Differential Equations Methods for the Monge–Kantorovich Mass Transfer Problem

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Abstract

We demonstrate that a solution to the classical Monge–Kantorovich problem of optimally rearranging the measure $\mu^+ = f^+ dx$ onto $\mu^- = f^- dy$ can be constructed by studying the p-Laplacian equation

$$-\mathrm{div}(|Du_p|^{p-2} Du_p) = f^+ - f^-$$

in the limit as $p \to \infty$. The idea is to show $u_p \to u$, where u satisfies

$$|Du| \leq 1, \; -\mathrm{div}(aDu) = f^+ - f^-$$

for some density $a \geq 0$, and then to build a flow by solving an ODE involving a, Du, f^+ and f^-.

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1 Introduction

This paper introduces some PDE and ODE methods for constructively solving one version of the Monge–Kantorovich mass transfer problem.

The basic issue is this. Given two nonnegative, summable functions $f^{\pm}$ on $\mathbb{R}^n$ satisfying the compatibility condition

$$\int_{\mathbb{R}^n} f^+ dx = \int_{\mathbb{R}^n} f^- dy, \tag{1.1}$$

we consider the corresponding measures $\mu^+ = f^+ dx$, $\mu^- = f^- dy$, and ask how we can optimally rearrange μ^+ onto μ^-. If $\mathbf{r} : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth, one-to-one mapping, the requirement is that $\mathbf{r}$ transfer μ^+ onto μ^-; that is,

$$f^+(x) = f^-(\mathbf{r}(x)) \ \det D\mathbf{r}(x) \qquad (x \in \mathbb{R}^n). \tag{1.2}$$

Denote by $\mathcal{A}$ the admissible class of smooth, one-to-one functions $\mathbf{r}$ satisfying (1.2). We then seek a mass transfer plan $\mathbf{s} \in \mathcal{A}$ which is optimal in the sense that

$$I[\mathbf{s}] = \min_{\mathbf{r} \in \mathcal{A}} I[\mathbf{r}], \tag{1.3}$$

where

$$I[\mathbf{r}] = \int_{\mathbb{R}^n} |x - \mathbf{r}(x)| f^+(x) \ dx = \int_{\mathbb{R}^{\propto}} |x - \mathbf{r}(x)| d\mu^+ \ . \tag{1.4}$$

This is a form of Monge's problem of the "déblais" and "remblais" (cf. Monge [M], Dupin [D], Appell [A]), dating from the early 1780's. The physical interpretation is that we are given a pile of soil or rubble (the "déblais"), with mass density f^+, which we wish to transport to an excavation or fill (the "remblais"), with mass density f^-. For a given transport scheme $\mathbf{r}$, condition (1.2) is conservation of mass. Furthermore, as each particle of soil moves a distance $|x - \mathbf{r}(x)|$, we can interpret $I[\mathbf{r}]$ as the total work involved. We consequently are looking for a way to rearrange $\mu^+ = f^+ dx$ onto $\mu^- = f^- dy$, which requires the least work.

This optimization problem, and its many, many variants and extensions (entailing for example more general measures on more general spaces, different cost functionals, etc.) has been intensively studied for over two hundred years. We review some of the principal discoveries.

a. Monge. Monge himself contributed the essential insight that an optimal transfer plan $\mathbf{s}$ should be in part determined by a *potential* u. More precisely, he deduced by heutistic, geometric arguments that if an optimal plan $\mathbf{s}$ exists, then there exists a scalar potential function u such that

$$\frac{\mathbf{s}(x)-x}{|\mathbf{s}(x)-x|} = -Du(x) \qquad (x \in X), \tag{1.5}$$

where $X = \operatorname{supp}(f^+)$. In other words the *direction* that each particle of soil should move is determined as the (opposite of the) gradient Du of u. Observe that necessarily then

$$|Du| = 1 \text{ in } X. \tag{1.6}$$

Monge's discovery, elaborated in the sources cited above, is deeply connected with the theory of developable surfaces, lines of curvature, etc. he and his students discovered. See also Struik [SD, Chapter 2, esp. pp. 95-96] for more background.

b. Appell. A hundred years later Appell [A] provided an analytic, but still formal, proof of (1.5). His idea was to assume a smooth mapping $\mathbf{s} \in \mathcal{A}$ minimizes $I[\cdot]$ and to work out the corresponding Euler–Lagrange equation, using a Lagrange multiplier to incorporate the constraint (1.2) for functions $\mathbf{r} \in \mathcal{A}$. We reproduce his result by computing the first variation in $\mathbf{s} = (s^1, \dots s^n)$ of the augmented work functional

$$\int_{\mathbb{R}^n} |x - \mathbf{s}(x)| f^+(x) + \lambda(x)[f^-(\mathbf{s}(x)) \operatorname{det} D\mathbf{s}(x) - f^+(x)]dx, \tag{1.7}$$

where the function λ is the Lagrange multiplier corresponding to the pointwise constraint

$$f^-(\mathbf{s}(x)) \operatorname{det} D\mathbf{s}(x) = f^+(x) \qquad (x \in \mathbb{R}^n). \tag{1.8}$$

We deduce

$$(\lambda f^-(\mathbf{s})(\operatorname{cof} D\mathbf{s})^k_i)_{x_i} = \frac{s^k(x) - x_k}{|\mathbf{s}(x) - x|} f^+(x) + \lambda f^-_{y_k}(\mathbf{s}) \operatorname{det} D\mathbf{s} \tag{1.9}$$

for $k = 1, \dots, n$, where $\operatorname{cof} D\mathbf{s}$ is the cofactor matrix of $D\mathbf{s}$. Here and afterwards we sum on repeated indices. Now $((\operatorname{cof} D\mathbf{s})^k_i)_{x_i} = 0 \;\; (k = 1, \dots, n)$ and

$$\begin{cases} s^l_{x_i}(\operatorname{cof} D\mathbf{s})^k_i = \delta_{kl}(\operatorname{det} D\mathbf{s}) & (1 \le k, l \le n) \\ s^k_{x_j}(\operatorname{cof} D\mathbf{s})^k_i = \delta_{ij}(\operatorname{det} D\mathbf{s}) & (1 \le i, j \le n)\ . \end{cases} \tag{1.10}$$

Consequently (1.9) simplifies to read

$$\lambda_{x_i} f^-(\mathbf{s})(\operatorname{cof} D\mathbf{s})^k_i = \frac{s^k(x) - x_k}{|\mathbf{s}(x) - x|} f^+.$$

Multiply now by $s^k_{x_j}$ and sum on k, recalling (1.8), (1.10) to deduce

$$\lambda_{x_j} = \frac{s^k(x) - x_k}{|\mathbf{s}(x) - x|} s^k_{x_j} \qquad (1 \le j \le n)$$

on X. Now define u by

$$u(\mathbf{s}(x)) = -\lambda(x). \tag{1.11}$$

Then

$$u_{y_k}(\mathbf{s}) s^k_{x_j} = -\lambda_{x_j} = -\frac{s^k(x) - x_k}{|\mathbf{s}(x) - x|}\ s^k_{x_j} \qquad (1 \le k \le n).$$

As $D\mathbf{s}$ is invertible on X, we see

$$Du(\mathbf{s}(x)) = -\frac{\mathbf{s}(x) - x}{|\mathbf{s}(x) - x|} \qquad (x \in X).$$

But $Du(\mathbf{s}(x)) = Du(x)$ and so Monge's assertion (1.5) follows.

We observe from (1.11) that the potential u *can be intepreted as (a transformation of) the Lagrange multiplier for the mass conservation constraint* (1.8).

c. Kantorovich. In the 1940's Kantorovich [K1] [K2] made this last statement rigorous. His first idea was that the minimization problem (1.3) can be relaxed, as follows. He proposed the new task of finding a measure $p \in \mathcal{M}$ solving

$$J[p] = \min_{q \in \mathcal{M}} J[q], \tag{1.12}$$

where

$$\begin{aligned} \mathcal{M} &= \{\text{Radon probability measures } q \text{ on } \mathbb{R}^n \times \mathbb{R}^n \mid \\ &\qquad \text{proj}_x q = \mu^+ = f^+(x)dx,\ \text{proj}_y q = \mu^- = f^-(y)dy\} \end{aligned} \tag{1.13}$$

and

$$J[q] = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x - y|\ dq(x, y) \qquad (q \in \mathcal{M}). \tag{1.14}$$

The point is that if Monge's original problem (1.3) has a minimizer $\mathbf{s}$, then the measure q defined by

$$q(E) = \int_{\{x \mid (x, \mathbf{s}(x)) \in E\}} f^+(x)dx \qquad (E \subset \mathbb{R}^n \times \mathbb{R}^n,\ E \text{ Borel}) \tag{1.15}$$

belongs to $\mathcal{M}$. Furthermore the relaxed cost functional J is linear in q and so simple compactness arguments show (1.12) has at least one solution.

In addition Kantorovich observed that problem (1.12)–(1.14) is an infinite dimensional linear programming minimization problem, which has a *dual maximization problem*, namely to find $u \in \mathcal{L}$ solving

$$K[u] = \max_{w \in \mathcal{L}} K[w], \tag{1.16}$$

where

$$\mathcal{L} = \left\{ w : \mathbb{R}^n \to \mathbb{R} \mid \ \mathrm{Lip}[w] = \sup_{x \neq y} \frac{|w(x) - w(y)|}{|x-y|} \leq 1 \right\} \tag{1.17}$$

and

$$K[w] = \int_{\mathbb{R}^n} w(f^+ - f^-)\ dz = \int_{\mathbb{R}^n} wf\ dz. \tag{1.18}$$

for $f = f^+ - f^-$. Consult for instance Vershik [V] for more explanation. As we will see, a solution u of (1.16)–(1.18) can be interpreted as Monge's transport potential and thus as the Lagrange multiplier for the original mass transfer problem.

Kantorovich's optimality principle states

$$\min_{q \in \mathcal{M}} J[q] = \max_{w \in \mathcal{L}} K[w]\ , \tag{1.19}$$

and an equivalent formulation of Kantorovich's principle is

$$\max \left\{ \int_X uf^+ dx + \int_Y vf^- dy \mid u(x) + v(y) \leq |x-y| \text{ for all } x \in X,\ y \in Y \right\} = \min_{q \in \mathcal{M}} J[q].$$

This identity is valid in far more general circumstances. Indeed there has developed a huge body of mathematics for explicitly computing

$$\max_{\mathrm{Lip}(w) \leq 1} \int_{\mathbb{R}^n} w(d\mu^+ - d\mu^-) = \ \mathrm{dist}(\mu^+, \mu^-)$$

for general probability measures μ^+, μ^- on $\mathbb{R}^n$ and general spaces. The resulting distance and its variants are called the Monge–Kantorovich–Rubinstein–Wasserstein–etc. metrics; see Rachev [R] for more information.

d. Sudakov. Left open in Kantorovich's work is the question as to whether a relaxed measure solution $p \in \mathcal{M}$ is representable by a one-to-one mapping $\mathbf{s} \in \mathcal{A}$, i.e. whether p has the form (1.15). Sudakov in [SV] shows that this is indeed the case, although the mapping $\mathbf{s}$ need not be smooth. Instead we replace (1.8) with the integral condition

$$\int_{\mathbb{R}^n} h(\mathbf{s}(x)) f^+(x) dx = \int_{\mathbb{R}^n} h(y) f^-(y) dy \ \text{ for each continuous function } h. \tag{1.20}$$

Sudakov's solution is an outgrowth of his study of "measure decompositions" of probability spaces.

Our paper presents a differential-equations-based alternative to Sudakov's approach. The primary new contribution is to identify an ODE flow which, taken for time one, generates an "essentially one-to-one" mapping $\mathbf{s}$, satisfying (1.20) and minimizing the Monge cost functional $I[\cdot]$ among all such maps. The trick is to study the p-Laplacian PDE

$$-\operatorname{div}(|Du_p|^{p-2}Du_p) = f^+ - f^- = f \qquad (n+1 \le p < \infty) \tag{1.21}$$

and then to let $p \to \infty$.

The diffusion coefficient in (1.21) is $|Du_p|^{p-2}$, which is very large in any region $\{|Du_p| > 1+\delta\}$ and very small in $\{|Du_p| < 1-\delta\}$ if $\delta > 0$. We can consequently interpret the limit of (1.21) as $p \to \infty$ as an "infinitely fast/infinitely slow" diffusion limit (cf. [A-E-W]). Since (1.21) is the Euler–Lagrange equation for the problem of minimizing

$$\int_{\mathbb{R}^n} \frac{1}{p}|Dw|^p - wf \; dz \;,$$

it is not very hard to prove that if $u_p \to u$, then $u \in \mathcal{L}$ and u maximizes

$$K[w] = \int_{\mathbb{R}^n} wf \; dz \;.$$

Thus u is a potential for the Monge–Kantorovich transport problem. Consequently, formally at least, we see from (1.5) that Du determines the direction of an optimal transfer plan $\mathbf{s}$.

But there is still another missing piece of information, namely the length $|x-\mathbf{s}(x)|$, which is not determined solely by u. Our primary new discovery is that the PDE (1.21) in fact contains in the limit $p \to \infty$ a "transport density", which will allow us to compute $|x-\mathbf{s}(x)|$. The procedure is this. If we set

$$\mathbf{A}_p = |Du_p|^{p-2}Du_p,$$

then $\mathbf{A}_p \rightharpoonup \mathbf{A}$ weakly $*$ in L^∞ and $\mathbf{A}$ has the form $\mathbf{A} = aDu$ for some nonnegative, bounded function a, called the *transport density.* Thus

$$-\operatorname{div}(aDu) = f^+ - f^- = f. \tag{1.22}$$

The function a is the Lagrange multiplier for the constraint that $|Du| \le 1$ a.e. We employ u and a to design an optimal mapping $\mathbf{s}$, by solving for a.e. point z_0 the ODE (cf. Dacorogna–Moser [D-M])

$$\begin{cases} \dot{\mathbf{z}}(t) = \dfrac{-a(\mathbf{z}(t))Du(\mathbf{z}(t))}{tf^-(\mathbf{z}(t)) + (1-t)f^+(\mathbf{z}(t))} & (0 \le t \le 1) \\ \mathbf{z}(0) = z_0. \end{cases} \tag{1.23}$$

Write $\mathbf{z}(t) = \mathbf{z}(t, z_0)$ to display the dependence on the initial point. We will then define

$$\mathbf{s}(z_0) = \mathbf{z}(1, z_0). \tag{1.24}$$

In other words, we claim that the time-one flow governed by the nonautonomous ODE (1.23) generates a solution to the Monge–Kantorovich problem, where the potential u and density a are related by (1.22). *Therefore the PDE (1.22)* (which is the Euler–Lagrange equation of Kantorovich's dual problem) *contains all the information needed to solve Monge's original problem, both the direction and the distance a.e. particle should move.*

It is not very hard to invent formal calculations justifying the foregoing claims (cf. §8, 10). However, a rigorous proof is really, really tricky, mainly because the functions $\mathbf{s}$, a and Du are not in general even continuous. In fact we expect typically quite a complicated pattern of optimal mass transport, as indicated schematically in the picture.

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The overall procedure will be first to identify "transport rays", which are segments joining X $(= \operatorname{supp}(f^+))$ to Y $(= \operatorname{supp}(f^-))$ on which u decreases linearly at rate one. We will show that for a "typical" such transport ray, the restriction of a to R is defined and can be interpreted as a Lipschitz continuous function along R. The reason is that we can think of u as known and then regard (1.22) as a first-order linear PDE for a, namely

$$a_\nu - a\Delta u = f \ , \qquad \text{where} \ \ \nu = -Du \ .$$

Assuming then that $f^\pm$ are Lipschitz, we will see that the ODE (1.23) (modified to avoid dividing by zero) has a unique solution, which moves the point z_0 "downhill" along the ray R. We will prove as well that the density a vanishes at the ends of the ray, and so the trajectory does not "overshoot" the end.

Establishing the measure preserving identity (1.20) for $\mathbf{s}$ defined by (1.23), (1.24) is much more problematical, as $\mathbf{s}$ is not in general continuous. We first mollify aDu and so obtain a smooth vector field to which the change of Jacobian calculations of Dacorogna–Moser apply. We build therefore an approximate transfer scheme $\mathbf{s}_\varepsilon$, and are faced with the basic task of showing $\mathbf{s}_\varepsilon \to \mathbf{s}$ a.e. as $\varepsilon \to 0$. This will be enough, as the integral form (1.20) of the measure preserving requirement is conserved under a.e. convergence. It is however quite subtle to verify the limit: a and Du, while well behaved on transport rays, are not in general continuous, and so the effects of the mollification must be carefully tracked. A main fact here (Proposition 4.1) is that u is basically $C^{1,1}$ along the interior of any transport ray. The corresponding estimate gives just enough control to show (Theorem 9.1) that the trajectories of the mollified vector fields do indeed converge to the trajectory determined by the ODE (1.23). Nonetheless the detailed proofs are extremely intricate.

We should call to the reader's attention to some interesting papers by Strang [S1], [S2] and Iri [I] on a somewhat related problem. These authors note that the question of minimizing

$$L[\boldsymbol{\xi}] = \int_{\mathbb{R}^n} |\boldsymbol{\xi}| dx \tag{1.25}$$

over

$$\mathcal{V} = \{\text{summable vector fields } \boldsymbol{\xi} : \mathbb{R}^n \to \mathbb{R}^n \mid -\text{div } \boldsymbol{\xi} = f\} \tag{1.26}$$

is another kind of convex dual problem to (1.16)–(1.18). The connection with our work is that $\boldsymbol{\eta} = aDu \in \mathcal{V}$ solves

$$L[\boldsymbol{\eta}] = \min_{\xi \in \mathcal{V}} L[\boldsymbol{\xi}]. \tag{1.27}$$

Strang and Iri physically interpret (1.16)–(1.18) and (1.25)–(1.27) in terms of certain maximal flow/minimal cut problems, but the context is not that of the Monge–Kantorovich problem. The difference seems to be that in the Iri–Strang problem mass can be thought of as being continuously added at the rate $f^+(x)$ at points $x \in X$, continuously being transported by the flow

$$\dot{\mathbf{z}}(t) = -a(\mathbf{z}(t))Du(\mathbf{z}(t)) \qquad (t > 0),$$

and continuously being removed at the rate $f^-(y)$ at points $y \in Y$. This situation differs from the Monge–Kantorovich requirement that a one-time mass transfer plan be devised.

Our paper has also been inspired in large part by Bhattacharya–DiBenedetto–Manfredi [B-D-M] and Janfalk [J]. We borrow from [B-D-M] the important observation that not only $|Du_p|$, but also $|Du_p|^p$, is bounded independently of p. We have taken up Janfalk's observation that the transport density a is supported within the collection of transport rays [J, p. 76-79]. He seems also to have been the first actually to compute the transport density [J, p. 93-96] and to note that it vanishes at the interior endpoints of transport rays. (Janfalk's notation differs from ours, as does the physical setting.)

We make in §2 a number of assumptions on f^+, f^-. We in particular assume f^+ and f^- are Lipschitz continuous. We need this hypothesis mostly in order to ensure the ODE (1.23) has a unique solution. In addition we suppose that $X = \text{supp}(f^+)$ and $Y = \text{supp}(f^-)$ are a positive distance apart. The later assumption is useful in excluding certain bad behavior (e.g. a transport ray entering Y, then entering X, then entering Y, etc., infinitely many times), but is presumably not essential. We hope to return to this point in future work.

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2 Uniform estimates on the p-Laplacian, limits as $p \to \infty$

In this section we set forth our hypotheses regarding the densities f^+, f^- and then obtain estimates, independent of p, on solutions of the corresponding p-Laplacian equations.

