is Neumann.

4 The 3D Equations

The algorithm below was implemented for incompressible Eulerian vortex motion in three spatial dimensions.

4.1 Algorithm

Initialize the level set function ϕ and the vortex sheet strength vector $\vec{\eta}$,
and do the following steps iteratively:

1. Find the vector potential $\vec{A}$, the solution of the Poisson equations:

$$\triangle \vec{A} = -P(\phi)\vec{\eta},$$

or

$$\triangle \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = -P(\phi) \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix},$$

with appropriate boundary conditions for $\vec{A}$.

2. Find the velocity vector $\vec{v} = (v_1, v_2, v_3)$, a solution to:

$$\vec{v} = \nabla \times \vec{A},$$

or

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \\ \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \\ \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{pmatrix}.$$

3. Evolve ϕ and $\vec{\eta}$ by:

$$\begin{aligned} \phi_t + \vec{v} \cdot \nabla \phi &= 0, \\ \vec{\eta}_t + \vec{v} \cdot \nabla \vec{\eta} - \nabla \vec{v} \vec{\eta} &= 0. \end{aligned}$$

We write out the terms for the second equation:

$$\vec{v} \cdot \nabla \vec{\eta} = \left(v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + v_3 \frac{\partial}{\partial z} \right) \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} v_1 \frac{\partial \eta_1}{\partial x} + v_2 \frac{\partial \eta_1}{\partial y} + v_3 \frac{\partial \eta_1}{\partial z} \\ v_1 \frac{\partial \eta_2}{\partial x} + v_2 \frac{\partial \eta_2}{\partial y} + v_3 \frac{\partial \eta_2}{\partial z} \\ v_1 \frac{\partial \eta_3}{\partial x} + v_2 \frac{\partial \eta_3}{\partial y} + v_3 \frac{\partial \eta_3}{\partial z} \end{pmatrix},$$

$$\nabla \vec{v} \vec{\eta} = \begin{pmatrix} \frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y} & \frac{\partial v_1}{\partial z} \\ \frac{\partial v_2}{\partial x} & \frac{\partial v_2}{\partial y} & \frac{\partial v_2}{\partial z} \\ \frac{\partial v_3}{\partial x} & \frac{\partial v_3}{\partial y} & \frac{\partial v_3}{\partial z} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial v_1}{\partial x}\eta_1 + \frac{\partial v_1}{\partial y}\eta_2 + \frac{\partial v_1}{\partial z}\eta_3 \\ \frac{\partial v_2}{\partial x}\eta_1 + \frac{\partial v_2}{\partial y}\eta_2 + \frac{\partial v_2}{\partial z}\eta_3 \\ \frac{\partial v_3}{\partial x}\eta_1 + \frac{\partial v_3}{\partial y}\eta_2 + \frac{\partial v_3}{\partial z}\eta_3 \end{pmatrix}.$$

Also note that we could have used the identity $\nabla \vec{v} \vec{\eta} = \vec{\eta} \cdot \nabla \vec{v}$.
The equations for $\vec{\eta}$ become:

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}_t + \begin{pmatrix} v_1 \frac{\partial \eta_1}{\partial x} + v_2 \frac{\partial \eta_1}{\partial y} + v_3 \frac{\partial \eta_1}{\partial z} \\ v_1 \frac{\partial \eta_2}{\partial x} + v_2 \frac{\partial \eta_2}{\partial y} + v_3 \frac{\partial \eta_2}{\partial z} \\ v_1 \frac{\partial \eta_3}{\partial x} + v_2 \frac{\partial \eta_3}{\partial y} + v_3 \frac{\partial \eta_3}{\partial z} \end{pmatrix} - \begin{pmatrix} \frac{\partial v_1}{\partial x}\eta_1 + \frac{\partial v_1}{\partial y}\eta_2 + \frac{\partial v_1}{\partial z}\eta_3 \\ \frac{\partial v_2}{\partial x}\eta_1 + \frac{\partial v_2}{\partial y}\eta_2 + \frac{\partial v_2}{\partial z}\eta_3 \\ \frac{\partial v_3}{\partial x}\eta_1 + \frac{\partial v_3}{\partial y}\eta_2 + \frac{\partial v_3}{\partial z}\eta_3 \end{pmatrix} = 0.$$

Therefore, in order to solve the 3D problem, four evolution equations (for ϕ and $\vec{\eta}$) and three potential equations need to be considered.

