Math 273a: Optimization
Linear programming

Instructor: Wotao Yin
Department of Mathematics, UCLA
Fall 2015

some material taken from the textbook Chong-Zak, 4th Ed.
### History

- The word “programming” used traditionally by planners to describe the process of operations planning and resource allocation.

- In 1930s–40s, this process could often be aided by solving LPs. Kantorovich: solutions to problems in production and transportation.

- The initial impetus came in the aftermath of World War II.

- In 1947, George Dantzig proposed the Simplex Method (poorly named great method\(^1\)). Made the solution of LPs practical. But, it has exponential worst-case complexity.

- Advance in computer technology expand the applications of LP. Bringing people to study and apply LP extensively.

---

\(^1\) One of the 10 algorithms with the greatest influence on the development and practice of science and engineering in the twentieth century
The Best of the 20th Century: Top 10 Algorithms
by Barry A. Cipra

- 1946, von Neumann, Ulam, and Metropolis: Monte Carlo method
- 1947, Dantzig: the Simplex method
- 1950, Hestenes, Stiefel, and Lanczos: Krylov subspace iteration methods
- 1951, Householder: decompositional approach to matrix computations
- 1957, Backus: Fortran optimizing compiler
- 1959–61: J.G.F. Francis of Ferranti Ltd.: QR algorithm
- 1962: Hoare: Quicksort
- 1965: Cooley and Tukey: the fast Fourier transform
- 1977, Ferguson and Forcadel: integer relation detection algorithm
- 1987, Greengard and Rokhlin: fast multipole algorithm
Modern period

- 1950s –, Applications
- 1960s, Large-scale optimization
- 1970s, Complexity theory
- Khachyan, 1979, the ellipsoid algorithm, first polynomial-time algorithm, but impractical
- Karmakar, 1984, interior-point algorithms, lead to later interior-point methods.
- CPLEX 1.0, 1988, research shifts to commercial
- CPLEX acquired by ILOG, which was later acquired by IBM
- Guroi, 2008
- Today: huge-scale, distributed, streaming LP
Example: the diet problem

- $n$ different foods, $j$th food sells at price $c_j$ per unit
- $m$ basic nutrients; for balanced diet, receive at least $b_i$ units of $i$th nutrient
- each unit of food $j$ contains $a_{ij}$ units of $i$th nutrient
- variable $x_j$: # units of food $j$ in diet
- total cost: $c_1 x_1 + \cdots + c_n x_n$
- nutritional constraints: $a_{i1} x_1 + \cdots + a_{in} x_n \geq b_i$, $i = 1, \ldots, m$.

\[
\text{minimize } c^T x \\
\text{subject to } Ax \geq b \\
x \geq 0.
\]
In the problem, a company manufactures two iPod player models, both with 3.5-inch LCD but have different memory capacities:

- 16GB – two 8GB chips
- 8GB – one 8GB chip

Weekly resources are limited to

- 800 units of 3.5-inch antiglare LCDs
- 1000 units of 8GB memory chips
- 50 hours of total labor time. It takes 3 minutes of labor for each 16GB player, and 4 minutes of labor for each 8GB player.
For marketing reasons,

- Total production cannot exceed 700
- # 16GB players cannot exceed # 8GB players by more than 350

Profit, while remaining within the marketing guidelines, can be computed as

- $16 each 16GB player
- $10 each 8GB player

The current weekly production plan consists of 450 16GB players and 100 8GB players, make a profit of $16*450 + $10*100 = $8200.

Management is seeking a new production plan that will increase the profit.
Variables:

- $x_1$: weekly produced units of 16GB players
- $x_2$: weekly produced units of 8GB players

Objective: to maximize the weekly profit $16x_1 + 10x_2$

Constraints:

- $x_1, x_2 \geq 0$
- LCD: $x_1 + x_2 \leq 8000$
- Memory: $2x_1 + x_2 \leq 1000$
- Labor: $3x_1 + 4x_2 \leq 3000$
- Marketing total: $x_1 + x_2 \leq 7000$
- Marketing mix: $x_1 - x_2 \leq 350$
Graphical optimization

- 2D plot of the variables, constraints, and level curves of the objective
- feasible region is a polyhedron, possibly empty or unbounded
- three types of feasible points: interior, boundary, and extreme points
- level curves of the objective are parallel lines
- if there is a solution, there is an extreme point solution
- it is possible that the problem is feasible but has an unbounded $-\infty$ optimal objective
- they can be infinitely many solutions
A set $S$ is convex if any $x, y \in S$, $\alpha x + (1 - \alpha)y \in S$, $\forall \alpha \in [0, 1]$

Let $S := \{x^1, \ldots, x^K\}$.