Assumptions on the mass densities f^+, f^-

We henceforth suppose

$$f^-, f^+ \text{ are nonnegative, Lipschitz functions on } \mathbb{R}^n \text{ with compact support,} \tag{2.1}$$

satisfying the compatibility condition:

$$\int_{\mathbb{R}^n} f^+ dx = \int_{\mathbb{R}^n} f^- dy. \tag{2.2}$$

We write

$$X = \text{ supp}(f^+), \; Y = \text{ supp}(f^-), \tag{2.3}$$

and assume as well

$$\begin{cases} X \cap Y = \emptyset, \\ f^+ > 0 \text{ on } X^0, \quad f^- > 0 \text{ on } Y^0, \\ \partial X, \; \partial Y \text{ are smooth,} \\ X, Y \subset B(0,S) \text{ for some } S > 0. \end{cases} \tag{2.4}$$

We have written X^0 for the interior of X, Y^0 for the interior of Y, and $B(0,S)$ for the closed ball with center 0, radius S. Hereafter set

$$f = f^+ - f^-. \tag{2.5}$$

As explained in §1, our intention is eventually to construct an optimal mass transport plan using information gleaned from the $p \to \infty$ limit of the p-Laplacian PDE

$$\begin{cases} -\text{div}(|Du_p|^{p-2}Du_p) = f & \text{in } B(0,R) \\ u_p = 0 & \text{on } \partial B(0,R). \end{cases} \tag{2.6}$$

Here $n + 1 \le p < \infty$ and the large radius $R > S$ will be selected below. The unique weak solution u_p of (2.5) is found as the minimizer of the functional

$$\int_{B(0,R)} \frac{1}{p}|Dw|^p dx - fw \; dz \tag{2.7}$$

over all $w \in W_0^{1,p}(B(0,R))$. We have $u_p \in C^{1,\alpha}(B(0,R))$ for some $\alpha = \alpha(p) > 0$, according to Uhlenbeck [U], Lieberman [LG], etc.

Lemma 2.1 *There exists a large radius $R > 0$ and a constant C_1 such that*

$$\max_{B(0,R)} |u_p| \leq C_1 \tag{2.8}$$

$$\max_{\partial B(0,R)} |Du_p| \leq \frac{1}{2} \tag{2.9}$$

for all $n + 1 \leq p < \infty$.

Proof. 1. To simplify notation we drop the subscript p. Thus, writing $u = u_p$, we have

$$\begin{cases} -\mathrm{div}(|Du|^{p-2}Du) = f & \text{in } B(0,R) \\ u = 0 & \text{on } \partial B(0,R) \end{cases} \tag{2.10}$$

in the weak sense. Consequently

$$\begin{aligned} \int_{B(0,R)} |Du|^p dz &= \int_{B(0,R)} fu\ dz \\ &= \int_{B(0,R)} f(u-\lambda) dz \end{aligned}$$

for any constant $\lambda \in \mathbb{R}$, according to (2.2),(2.5). Selecting $\lambda = ⨍_{B(0,S)} u\ dz = \frac{1}{|B(0,S)|} \int_{B(0,S)} u\ dz$, we deduce

$$\begin{aligned} \int_{B(0,R)} |Du|^p dz &\leq \|f\|_{L^\infty} \int_{B(0,S)} |u-\lambda| dz \\ &\leq C \int_{B(0,S)} |Du| dz \\ &\leq C \left(\int_{B(0,R)} |Du|^p dz \right)^{1/p}. \end{aligned}$$

Hence

$$\int_{B(0,R)} |Du|^p dz \leq C, \tag{2.11}$$

and the constant C does not depend on p or R.

2. We *claim* next

$$\text{there exists a point } x_0 \in \partial B(0,S) \text{ such that } u(x_0) = 0. \tag{2.12}$$

Suppose not. Then without loss we may assume

$$u \geq \theta > 0 \text{ on } \partial B(0,S)$$

for some constant $\theta > 0$. Solve the PDE

$$\begin{cases} -\mathrm{div}(|D\tilde{u}|^{p-2}D\tilde{u}) = 0 & \text{in } B(0,R) - B(0,S) \\ \tilde{u} = \theta & \text{on } \partial B(0,S) \\ \tilde{u} = 0 & \text{on } \partial B(0,R). \end{cases}$$

Then $u \geq \tilde{u}$ in $B(0,R) - B(0,S)$. But $\tilde{u}$ has the form

$$\tilde{u}(x) = -\alpha |x|^{\frac{p-n}{p-1}} + \beta \qquad (S \leq |x| \leq R)$$

for appropriate constants α, β, with $\alpha > 0$ (cf. Kawohl [KB]). Thus

$$\frac{\partial \tilde{u}}{\partial \nu} < 0 \text{ on } \partial B(0,R),$$

ν denoting the outward unit normal. Since $u \geq \tilde{u}$, we conclude

$$\frac{\partial u}{\partial \nu} < 0 \text{ on } \partial B(0,R) \tag{2.13}$$

as well. But the PDE (2.10) and the compatiblity condition (2.2) together imply

$$\int_{\partial B(0,R)} |Du|^{p-2} \, \frac{\partial u}{\partial \nu} d\mathcal{H}^{\backslash - \infty} = \prime,$$

$\mathcal{H}^{\backslash - \infty}$ being $(n-1)$-dimensional Hausdorff measure. Since u is $C^{1,\alpha}$ up to $\partial B(0,R)$, the boundary integral makes sense. This equality however contradicts (2.13), and so assertion (2.12) is proved.

3. In light of (2.11) we have

$$\int_{B(0,S)} |Du|^{n+1} dz \leq C;$$

and this bound together with (2.12) imply

$$\max_{B(0,S)} |u| \leq C_1,$$

the constant C_1 independent of $p \geq n+1$ and R. The maximum principle implies the same bound for $|u|$ over $B(0,R)$. This establishes estimate (2.8).

4. Now take any point $x \in \partial B(0,R)$. We may as well assume $x = -\mathrm{R}e_n$ ($e_n = (0, \ldots, 1)$). Define

$$v(x) = \tfrac{1}{2}(x_n + R) \qquad (x \in B(0,R)).$$

Then

$$\begin{cases} -\mathrm{div}(|Dv|^{p-2} Dv) = 0 & \text{in } B(0,R) - B(0,S) \\ v \geq 0 & \text{on } \partial B(0,R) \\ v \geq \frac{1}{2}(R-S) & \text{on } \partial B(0,S). \end{cases}$$

Fix R so large that

$$\tfrac{1}{2}(R-S) \geq C_1.$$

Then

$$v \geq \pm u \text{ on } \partial(B(0,R) - B(0,S)),$$

and therefore

$$|u| \leq v \text{ in } B(0,R) - B(0,S).$$

Consequently

$$|Du(x)| = |u_{x_n}| \leq v_{x_n} = \tfrac{1}{2}.$$

This bound gives (2.9). □

Next we estimate the sup-norm of $|Du_p|^p$. The following bound follows from Bhattacharya–DiBenedetto–Manfredi [B-D-M, Part III, Propositions 1.1, 2.1]. For variety and to keep our presentation self-contained, we provide a different proof (which however is not so good as that in [B-D-M], as we require $\max|Df|$ be finite).

Proposition 2.1 *There exists a constant C_2 such that*

$$\max_{B(0,R)} |Du_p|^p \leq C_2 \tag{2.14}$$

for $n+1 \leq p < \infty$.

Proof. 1. Fix $0 < \varepsilon \leq \frac{1}{2}$ and write $\phi(q) = \phi_\varepsilon(q) = (|q|^2 + \varepsilon^2)^{1/2}$ $(q \in \mathbb{R}^n)$. Assume temporarily $f = f^+ - f^-$ is smooth. We approximate u_p by the smooth solution $u = u_p^\varepsilon$ of the PDE

$$\begin{cases} -\text{div}(\phi(Du)^{p-2}Du) = f & \text{in } B(0,R) \\ u = 0 & \text{on } \partial B(0,R). \end{cases} \tag{2.15}$$

As in Lemma 2.1 we have the estimates:

$$\max_{B(0,R)} |u| \leq C_1, \ \max_{\partial B(0,R)} |Du| \leq \tfrac{1}{2}. \tag{2.16}$$

Differentiate the PDE (2.15) with respect to x_k:

$$-(\phi^{p-2}u_{x_i x_k} + (p-2)\phi^{p-4}u_{x_i}u_{x_j}u_{x_j x_k})_{x_i} = f_{x_k}.$$

Multiply by u_{x_k} and sum $k = 1, \ldots, n$ to deduce

$$\begin{aligned} &-(\phi^{p-2}u_{x_i x_k}u_{x_k} + (p-2)\phi^{p-4}u_{x_i}u_{x_j}u_{x_j x_k}u_{x_k})_{x_i} \\ &\quad = Df \cdot Du - \phi^{p-2}|D^2u|^2 - \tfrac{p-2}{4}\phi^{p-4}|D|Du|^2|^2 \\ &\quad \leq C|Du| - \tfrac{p-2}{4}\ \phi^{p-4}|D|Du|^2|^2. \end{aligned} \tag{2.17}$$

2. Define

$$a = \phi(Du)^p.$$

Then

$$a_{x_i} = p\phi^{p-2} u_{x_k} u_{x_k x_i} \qquad (1 \le i \le n). \tag{2.18}$$

Consequently (2.17) implies

$$-\left(\frac{a_{x_i}}{p} + \frac{(p-2)}{p}\,\frac{u_{x_i} u_{x_j}}{\phi^2} a_{x_j}\right)_{x_i} \le C|Du| - \frac{p-2}{4}\phi^{p-4}|D|Du|^2|^2. \tag{2.19}$$

Next write

$$b = a\Phi(u),$$

the smooth, positive function $\Phi : \mathbb{R} \to \mathbb{R}$ to be chosen later. Observe

$$b_{x_i} = a_{x_i}\Phi + a\Phi' u_{x_i} \qquad (1 \le i \le n). \tag{2.20}$$

3. Suppose now b attains its maximum over $B(0,R)$ at an interior point x_0. Then at this point x_0 we have

$$
\begin{aligned}
0 \quad &\leq -\left(\tfrac{b_{x_i}}{p} + \tfrac{p-2}{p}\tfrac{u_{x_i}u_{x_j}}{\phi^2}b_{x_j}\right)_{x_i} \\
&= -\left(\Phi\left(\tfrac{a_{x_i}}{p} + \tfrac{p-2}{p}\tfrac{u_{x_i}u_{x_j}}{\phi^2}a_{x_j}\right)\right)_{x_i} \\
&\quad -\left(\Phi' a\left(\tfrac{u_{x_i}}{p} + \tfrac{p-2}{p}\tfrac{u_{x_i}u_{x_j}}{\phi^2}u_{x_j}\right)\right)_{x_i} \qquad \text{according to (2.20)} \\
&= -\Phi\left(\tfrac{a_{x_i}}{p} + \tfrac{p-2}{p}\tfrac{u_{x_i}u_{x_j}}{\phi^2}a_{x_j}\right)_{x_i} \qquad (2.21) \\
&\quad -\Phi' u_{x_i}\left(\tfrac{a_{x_i}}{p} + \tfrac{p-2}{p}\tfrac{u_{x_i}u_{x_j}}{\phi^2}a_{x_j}\right) \\
&\quad -\Phi'(\Lambda a u_{x_i})_{x_i} \\
&\quad -\Phi'' a|Du|^2\Lambda \\
&\equiv A + B + C + D.
\end{aligned}
$$

Here we have written

$$\Lambda = \frac{1}{p} + \frac{p-2}{p}\ \frac{|Du|^2}{\phi^2}. \tag{2.22}$$

Owing to (2.19), we have

$$A \leq C\Phi|Du| - \Phi\ \frac{p-2}{4}\ \phi^{p-4}|D|Du|^2|^2. \tag{2.23}$$

Utilizing (2.20) and the fact $Db = 0$ at x_0, we deduce as well that

$$B = \frac{(\Phi')^2}{\Phi}\ a|Du|^2\Lambda. \tag{2.24}$$

By definition:

$$D = -\Phi'' a|Du|^2\Lambda. \tag{2.25}$$

It remains to estimate the term:

$$\begin{aligned} C &= -\Phi'(\Lambda a u_{x_i})_{x_i} \\ &= -\Phi'(\Lambda \phi^2 \phi^{p-2} u_{x_i})_{x_i} \\ &= -\tfrac{\Phi'}{p}([(p-1)|Du|^2+\varepsilon^2]\phi^{p-2}u_{x_i})_{x_i} \\ &= -\tfrac{\Phi'}{p}[(p-1)|Du|^2+\varepsilon^2](\phi^{p-2}u_{x_i})_{x_i} \\ &\quad -\tfrac{p-1}{p}\Phi'\phi^{p-2}(D|Du|^2\cdot Du) \\ &\equiv E+F. \end{aligned} \tag{2.26}$$

The PDE (2.15) implies then

$$E \leq C|\Phi'|(1+|Du|^2). \tag{2.27}$$

Furthermore:

$$\begin{aligned} F &\leq |\Phi'|\phi^{p-1}|D|Du|^2| \\ &\leq \Phi\tfrac{p-2}{4}\ \phi^{p-4}|D|Du|^2|^2 + \tfrac{(\Phi')^2}{\Phi}\ \tfrac{\phi^{p+2}}{p-2}. \end{aligned} \tag{2.28}$$

Combining (2.21)–(2.28) yields the inequality:

$$\Phi'' a|Du|^2\Lambda \leq C\Phi|Du| + C|\Phi'|(1+|Du|^2) + \frac{(\Phi')^2}{\Phi}\left[a|Du|^2\Lambda + \frac{\phi^{p+2}}{p-2}\right]\ . \tag{2.29}$$

4. To extract useful information from this estimate let us take

$$\Phi(z) = e^{\mu z^2} \qquad (z\in\mathbb{R}),$$

$\mu > 0$ to be adjusted below. Then

$$\frac{(\Phi')^2}{\Phi} = 4\mu^2 z^2\Phi,\ \ \Phi'' - \frac{(\Phi')^2}{\Phi} = 2\mu\Phi.$$

Consequently (2.29) implies

$$2\mu\Phi a|Du|^2\Lambda \leq C\Phi|Du| + C|2\mu u|\Phi(1+|Du|^2) + \frac{4\mu^2u^2\Phi}{(p-2)}a(|Du|^2+\varepsilon^2). \tag{2.30}$$

We may assume without loss $|Du(x_0)| \geq 1 \geq \varepsilon$, in which case

$$\Lambda \geq \frac{1}{p} + \frac{p-2}{2p} = \frac{1}{2}.$$

Accordingly if we fix $\mu > 0$ small enough, (2.30) implies

$$a = \phi^p \leq C \text{ at } x_0.$$

Since $b = a\Phi$ and $\Phi \geq 1$, we infer

$$\max_{B(0,R)} |Du|^p \leq C. \tag{2.31}$$

The constant C does not depend on p on ε. Estimate (2.31) is valid provided $b = a\Phi$ attains its maximum at an interior point of $B(0,R)$. Should b instead attain its maximum only at a boundary point, we invoke the second inequality in (2.16).

The estimate (2.31) holds for the solution $u = u_p^\varepsilon$ of (2.15). By an approximation the same estimate holds if $f^\pm$ are only Lipschitz. We then send $\varepsilon \to 0$ to obtain inequality (2.14). □

Remark. Related computations are in Kawohl [KB] and Payne–Phillippin [P-P]. □

Making use of the estimates provided by Lemmas 2.1, 2.2, we can now extract a subsequence $p_k \to \infty$ so that

$$\begin{cases} u_{p_k} \to u & \text{uniformly} \\ Du_{p_k} \overset{*}{\rightharpoonup} Du & \text{weakly } * \text{ in } L^\infty(B(0,R);\mathbb{R}^\kappa). \end{cases} \tag{2.32}$$

for some function $u \in C^{0,1}(B(0,R))$, with $u = 0$ on $\partial B(0,R)$. Recall from Rademacher's Theorem that Du exists a.e. The next Theorem and the following Remark characterize u.

Theorem 2.1 (i) *There exists function $a \in L^\infty(B(0,R))$ such that*

$$-\operatorname{div}(aDu) = f \text{ in } B(0,R) \tag{2.33}$$

in the weak sense. In addition

$$|Du| \leq 1 \text{ a.e., } a \geq 0 \text{ a.e.} \tag{2.34}$$

and

$$\text{for a.e. } z,\ a(z) > 0 \text{ implies } |Du(z)| = 1. \tag{2.35}$$

(ii) *Furthermore,*

$$\int_{B(0,R)} uf\, dz = \max_{|Dw|\leq 1 \text{ a.e.}} \int_{B(0,R)} wf\, dz. \tag{2.36}$$

We hereafter call u the *potential* and a the *transport density*.

Proof. 1. Since estimate (2.14) implies $|Du_{p_k}| \le C_2^{1/p_k}$, we deduce from (2.32) that $|Du| \le 1$ a.e.

2. Define

$$\mathbf{A}_p = |Du_p|^{p-2} Du_p;$$

then

$$\int_{B(0,R)} \mathbf{A}_p \cdot Dv \; dz = \int_{B(0,R)} vf \; dz$$

for each $v \in C^{0,1}(B(0,R))$ with $v = 0$ on $\partial B(0,R)$. According to Lemma 2.2 the vector fields $\{\mathbf{A}_{p_k}\}_{k=1}^{\infty}$ are bounded in L^∞; and so, passing if necessary to a further subsequence, we have

$$\mathbf{A}_{p_k} \overset{*}{\rightharpoonup} \mathbf{A} \text{ weakly } * \text{ in } L^\infty.$$

Thus

$$\int_{B(0,R)} \mathbf{A} \cdot Dv \; dz = \int_{B(0,R)} vf \; dz. \tag{2.37}$$

2. In addition

$$\begin{aligned} \textstyle\int_{B(0,R)} |Du_{p_k}|^{p_k} dz \quad &= \textstyle\int_{B(0,R)} u_{p_k} f \; dz \\ &\to \textstyle\int_{B(0,R)} uf \; dz \\ &= \textstyle\int_{B(0,R)} \mathbf{A} \cdot Du \; dz. \end{aligned} \tag{2.38}$$

We compute

$$\begin{aligned} \textstyle\int_{B(0,R)} |\mathbf{A}| \; dz \quad &\le \liminf_{k\to\infty} \textstyle\int_{B(0,R)} |\mathbf{A}_{p_k}| \; dz \\ &\le \lim_{k\to\infty} \left(\textstyle\int_{B(0,R)} |Du_{p_k}|^{p_k} \; dz \right)^{1-\frac{1}{p_k}} (B(0,R))^{\frac{1}{p_k}} \\ &= \textstyle\int_{B(0,R)} \mathbf{A} \cdot Du \; dz, \text{ by (2.38).} \end{aligned}$$

Since $|Du| \le 1$ a.e., it follows that

$$|\mathbf{A}| = \mathbf{A} \cdot Du \text{ a.e.} \tag{2.39}$$

Thus for a.e. z we can write

$$\mathbf{A}(z) = a(z) Du(z), \tag{2.40}$$

where $a \in L^\infty$, $a \ge 0$. For a.e. point z such that $Du(z)$ exists and $|Du(z)| < 1$, (2.39) implies $a(z) = 0$. Finally (2.40) and (2.37) show $-\text{div}(a \; Du) = f$ in the weak sense. Assertion (i) is proved.

3. To verify assertion (ii) we first take any $w \in C^{0,1}(B(0,R))$ with $w = 0$ on $\partial B(0,R)$, $|Dw| \leq 1$ a.e. In light of (2.6)

$$-\int_{B(0,R)} f u_p \, dz \leq \int_{B(0,R)} \frac{1}{p}|Du_p|^p - f u_p \, dz \leq \int_{B(0,R)} \frac{1}{p}|Dw|^p - f w \, dz.$$

Let $p = p_k \to \infty$:

$$\int_{B(0,R)} f u \, dz \geq \int_{B(0,R)} f w \, dz.$$

If $|Dw| \leq 1$ a.e. but we do not have $w = 0$ on $\partial B(0,R)$, we can modify w on $B(0,R) - B(0,S)$ (i.e., outside the support of f) to reduce to the previous case. □

Remark. In the language of convex analysis (cf. Ekeland–Temam [E-T]) we have shown that

$$f^+ - f^- = f \in \partial I_\infty[u], \tag{2.41}$$

where the convex function $I_\infty : L^2(B(0,R)) \to [0, \infty]$ is defined by

$$I_\infty[w] = \begin{cases} 0 & \text{if } \operatorname{Lip}[w] \leq 1 \\ +\infty & \text{otherwise,} \end{cases}$$

and ∂I_∞ denotes the subdifferential of I_∞. Note carefully that (2.41) does not have a unique solution: we can modify u in any open region where $|Du| \leq 1 - \delta$ to create a new function $\tilde{u}$ still satisfying $|D\tilde{u}| \leq 1$ a.e., $f \in \partial I_\infty[\tilde{u}]$.