4.2 Example: Vortex Sheets in 3D

Function P is defined as

$$P(\phi) = \delta(\phi) = \begin{cases} \frac{1}{2\varepsilon}\left(1+\cos\left(\frac{\pi\phi}{\varepsilon}\right)\right) & \text{if } |\phi| \le \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

The initial conditions are

$$\begin{aligned} \phi_0(x,y,z) &= y + 0.05\sin(\pi x), \\ \eta_0(x,y,z) &= (0,0,1). \end{aligned}$$

The boundary conditions for the vector potential are:

$$\begin{aligned} A_1(x,\pm1,z) &= 0, \\ \partial_y A_2(x,\pm1,z) &= 0, \\ A_3(x,\pm1,z) &= 0, \\ A_1, A_2, A_3 &\ \text{ periodic in } x,\ z. \end{aligned}$$

Similar to two-dimensional problem, the boundary conditions for ϕ are periodic in x, z and Neumann in y.
The vortex sheet strength vector $\vec{\eta}$ is periodic in all directions.

The figures of the results of the above example may be found at the end of this report. The movies of this and other examples may be found at the following address:

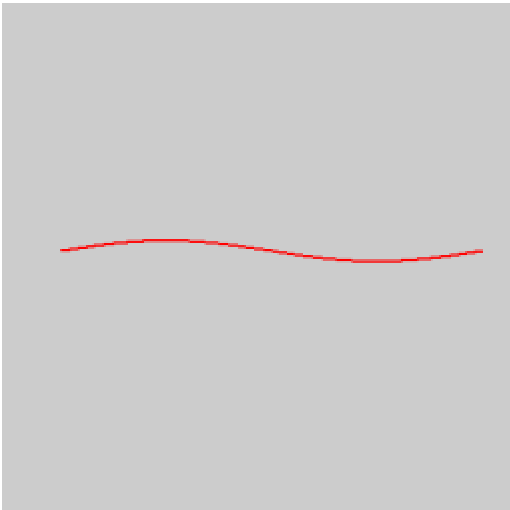
http://www.math.ucla.edu/~yanovsky/VortexSheets.htm

5 References

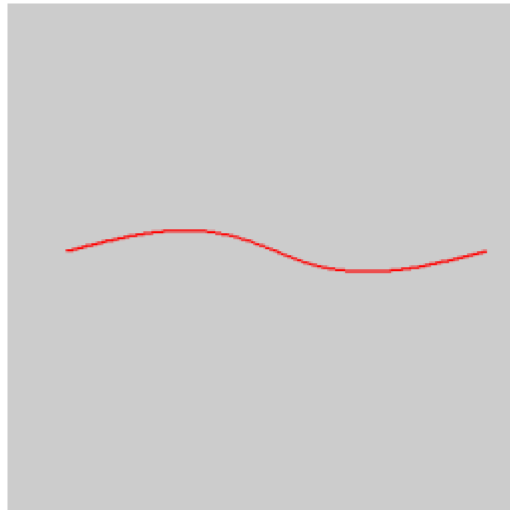
1. Harabetian, E., Osher, S., Shu, C.-W., *An Eulerian Approach for Vortex Motion Using a Level Set Regularization Procedure*, Journal of Computational Physics, 127, 15-26 (1996).
2. Jiang, G.-S., Peng, D., *Weighted ENO Schemes for Hamilton-Jacobi Equations*, SIAM J. Sci. Comput. 21, 2126-2143 (2000).
3. Osher, S. and Sethian, J., *Fronts Propagating with Curvature Dependent Speed: Algorithms Based on Hamilton-Jacobi Formulations*, Journal of Computational Physics, 79, 12-49 (1988).
4. Osher, S. and Shu, C.-W., *High-Order Essentially Nonoscillatory Schemes for Hamilton-Jacobi Equations*, SIAM J. Numer. Anal. 28, 907-922 (1991).