- $\text{span}(S) = \{ \sum_{k=1}^{K} \lambda_k x^k : \lambda_k \in \mathbb{R}, \forall k \}$
- $\text{aff}(S) = \{ \sum_{k=1}^{K} \lambda_k x^k : \sum_{k=1}^{K} \lambda_k = 1, \lambda_k \in \mathbb{R}, \forall k \}$, called affine hull
- $\text{cone}(S) = \{ \sum_{k=1}^{K} \lambda_k x^k : \lambda_k \in \mathbb{R}^+, \forall k \}$, called convex cone
- $\text{convex}(S) = \{ \sum_{k=1}^{K} \lambda_k x^k : \sum_{k=1}^{K} \lambda_k = 1, \lambda_k \in \mathbb{R}^+, \forall k \}$, called convex hull

The intersection of convex sets is a convex set
- consider $\mathbb{R}^n$

- $\{x : a^\top x = b\}$ is called a hyperplane

- $\{x : a^\top x \geq b\}$ is called a halfspace

- The intersection of finitely many halfspaces is called a polyhedron
Consider the polyhedron $P := \{ x : Ax \geq b \} \subset \mathbb{R}^n$.

- $x \in P$ is an **extreme point** of $P$ if $\nexists \ y, z \in P, y \neq x, z \neq x, 0 < \lambda < 1$, such that $x = \lambda y + (1 - \lambda)z$.

- an extreme point is not strictly within the line segment connecting two other points in $P$
- $x \in P$ is a vertex of $P$ if $\exists c, \exists c^T x < c^T z, \forall z \in P \setminus \{x\}$.

- A vertex is the *unique* minimizer of some linear function over $P$. 
The standard simplex in $\mathbb{R}^3$:

$$P := \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1, \ x_1, x_2, x_3 \geq 0\}.$$

- Points A, B, C: each has 3 active (i.e., “=” ) constraints
- Point E: 2 active constraints. If add a constraint: $2x_1 + 2x_2 + 2x_3 = 2$. Then, 3 constraints are active at E, but they are not linearly independent.

A vertex or extreme point has $n$ linearly independent active constraints
Standard form

- variable \( x \in \mathbb{R}^n \)
- cost vector \( c \in \mathbb{R}^n \)
- right-hand side vector \( b \in \mathbb{R}^m \)
- coefficient matrix \( A \in \mathbb{R}^{m \times n} \)

standard form

\[
\text{minimize } c^T x \\
\text{subject to } Ax = b \\
x \geq 0.
\]

Any non-standard form LP can be reformulated to the standard form. The standard form simplifies algorithms and unifies analysis.
Conversion to the standard form

Consider

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \geq b \\
& \quad x \geq 0.
\end{align*}
\]

Introduce \textit{surplus} or \textit{dummy variables} \( s_i \).

\[
a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i \iff a_{i1}x_1 + \cdots + a_{in}x_n - s_i = b_i, \quad s_i \geq 0
\]

New form

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad [A, -I_m] \begin{bmatrix} x \\ s \end{bmatrix} = b \\
& \quad x \geq 0, \quad s \geq 0.
\end{align*}
\]
General methods:

- “maximize” objective: minimize its negative
- $\leq$ constraint: add nonnegative slack variable
- $\geq$ constraint: subtract nonnegative slack variable
- $x_i \leq 0$: substitute $x_i$ by $-x_i$ throughout
- free $x_i$: introduce $u_i, v_i \geq 0$ and substitute $x_i$ by $u_i - v_i$ throughout
- constraint $|x_i| \leq b_i$: replace by $x_i \leq b_i$ and $-x_i \leq b_i$
- objective $|x_i|$: introduce $u_i, v_i \geq 0$ and substitute
  - $x_i$ by $u_i - v_i$
  - $|x_i|$ by $u_i + v_i$
Example

maximize $x_2 - x_1$

subject to $3x_1 = x_2 - 5$

$|x_2| \leq 2$

$x_1 \leq 0$.

Steps:

1. change to minimize $x_1 - x_2$

2. substitute $x_1$ by $-x_1$

3. write $|x_2| \leq 2$ by $x_2 \leq 2$ and $-x_2 \leq 2$

4. introduce $s_1$ and $s_2$ and rewrite $x_2 + s_1 = 2$ and $-x_2 + s_2 = 2$

5. split $x_2 = u_2 - v_2$, $u_2, v_2 \geq 0$. 
We obtain:

\[
\begin{align*}
\text{minimize } & x_1 - x_2 \\
\text{subject to } & 3x_1 + u_2 - v_2 = 5 \\
& u_2 - v_2 + s_1 = 2 \\
& v_2 - u_2 + s_2 = 2 \\
& x_1, u_2, v_2, s_1, s_2 \geq 0.
\end{align*}
\]