The particular solution u we obtain from (2.32) is a viscosity solution of the "∞-Laplacian" PDE

$$-\Delta_\infty u = -u_{x_i} u_{x_j} u_{x_i x_j} = 0 \text{ in } B(0,R) - (X \cup Y) :$$

see Jensen [JR]. □

3 The transport set and transport rays

Our goal for this and the next three sections is to examine very carefully properties of the potential u and the transport density a (obtained in Theorem 2.1). We will eventually need all this information to make sense of the ODE (1.23).

We will almost exclusively focus our attention upon the *transport set*

$$\begin{aligned} T \; &= \{z \in B(0,R) \mid u(z) = u(x) - |x - z| \text{ for some point} \\ &\quad x \in X \text{ and } u(z) = u(y) + |z - y| \text{ for some point } y \in Y\}. \end{aligned} \tag{3.1}$$

This is the set in which the trajectories of the ODE will lie.

Remark. Observe that if $z \in T$, then *the corresponding points* x, y, z *are colinear.* Indeed, adding the identities $u(x) - u(z) = |x - z|$, $u(z) - u(y) = |z - y|$, we deduce

$$|x - z| + |z - y| = u(z) - u(y) \leq |x - y|.$$

Note also T *is closed.* □

We intend to study T and the behavior of u, a restricted to T. It will be convenient to define as well the *upper envelope*

$$u^*(z) = \min_{y \in Y}\{u(y) + |z - y|\} \qquad (z \in B(0,R)) \tag{3.2}$$

and the *lower envelope*

$$u_*(z) = \max_{x \in X}\{u(x) - |z - x|\} \qquad (z \in B(0,R)). \tag{3.3}$$

Observe that $|Du^*| = 1$, a.e. in $B(0,R) - Y$, $|Du_*| = 1$ a.e. on $B(0,R) - X$.

Lemma 3.1 (i) *We have*

$$u_* \leq u \leq u^* \text{ on } B(0,R). \tag{3.4}$$

(ii) *Furthermore,*

$$T = \{z \in B(0,R) \mid u_*(z) = u(z) = u^*(z)\} \tag{3.5}$$

and

$$T \supseteq X \cup Y. \tag{3.6}$$

Proof. 1. Let $z \in B(0,R)$. Then since $\text{Lip}[u] \leq 1$,

$$u(z) - u(y) \leq |z - y| \text{ for all } y \in Y,$$

and so

$$u(z) \leq \min_{y \in Y}\{u(y) + |z - y|\} = u^*(z).$$

Similarly

$$u(x) - u(z) \leq |z - x| \text{ for all } x \in X;$$

whence

$$u(z) \geq \max_{x \in X}\{u(x) - |z - x|\} = u_*(z).$$

2. To prove (3.5) let us first select any point $z \in T$. Then there exists a point $x \in X$ with

$$u(z) = u(x) - |x - z| \leq \max_{w \in X}\{u(w) - |z - w|\} = u_*(z).$$

Similarly, for some point $y \in Y$ we have

$$u(z) = u(y) + |z - y| \geq \min_{w \in Y}\{u(w) + |z - w|\} = u^*(z).$$

In light of (3.4) then, we see $z \in T$ implies $u_* = u = u^*$ at z. Conversely, assume $u_*(z) = u(z) = u^*(z)$. Then $u(z) = u(y) + |z - y|$ for some $y \in Y$, and $u(z) = u(x) - |z - x|$ for some point $x \in X$. Thus $z \in T$.

3. Next observe that trivially $u = u_*$ on X, $u = u^*$ on Y, since $\text{Lip}[u] \leq 1$. To show $u = u^*$ on X, suppose instead that

$$u < u^* \text{ in some open subset of } X^0. \tag{3.7}$$

Then

$$\begin{aligned} \int_{B(0,R)} u^* \ f \ dz \ &= \int_X u^* \ f^+ \ dx - \int_Y u^* \ f^- \ dy \\ &> \int_X u \ f^+ \ dx - \int_Y u \ f^- \ dy \\ &= \int_{B(0,R)} uf \ dz, \end{aligned}$$

where we used (3.7) and the fact $f^+ > 0$ in X^0, $u = u^*$ on Y. This inequality however contradicts the maximization principle (2.36). Therefore $u_* = u = u^*$ in X and, similarly, on Y. □

We introduce more terminology by defining for each point $z_0 \in T$, the set

$$R_{z_0} = \{z \in B(0,R) \mid |u(z_0) - u(z)| = |z_0 - z|\}.$$

This is the set containing z_0 along which u changes at the maximum rate 1. Observe that *if u is differentiable at z_0, then R_{z_0} is a line segment.* We then call R_{z_0} the *transport ray* through z_0. Below we show that outside of T there are no other segments along which u grows or decreases with the maximum slope 1. Consequently one end of R_{z_0}, call it a_0, lies in X and the other end, call it b_0, lies in Y. We think of R_{z_0} as pointing "downhill" from a_0 to b_0.

=750 Fig3.1

Call a_0 the *upper end* of R_{z_0}, b_0 the *lower end*, and $R_{z_0} - \{a_0, b_0\}$ the *relative interior* of the transport ray. The potential u *decreases* at the rate one as we move along R_{z_0} from a_0 to b_0.

Lemma 3.2 (i) *Assume $z \in B(0,R)$ and $u(z) = u(w) - |z - w|$ for some other point $w \in B(0,R)$. Then $z \in T$.*

(ii) *Similarly, if $u(z) = u(w) + |z - w|$ for some other point $w \in B(0,R)$, then $z \in T$.*

Remark. In particular we are asserting that any line segment leaving X (resp. Y) on which u decreases (resp. increases) at rate one must terminate in Y (resp. X). The union of such segments is the transport set T. □

Proof. Assume $u(z) = u(w) \pm |z - w|$, $w \neq z$. Denote by L the longest line segment in $B(0, R)$ containing w, z along which u changes at rate one. Let x, y denote the end points of L, with $u(x) > u(y)$.

2. We *claim* first that $x, y \notin B^0(0, R) - (X \cup Y)$; that is, the segment L cannot terminate except within X, Y or $\partial B(0, R)$.

To verify the claim, suppose instead that $y \in B^0(0, R) - (X \cup Y)$. We may then rescale and rotate coordinates to arrive at the following situation:

$$\begin{cases} B(0,1) \subset B(0,R)^0 - (X \cup Y) & \\ u(\lambda e_n) = \lambda & \text{for } -\frac{1}{2} \leq \lambda \leq 1 \\ u(-e_n) > -1. & \end{cases} \tag{3.8}$$

In other words, we are taking L to be along the z_n axis, with $y = -\frac{1}{2}e_n$.

Note that in fact

$$u|_{\partial B(0,1)} \geq -\eta \ \text{ for some } \ \eta < 1. \tag{3.9}$$

Otherwise there would exist a point $z \in \partial B(0, 1)$ at which $u(z) = -1$. But then u decreases at rate one on the segment from 0 to z, a contradiction, as this segment does not point in the same direction as the segment from 0 to $-e_n$. Define now $v(x) = -\eta|x|$ ($x \in B(0, 1)$). We have $v = u = 0$ at $x = 0$ and $u \geq v$ on $\partial B(0, 1)$. In addition $-\Delta_\infty v = -v_{x_i} v_{x_j} v_{x_i x_j} = 0$ in $B(0, 1) - \{0\}$. As $-\Delta_\infty u = 0$ in $B(0, 1) - \{0\}$, we deduce from Jensen's comparison principle [JR] that $u \geq v = -\eta|x|$ in $B(0, 1)$. But this is impossible since $u(\lambda e_n) = \lambda$ for $-\frac{1}{2} \leq \lambda \leq 0$.

Thus it is impossible that $y \in B^0(0, R) - (X \cup Y)$. Likewise, we cannot have $x \in B^0(0, R) - (X \cup Y)$.

3. Consequently $x, y \in X \cup Y \cup \partial B(0, R)$. Since $u = 0$ on $\partial B(0, R)$, we cannot have both $x, y \in \partial B^0(0, R)$. Furthermore since u is bounded and R is large, it is in fact impossible that either x or y belong to $\partial B(0, R)$. Hence $x, y \in X \cup Y$.

4. We *claim* finally $x \in X$, $y \in Y$, and so $z \in T$. Suppose instead that, say $x, y \in X$. Then by Lemma 3.1 there is a line segment $\tilde{L}$ connecting y to a point $w \in Y$, along which u decreases at rate one. As the segment L terminated at y, it cannot be that $L, \tilde{L}$ are colinear. But this is a contradiction, since $\text{Lip}[u] \leq 1$. Thus, $x, y \in X$ and likewise $x, y \in Y$ is impossible. We similarly exclude the possibility $y \in X$, $x \in Y$. □

The importance of the transport set T is that the transport density a is supported within T:

Proposition 3.1 *We have*

$$\mathrm{supp}(a) \subseteq T. \tag{3.10}$$

More precisely, $a = 0$ *a.e. in* $B(0,R) - T$.

Proof. 1. We modify some clever calculations due to Janfalk [J, p. 76-79], being careful since our potential u (unlike his) can change sign. Select a constant γ so large that $\pm u + \gamma > 0$ everywhere on $B(0,R)$. Define then

$$v(z) = \begin{cases} u(z) \max_{w \in X \cup Y} \left[\dfrac{u(w)+\gamma}{u(z)+|w-z|+\gamma} \right] & \text{if } \ u(z) \geq 0 \\ \\ u(z) \max_{w \in X \cup Y} \left[\dfrac{\gamma - u(w)}{-u(z)+|w-z|+\gamma} \right] & \text{if } \ u(z) \leq 0\ . \end{cases} \tag{3.11}$$

2. Clearly $v = 0$ on $\partial B(0,R)$, and we *claim* as well that

$$|Dv| \leq 1 \text{ a.e. in } \ B(0,R)\ . \tag{3.12}$$

To prove this, first take any two points $z, \hat{z} \in B(0,R)$ with

$$v(z) \geq 0,\ v(\hat{z}) \geq 0. \tag{3.13}$$

Note that therefore $u(z) \geq 0$, $u(\hat{z}) \geq 0$. Then, interchanging z and $\hat{z}$ if needs be, we may assume

$$v(z) \geq v(\hat{z}). \tag{3.14}$$

There exists a point $w \in X \cup Y$ such that

$$v(z) = u(z) \frac{u(w)+\gamma}{u(z)+|z-w|+\gamma}.$$

Since $u(\hat{z}) \geq 0$, we have

$$v(\hat{z}) \geq u(\hat{z}) \frac{u(w)+\gamma}{u(\hat{z})+|\hat{z}-w|+\gamma}.$$

Consequently,

$$\begin{aligned}|v(z)-v(\hat{z})| &= v(z)-v(\hat{z}) \leq (u(w)+\gamma)\left[\frac{u(z)}{u(z)+|z-w|+\gamma}-\frac{u(\hat{z})}{u(\hat{z})+|\hat{z}-w|+\gamma}\right]\\ &= (u(w)+\gamma)\frac{[u(z)(|\hat{z}-w|+\gamma)-u(\hat{z})(|z-w|+\gamma)]}{(u(z)+|z-w|+\gamma)(u(\hat{z})+|\hat{z}-w|+\gamma)}\\ &= (u(w)+\gamma)\frac{[(u(z)-u(\hat{z}))(|\hat{z}-w|+\gamma)+u(\hat{z})(|\hat{z}-w|-|z-w|)]}{(u(z)+|z-w|+\gamma)(u(\hat{z})+|\hat{z}-w|+\gamma)}\\ &\leq (u(w)+\gamma)\frac{|z-\hat{z}|(|\hat{z}-w|+\gamma)+u(\hat{z})|z-\hat{z}|}{(u(z)+|z-w|+\gamma)(u(\hat{z})+|\hat{z}-w|+\gamma)},\end{aligned}$$

since $u(\hat{z}) \geq 0$. Thus

$$\begin{aligned}|v(z)-v(\hat{z})| &\leq |z-\hat{z}|\frac{u(w)+\gamma}{u(z)+|z-w|+\gamma}\\ &\leq |z-\hat{z}|.\end{aligned} \tag{3.15}$$

3. Next suppose instead of (3.13) that

$$v(z) \leq 0, \ v(\hat{z}) \leq 0, \tag{3.16}$$

and so $u(z) \leq 0$, $u(\hat{z}) \leq 0$. We may assume (3.14) holds. There exists a point $\hat{w} \in X \cup Y$ such that

$$v(\hat{z}) = u(\hat{z})\frac{\gamma-u(\hat{w})}{-u(\hat{z})+|\hat{z}-\hat{w}|+\gamma}.$$

Since $u(z) \leq 0$,

$$v(z) \leq u(z)\frac{\gamma-u(\hat{w})}{-u(z)+|z-\hat{w}|+\gamma}.$$

Computing now as in Step 2 and recalling $u(\hat{z}) \leq 0$, we deduce

$$\begin{aligned}|v(z)-v(\hat{z})| &\leq |z-\hat{z}|\frac{\gamma-u(\hat{w})}{-u(z)+|z-\hat{w}|+\gamma}\\ &\leq |z-\hat{z}|.\end{aligned} \tag{3.17}$$

4. If finally

$$v(z) \geq 0, \ v(\hat{z}) \leq 0 \tag{3.18}$$

(or vice versa), there is a point z^* on the line segment connecting z and $\hat{z}$, where $v(z^*) = 0$. Thus

$$\begin{aligned}|v(z)-v(\hat{z})| &\leq |v(z)-v(z^*)|+|v(z^*)-v(\hat{z})|\\ &\leq |z-z^*|+|z^*-\hat{z}| = |z-\hat{z}|.\end{aligned} \tag{3.19}$$

This observation completes the proof of the claim (3.12).

5. Now select a closed ball $B \subset B(0,R)$ which is a positive distance σ away from the transport set T. We will show $a = 0$ a.e. in B. According to (3.15) if $\hat{z}, z \in B$ with $v(z) \geq 0$, $v(\hat{z}) \geq 0$, we have

$$|v(z) - v(\hat{z})| \leq |z - \hat{z}| \max_{\substack{z \in B \\ w \in X \cup Y}} \left[\frac{u(w) + \gamma}{u(z) + |z - w| + \gamma} \right]. \tag{3.20}$$

But since $B \cap T = \emptyset$, it follows that

$$\max_{\substack{z \in B \\ w \in X \cup Y}} \left[\frac{u(w) + \gamma}{u(z) + |z - w| + \gamma} \right] \leq 1 - \delta \tag{3.21}$$

for some $\delta > 0$. Indeed, there would otherwise exist points $z_k \in B$ and $w_k \in X \cup Y$ $(k = 1, \dots)$ such that

$$u(w_k) + \gamma \geq (1 - 1/k)(u(z_k) + |z_k - w_k| + \gamma).$$

Pass as necessary to a subsequence such that $z_k \to z \in B$, $w_k \to w \in X \cup Y$, and deduce

$$u(w) \geq u(z) + |z - w|.$$

Thus

$$u(z) = u(w) - |z - w| \qquad (w \in X \cup Y)\ .$$

But according to Lemma 3.2, this implies the contradiction $z \in T$. Estimate (3.21) is consequently proved, and so

$$|v(z) - v(\hat{z})| \leq (1 - \delta)|z - \hat{z}|\ .$$

Similarly if $\hat{z}, z \in B$ and $v(z) \leq 0$, $v(\hat{z}) \leq 0$, we have

$$|v(z) - v(\hat{z})| \leq (1 - \delta)|z - \hat{z}|.$$

Thus

$$\operatorname*{ess\,sup}_{B} |Dv| \leq 1 - \delta. \tag{3.22}$$

Observe lastly that

$$u = v \text{ on } X \cup Y,$$

since we can then take $w = z$ in computing the max in (3.11).

6. As u_p solves the p-Laplacian PDE (2.6) and $v = 0$ on $\partial B(0,R)$, we have

$$\int_{B(0,R)} fv\ dz = \int_{B(0,R)} |Du_p|^{p-2} Du_p \cdot Dv\ dz.$$

Since $v = u$ on $X \cup Y = \operatorname{supp}(f)$ and $|Dv| \le 1$, $|Dv| \le 1 - \delta$ on B, we discover:

$$\int_{B(0,R)} fu\ dz \le \int_{B(0,R)-B} |Du_p|^{p-1} dz + (1-\delta) \int_B |Du_p|^{p-1} dz.$$

Therefore

$$\delta \int_B |Du_p|^{p-1} dz \le \left(\int_{B(0,R)} |Du_p|^p dz \right)^{1-\frac{1}{p}} |B(0,R)|^{\frac{1}{p}} - \int_{B(0,R)} fu\ dz.$$

Let $p = p_k \to \infty$ and recall:

$$\int_{B(0,R)} |Du_{p_k}|^{p_k} dz = \int_{B(0,R)} fu_{p_k}\ dz \to \int_{B(0,R)} fu\ dz.$$

We conclude that

$$\lim_{k\to\infty} \int_B |Du_{p_k}|^{p_k - 1} dz = 0.$$

Using then the notation from the proof of Theorem 2.1, we deduce

$$\int_B a\ dz = \int_B |\mathbf{A}| dz \le \liminf_{k\to\infty} \int_B |Du_{p_k}|^{p_k} dz = 0.$$

As $a \ge 0$, we conclude $a = 0$ a.e. in B. □

4 Differentiability and smoothness properties of the potential

Next we study Du and D^2u on the transport set T.

Lemma 4.1 (i) *We have*

$$|Du| = 1 \ \text{a.e. on } T. \tag{4.1}$$

(ii) *If z lies in the relative interior of some transport ray, then*

$$u \text{ is differentiable at } z \text{ and } |Du(z)| = 1. \tag{4.2}$$

(iii) *For each* $\delta > 0$ *there exists a constant* C_δ *such that*

$$D^2 u \leq C_\delta I \ \textit{ on } T - Y_\delta \tag{4.3}$$

$$D^2 u \geq -C_\delta I \ \textit{ on } T - X_\delta, \tag{4.4}$$

where X_δ *(resp.* Y_δ*) denotes the* δ*-neighborhood of* X *(resp.* Y*).*

Remark. Assertions (4.3), (4.4) mean

$$\begin{cases} u - \frac{C_\delta}{2}|z|^2 \text{ is concave on } T - Y_\delta, \\ u + \frac{C_\delta}{2}|z|^2 \text{ is convex on } T - X_\delta. \end{cases}$$

We say u is *semiconcave* on $T - Y_\delta$, *semiconvex* on $T - X_\delta$. □

Proof. 1. We note that the upper and lower envelopes u^*, u_* and u itself are Lipschitz and thus differentiable a.e. Thus at a.e. point $z \in T$, $Du^*(z)$, $Du_*(z)$ and $Du(z)$ exist. As Lemma 3.1(i) asserts $u_* \leq u \leq u^*$ everywhere, whereas $u_* = u = u^*$ at z, it follows that

$$Du_*(z) = Du(z) = Du^*(z).$$

But $|Du^*| = 1$ a.e. on $B(0,R) - Y$, $|Du_*| = 1$ a.e. on $B(0,R) - X$.

2. Take a point z_0 in the relative interior of some transport ray. We may assume $z_0 = 0$, $u(z_0) = 0$, and the ray is along the z_n-axis. Hence for some small constant r_0, we have

$$u(te_n) = t \quad \text{ if } -r_0 \leq t \leq r_0. \tag{4.5}$$

We rescale by setting

$$u^r(z) = \frac{u(rz)}{r} \qquad (0 < r \leq r_0). \tag{4.6}$$

Obviously $|Du^r| \leq 1$ a.e. Hence for some sequence $r_k \to 0$, we have

$$u^{r_k} \to v \text{ locally uniformly on } \mathbb{R}^n,$$

where

$$|Dv| \leq 1 \text{ a.e. on } \mathbb{R}^n, \ v(te_n) = t \text{ for all } t \in \mathbb{R}, \tag{4.7}$$

according to (4.5), (4.6). But (4.7) in fact implies

$$v(z) = z_n \text{ for all } z \in \mathbb{R}^n. \tag{4.8}$$

To see this, observe for each $z \in \mathbb{R}^n$ that

$$|v(z) - t| = |v(z) - v(te_n)| \le |z - te_n|$$

for each $t \in \mathbb{R}$. Square and then simplify to discover

$$2t[z_n - v(z)] \le |z|^2 - v(z)^2.$$

We send $t \to \pm\infty$ and obtain a contradiction unless (4.8) is valid. Thus we conclude that

$$\lim_{r\to 0} \frac{u(rz)}{r} = z_n, \quad \text{uniformly for } \; z \in B(0,1).$$

Thus u is differentiable at 0, with $Du(0) = e_n$.