Two-Dimensional Examples

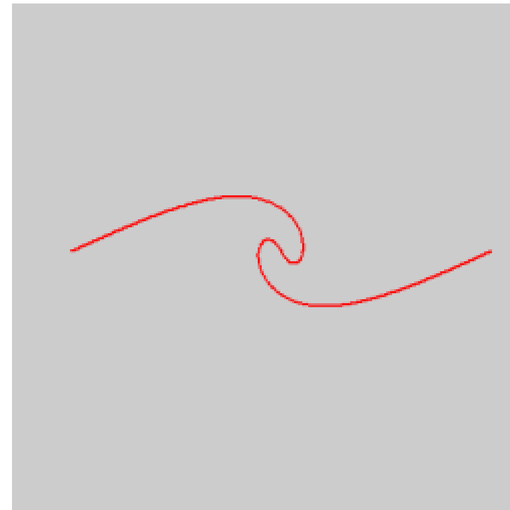
Vortex Sheets



t = 0



t = 1.0



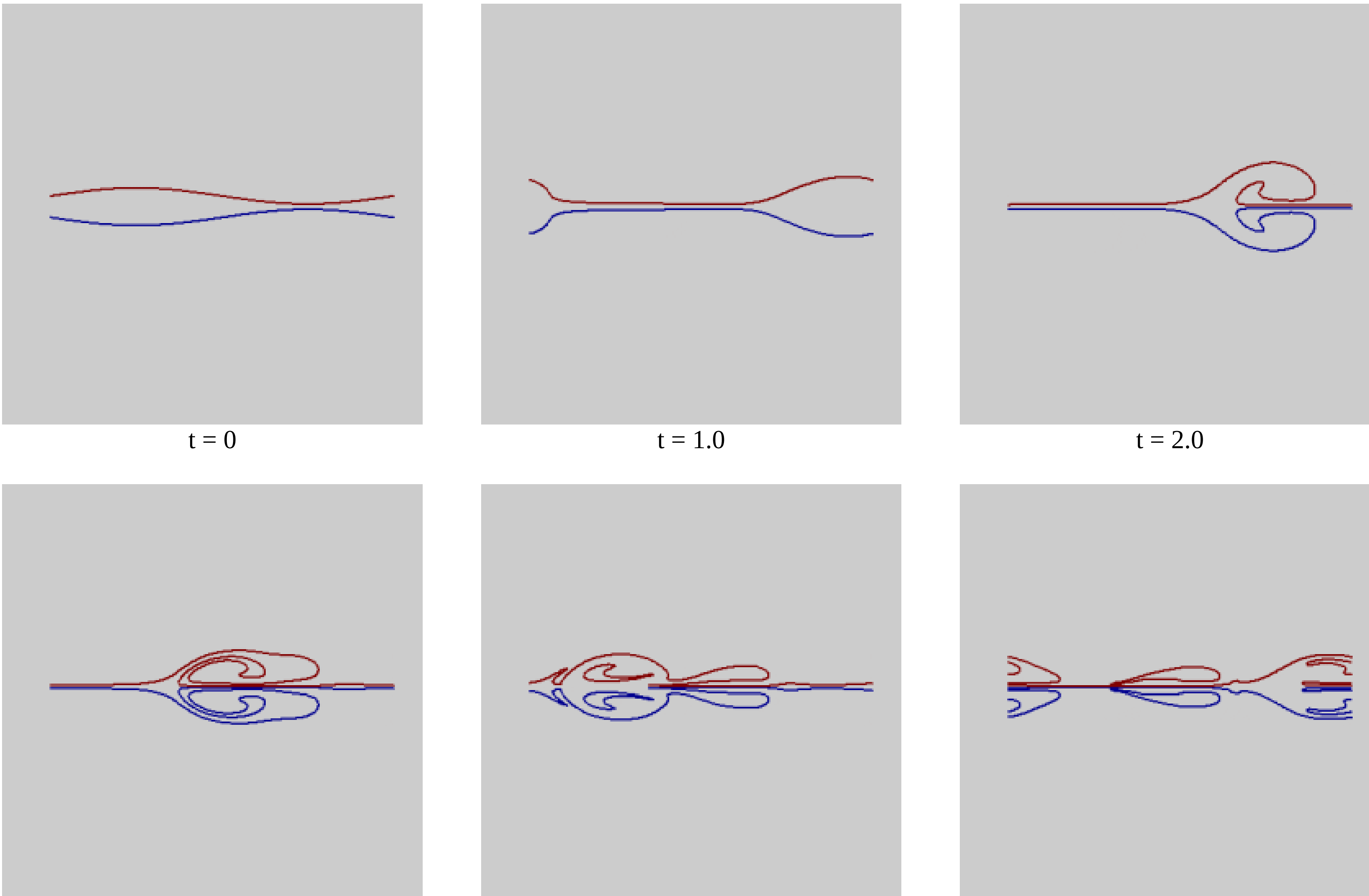
t = 2.0

t = 3.0