3. Take $\delta > 0$. Pick $z \in B(0,R) - Y_\delta$, and choose $y \in Y$ for which

$$u^*(z) = u(y) + |z - y|.$$

Then if $|w| \le \frac{\delta}{2}$, we have

$$\begin{aligned} u^*(z+w) - 2u^*(z) + u^*(z-w) &\le |z+w-y| - 2|z-y| + |z-w-y| \\ &\le \tfrac{C|w|^2}{\delta^2} = C_\delta |w|^2. \end{aligned}$$

If $z \in T - Y_\delta$, then $u(z) = u^*(z)$ and $u(z \pm w) \le u^*(z \pm w)$. Hence

$$u(z+w) - 2u(z) + u(z-w) \le C_\delta |w|^2 \tag{4.9}$$

for all $|w| \le \frac{\delta}{2}$. This inequality implies (4.3) and the proof of (4.4) is similar. □

We next refine Lemma 4.1(i):

Lemma 4.2 *The potential u satisfies*

$$|Du| = 1 \; in \; X^0 \tag{4.10}$$

$$-|Du| = -1 \; in \; Y^0, \tag{4.11}$$

in the sense of viscosity solutions.

See, for instance, Fleming–Soner [F-S] for relevant definitions, and Bhattacharya–DiBenedetto–Manfredi [B-D-M] for related statements.

Assertions (4.10), (4.11) in fact imply $|Du| = 1$ a.e. in X^0, Y^0, but are much stronger, as they limit the types of singularities of u allowed within X^0, Y^0. The strange difference with the minus sign between (4.10), (4.11) records the different types of singularities in the two regions (cf. (4.3) versus (4.4)).

One direct proof uses the fact that $u = u^*$ in X^0, $u = u_*$ in Y_0. For variety we provide a different demonstration, based on the p-Laplacian approximations from §2.

Proof. 1. Let ϕ be a smooth function and suppose $u - \phi$ has a strict local maximum (resp. minimum) at a point $x_0 \in X^0$. We must prove $|D\phi| \leq 1$ (resp. ≥ 1) at x_0.

2. As $|Du| \leq 1$ a.e., we deduce at once $|D\phi(x_0)| \leq 1$ in the first case. Suppose instead $u - \phi$ has a strict local minimum at x_0, but

$$|D\phi(x_0)| < 1. \tag{4.12}$$

Since $u_{p_k} \to u$ uniformly near x_0, there exist points x_{p_k} such that $u_{p_k} - \phi$ has a minimum at x_{p_k}. Now u_{p_k} is a viscosity solution of (2.6), as we see by approximating u_p as in (2.15). Consequently

$$-|D\phi(x_{p_k})|^{p_k-2}\left[\Delta\phi(x_{p_k}) - (p_k-2)\frac{\phi_{x_i}(x_{p_k})\phi_{x_j}(x_{p_k})}{|D\phi(x_{p_k})|^2}\phi_{x_ix_j}(x_{p_k})\right] \geq f(x_{p_k}). \tag{4.13}$$

But $|D\phi(x_{p_k})| \to |D\phi(x_0)| < 1$, and so the left hand side of (4.13) goes to zero as $p_k \to \infty$. However, since $f(x_{p_k}) \to f(x_0) > 0$, since $x_0 \in X^0$ and $f^+ > 0$ on X^0, by (2.4). This contradiction proves that in fact $|D\phi(x_0)| \geq 1$, as required. We have verified (4.10).

The proof of (4.11) is similar, except now we must show that $|D\phi| \geq 1$ (resp. ≤ 1) if $u - \phi$ has a strict local maximum (resp. minimum) at a point $y_0 \in Y^0$. □

Lemma 4.1(iii) provides "one-sided" estimates on D^2u in various regions. We next show that we in effect have "two-sided" control of D^2u along the relative interior of any transport ray.

In the following Lemma we write $R = R_{z_0}$ to denote some transport ray passing through z_0 and joining its upper end $a_0 \in X$ with its lower end $b_0 \in Y$. For each $\sigma > 0$ we write

$$R^\sigma = R - [B(a_0, \sigma) \cup B(b_0, \sigma)] \tag{4.14}$$

to denote the points on the ray of distance at least σ from the ends.

Proposition 4.1 *For each transport ray R as above and each $\sigma > 0$, there exist a constant $C = C_\sigma$ and a tubular neighborhood N of R^σ such that*

$$|Du(z) - Du(\hat{z})| \leq C|z - \hat{z}| \tag{4.15}$$

for each point $z \in N \cap T$ at which $Du(z)$ exists. Here $\hat{z}$ denotes the projection of z onto R.

=750 Fig4.1

We may informally interpret this result as saying "u is $C^{1,1}_{\text{loc}}$ along the transport ray R".

Proof. 1. We may without loss suppose $a_0 = 0 \in X$, $b_0 = le_n \in Y$, where $e_n = (0, 0, \ldots, 1)$ and $l > 0$. Fix a point $\hat{w} \in R - (X \cup Y)$ such that the segment $[0, \hat{w}]$ does not intersect Y. This is possible since X and Y are a positive distance apart, according to (2.4).

=750 Fig4.2

Write

$$\hat{w} = re_n \qquad (0 < r < l). \tag{4.16}$$

We may as well assume

$$u(\hat{w}) = u^*(\hat{w}) = 0. \tag{4.17}$$

Define

$$\Gamma_0^* = \{w \in B(0, R) \mid u^*(w) = 0\}. \tag{4.18}$$

Next select $\hat{z} \in R$ such that $\hat{z}$ lies between 0 and $\hat{w}$, at a distance greater than σ from both 0, $\hat{w}$. This is possible if $\sigma > 0$ is small enough. Then

$$\hat{z} = de_n, \qquad \sigma \leq d \leq r - \sigma. \tag{4.19}$$

2. We *claim* first that there exists a small ball $B = B(\hat{z}, \rho)$, such that

$$u(z) = \operatorname{dist}(z, \Gamma_0^*) \text{ if } z \in T \cap B. \tag{4.20}$$

To see this, first select B so small that $B \cap Y = \emptyset$, $B \cap \Gamma_0^* = \emptyset$. Observe next that if the radius ρ is sufficiently small, then any transport ray R_z passing through B must intersect Γ_0^* near $\hat{w}$. (If not, there would exist points $z_k \to \hat{z}$ such that u would be decreasing at rate one along transport rays R_k through z_k, but these R_k would not in the limit be pointing in the direction e_n. This contradicts the fact (Lemma 4.1(ii)) that u is differentiable at $\hat{z}$.)

Thus if $z \in T \cap B$ lies on a transport ray R_z, we see

$$\begin{cases} u(z) = u^*(z) & = \text{ distance from } z \text{ to } \Gamma_0^* \text{ along } R_z. \\ & = |z - w| \text{ for some point } w \in \Gamma_0^*. \end{cases} \tag{4.21}$$

We next observe that

$$|z - w| = \operatorname{dist}(z, \Gamma_0^*). \tag{4.22}$$

Were this false, we would have $|z - w^*| < |z - w|$ for some other point $w^* \in \Gamma_0^*$. But since $|Du^*| \le 1$ a.e., this would at once imply:

$$u(z) = u^*(z) - u^*(w^*) \le |z - w^*| < |z - w|,$$

a contradiction to (4.21). Thus (4.22) holds and so (4.21) establishes the claim (4.20).

3. Keeping the notation as in Step 2, let us suppose in addition that $z \in T \cap B$ is such that $\hat{z}$ is the projection of z onto R. Assume also $Du(z)$ exists, and so the transport ray R_z through z is unique. We assert next that if $\rho > 0$ is sufficiently small, then

$$|w - \hat{w}| \le C|z - \hat{z}| \tag{4.23}$$

for some constant $C = C(\rho, \sigma)$.

To prove this estimate, we introduce first the notation

$$\hat{z} = (0, d),\ \hat{w} = (0, r),\ z = (z', d),\ w = (w', w_n) \tag{4.24}$$

(cf. (4.16), (4.19)). Here $z', w' \in \mathbb{R}^{n-1}$. As $\hat{w} \in \Gamma_0^*$, (4.22) implies

$$|z - w| \le |z - \hat{w}|.$$

Squaring, we deduce

$$|z' - w'|^2 + (w_n - d)^2 \le \varepsilon^2 + (r - d)^2, \tag{4.25}$$

where

$$\varepsilon = |z - \hat{z}| = |z'|. \tag{4.26}$$

Now by hypotheses u grows at rate one on the segment joining $\hat{w} \in \Gamma_0^*$ to 0. Thus $u(0) = |\hat{w}| = r$. As $u^*(0) = u(0) = r$ and $|Du^*| \le 1$ a.e., we conclude that

$$B(0, r)^0 \cap \Gamma_0^* = \emptyset.$$

In particular

$$w_n \ge (r^2 - |w'|^2)^{1/2}.$$

Thus

$$\begin{aligned} w_n &\ge r\left(1 - \tfrac{|w'|^2}{r^2}\right)^{1/2} \\ &= r\left(1 - \tfrac{|w'|^2}{2r^2} + o(|w'|^2)\right) \\ &= r - \tfrac{|w'|^2}{2r} + o(|w'|^2) \quad \text{as} \quad w' \to 0\ . \end{aligned} \tag{4.27}$$

Consequently

$$\begin{aligned} 0 < r - d &= w_n - d + r - w_n \\ &\le (w_n - d) + \tfrac{|w'|^2}{2r} + o(|w'|^2), \end{aligned}$$

and so

$$(r-d)^2 \le (w_n - d)^2 + |w_n - d| \frac{|w'|^2}{r} + o(|w'|^2) \quad \text{as} \quad w' \to 0 \ . \tag{4.28}$$

According to (4.25)

$$|w_n - d| \le (\varepsilon^2 + (r-d)^2)^{1/2} = (r-d)(1 + o(1)) \quad \text{as} \quad w' \to 0 \ .$$

Thus (4.28) yields

$$\begin{aligned} (r-d)^2 &\le (w_n - d)^2 + (r-d)\tfrac{|w'|^2}{r} + o(|w'|^2) \\ &\le (w_n - d)^2 + \theta |w'|^2 \end{aligned} \tag{4.29}$$

for $\theta = 1 - \frac{d}{2r} < 1$, provided ε is small enough. Inserting estimate (4.29) into (4.25) gives us

$$|z' - w'|^2 \le \varepsilon^2 + \theta |w'|^2. \tag{4.30}$$

Since $|w'| \le |w' - z'| + \varepsilon$, we have

$$\begin{aligned} |w'|^2 &\le |w' - z'|^2 + 2|w' - z'|\varepsilon + \varepsilon^2 \\ &\le (1+\mu)|w' - z'|^2 + \left(\tfrac{1}{\mu} + 1\right)\varepsilon^2. \end{aligned} \tag{4.31}$$

We fix $\mu > 0$ so small that $\theta(1+\mu) < 1$. Then (4.30), (4.31) and (4.26) imply

$$|w'|^2 \le C\varepsilon^2 = C|z - \hat{z}|^2. \tag{4.32}$$

Finally observe from (4.27) that

$$r - w_n \le C|w'|^2 \le C|z - \hat{z}|^2. \tag{4.33}$$

Also, since $u(\hat{z}) = r - d$ and $u(z) = |z - w|$, we have

$$w_n - d \le |z - w| = r - d + u(z) - u(\hat{z}) \le r - d + \varepsilon.$$

Consequently

$$w_n - r \le \varepsilon = |z - \hat{z}|.$$

This bound and (4.33), (4.32) imply

$$|w - \hat{w}| \le C|z - \hat{z}|,$$

as asserted in (4.23).

4. Now for $z, \hat{z}, w, \hat{w}$ as above, we have

$$Du(\hat{z}) = \frac{\hat{z} - \hat{w}}{|\hat{z} - \hat{w}|} = -e_n, \; D(z) = \frac{z - w}{|z - w|}.$$

Then

$$|Du(z) - Du(\hat{z})| \leq \frac{1}{|z - w| \, |\hat{z} - \hat{w}|} \left| (z - w)|\hat{z} - \hat{w}| - (\hat{z} - \hat{w})|z - w| \right|.$$

Add and subtract $(z - w)|z - w|$ on the right and estimate, using (4.23) to conclude:

$$|Du(z) - Du(\hat{z})| \leq \frac{|z - \hat{z}| + |w - \hat{w}|}{|\hat{z} - \hat{w}|} \leq C|z - \hat{z}| \tag{4.34}$$

since $|\hat{z} - \hat{w}| \geq \sigma$.

5. As the sets X and Y are a finite distance apart, the ray R can leave X, enter Y and then re-enter X (or vice versa) only a finite number of times. We can therefore subdivide R into finitely many pieces on each of which an argument similar to that leading to (4.34) applies. Combining the resulting estimates leads us at last to (4.15). □

Remark. The calculations above should be compared with those in Caffarelli–Friedman [C-F], Evans–Harris [E-H], etc. □

5 Generic properties of transport rays

As our ultimate intention is to construct an optimal mass transfer scheme by moving points in X along corresponding transfer rays, we must show that the rays are well behaved, at least "generically".

First we demonstrate that a.e. point $x_0 \in X$ and a.e. point $y_0 \in Y$ are *not* endpoints of a transfer ray. To see this, let us first of all define the set of endpoints

$$E = \{z \in X \cup Y \mid z \text{ is an endpoint of some transfer ray}\}.$$

Proposition 5.1 *We have*

$$|E| = 0. \tag{5.1}$$

Thus for a.e. point $z \in T$, z is not the endpoint of the unique transport ray passing through z.

The proof is complicated because of the possibility that u is differentiable at an endpoint of a ray.

Proof. 1. We note first that E is Lebesgue measurable. Indeed, $E \cap \{z \mid Du(z)$ does not exist$\}$ has measure zero, by Rademacher's Theorem, and so is Lebesgue measurable. Since $|\partial X| = |\partial Y| = 0$, $E \cap \partial X$ and $E \cap \partial Y$ are measurable as well. Also

$$E \cap \{x \in X^0 \mid Du(x) \text{ exists}\} = \cap_{k=1}^{\infty} \left\{ x \in X^0 \mid Du(x) \text{ exists}, \ u\left(x + \frac{1}{k} Du(x)\right) < u(x) + \frac{1}{k} \right\}.$$

Since u is continuous, $x \mapsto u\left(x + \frac{1}{k}Du(x)\right)$ is measurable. Thus $E \cap \{x \in X^0 \mid Du(x)$ exists$\}$ is measurable, as is $E \cap \{y \in Y^0 \mid Du(y)$ exists$\}$.

2. Suppose now, say,

$$|E \cap X^0| > 0.$$

Set

$$F = \{x \in E \cap X^0 \mid Du(x) \text{ exists}\}.$$

Then

$$|F| = |E| > 0. \tag{5.2}$$

We will force a contradiction from (5.2). Observe that if $x \in F$, then x is the upper endpoint of exactly one transport ray (pointing in the direction $-Du(x)$). Otherwise u would not be differentiable at x.

3. Since u is semiconcave in X (Lemma 4.1(iii)), Proposition A.1 in the Appendix asserts that for each $\varepsilon > 0$ there exists a measurable set $X_\varepsilon \subset X$ and a C^2 function $\tilde{u} = \tilde{u}_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ such that

$$|X - X_\varepsilon| < \varepsilon$$

and

$$\tilde{u} = u, \ D\tilde{u} = Du \ \text{ on } \ X_\varepsilon \ . \tag{5.3}$$

Fix $\varepsilon > 0$ so small that $F_\varepsilon = F \cap X_\varepsilon$ has positive measure:

$$|F_\varepsilon| > 0.$$

Recall from Lemma 4.1(i) that $|Du| = 1$ a.e. in X. Hence the Coarea Formula (see, e.g., [E-G, Chapter 3]) implies

$$\begin{aligned} |F_\varepsilon| &= \textstyle\int_{F_\varepsilon} |Du| dx \\ &= \textstyle\int_{-\infty}^{\infty} \mathcal{H}^{\setminus - \infty}(\mathcal{F}_\varepsilon \cap \Gamma_\sqcup)\lceil\sqcup \ , \end{aligned} \tag{5.4}$$

where $\Gamma_t = \{x \in \mathbb{R}^n \mid u(x) = t\}$.

Let $\theta > 0$ be a small positive number, to be selected below, depending only on $\tilde{u}$. Then since $|F_\varepsilon| > 0$, there exists $t_0 > 0$ such that

$$0 < \int_{t_0}^{t_0+\theta} \mathcal{H}^{\backslash-\infty}(\mathcal{F}_\varepsilon \cap \Gamma_\sqcup)\lceil\sqcup. \tag{5.5}$$

4. Fix a level $t_0 - \theta \leq s_0 \leq t_0$ such that

$$\mathcal{H}^{\backslash-\infty}(\Gamma_{\int_\prime}) < \infty. \tag{5.6}$$

Now for each $t_0 \leq t \leq t_0 + \theta$ define the C^1 mapping

$$\phi_t(x) = x - (t - s_0)D\tilde{u}(x) \qquad (x \in B(0, R)). \tag{5.7}$$

Since $u = \tilde{u}$ and $|D\tilde{u}| = |Du| = 1$ on $F_\varepsilon \subset X_\varepsilon$, we observe

$$\phi_t : F_\varepsilon \cap \Gamma_t \to \Gamma_{s_0}. \tag{5.8}$$

Indeed, the effect of ϕ_t on $F_\varepsilon \cap \Gamma_t$ is to move each point x a distance $t - s_0$ along the transport ray beginning at x. The value of u consequently decreases from t to $t - (t - s_0) = s_0$.

We next note that the area of the image of ϕ_t is

$$\mathcal{H}^{\backslash-\infty}(\phi_\sqcup(\mathcal{F}_\varepsilon \cap \Gamma_\sqcup)) = \int_{\mathcal{F}_\varepsilon \cap \Gamma_\sqcup} [\infty - (\sqcup - \int_\prime)\mathcal{H} + \wr(\sqcup - \int_\prime)]\lceil\mathcal{H}^{\backslash-\infty},$$

where $H = \operatorname{div}(\frac{D\tilde{u}}{|D\tilde{u}|}) = \Delta\tilde{u}$ is the mean curvature. Hence

$$\mathcal{H}^{\backslash-\infty}(\phi_\sqcup(\mathcal{F}_\varepsilon \cap \Gamma_\sqcup)) \geq \frac{\infty}{\in}\mathcal{H}^{\backslash-\infty}(\mathcal{F}_\varepsilon \cap \Gamma_\sqcup) \qquad \text{for all } \sqcup_\prime \leq \sqcup \leq \sqcup_\prime + \theta \ , \tag{5.9}$$

provided $\theta > 0$ is now fixed small enough. But observe also that

$$\phi_{t_1}(F_\varepsilon \cap \Gamma_{t_1}) \cap \phi_{t_2}(F_\varepsilon \cap \Gamma_{t_2}) = \emptyset \tag{5.10}$$

if $t_0 \leq t_1,\ t_2 \leq t_0 + \theta$, with $t_1 \neq t_2$. For if this intersection were non-empty, it would contain a point lying on two transport rays, one with upper endpoint in $F_\varepsilon \cap \Gamma_{t_1}$ and the other with upper endpoint in $F_\varepsilon \cap \Gamma_{t_2}$. This is not possible.

5. In view of (5.6), (5.8)–(5.10), we see that for any choices $t_0 \leq t_1 < t_2 < \cdots < t_m \leq t_0 + \theta$, we have

$$\tfrac{1}{2}\textstyle\sum_{j=1}^m \mathcal{H}^{\backslash-\infty}(\mathcal{F}_\varepsilon \cap \Gamma_{\sqcup_|}) \leq \mathcal{H}^{\backslash-\infty}(\Gamma_{\int_\prime}) < \infty.$$

This is only possible if

$$\mathcal{H}^{\backslash-\infty}(\mathcal{F}_\varepsilon \cap \Gamma_\sqcup) = \prime \qquad (\sqcup_\prime \leq \sqcup \leq \sqcup_\prime + \theta),$$

except for possibly countably many values of t. This conclusion is however at variance with (5.5). □

Now we prove that if N is a set of Lebesgue measure zero, then a "generic" transport ray will intersect N only on a set of one-dimensional Hausdorff measure zero.

Proposition 5.2 *Let $N \subset B(0,R)$ have Lebesgue measure zero. Then for a.e. $z_0 \in T$,*

$$\mathcal{H}^\infty(\mathcal{R}_{\ddagger\prime} \cap \mathcal{N}) = \prime. \tag{5.11}$$

Proof. 1. First of all we may as well assume

$$N \text{ is a } G_\delta \text{ set }, \tag{5.12}$$

as we could otherwise apply the following reasoning with $\tilde{N}$ replacing N, $\tilde{N}$ denoting a G_δ set such that

$$N \subseteq \tilde{N}\ , \quad |\tilde{N}| = 0\ .$$

2. We *claim* that

$$\text{the mapping} \quad z \mapsto \mathcal{H}^\infty(\mathcal{N} \cap \mathcal{R}_\ddagger) \text{ is Borel measurable from } T \text{ to } [\prime, \infty)\ . \tag{5.13}$$

To see this, write as above E to denote the set of endpoints of the transfer rays comprising T. We may assume E is Borel measurable, as we could otherwise replace E by a G_δ set $\tilde{E}$ with $E \subseteq \tilde{E}$, $|\tilde{E}| = 0$. If U is open and $U \supset E$, the mapping $z \mapsto \mathcal{H}^\infty(\mathcal{C} \cap \mathcal{R}_\ddagger)$ is upper semicontinuous on $T - U$, and thus is Borel measurable, for each closed set C. Since Proposition 5.1 asserts $|E| = 0$, we can find open sets $U_k \supset E$ $(k = 1, \dots)$ with $|U_k| \to 0$. Thus $z \mapsto \mathcal{H}^\infty(\mathcal{C} \cap \mathcal{R}_\ddagger)$ is Borel measurable on T for each closed set C. If V is open, then $\mathcal{H}^\infty(\mathcal{V} \cap \mathcal{R}_\ddagger) = \mathcal{H}^\infty(\mathcal{R}_\ddagger) - \mathcal{H}^\infty((\mathbb{R}^\ltimes - \mathbb{V}) \cap \mathbb{R}_\digamma)$ and so $z \mapsto \mathcal{H}^\infty(\mathcal{V} \cap \mathcal{R}_\ddagger)$ is Borel measurable on T. Finally, in light of (5.12) we can find open sets $\{V_k\}_{k=1}^\infty$ such that

$$V_k \supseteq V_{k+1} \quad (k = 1, \dots)\ , \qquad N = \bigcap_{k=1}^\infty V_k\ .$$

Hence $\mathcal{H}^\infty(\mathcal{N} \cap \mathcal{R}_\ddagger) = \lim_{\parallel \to \infty} \mathcal{H}(\mathcal{V}_\parallel \cap \mathcal{R}_\ddagger)$ $(\ddagger \in \mathcal{T})$, and (5.13) follows.

3. According to the Coarea Formula,

$$0 = \int_N |Du| dx = \int_{-\infty}^{\infty} \mathcal{H}^{\setminus - \infty}(\mathcal{N} \cap \Gamma_\sqcup) \lceil \sqcup,$$

where, as before, $\Gamma_t = \{u = t\}$. Therefore

$$\mathcal{H}^{\setminus - \infty}(\mathcal{N} \cap \Gamma_\sqcup) = \prime \text{ for a.e. } \sqcup\ . \tag{5.14}$$

4. If the Lemma is false, then we may without loss assume that for some $\delta > 0$, we have $\mathcal{H}^{\infty}(\mathcal{R}_{\ddagger} \cap \mathcal{N}) > \prime$ for z belonging to some subset of $T - Y_\delta$ having positive Lebesgue measure. (Otherwise we consider $T - X_\delta$.) Since u is semiconcave on $T - Y_\delta$ (Lemma 4.1(iii)), there exists according to the Appendix for each $\varepsilon > 0$ a measurable set $T_\varepsilon \subset T - Y_\delta$ and a C^2 function $\tilde{u} = \tilde{u}_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ such that

$$|(T - Y_\delta) - T_\varepsilon| < \varepsilon$$

and

$$\tilde{u} = u, \ D\tilde{u} = Du \ \text{ on } \ T_\varepsilon \ . \tag{5.15}$$

(More precisely, we apply the approximation described in the Appendix to u^*, which is semiconcave in $B(0, R) - Y_\delta$. As $u \le u^*$ on $B(0, R)$ and $u = u^*$ on T, (5.15) follows.)

Our assumption is that for some $\sigma > 0$, the set

$$F_\sigma = \{z \in T - Y_\delta \mid Du(z) \text{ exists}, \mathcal{H}^{\infty}(\mathcal{R}_{\ddagger} \cap \mathcal{N}) \ge \sigma\} \tag{5.16}$$

has positive measure. We may consequently fix $\varepsilon > 0$ so small that

$$|F_\sigma \cap T_\varepsilon| > 0.$$

5. Let $z_0 \in F_\sigma \cap T_\varepsilon$ be a point of density one of $F_\sigma \cap T_\varepsilon$. We may as well assume $u(z_0) = 0$. Then, since $|Du| = 1$ a.e. on T,

$$\begin{aligned} 0 \ &< |B(z_0, r) \cap F_\sigma \cap T_\varepsilon| \\ &= \textstyle\int_{-r}^{r} \mathcal{H}^{\setminus - \infty}(\mathcal{B}(\ddagger_\prime, \nabla) \cap \mathcal{F}_\sigma \cap \mathcal{T}_\varepsilon \cap \Gamma_\int) \, \lceil\int, \end{aligned} \tag{5.17}$$

for all $r > 0$.

Since $z_0 \in T_\varepsilon$, $|D\tilde{u}(z_0)| = |Du(z_0)| = 1$. Thus the level sets $\tilde{\Gamma}_s = \{\tilde{u} = s\}$ are C^2 hypersurfaces inside $B(z_0, r)$, provided we hereafter fix $r > 0$ sufficiently small. Observe $T_\varepsilon \cap \Gamma_s = T_\varepsilon \cap \tilde{\Gamma}_s$.

6. As in the previous proof we introduce next the C^1 mappings

$$\phi_t(z) = z - tD\tilde{u}(z) \qquad (z \in \mathbb{R}^n)$$

for each $t \in \mathbb{R}$. Note

$$\phi_t(z) = z - tDu(z) \quad \text{ if } z \in T_\varepsilon. \tag{5.18}$$

For each point $z \in B(z_0, r) \cap F_\sigma \cap T_\varepsilon \cap \Gamma_s$, let R_z denote the transport ray through z, and set

$$\begin{cases} r(z) = \max\{t \ge 0 \mid z - tDu(z) \in R_z\} \\ s(z) = \max\{s \ge 0 \mid z + sDu(z) \in R_z\}. \end{cases} \tag{5.19}$$

Thus $r(z)$ is the distance z can be moved forward along R_z and $s(z)$ is the distance z can be moved backward along R_z. The mappings $z \mapsto r(z), s(z)$ are upper semicontinuous and are thus Borel measurable (cf. Evans–Harris [E-H, Proposition 3.7]). Finally, if $k, l \in \{0, 1, 2, \ldots\}$ and $-r \le s \le r$, write

$$G_{kl} = G^s_{kl} = \left\{ z \in B(z_0, r) \cap F_\sigma \cap T_\varepsilon \cap \Gamma_s \mid \frac{k\sigma}{4} < r(z) \le \frac{(k+1)\sigma}{4}, \frac{l\sigma}{4} < s(z) \le (l+1)\frac{\sigma}{4} \right\}, \tag{5.20}$$

where $\sigma > 0$ is the constant from (5.16). Consequently (5.18) implies

$$\phi_t(G_{kl}) \subset \Gamma_{s-t} \text{ if } -\frac{l\sigma}{4} \le t \le \frac{k\sigma}{4}. \tag{5.21}$$

Since $\phi_t|_{G_{kl}}$ is one-to-one for t in this range, we have

$$\int_{G_{kl}} f(\phi_t(z)) J_t\phi(z)\ d\mathcal{H}^{\backslash-\infty}(\ddagger) = \int_{\phi_\sqcup(\mathcal{G}_{\|\updownarrow})} \{(\sqsupseteq)\ \lceil\mathcal{H}^{\backslash-\infty}(\sqsupseteq)$$

for each Borel measurable function f, $J\phi_t > 0$ denoting the Jacobian of ϕ_t. Let $f = \chi_N$. Then

$$\begin{aligned} \textstyle\int_{G_{kl}} \chi_N(\phi_t(z)) J\phi_t(z) d\mathcal{H}^{\backslash-\infty}(\ddagger) &= \mathcal{H}^{\backslash-\infty}(\mathcal{N} \cap \phi_\sqcup(\mathcal{G}_{\|\updownarrow})) \\ &\le \mathcal{H}^{\backslash-\infty}(\mathcal{N} \cap \Gamma_{\int-\sqcup}) \text{ by (5.21)} \\ &= 0 \text{ for a.e. } t \in \left[-\tfrac{l\sigma}{4}, \tfrac{k\sigma}{4}\right], \end{aligned}$$

the last equality holding according to (5.14). Consequently

$$\int_{G_{kl}} \left(\int_{-\frac{l\sigma}{4}}^{\frac{k\sigma}{4}} \chi_N(\phi_t(z)) J\phi_t(z) dt \right) d\mathcal{H}^{\backslash-\infty}(\ddagger) = \prime. \tag{5.22}$$

In particular, for $\mathcal{H}^{\backslash-\infty}$-a.e. $z \in G_{kl}$,

$$\int_{-\frac{l\sigma}{4}}^{\frac{k\sigma}{4}} \chi_N(\phi_t(z)) J\phi_t(z)\ dt = 0.$$

As $J\phi_t(z) > 0$ for t as above, we see:

$$\int_{-\frac{l\sigma}{4}}^{\frac{k\sigma}{4}} \chi_N(z - tDu(z))\ dt = 0 \tag{5.23}$$

for $\mathcal{H}^{\backslash-\infty}$-a.e. $z \in G_{kl}$. But such a point z lies in F_σ and so, according to (5.16),

$$\int_{-s(z)}^{r(z)} \chi_N(z - tDu(z))\ dt \ge \sigma. \tag{5.24}$$

However the definition (5.20) of G_{kl} implies

$$\int_{\frac{k\sigma}{4}}^{r(z)} \chi_N(z - tDu(z))\ dt \leq \frac{\sigma}{4}, \tag{5.25}$$

$$\int_{-s(z)}^{-\frac{l\sigma}{4}} \chi_N(z - tDu(z))\ dt \leq \frac{\sigma}{4}. \tag{5.26}$$

The inequalities (5.24)–(5.26) and the equality (5.23) are inconsistent. Thus in fact

$$\mathcal{H}^{n-1}(\mathcal{G}_{kl}) = 0.$$

Since

$$B(z_0, r) \cap F_\sigma \cap T_\varepsilon \cap \Gamma_s = \bigcup_{k,l=0}^{\infty} G_{kl},$$

we deduce

$$\mathcal{H}^{n-1}(\mathcal{B}(z_0, r) \cap \mathcal{F}_\sigma \cap \mathcal{T}_\varepsilon \cap \Gamma_s) = 0$$

for all $-r \leq s \leq r$. This equality is however a contradiction to (5.17). $\square$

Finally we need to prove that a generic transport ray penetrates the interiors of X and Y. This will be a consequence of the following Lemma.

We call a measurable set $A \subset B(0, R)$ a *transport set* if $z \in A$ implies $R_z \subseteq A$. In other words, a transport set is the union of all transport rays through its points.

Lemma 5.1 (Mass balance) *Let A be a transport set, as above. Then*

$$\int_A f^+(x)dx = \int_A f^-(y)dy. \tag{5.27}$$

This Lemma asserts that the mass of $\mu^+ = f^+dx$ within A equals the mass of $\mu^- = f^-dy$. This is reasonable since we intend to move mass along the transport rays in A.

Proof. 1. Suppose first A is closed. Write $h = \chi_A$ and set

$$u_\varepsilon(z) = u(z) + \varepsilon h(z) \qquad (z \in B(0, R))\ , \tag{5.28}$$

$$v_\varepsilon(w) = \min_{z \in B(0,R)} \{|z - w| - u_\varepsilon(z)\} \qquad (w \in B(0, R)). \tag{5.29}$$

Also write

$$v(w) = \min_{z \in B(0,R)} \{|z - w| - u(z)\} = -u(w). \tag{5.30}$$

2. Now

$$u_\varepsilon(z) + v_\varepsilon(w) \le |z-w| \text{ for } w, z \in B(0,R),$$

and so, according to Kantorovich's principle as discussed in §1,

$$\int_X u_\varepsilon(x) f^+(x)dx + \int_Y v_\varepsilon(y) f^-(y)dy \le \int_X u(x) f^+(x)dy + \int_Y v(y) f^-(y)dy.$$

Thus

$$\begin{aligned} \int_X \chi_A f^+ dx &= \int_X \left(\tfrac{u_\varepsilon - u}{\varepsilon}\right) f^+ dx \\ &\le \int_Y \left(\tfrac{v - v_\varepsilon}{\varepsilon}\right) f^- dy. \end{aligned} \tag{5.31}$$

Now if $y \in A \cap Y$,

$$\begin{aligned} v^\varepsilon(y) &= \min_{z \in B(0,R)} \{|z-y| - u(z) - \varepsilon \chi_A(z)\} \\ &= -u(y) - \varepsilon \\ &= v(y) - \varepsilon \chi_A(y). \end{aligned}$$

Consequently

$$\frac{v(y) - v^\varepsilon(y)}{\varepsilon} = 1 = \chi_A(y) \qquad (y \in A \cap Y).$$

If $y \in Y - A$, then

$$\min_{z \in A} \{|z-y| + u(y) - u(z)\} = \theta > 0$$

for some constant $\theta = \theta(y)$, since A is a transport set and is closed. Therefore

$$\min_{z \in A} \{|z-y| - u(z) - \varepsilon \chi_A(z)\} = \theta - \varepsilon - u(y).$$

On the other hand,

$$\inf_{z \in B(0,R) - A} \{|z-y| - u(z) - \varepsilon \chi_A(z)\} = -u(y).$$

Thus if $\varepsilon = \varepsilon(y) > 0$ is so small that $\varepsilon < \theta$, then

$$v^\varepsilon(y) = -u(y) = v(y),$$

and so

$$\frac{v(y) - v^\varepsilon(y)}{\varepsilon} = 0 = \chi_A(y).$$

Hence

$$\lim_{\varepsilon \to 0^+} \frac{v - v^\varepsilon}{\varepsilon} = \chi_A \qquad (y \in Y). \tag{5.32}$$

Since $0 \leq \frac{v-v^\varepsilon}{\varepsilon} \leq 1$ a.e., we deduce from (5.31),(5.32) that

$$\int_A f^+ dx \leq \int_A f^- dy.$$

The opposite inequality follows by symmetry.

Suppose now A is not closed. We have

$$\int_B f^+ dx = \int_B f^- dy$$

for each closed transport set $B \subset A$. Taking the supremum over all such sets gives (5.27).

□

Lemma 5.2 (i) *For a.e.* $x_0 \in X$ *the transport ray* R_{x_0} *intersects* Y^0.

(ii) *Similarly, for a.e.* $y_0 \in Y$ *the transport ray* R_{y_0} *intersects* X^0.

Proof. Let $A = \{z \in T \mid Du(z) \text{ exists}, R_z \cap Y^0 = \emptyset\}$. A is measurable, since

$$A = T \cap \{z \mid Du(z) \text{ exists}\} \cap \bigcap_{k=1}^{\infty} \{z \mid u(z) < \min_{\substack{y \in Y^0 \\ \text{dist}(y, \partial Y) \geq 1/k}} (u(y) + |y-z|)\}.$$

Note that A is a transport set, up to a set of Lebesgue measure zero. Then Proposition 5.3 implies

$$\int_A f^+ dx = \int_A f^- dy = 0.$$

As $f^+ > 0$ in X^0, we obtain $|A \cap X^0| = 0$. Assertion (i) follows, as by symmetry does assertion (ii). □

6 Behavior of the transport density along rays

This section scrutinizes closely the behavior of the transport density restricted to a generic transport ray.

First of all let us observe from Proposition 5.2 that for a.e. z_0, the transport ray R_{z_0} intersects the set of Lebesgue points of a on a set of full $\mathcal{H}^\infty$ measure. The *restriction* of a to R_{z_0}, denoted $a|_{R_{z_0}}$, is thus defined $\mathcal{H}^\infty$ a.e.

We intend to show that this restriction of the density a is locally Lipschitz along R_{z_0}, is positive on certain subintervals of R_{z_0}, and vanishes at the endpoints of R_{z_0}. The main idea in proving all this is to regard u as known and to think of the PDE $-\text{div}(aDu) = f$ as a *linear* first-order ODE for a.

Proposition 6.1 *For a.e. point $z_0 \in T$,*

(i) $a|_{R_{z_0}}$ *is locally Lipschitz along R_{z_0}.*

(ii) *If I is a subinterval of $R_{z_0} - (X^0 \cup Y^0)$, then $a|_I$ is either identically zero or else everywhere positive.*

(iii) *Furthermore*

$$a|_{R_{z_0} \cap X^0} \quad \textit{and} \quad a|_{R_{z_0} \cap Y^0}$$

are both positive, except possibly at the end points a_0, b_0.

We will later prove (Proposition 7.1) that in fact a vanishes at a_0, b_0.

=750 Fig6.1

Proof. 1. Select z_0 so that the ray $R = R_{z_0}$ intersects the set of Lebesgue points of a on a set of full $\mathcal{H}^\infty$ measure. A.e. $z_0 \in T$ will do, according to Proposition 5.2. We may as well assume that a_0, the upper endpoint of R, is 0 and b_0, the lower endpoint, is le_n, where $l > 0$. Hence $Du = -e_n$ on R. As before, we write

$$R^\sigma = R - [B(a_0, \sigma) \cup B(b_0, \sigma)]$$

to denote the points on R of distance at least $\sigma > 0$ away from the ends.

Define the cylinder

$$C_\varepsilon = B'(0, \varepsilon) \times [\sigma, l - \sigma] \qquad (\varepsilon > 0),$$

B' here and hereafter denoting a ball in $\mathbb{R}^{n-1}$.

We select $0 < \varepsilon \leq \frac{\varepsilon_0}{2}$, the radius ε_0 adjusted so small that

$$C_{\varepsilon_0} \subseteq N,$$

N the tubular neighborhood of R^σ from Proposition 4.1. Thus estimate (4.15) tells us

$$|Du(z) - Du(\hat{z})| = |Du(z) + e_n| \leq C|z - \hat{z}| \leq C\varepsilon \tag{6.1}$$

for each point $z \in C_\varepsilon \cap T$ at which $Du(z)$ exists, $\hat{z}$ denoting the projection of z onto R.

2. Let ϕ_ε, ψ be smooth, nonnegative functions such that $\phi_\varepsilon = \phi_\varepsilon(x')$ has compact support in $B'(0, \varepsilon)$ and $\psi = \psi(x_n)$ has compact support in $[\sigma, l - \sigma]$. Then, since u is a weak solution of $-\operatorname{div}(aDu) = f$ according to Theorem 2.1(i), we have

$$\begin{aligned} \int_{C_\varepsilon} f\psi\phi_\varepsilon dz &= \int_{C_\varepsilon} aDu \cdot D(\phi_\varepsilon\psi)\, dz \\ &= \int_{C_\varepsilon} a(Du \cdot D\phi_\varepsilon)\psi\, dz + \int_{C_\varepsilon} a(Du \cdot D\psi)\phi_\varepsilon\, dz. \end{aligned} \tag{6.2}$$

Define also for a.e. $\sigma \le z_n \le l - \sigma$,

$$\alpha_\varepsilon(z_n) = \int_{B'(0,\varepsilon)} a(z', z_n)\phi_\varepsilon(z')\ dz'. \tag{6.3}$$

Let us also take

$$\phi_\varepsilon(z') = \frac{1}{\varepsilon^{n-1}}\phi\left(\frac{z'}{\varepsilon}\right) \qquad (z' \in \mathbb{R}^{\ltimes - \nVdash})\ ,$$

where

$$\begin{cases} \phi : \mathbb{R}^{n-1} \to \mathbb{R} \text{ is smooth, } \phi \ge 0, \\ \int_{\mathbb{R}^{n-1}} \phi\ dz' = 1,\ \operatorname{supp}(\phi) \subset B'(0,1) \end{cases} \tag{6.4}$$

and

$$\phi > 0 \text{ on } B'(0, \tfrac{1}{2}). \tag{6.5}$$

Consequently we see that for a.e. z_n, $\alpha_\varepsilon(z_n)$ is a weighted average of $a(\cdot, z_n)$ over the $(n-1)$-dimensional disk $B'(0,\varepsilon) \times \{z_n\}$.