t = 4.0

t = 5.0

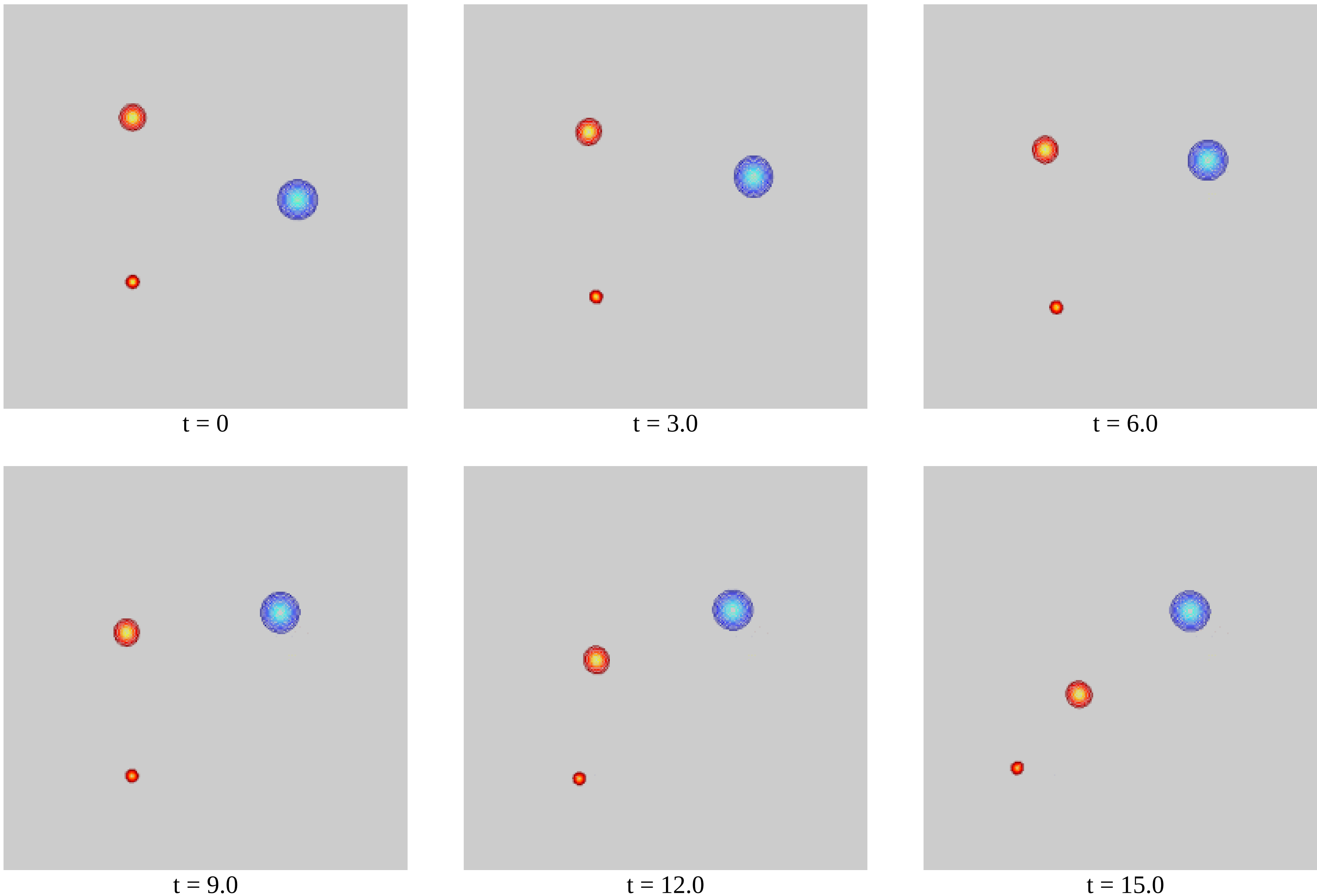
Vortex Dipole



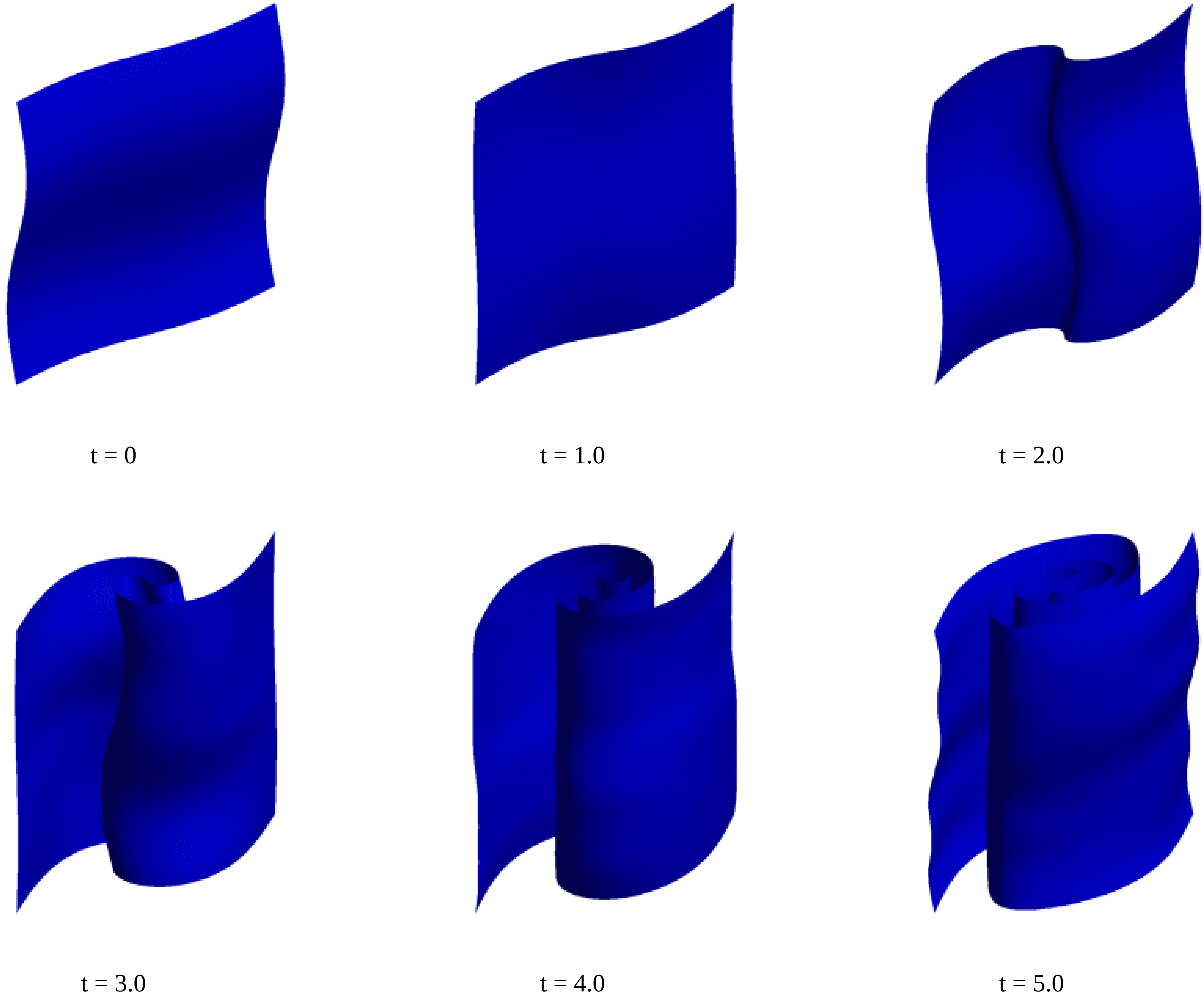
t = 0 t = 1.0 t = 2.0

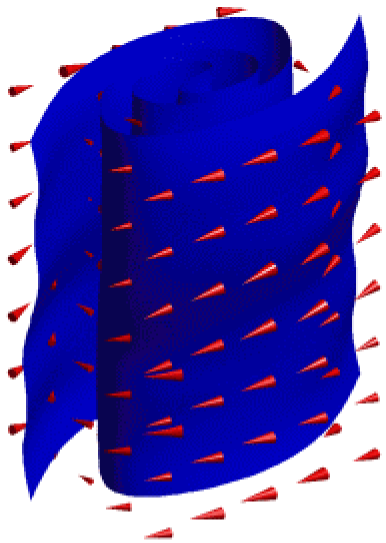