3. We will show that this average approximately solves an ODE in the variable z_n. Since $Du = -e_n$ on R and $\psi = \psi(z_n)$, we have:

$$\begin{aligned} \int_{C_\varepsilon} a(Du \cdot D\psi)\phi_\varepsilon\ dz &= \int_\sigma^{l-\sigma} \int_{B'(0,\varepsilon)} a u_{z_n} \psi' \phi_\varepsilon\ dz' dz_n \\ &= -\int_\sigma^{l-\sigma} \psi' \alpha_\varepsilon\ dz_n \\ &+ \int_\sigma^{l-\sigma} \int_{B'(0,\varepsilon)} a(u_{z_n} + 1)\psi' \phi_\varepsilon\ dz' dz_n. \end{aligned} \tag{6.6}$$

According to (6.1) we have $|u_{z_n} + 1| \le |Du + e_n| \le C\varepsilon$ for points z lying in $T \cap C_\varepsilon$. As $\operatorname{supp}(a) \subseteq T$ (Proposition 3.1) and $a \ge 0$, we deduce

$$\left| \int_\sigma^{l-\sigma} \int_{B'(0,\varepsilon)} a(u_{z_n} + 1)\psi' \phi_\varepsilon\ dz' dz_n \right| \le C\varepsilon \int_\sigma^{l-\sigma} |\psi'| \alpha_\varepsilon\ dz_n. \tag{6.7}$$

In addition, since $\phi_\varepsilon = \phi_\varepsilon(z')$, we have

$$\begin{aligned} \left| \int_{C_\varepsilon} a(Du \cdot D\phi_\varepsilon)\psi\ dz \right| &= \left| \int_\sigma^{l-\sigma} \int_{B'(0,\varepsilon)} a(Du + e_n) \cdot D\phi_\varepsilon \psi\ dz' dz_n \right| \\ &\le C\varepsilon \int_\sigma^{l-\sigma} \int_{B'(0,\varepsilon)} a|D\phi_\varepsilon|\psi\ dz' dz_n \quad \text{by (6.1)} \\ &\le C \int_\sigma^{l-\sigma} \psi \alpha_{2\varepsilon}\ dz_n \text{ by (6.4), (6.5).} \end{aligned} \tag{6.8}$$

Finally, let us write

$$\beta_\varepsilon(z_n) = \int_{B'(0,\varepsilon)} f(z', z_n)\phi_\varepsilon(z')\ dz';$$

whence

$$\int_{C_\varepsilon} f\psi\phi_\varepsilon\ dz = \int_\sigma^{l-\sigma} \psi\beta_\varepsilon\ dz_n. \tag{6.9}$$

4. Combine (6.2), (6.6)–(6.9):

$$\left|\int_\sigma^{l-\sigma} \psi'\alpha_\varepsilon + \psi\beta_\varepsilon\ dz_n\right| \le C\int_\sigma^{l-\sigma} \psi\alpha_{2\varepsilon} dz_n + C\varepsilon\int_\sigma^{l-\sigma} |\psi'|\alpha_\varepsilon\ dz_n.$$

Given $\sigma < s < t < l-\sigma$ and small $\delta > 0$, we define

$$\psi(z_n) = \begin{cases} 0 & \sigma \le z_n \le s \\ \text{linear} & s \le z_n \le s+\delta \\ 1 & s+\delta \le z_n \le t-\delta \\ \text{linear} & t-\delta \le z_n \le t \\ 0 & t \le z_n \le l-\sigma. \end{cases}$$

Substituting this choice of ψ above and sending $\delta \to 0^+$, we deduce that for a.e. $\sigma < s < t < l-\sigma$:

$$\left|\alpha_\varepsilon(s) - \alpha_\varepsilon(t) + \int_s^t \beta_\varepsilon dz_n\right| \le C\int_s^t \alpha_{2\varepsilon}\ dz_n + C\varepsilon. \tag{6.10}$$

Next define for $\sigma+\varepsilon \le t \le l-\sigma-\varepsilon$:

$$\begin{cases} a_\varepsilon(t) &= \frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} \alpha_\varepsilon(z_n)\ dz_n, \\ f_\varepsilon(t) &= \frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} \beta_\varepsilon(z_n)\ dz_n. \end{cases} \tag{6.11}$$

Thus $a_\varepsilon(t)$ is a weighted average of a over the n-dimensional cylinder $B'(0,\varepsilon)\times[t-\varepsilon, t+\varepsilon]$, and $f_\varepsilon(t)$ is a similar average of f.

5. We next claim

$$\left|a_\varepsilon(s) - a_\varepsilon(t) + \int_s^t f_\varepsilon\ dr\right| \le C\int_s^t a_{2\varepsilon}\ dr + C\varepsilon \tag{6.12}$$

for $\sigma+\varepsilon \le s < t \le l-\sigma-\varepsilon$. To see this, write the left hand side of (6.12) as

$$\left|\int_{\mathbb{R}^{\mathbb{K}}} \eta^\varepsilon(r)[\alpha_\varepsilon(s-r) - \alpha_\varepsilon(t-r) + \int_s^t \beta_\varepsilon(z_n - r)\ dz_n]\ dr\right|\ ,$$

where $\eta_\varepsilon = \frac{1}{2\varepsilon}\chi_{[-\varepsilon,\varepsilon]}$. According to (6.10) the foregoing expression is less than or equal to

$$\begin{aligned} &C \textstyle\int_{\mathbb{R}} \eta^\varepsilon(r) \int_{s-r}^{t-r} \alpha_{2\varepsilon}(z_n)\ dz_n dr + C\varepsilon \\ &= C \textstyle\int_s^t \int_{\mathbb{R}} \eta^\varepsilon(r)\alpha_{2\varepsilon}(z_n - r)\ dr dz_n + C\varepsilon \\ &= C \textstyle\int_s^t a_{2\varepsilon}(z_n)\ dz_n + C\varepsilon. \end{aligned}$$

6. Since a and f are bounded, we conclude from (6.12) first of all that

$$|a_\varepsilon(s) - a_\varepsilon(t)| \leq C(t-s) + C\varepsilon \tag{6.13}$$

for $\sigma + \varepsilon \leq s \leq t \leq l - \sigma - \varepsilon$. Hence if $\varepsilon_k \to 0$, the functions $\{a_{\varepsilon_k}(\cdot)\}_{k=1}^\infty$ are equicontinuous on compact subsets of $(\sigma, l-\sigma)$. But

$$a_\varepsilon(s) = \frac{1}{2\varepsilon}\int_{s-\varepsilon}^{s+\varepsilon}\int_{B'(0,\varepsilon)} a\phi_\varepsilon\ dz' dz_n \to a(0, se_n) = a(s) \tag{6.14}$$

as $\varepsilon \to 0$ for a.e. $\sigma \leq s \leq l - \sigma$. This is true since $\mathcal{H}^\infty$-a.e. point along the ray R is a Lebesgue point for the density a. Hence in fact

$$a_\varepsilon(s) \to a(s) \textit{uniformly} \text{ on } [\sigma, l-\sigma] \quad \text{as} \quad \varepsilon \to 0\ . \tag{6.15}$$

In view of (6.13) we see that the mapping $s \mapsto a(s)$ is Lipschitz for $\sigma \leq s \leq l - \sigma$. This proves assertion (i).

7. Let $\varepsilon \to 0$ in (6.12) and write $f(re_n) = f(r)$:

$$\left| a(s) - a(t) + \int_s^t f(r)\ dr \right| \leq C\int_s^t a(r)\ dr$$

for each $\sigma \leq s < t \leq l - \sigma$. Thus

$$|a'(s) - f(s)| \leq Ca(s) \qquad (\,' \ = \frac{d}{ds}) \tag{6.16}$$

for a.e. $\sigma \leq s \leq l - \sigma$. In particular,

$$(e^{Cs}a(s))' \geq e^{Cs}f(s) \tag{6.17}$$

and

$$(e^{-Cs}a(s))' \leq e^{-Cs}f(s). \tag{6.18}$$

Assume $a(s_0) = a(s_0 e_n) = 0$ and $s_0 e_n$ lies in a subinterval $I \subset R_{z_0} - (X^0 \cup Y^0)$. Then $f = 0$ along I. According to (6.16)

$$|a'(s)| \leq Ca(s)$$

for all s such that $se_n \in I$, also we deduce from Gronwall's inequality that $a(s) = 0$ for all such s. This is assertion (ii).

8. Let J denote a subinterval of $R_{z_0} \cap X^0$ and take s_1 to be the smallest number such that $s_1 e_n \in \bar{J}$. Then for $s > s_1$ such that $se_n \in J$ we deduce from (6.17) that

$$e^{Cs} a(s) \geq e^{Cs_1} a(s_1) + \int_{s_1}^{s} e^{Cr} f(r) dr > 0,$$

as $a(s_1) \geq 0$ and $f = f^+ > 0$ in X^0. Similarly, if J now denotes a subinterval of $R_{z_0} \cap Y^0$ and s_2 is the largest number such that $s_2 e_n \in \bar{J}$, then (6.28) implies for $s < s_2$ such that $se_n \in J$,

$$e^{-Cs} a(s) \geq e^{-Cs_2} a(s_2) - \int_{s}^{s_2} e^{-Cr} f(r) dr > 0,$$

since $a(s_2) \geq 0$ and $f = f^- < 0$ in Y^0. □

Remark. The previous proposition is based upon the formal observation that

$$f = -\operatorname{div}(aDu) = -Da \cdot Du - a\Delta u. \tag{6.19}$$

Thus along the vertical ray R in the preceding proof we have $Du = -e_n$, and so

$$a_{z_n} - a\Delta u = f. \tag{6.20}$$

Since Proposition 4.1 implies that, heuristically at least, Δu is bounded on R (away from the endpoints), (6.20) can be thought of as a linear ODE for a, which implies a bound on $|a_{z_n}|$. Statements (ii), (iii) in Proposition 6.1 follow formally as well. □

Remark. For use later (in §9) we record a technical fact established in the proof above, namely

$$\begin{cases} a_\varepsilon(s) = \frac{1}{2\varepsilon} \int_{s-\varepsilon}^{s+\varepsilon} \int_{\mathbb{R}^{n-1}} a\phi_\varepsilon \, dz' dz_n \\ \to a(s) = a(se_n) \text{ as } \varepsilon \to 0, \\ \textit{uniformly} \text{ for } \sigma \leq s \leq l - \sigma. \end{cases} \tag{6.21}$$

Note that (6.21) is valid for even if ϕ does not satisfy condition (6.5). (We used (6.5) only to get the term "$\alpha_{2\varepsilon}$" on the right hand of (6.8). To prove the equicontinuity statement (6.13) it is enough simply to estimate

$$⨍_{B'(0,\varepsilon)} a dz' \leq C$$

in (6.8).) □

We next interpret the ODE (6.20) rigorously. Let us recall from Lemma 4.1(iii) that for each $\delta > 0$, $D^2 u \leq C_\delta$ on $T - Y_\delta$; that is,

$$u - \frac{C_\delta}{2}|x|^2 \text{ is concave on } T - Y_\delta.$$

Consequently we can interpret the entries of the Hessian matrix $D^2 u$ as signed Radon measures on $T - Y_\delta$: see the Appendix. We write

$$d[D^2 u] = [D^2 u]_{ac} dx + d[D^2 u]_s, \tag{6.22}$$

$[D^2 u]_{ac}$ denoting the absolutely continuous part of $D^2 u$ (with respect to n-dimensional Lebesgue measure) and $[D^2 u]_s$ the singular part. In particular, if ϕ is smooth and has compact support in $T - Y_\delta$,

$$\int_{B(0,R)} D\phi \cdot Du \, dz = -\int_{B(0,R)} \phi [\Delta u]_{ac} \, dz - \int_{B(0,R)} \phi \; d[\Delta u]_s \; , \tag{6.23}$$

where $[\Delta u]_{ac} = \text{trace}[D^2 u]_{ac}$, $d[\Delta u]_s = \text{trace } d[D^2 u]_s$.

Proposition 6.2 *For a.e. $z_0 \in T$,*

$$a_\nu - [\Delta u]_{ac} a = f^+ \qquad \mathcal{H}^\infty \textit{ a.e. on } \mathcal{R}_{\ddagger'} \cap \mathcal{X}', \tag{6.24}$$

where $\nu = -Du(z_0)$. Similarly,

$$a_\nu - [\Delta u]_{ac} a = f^- \qquad \mathcal{H}^\infty \textit{ a.e. on } \mathcal{R}_{\ddagger'} \cap \mathcal{Y}'.$$

Proof. 1. In view of Proposition 5.2 we may as before suppose z_0 is selected so that the ray $R = R_{z_0}$ intersects the set of Lebesgue points of a and the set of Lebesgue points of $[D^2 u]_{ac}$ on a set of full $\mathcal{H}^\infty$-measure. We may also suppose that

$$\lim_{\varepsilon \to 0} \frac{\|[D^2 u]_s\|(B(z,\varepsilon))}{\varepsilon^n} = 0 \tag{6.25}$$

for $\mathcal{H}^\infty$ a.e. $z \in R \cap X^0$, and, finally, that u is twice differentiable at $\mathcal{H}^\infty$ a.e. point $z \in R \cap X^0$.

As before, we take $a_0 = 0$, $b_0 = le_n$, $Du = -e_n$ on R, $\nu = e_n$. We suppose as well that the open interval $(0, ke_n)$ lies in X^0 for some $0 < k < l$.

Define R^σ, $\phi_\varepsilon = \phi_\varepsilon(z')$, $\psi = \psi(z_n)$, etc. as in the previous proof, with k replacing l. Thus we now write $C_\varepsilon = B'(0,\varepsilon) \times [\sigma, k - \sigma]$.

2. Recall equality (6.2), which asserts

$$\int_{C_\varepsilon} f\psi\phi_\varepsilon dz = \int_{C_\varepsilon} a(Du\cdot D\phi_\varepsilon)\psi\ dz + \int_{C_\varepsilon} a(Du\cdot D\psi)\phi_\varepsilon\ dz. \tag{6.26}$$

¿From Step 3 in the proof of Proposition 6.1 we recall as well that

$$\left|\int_{C_\varepsilon} a(Du\cdot D\psi)\phi_\varepsilon dz + \int_\sigma^{k-\sigma} \psi'\alpha_\varepsilon\ dz_n\right| \le C\varepsilon, \tag{6.27}$$

and

$$\int_{C_\varepsilon} f\psi\phi_\varepsilon dz = \int_\sigma^{k-\sigma} \psi\beta_\varepsilon\ dz_n. \tag{6.28}$$

3. We more carefully examine the first term on the right hand side of (6.26). We have

$$\begin{aligned}\textstyle\int_{C_\varepsilon} a(Du\cdot D\phi_\varepsilon)\psi\ dz &= \textstyle\int_\sigma^{k-\sigma} a(0,z_n)\psi(z_n)\left(\int_{B'(0,\varepsilon)} Du\cdot D\phi_\varepsilon dz'\right) dz_n\\ &\quad+\textstyle\int_\sigma^{k-\sigma}\psi(z_n)\int_{B'(0,\varepsilon)}(Du\cdot D\phi_\varepsilon)[a(z',z_n)-a(0,z_n)]\ dz'dz_n\\ &\equiv A_1+A_2.\end{aligned} \tag{6.29}$$

Since ϕ_ε does not depend on z_n, we may estimate:

$$\begin{aligned}|A_2| &\le C\textstyle\int_\sigma^{k-\sigma}\psi\int_{B'(0,\varepsilon)}|(Du+e_n)\cdot D\phi_\varepsilon|\,|a(z',z_n)-a(0,z_n)|\ dz'dz_n\\ &\le C\textstyle\int_\sigma^{k-\sigma}\psi ⨍_{B'(0,\varepsilon)}|a(z',z_n)-a(0,z_n)|\ dz'dz_n.\end{aligned} \tag{6.30}$$

This inequality holds since

$$|Du(z)+e_n|\le C\varepsilon \text{ a.e. in } T\cap N\supset B'(0,\varepsilon)\times\{z_n\}$$

(since $B'(0,\varepsilon)\times\{z_n\}\subset X^0$ for $\varepsilon>0$ small enough and $\sigma\le z_n\le k-\sigma$).

4. Next, let

$$\lambda_\varepsilon(z_n) = -\int_{B'(0,\varepsilon)} Du\cdot D\phi_\varepsilon\ dz'. \tag{6.31}$$

Then select ψ as in the proof of Proposition 6.1 (k replacing l) and let $\delta\to 0$ in (6.26) to deduce for a.e. $\sigma\le s\le t\le k-\sigma$ that

$$\begin{aligned}&\left|\alpha_\varepsilon(s)-\alpha_\varepsilon(t)+\textstyle\int_s^t\beta_\varepsilon(z_n)dz_n+\int_s^t a(0,z_n)\lambda_\varepsilon(z_n)dz_n\right|\\ &\quad\le C\varepsilon+\textstyle\int_s^t ⨍_{B'(0,\varepsilon)}|a(z',z_n)-a(0,z_n)|\ dz'dz_n.\end{aligned} \tag{6.32}$$

Define $a_\varepsilon, f_\varepsilon$ as before, and similarly set

$$\begin{aligned} l_\varepsilon(t) &= \tfrac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} a(0,z_n)\lambda_\varepsilon(z_n)\ dz_n, \\ v_\varepsilon(t) &= \tfrac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} ⨍_{B'(0,\varepsilon)} |a(z',z_n)-a(0,z_n)|\ dz'dz_n. \end{aligned}$$

Then (6.32) implies

$$\left| a_\varepsilon(s) - a_\varepsilon(t) + \int_s^t f_\varepsilon(r)\ dr + \int_s^t l_\varepsilon(r)\ dr \right| \le C\varepsilon + \int_s^t v_\varepsilon(r)\ dr, \tag{6.33}$$

for $\sigma+\varepsilon \le s \le t \le k-\sigma-\varepsilon$. (This is proved in the same way that (6.12) follows from (6.10) in the previous proof.) Now

$$|l_\varepsilon(t)| \le \frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} a(0,z_n)|\lambda_\varepsilon(z_n)| dz_n$$

and

$$\begin{aligned} |\lambda_\varepsilon(z_n)| &= \left| \int_{B'(0,\varepsilon)} Du\cdot D\phi_\varepsilon\ dz' \right| \\ &= \left| \int_{B'(0,\varepsilon)} (Du+e_n)\cdot D\phi_\varepsilon\ dz' \right| \le C, \end{aligned}$$

since $|Du+e_n| \le C\varepsilon$ a.e. in $B'(0,\varepsilon)\times\{z_n\}$. Thus

$$|l_\varepsilon(t)| \le C \qquad (\sigma+\varepsilon \le t \le k-\sigma-\varepsilon). \tag{6.34}$$

Furthermore for $\sigma+\varepsilon \le t \le k-\sigma-\varepsilon$:

$$\begin{aligned} \tfrac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} \lambda_\varepsilon(z_n) dz_n &= -\tfrac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}\int_{B'(0,\varepsilon)} Du\cdot D\phi_\varepsilon\ dz'dz_n \\ &= \tfrac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}\int_{B'(0,\varepsilon)} \phi_\varepsilon [\Delta' u]_{ac}\ dz'dz_n \\ &+ \tfrac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}\int_{B'(0,\varepsilon)} \phi^\varepsilon\ d[\Delta' u]_s, \end{aligned}$$

where $\Delta' u = \sum_{i=1}^{n-1} u_{x_i x_i}$. Now (6.25) implies

$$\left| \frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}\int_{B'(0,\varepsilon)} \phi^\varepsilon\ d[\Delta' u]_s \right| \to 0$$

as $\varepsilon \to 0$, for a.e. $\sigma \le t \le k-\sigma$. Furthermore, as $\mathcal{H}^\infty$ a.e. point of R is a Lebesgue point of $[D^2 u]_{ac}$, we have

$$\frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}\int_{B'(0,\varepsilon)} \phi_\varepsilon [\Delta' u]_{ac} dz'dz_n \to [\Delta' u]_{ac}(te_n) = [\Delta u]_{ac}(te_n) \text{ for a.e. } \sigma \le t \le l-\sigma.$$

Since $z_n \mapsto a(0, z_n)$ is Lipschitz, we conclude

$$l_\varepsilon(t) \to [\Delta u]_{ac}(te_n)a(t) \quad \text{for a.e.} \quad \sigma \leq t \leq l - \sigma.$$

Consequently (6.34) and the Dominated Convergence Theorem imply

$$\int_s^t l_\varepsilon(r)dr \to \int_s^t [\Delta u]_{ac}(re_n)a(r)\ dr. \tag{6.35}$$

Likewise

$$|v_\varepsilon(t)| \leq C \qquad (\sigma \leq t \leq l - \sigma)$$

and

$$v_\varepsilon \to 0 \quad \text{for a.e.} \quad \sigma \leq t \leq l - \sigma,$$

since $\mathcal{H}^\infty$-a.e. point along R is a Lebesgue point of a. Thus

$$\int_s^t v_\varepsilon(r)\ dr \to 0.$$

Since $a_\varepsilon(t) \to a(te_n) = a(t)$ for a.e. t, we conclude upon passing to limits as $\varepsilon \to 0$ in (6.33) that for $\sigma \leq s \leq t \leq k - \sigma$ we have:

$$a(t) - a(s) = \int_s^t f(r) + [\Delta u]_{ac}(re_n)a(r)\ dr.$$

This identity proves (6.24). □

Remark. Note carefully in Proposition 6.2 that we are asserting the ODE $a_\nu - [\Delta u]_{ac}a = f$ holds in the *interior* regions X^0, Y^0. The reason is that we need the transport set T to completely fill the cylinder C_ε to carry out estimate (6.30). □

7 Vanishing of the transport density at the ends of rays

Finally we assert that the transport density a, restricted to a generic transport ray R_{z_0}, goes to zero at the endpoints a_0, b_0 of R_{z_0}.

Proposition 7.1 *For a.e.* $z_0 \in T$,

$$\lim_{\substack{z \to a_0 \\ z \in R_{z_0}}} a(z) = 0 \tag{7.1}$$

and

$$\lim_{\substack{z \to b_0 \\ z \in R_{z_0}}} a(z) = 0. \tag{7.2}$$

Proof. 1. As before, fix z_0 so that $R = R_{z_0}$ intersects the set of Lebesgue points of a on a set of full $\mathcal{H}^\infty$-measure. Also, recall that u^* (resp. u_*) is locally semiconcave (resp. semiconvex) on $B(0,R) - Y$ (resp. $B(0,R) - X$). Thus u^* (resp. u_*) is twice differentiable a.e. on $B(0,R) - Y$ (resp. $B(0,R) - X$) by Alexandrov's Theorem ([E-G], §6.4). We may consequently assume that z_0 is selected so that R intersects the set of twice differentiably of u^* on $B(0,R) - Y$ and u_* on $B(0,R) - X$ on a set of full $\mathcal{H}^\infty$-measure (cf. Proposition 5.2). We also take (as in earlier proofs) $a_0 = 0$, $b_0 = le_n$, $Du = -e_n$ on R. We will prove

$$\lim_{\substack{z \in R \\ z \to 0}} a(z) = 0. \tag{7.3}$$

As in the proof of Proposition 4.1, let us select a point $\hat{w} \in R - (X \cup Y)$ such that the segment $[0, \hat{w}]$ does not intersect Y. We write

$$\hat{w} = re_n \qquad (0 < r < l). \tag{7.4}$$

We also assume

$$u(\hat{w}) = u^*(\hat{w}) = 0, \tag{7.5}$$

and define

$$\Gamma_0^* = \{u^* = 0\}.$$

We may suppose as well u^* is twice differentiable at $\hat{w}$.

Observe

$$u(0) = r. \tag{7.6}$$

We must consider next these various possibilities: Does the endpoint $a_0 = 0$ belong to X^0 or ∂X? Is u differentiable at $a_0 = 0$, or not? We will establish (7.3) in each situation, but the reasoning will be different.

2. Let us first suppose

$$0 \in X^0 \;\text{ and }\; Du(0) \;\text{ exists.} \tag{7.7}$$

The idea is now to show that $[\Delta u]_{ac}(z) \to -\infty$ as $z \to 0$, $z \in R$. Then the ODE (6.24) will force $a(z) \to 0$ as $z \to 0$, $z \in R$.

Recall u^* is twice differentiable at $\hat{w}$ and $Du^*(\hat{w}) = -e_n \neq 0$. Hence for some $\rho > 0$

$$\Delta = \Gamma_0^* \cap B(\hat{w}, \rho)$$

is the graph of a Lipschitz function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$, which is twice differentiable at $w' = 0$. Thus, in appropriate coordinates,

$$\Delta = \{z \mid z_n = r - \frac{1}{2}\sum_{i=1}^{n-1} \kappa_i z_i^2 + o(|z'|^2)\} \text{ near } \hat{w}. \tag{7.8}$$

The $\{\kappa_i\}_{i=1}^{n-1}$ are the principal curvatures of Δ at $\hat{w}$. We record also the observation that

$$\begin{cases} u(z) = \text{ dist}(z, \Delta) \text{ for all } z \text{ in some} \\ \text{neighborhood of the } \textit{closed} \text{ interval } [0, \hat{w}]. \end{cases} \tag{7.9}$$

If this were false, there would exist points $z_k \to z \in [0, \hat{w}]$ such that u would be decreasing at rate one along transport rays R_k through z_k, but the R_k would not in the limit be pointing in the direction e_n. This contradicts the fact (Lemma 4.1(ii) and (7.7)) that u is differentiable at z.

3. Now set $\kappa = \max_{1\leq i\leq n-1} \kappa_i$. We will next verify

$$\kappa > 0, \ r = \frac{1}{\kappa}. \tag{7.10}$$

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To see this, define for small $\varepsilon > 0$

$$\underline{\gamma}(z') = r - \frac{(\kappa+\varepsilon)}{2}|z'|^2 \qquad (z' \in \mathbb{R}^{n-1}),$$

and let $\underline{\Delta}$ denote the graph of $\underline{\gamma}$ near $\hat{w}$. Write

$$\underline{u}(z) = \text{ dist}(z, \underline{\Delta}).$$

Since $\underline{\gamma} \leq \gamma$ for small $|z'|$, we see from (7.9) that

$$\underline{u} \leq u \text{ near the } \textit{closed} \text{ interval } [0, \hat{w}]. \tag{7.11}$$

If $\kappa + \varepsilon \leq 0$, then

$$\underline{u}(te_n) = r - t \quad \text{for all} \quad t \leq r\ . \tag{7.12}$$

If $\kappa + \varepsilon > 0$, then

$$\underline{u}(te_n) = r - t \tag{7.13}$$

so long as

$$0 \leq r - t \leq \frac{1}{\kappa + \varepsilon}. \tag{7.14}$$

If (7.12) holds, then (7.11) implies

$$u(te_n) = r - t \text{ for some small } t < 0.$$

This however is a contradiction, since $a_0 = 0$ is the upper endpoint of the ray R. Suppose instead (7.13), (7.14) are valid. Again we have a contradiction if $t < 0$. Consequently

$$r - t > \frac{1}{\kappa + \varepsilon} \text{ for each } t < 0,\ \varepsilon > 0;$$

and so

$$\kappa > 0,\ r \geq \frac{1}{\kappa}. \tag{7.15}$$

On the other hand if $z = (z', z_n) \in \Delta$, then

$$z_n \geq (r^2 - |z'|^2)^{1/2} = r - \frac{|z'|^2}{2r} + o(|z'|^2).$$

As $z_n = r - \frac{1}{2}\sum_{i=1}^{n-1} \kappa_i z_i^2 + o(|z'|^2)$, we deduce

$$\frac{1}{r} \geq \kappa_i \qquad (i = 1, \dots, n-1).$$

Thus

$$\frac{1}{r} \geq \kappa,$$

and so

$$\frac{1}{\kappa} \geq r \tag{7.16}$$

since $\kappa > 0$. This inequality and (7.15) complete the proof of (7.10).

4. Now we estimate $[\Delta u]_{ac}$ from above along the segment $[0, \hat{w}]$. Suppose the segment $[0, ke_n]$ lies in X^0. Now fix a point $\hat{z} = te_n$, where $0 < t \leq k$. We may assume $u = u^*$ is twice differentiable at $\hat{z}$, whence

$$u(z) = u(\hat{z}) + Du(\hat{z}) \cdot (z - \hat{z}) + \frac{1}{2}(z - z)^T [D^2 u(\hat{z})]_{ac}(z - \hat{z}) + o(|z - \hat{z}|^2)\ . \tag{7.17}$$

In view of Lemma 4.1(iii),

$$[D^2 u(\hat{z})]_{ac} \leq C. \tag{7.18}$$

Define

$$\bar{\gamma}(z') = r - \frac{1}{2}\sum_{i=1}^{n-1}(\kappa_i - \varepsilon) z_i^2 \qquad (z' \in \mathbb{R}^{\ltimes - \nVdash})$$

and let $\bar{\Delta}$ denote the graph of $\bar{\gamma}$ near $\hat{w}$. Write

$$\bar{u}(z) = \text{ dist}(z, \bar{\Delta}).$$

Then

$$u(\hat{z}) = \bar{u}(\hat{z}) \ ,$$

and, since $\gamma \leq \bar{\gamma}$ for small $|z'|$,

$$u \leq \bar{u} \ \text{ in some neighborhood of the closed line segment } \ [0, \hat{w}]. \tag{7.19}$$

Assuming $\kappa = \max_{1 \leq i \leq n-1} \kappa_i = \kappa_1$, we note

$$\bar{u}_{x_1 x_1}(\hat{z}) = -\frac{(\kappa - \varepsilon)}{1 - (\kappa - \varepsilon)(r - t)}.$$

(See for instance, Gilbarg–Trudinger [G-T, §14.6].) Hence if $h \neq 0$ is small,

$$\begin{aligned} \frac{u(\hat{z} + he_1) - 2u(\hat{z}) + u(\hat{z} - he_1)}{h^2} &\leq \frac{\bar{u}(\hat{z} + he_1) - 2\bar{u}(\hat{z}) + \bar{u}(\hat{z} - he_1)}{h^2} \\ &\to -\frac{(\kappa - \varepsilon)}{1 - (\kappa - \varepsilon)(r - t)} \ \text{ as } h \to 0. \end{aligned}$$

Consequently (7.17) and (7.18) imply

$$[\Delta u]_{ac}(te_n) \leq C - \frac{\kappa}{1 - \kappa(r - t)} \text{ for a.e. } 0 < t < k.$$

Since $r = \frac{1}{\kappa}$ according to (7.10), we deduce:

$$[\Delta u]_{ac}(te_n) \leq C - \frac{1}{t} \text{ for a.e. } 0 < t < k. \tag{7.20}$$

Next recall from Proposition 6.2 the ODE for a along the ray R:

$$a' - [\Delta u]_{ac} a = f \text{ for a.e. } 0 < t < k,$$

where $a(t) = a(te_n)$, and $te_n \in X^0$ for $0 \leq t \leq k$. In light of (7.20) therefore,

$$a' \leq C - \frac{a}{t}.$$

Hence

$$\begin{aligned} (ta)' &= ta' + a \\ &\leq tC \text{ a.e.}, \end{aligned}$$

and so

$$ta(t) \leq \frac{Ct^2}{2}.$$

Thus

$$0 \leq a(t) \leq \frac{Ct}{2} \text{ for } 0 < t \leq k.$$

Obviously then (7.3) is valid.

5. Consider next the possibility

$$0 \in X^0 \text{ and } Du(0) \text{ does not exist.} \tag{7.21}$$

Define Δ as above, so that $u(z) = \operatorname{dist}(z, \Delta)$ for z near the *half-open* interval $(0, \hat{w}]$. Now if $\kappa > 0$, then as in Step 3 we must have $r \leq \frac{1}{\kappa}$. If

$$r = \frac{1}{\kappa}\ ,$$

then the ODE calculations in Step 4 imply (7.3). Suppose instead

$$r < \frac{1}{\kappa} = r_0 \text{ or else } \kappa \leq 0. \tag{7.22}$$

Then, since u is not differentiable at 0, it must be that

$$\begin{cases} a_0 = 0 \text{ is the upper endpoint of at least one other} \\ \text{transport ray } S \neq R. \end{cases} \tag{7.23}$$

Also, observe from (7.22) that $v(z) = \operatorname{dist}(z, \Delta)$ satisfies the conclusion of Proposition 4.1 near compact subsets of the segment $((r - r_0)e_n, \hat{w}] \supseteq [0, \hat{w}]$. In particular for some neighborhood N of the *closed* interval $[0, \hat{w}]$, we have the estimate

$$|Dv(z) - Dv(\hat{z})| \leq C|z - \hat{z}| \text{ if } z \in T \cap N, \tag{7.24}$$

provided $Dv(z)$ exists. Here $\hat{z}$ is the projection of z onto $[0, \hat{w}]$ and T temporarily denotes the transport set for v. The same conclusion holds if $\kappa \leq 0$.

Next we *claim*

$$\begin{cases} \text{there exists a truncated spherical cone } C\text{, as drawn,} \\ \text{with axis } e_n \text{ and vertex 0, such that} \\ u(z) = v(z) = \operatorname{dist}(z, \Delta) \text{ for } z \in C \cap B(0, r). \end{cases} \tag{7.25}$$

In other words, we are saying that although $u(z)$ does not equal $\operatorname{dist}(z, \Delta)$ in a full neighborhood of 0, we do have $u(z) = \operatorname{dist}(x, \Delta)$ for z lying in a cone around the e_n-axis, with vertex 0. To verify this statement, note first that (7.22) implies

$$\hat{w} = \partial B(0, r) \cap \{z_n \geq \alpha |z'|\} \cap \Gamma_0^*$$

for some constant $\alpha > 0$. Recall $B(0,r)^0 \cap \Gamma_0^* = \emptyset$. Select $\beta > 0$ such that the cone $\{z_n \geq \alpha|z'|\}$ has twice the opening at 0 as the cone $\{z_n \geq \beta|z'|\}$. If

$$z \in B(0,r) \text{ and } z_n \geq \beta|z'|, \tag{7.26}$$

then $\text{dist}(z, \Gamma_0^* - \Delta) > \text{dist}(z, \hat{w}) \geq \text{dist}(z, \Delta)$. The inequality in (7.26) determines the desired cone.

=750 Fig7.2

Combining (7.24), (7.25) we see

$$|Du(z) - Du(\hat{z})| \leq C|z - \hat{z}| \text{ if } z \in T \cap C \tag{7.27}$$

provided $Du(z)$ exists.

6. The plan next is to show

$$\lim_{\varepsilon \to 0^+} ⨍_{B(0,\varepsilon)} a\, dz = 0, \tag{7.28}$$

and this we accomplish by a blow-up. Define the rescaled functions

$$\begin{cases} u^\varepsilon(z) = \frac{u(\varepsilon z) - u(0)}{\varepsilon}, \\ a^\varepsilon(z) = a(\varepsilon z), \ f^\varepsilon(z) = f(\varepsilon z). \end{cases} \tag{7.29}$$

Then $u^\varepsilon(0) = 0$, $|Du^\varepsilon| \leq 1$ a.e., and so we may extract a subsequence $\varepsilon_j \to 0$ and a Lipschitz function $u^0 : \mathbb{R}^n \to \mathbb{R}$ such that

$$u^{\varepsilon_j} \to u^0 \text{ locally uniformly on } \mathbb{R}^n. \tag{7.30}$$

Since $|Du^{\varepsilon_j}| = 1$ in the viscosity sense near $0 \in X^0$ (Lemma 4.2), we deduce from (7.30) that

$$|Du^0| = 1 \text{ in the viscosity sense in } \mathbb{R}^n.$$

In particular

$$|Du^0| = 1 \text{ a.e. in } \mathbb{R}^n. \tag{7.31}$$

Consequently for each compact set $K \subset \mathbb{R}^n$

$$\begin{aligned} \limsup_{j\to\infty} \int_K |Du^{\varepsilon_j}|^2 dx &= |K| \\ &= \int_K |Du^0|^2 dx \end{aligned}$$

and hence

$$Du^{\varepsilon_j} \to Du^0 \text{ strongly in } L^2_{\text{loc}}. \tag{7.32}$$

Also

$$D^2 u^{\varepsilon_j}(z) = \varepsilon_j D^2 u(\varepsilon_j z) \leq C\varepsilon_j$$

and so

$$u^0 \text{ is concave.} \tag{7.33}$$

Recall now from (7.23) that $a_0 = 0$ is the upper endpoint of (at least) two transport rays R and S. Let us temporarily rotate to new coordinates, so that these rays make an equal angle with the z_n-axis. In these coordinates $R = \{ta_1 \mid t \geq 0\}$, $S = \{ta_2 \mid t \geq 0\}$, where

$$a_1 = \lambda e_n + \gamma e', \ a_2 = \lambda e_n - \gamma e' \tag{7.34}$$

for $1 > \lambda \geq 0$, $\gamma > 0$, $\lambda^2 + \gamma^2 = 1$, and $e' \neq 0$ some unit vector perpendicular to e_n.

Recall that the function u decreases linearly at rate one along R and S. Thus from (7.29), (7.30) we deduce

$$u^0(ta_1) = -t, \ u^0(ta_2) = -t \qquad \text{for } t \geq 0. \tag{7.35}$$

Now observe that

$$u^0(z) \leq -z \cdot a_1, \ u^0(z) \leq -z \cdot a_2 \qquad (z \in \mathbb{R}^n). \tag{7.36}$$

Indeed if $t > 0$, $z \in \mathbb{R}^n$

$$\begin{aligned} |u^0(z) + t|^2 &= |u^0(z) - u^0(ta_i)|^2 \\ &\leq |z - ta_i|^2 \qquad (i = 1, 2). \end{aligned}$$

Consequently

$$u^0(z)^2 + 2tu^0(z) + t^2 \leq |z|^2 - 2ta_i \cdot z + t^2,$$

and so

$$2[u^0(z) + a_i \cdot z] \leq \frac{|z|^2}{t} \qquad (i = 1, 2)\ .$$

Letting $t \to \infty$ we deduce (7.36).

Using (7.36) we compute

$$\begin{aligned} u^0(z) &\leq \min(-z \cdot a_1, -z \cdot a_2) \\ &= \min(-\lambda z_n - \gamma(z \cdot e'), -\lambda z_n + \gamma(z \cdot e')) \\ &= -\lambda z_n - \gamma |z \cdot e'|. \end{aligned}$$

Hence

$$u^0(z) \leq -\lambda z_n \qquad (z \in \mathbb{R}^n). \tag{7.37}$$

Now take $\theta > 0$ so small that

$$\lambda + \theta < 1, \tag{7.38}$$

fix $\rho \gg 1$, and define

$$v(z) = -\lambda z_n + \theta|z| - \rho \qquad (z \in \mathbb{R}^{\ltimes})\ . \tag{7.39}$$

Then $v < u^0$ in some large neighborhood of 0. However we also note from (7.37) that

$$u^0(z) \leq v(z) \text{ if } |z| \text{ is large enough.} \tag{7.40}$$

We will employ the auxiliary function v below.

7. We next blow up the PDE $-\text{div}(aDu) = f$. From the scaling (7.29) it follows that

$$-\operatorname{div}(a^\varepsilon Du^\varepsilon) = \varepsilon f^\varepsilon \tag{7.41}$$

in the weak sense. We may assume

$$a^{\varepsilon_j} \rightharpoonup a^0 \text{ weakly} * \text{ in } L^\infty_{\text{loc}}, \tag{7.42}$$

where $a^0 \geq 0$. Since $Du^{\varepsilon_j} \to Du^0$ strongly in L^2_{loc}, we may pass to limits in (7.41) and deduce

$$-\operatorname{div}(a^0 Du^0) = 0 \text{ in } \mathbb{R}^n, \tag{7.43}$$

in the weak sense.