t = 3.0 t = 4.0 t = 5.0

Point Vortices

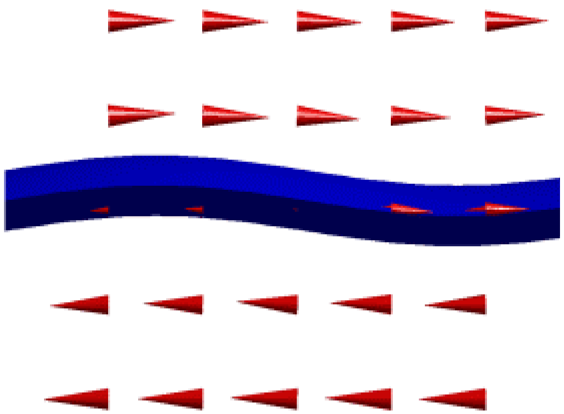


Three-Dimensional Examples

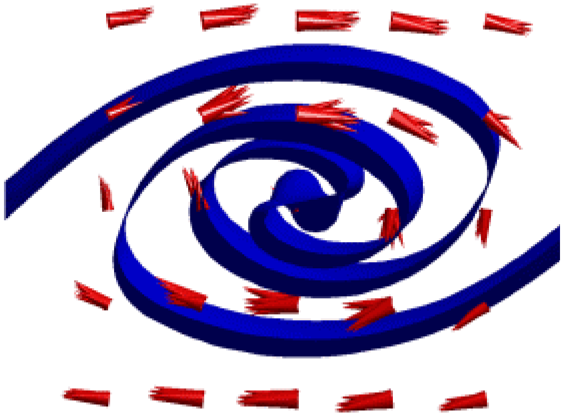




t = 5.0, vector field



t = 0, vector field; a view from the top



t = 5.0, vector field; a view from the top

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