We can finally establish the key assertion, namely that (7.43) and (7.31) together force a^0 to be identically zero. To prove this, let $M > 0$ be given and then select $\rho > 0$ so large that

$$v < u^0 \text{ in } B(0, M).$$

We take $w = (u^0 - v)^+$ as a test function in the weak formulation of (7.43). Note that according to (7.40) w has compact support in $\mathbb{R}^n$. Consequently

$$\begin{aligned} 0 &= \textstyle\int_{\mathbb{R}^n} a^0 Du^0 \cdot D(u^0 - v)^+ \, dz \\ &= \textstyle\int_{\{u^0 > v\}} a^0 [|Du^0|^2 - Du^0 \cdot Dv] \, dz. \end{aligned}$$

But $|Du^0| = 1$ a.e. and $|Dv| \leq \lambda + \theta < 1$. Hence

$$0 \geq \int_{\{u^0 > v\}} a^0 (1 - (\lambda + \theta)) \, dz$$

and so $a^0 = 0$ a.e. on $B(0, M) \subset \{u^0 > v\}$. This is true for each $M > 0$ and thus

$$a^0 = 0 \text{ a.e. in } \mathbb{R}^n. \tag{7.44}$$

But then

$$\begin{aligned} ⨍_{B(0,\varepsilon_j)} a \, dz &= ⨍_{B(0,1)} a^{\varepsilon_j} \, dz \\ &\to ⨍_{B(0,1)} a^0 \, dz = 0. \end{aligned}$$

This is true for any subsequence $\varepsilon_j \to 0$ and therefore assertion (7.28) is proved.

8. Return now to the original coordinate system, in which R lies along the positive z_n-axis. We will utilize (7.27), (7.28) finally to prove (7.3). Recall the cone C introduced in statement (7.25).

Fix $\mu > 0$ so small that the balls $B(te_n, 2\mu t)$ lie in the cone C for all $t > 0$. Let $\delta > 0$. Then (7.28) implies

$$⨍_{B(0,\varepsilon)} a \, dx \leq \delta$$

if $\varepsilon > 0$ is small enough. Hence

$$⨍_{B(\frac{\varepsilon}{2} e_n, \mu\varepsilon)} a \, dx \leq C\delta.$$

Thus

$$|a_{C\varepsilon}(\frac{\varepsilon}{2})| \leq C\delta, \tag{7.45}$$

where we are using the notation (6.11) from the proof of Proposition 6.1. But recall also estimate (6.13) from that proof, namely

$$|a_{C\varepsilon}(s) - a_{C\varepsilon}(t)| \leq C|t - s| + C\varepsilon \qquad (0 < s, t < k). \tag{7.46}$$

The constants C here depend only on the constant C from (7.27), and so in particular this estimate is valid for s, t *all the way down to zero.* Let $s = \frac{\varepsilon}{2}$ in (7.46), recall (7.45), and then send $\varepsilon \to 0$:

$$|a(t)| \leq Ct + C\delta \qquad 0 < t \leq k.$$

This is true for each $\delta > 0$ and so

$$a(t) \leq Ct \qquad (0 < t \leq k).$$

This inequality proves (7.3), provided (7.21) holds.

9. The last possibility is

$$0 \in \partial X. \tag{7.47}$$

In this situation $a_0 = 0$ could be the upper endpoint of R for either of the reasons discussed above in Steps 2–4 or Steps 5–8, or else because extending the ray R farther would exit X. We can treat all these possibilities at once by replacing u with u^* in $B(0,R) - Y$. Since $\text{supp}(a) \subset T$ and $u = u^*$, $Du = Du^*$ on T, we have

$$-\text{div}(aDu^*) = f \quad \text{in} \quad B(0,R) - Y$$

in the weak sense. Note that the transport set T^* for u^* fills all of $B(0,R) - Y$. Let $R^* \supseteq R$ be the corresponding transport ray for u^*. If the upper endpoint of R^* is still 0, we apply Steps 2–4 or 5–8 above to deduce (7.3). The remaining possibility is that the upper endpoint of R^* is

$$a_0^* = -t_0 e_n \quad \text{for some} \quad t_0 > 0.$$

Now $[a_0^*, 0) \cap T = \emptyset$, as otherwise $a_0 = 0$ would not have been the upper endpoint of R. If $a_0^* \in B(0,R)^0$, we deduce using Steps 2–4 or 5–8 above that

$$\lim_{\substack{z \to a_0^* \\ z \in R^*}} a = 0.$$

As $a|_{R^*}$ is Lipschitz and $|a'(t)| \leq Ca(t)$ for a.e. $-t_0 \leq t \leq 0$, since $[a_0^*, 0) \cap X = \emptyset$, we deduce $a(t) \equiv 0$ on $[a_0^*, 0]$ and so (7.3) follows.

The remaining possibility is $a_0 \in \partial B(0,R)$. But since $\text{supp}(a) \subset T \subset B(0,S)$, clearly

$$a|_{R^* \cap [B(0,R) - B(0,S)]} = 0.$$

□

8 Approximate mass transfer plans

The lengthy and intricate analysis of the fine properties of u and a now done with, we turn to the primary task, namely, building an optimal mass transfer plan $\mathbf{s}$. In this section we construct an approximate transfer plan $\mathbf{s}_{\varepsilon,\delta}$, and in §9, 10 study the limits as $\varepsilon \to 0$, then $\delta \to 0$.

Definition of the smooth approximate mass transfer plan $\mathbf{s}_{\varepsilon,\delta}$. Hereafter fix $0 < \varepsilon, \delta < 1$. Select a function $\eta : B(0,1) \to \mathbb{R}$ such that

$$\begin{cases} \eta \in C^\infty(B(0,1)), \eta \text{ has compact support in } B(0,1), \\ \eta \geq 0, \ \int_{B(0,1)} \eta \, dx = 1, \ \eta \text{ is radial}, \ \eta > 0 \text{ on } B(0, \frac{1}{2}) \ . \end{cases} \tag{8.1}$$

Next define the mollifier

$$\eta_\varepsilon(z) = \frac{1}{\varepsilon^n} \eta(\frac{z}{\varepsilon}) \qquad (z \in \mathbb{R}^n), \tag{8.2}$$

and write

$$(aDu)_\varepsilon(z) \begin{array}{l} = \eta_\varepsilon * (aDu)(z) \\ = \int_{\mathbb{R}^n} \eta_\varepsilon(z-w)a(w)Du(w)dw \qquad (z \in B(0,R)). \end{array} \tag{8.3}$$

Define as well for $z \in B(0,R)$

$$a_\varepsilon(z) = (\eta_\varepsilon * a)(z), \tag{8.4}$$

and

$$\nu_\varepsilon(z) = \begin{cases} \frac{-(aDu)_\varepsilon(z)}{a_\varepsilon(z)} & \text{if } a_\varepsilon(z) \neq 0 \\ 0 & \text{if } a_\varepsilon(z) = 0. \end{cases} \tag{8.5}$$

Then

$$-(aDu)_\varepsilon = a_\varepsilon \nu_\varepsilon. \tag{8.6}$$

Now since

$$-\mathrm{div}(aDu) = f$$

in the weak sense, we have

$$\mathrm{div}(a_\varepsilon \nu_\varepsilon) = f_\varepsilon = f_\varepsilon^+ - f_\varepsilon^-. \tag{8.7}$$

Next introduce the smooth, time-varying vector field

$$\mathbf{b}_\varepsilon(z,t) = \mathbf{b}_{\varepsilon,\delta}(z,t) = \frac{a_\varepsilon(z)\nu_\varepsilon(z)}{tf_\varepsilon^-(z) + (1-t)f_\varepsilon^+(z) + \delta} \qquad (0 \leq t \leq 1,\ z \in B(0,R)). \tag{8.8}$$

For a given point $z_0 \in B(0,R)$, let $\mathbf{z}_\varepsilon(\cdot) = \mathbf{z}_{\varepsilon,\delta}(\cdot)$ solve the nonautonomous ODE

$$\begin{cases} \dot{\mathbf{z}}_\varepsilon(t) = \mathbf{b}_\varepsilon(\mathbf{z}_\varepsilon(t),t) \quad (0 \leq t \leq 1) \\ \mathbf{z}_\varepsilon(0) = z_0. \end{cases} \tag{8.9}$$

Write $\mathbf{z}_\varepsilon(t) = \mathbf{z}_\varepsilon(t,z_0)$ to display the initial condition. Finally, let us define

$$\mathbf{s}_{\varepsilon,\delta}(z_0) = w_0\ , \tag{8.10}$$

where

$$w_0 = \mathbf{z}_\varepsilon(1,z_0).$$

The mapping $\mathbf{s}_{\varepsilon,\delta}$ is the *approximate mass transfer plan* (corresponding to $\varepsilon, \delta > 0$). Observe that since $a = 0$ on $B(0,R) - B(0,S)$, $\mathbf{s}_{\varepsilon,\delta}(z_0) = z_0$ unless z_0 is close to the ball $B(0,S)$.

We show now that the mapping $z \mapsto \mathbf{s}_{\varepsilon,\delta}(z)$ approximately transforms the measure $\mu^+ = f^+dx$ to $\mu^- = f^-dy$.

Lemma 8.1 *We have*

$$f^+_\varepsilon(z)+\delta=(f^-_\varepsilon(\mathbf{s}_\varepsilon(z))+\delta)\ \det\ D\mathbf{s}_{\varepsilon,\delta}(z)\qquad (z\in B(0,R)), \tag{8.11}$$

and so

$$\int_{B(0,R)} h(\mathbf{s}_{\varepsilon,\delta}(z))(f^+_\varepsilon(z)+\delta)dz=\int_{B(0,R)} h(w)(f^-_\varepsilon(w)+\delta)dw \tag{8.12}$$

for each $h\in C(B(0,R))$.

Proof. To simplify notation, drop the subscripts δ. Let us write

$$\mathbf{s}_\varepsilon(t,z)=\mathbf{z}_\varepsilon(t,z)\qquad (z\in B(0,R),\ 0\le t\le 1),$$

where $\mathbf{z}_\varepsilon(\cdot)$ solves the ODE (8.9). Let

$$J_\varepsilon(t,z)=\det\ D_z\mathbf{s}_\varepsilon(t,z).$$

Then Euler's formula states

$$J_{\varepsilon,t}=(\text{div}\ \mathbf{b}_\varepsilon)J_\varepsilon. \tag{8.13}$$

Following Dacorogna–Moser [D-M] we compute

$$\begin{aligned}
&\tfrac{\partial}{\partial t}[(tf^-_\varepsilon(\mathbf{s}_\varepsilon(t,z))+(1-t)f^+_\varepsilon(\mathbf{s}_\varepsilon(t,z))+\delta)J_\varepsilon]\\
&\quad=(f^-_\varepsilon-f^+_\varepsilon)J_\varepsilon+(tDf^-_\varepsilon\cdot\mathbf{s}_{\varepsilon,t}+(1-t)Df^+_\varepsilon\cdot\mathbf{s}_{\varepsilon,t})J_\varepsilon\\
&\qquad+(tf^-_\varepsilon+(1-t)f^+_\varepsilon+\delta)J_{\varepsilon,t}\\
&\quad=[(f^-_\varepsilon-f^+_\varepsilon)+(tDf^-_\varepsilon\cdot\mathbf{b}_\varepsilon+(1-t)Df^+_\varepsilon\cdot\mathbf{b}_\varepsilon)\\
&\qquad+(tf^-_\varepsilon+(1-t)f^+_\varepsilon+\delta)(\text{div}\ \mathbf{b}_\varepsilon)]J_\varepsilon,
\end{aligned} \tag{8.14}$$

according to (8.9), (8.13).

Now

$$\begin{aligned}
\text{div}\ \mathbf{b}_\varepsilon\ &=\text{div}\left(\frac{a_\varepsilon\nu_\varepsilon}{tf^-_\varepsilon+(1-t)f^+_\varepsilon+\delta}\right)\\
&=\frac{f^+_\varepsilon-f^-_\varepsilon}{tf^-_\varepsilon+(1-t)f^+_\varepsilon+\delta}-\frac{(tDf^-_\varepsilon+(1-t)Df^+_\varepsilon)\cdot(a_\varepsilon\nu_\varepsilon)}{(tf^-_\varepsilon+(1-t)f^+_\varepsilon+\delta)^2}\qquad\text{by (8.7).}
\end{aligned}$$

We insert this identity into (8.14) and deduce:

$$\frac{\partial}{\partial t}[(tf_\varepsilon^- + (1-t)f_\varepsilon^+ + \delta)J_\varepsilon] = 0 \qquad (0 \le t \le 1).$$

Taking $t = 0, 1$ we obtain (8.11). □

We must next understand what happens to $\mathbf{s}_{\varepsilon,\delta}$ as $\varepsilon \to 0$, $\delta > 0$ being fixed.

Definition of the approximate mass transfer plan $\mathbf{s}_\delta$. Write $\nu = -Du$. Then since $a_\varepsilon \to a$, $\nu_\varepsilon \to \nu$ a.e. on $\{a > 0\}$, it is reasonable to guess $\mathbf{s}_{\varepsilon,\delta} \to \mathbf{s}_\delta$ as $\varepsilon \to 0$, $\mathbf{s}_\delta$ built as follows.

First we select a "typical" point $z_0 \in T$, satisfying these conditions:

$$\left\{ \begin{array}{l} Du(z_0) \text{ exists and thus there exists a unique transport ray } R_{z_0} \text{ through} \\ z_0 \text{ connecting its upper endpoint } a_0 \in X \text{ to its lower endpoint } b_0 \in Y. \end{array} \right. \tag{8.15}$$

$$z_0 \ne a_0, b_0, \qquad z_0 \notin \partial X \cup \partial Y. \tag{8.16}$$

$$\left\{ \begin{array}{l} \mathcal{H}^{\infty}(\mathcal{R}_{\ddagger,} \cap \mathcal{A}) = |\dashv, - \lfloor,|, \text{ where } A \text{ denotes} \\ \text{the set of Lebesgue points of } a. \end{array} \right. \tag{8.17}$$

$$a \text{ is continuous and nonnegative on } R_{z_0} = [a_0, b_0], \text{ with } \ a(a_0) = a(b_0) = 0. \tag{8.18}$$

$$a\big|_{R_{z_0}} \text{ is locally Lipschitz.} \tag{8.19}$$

We observe that a.e. point $z_0 \in T$ satisfies:

(8.15) by Radamacher's Theorem,

(8.16) by Proposition 5.1 and (2.4),

(8.17) by Propostion 5.2,

(8.18) by Proposition 7.1,

(8.19) by Proposition 6.1.

For such a point z_0, we set $\nu = -Du(z_0)$ and define

$$\mathbf{b}_\delta(z,t) = \frac{a(z)\nu}{tf^-(z) + (1-t)f^+(z) + \delta} \qquad (0 \le t \le 1, \ z \in R_{z_0}). \tag{8.20}$$

Consider next the ODE

$$\begin{cases} \dot{\mathbf{z}}_\delta(t) = b_\delta(\mathbf{z}_\delta(t), t) & (0 \le t \le 1) \\ \mathbf{z}(0) = z_0. \end{cases} \tag{8.21}$$

Since f^+, f^- and $a|_{[z_0,b_0)}$ are locally Lipschitz (according to (2.1), (8.19)), *this ODE has a unique solution* $\mathbf{z}_\delta$. Furthermore, since $a(b_0) = 0$, we see

$$\mathbf{z}_\delta(t) \in [z_0, b_0) \qquad (0 \le t \le 1). \tag{8.22}$$

Write $\mathbf{z}_\delta(t) = \mathbf{z}_\delta(t, z_0)$. We define

$$\mathbf{s}_\delta(z_0) = w_0, \tag{8.23}$$

where

$$w_0 = \mathbf{z}_\delta(1, z_0). \tag{8.24}$$

We also set

$$\mathbf{s}_\delta(z_0) = z_0 \text{ if } z_0 \notin T. \tag{8.25}$$

Definition of the optimal mass transfer plan s. Finally observe that for a.e. $z_0 \in T$ is as above, the points $\{\mathbf{s}_\delta(z_0)\}_{0<\delta\le 1}$ are arranged monotonically along the segment $[z_0, b_0)$, since $a, f^+, f^- \ge 0$. If $z_0 \notin T$, then $\mathbf{s}_\delta(z_0) = z_0$. In both cases the limit

$$\lim_{\delta \to 0} \mathbf{s}_\delta(z_0) = \mathbf{s}(z_0) \tag{8.26}$$

exists. We call $\mathbf{s}$ the *optimal mass transfer plan generated by the potential u and transport density a*. It is in particular defined for a.e. point in X.

The remainder of the paper is devoted to showing that

$$\lim_{\varepsilon \to 0} \mathbf{s}_{\varepsilon,\delta}(z_0) = \mathbf{s}_\delta(z_0) \text{ for a.e. } z_0 \in T \text{ as above,} \tag{8.27}$$

and that $\mathbf{s}$ is indeed optimal.

9 Passage to limits a.e.

In this section we confirm that the smooth mappings $\mathbf{s}_\varepsilon = \mathbf{s}_{\varepsilon,\delta}$ converge a.e. as $\varepsilon \to 0$.

Theorem 9.1 *For a.e.* $z_0 \in B(0,R)$ *we have*

$$\lim_{\varepsilon\to 0} \mathbf{s}_{\varepsilon,\delta}(z_0) = \mathbf{s}_\delta(z_0). \tag{9.1}$$

Furthermore, the mapping $\mathbf{s}_\delta$ *satisfies*

$$\int_{B(0,R)} h(\mathbf{s}_\delta(z))(f^+(z)+\delta)dz = \int_{B(0,R)} h(w)(f^-(w)+\delta)dw \tag{9.2}$$

for each $h \in C(B(0,R))$.

Proof. We will need to consider various possibilities as to the location of z_0 and whether or not $a(z_0) > 0$.

1. Case 1: $z_0 \notin T$.

Then since T is closed and $\text{supp}(a) \subset T$ (Proposition 3.1), we have $a = 0$ a.e. in some ball near z_0. Thus $(aDu)_\varepsilon = 0$ near z_0, and so the solution $\mathbf{z}_\varepsilon(\cdot)$ of (8.9) is constant in t, for all small $\varepsilon > 0$. Clearly then

$$\mathbf{s}_\varepsilon(z_0) = \mathbf{s}_\delta(z_0) = z_0 \text{ for all sufficiently small } \varepsilon.$$

2. Case 2: $z_0 \in T$.

We may assume that z_0 satisfies conditions (8.15)–(8.19); a.e. point in T does so. The transfer ray R_{z_0} has upper endpoint $a_0 \in X$, lower endpoint $b_0 \in Y$.

3. Subcase A: $a(z_0) = 0$.

Then according to Proposition 6.1(iii), we have $z_0 \notin X^0 \cup Y^0$. Futhermore, Proposition 6.1(ii) implies there exists a point $c_0 \in (z_0, b_0)$ such that

$$a\big|_I = 0, \tag{9.3}$$

I denoting the interval $[z_0, c_0]$.

We must show that

$$\lim_{\varepsilon\to 0} \mathbf{s}_\varepsilon(z_0) = \mathbf{s}_\delta(z_0) = z_0. \tag{9.4}$$

We may as well suppose $z_0 = 0$, $c_0 = le_n$. According to Proposition 4.1 there exists a tubular neighborhood N of I such that

$$|Du(z) + e_n| \le C|z'|, \tag{9.5}$$

provided $z \in T \cap N$, $Du(z)$ exists, and $z = (z', z_n)$. Fix any index $i \in \{1, \dots, n-1\}$. Then (8.8),(8.9) say

$$\dot{z}_{\varepsilon,i}(t) = \frac{-(au_{x_i})_\varepsilon(\mathbf{z}_\varepsilon(t))}{tf_\varepsilon^- + (1-t)f_\varepsilon^+ + \delta} \qquad (0 \le t \le 1). \tag{9.6}$$

Let t_ε denote the minimum of 1 and the first time (if any) that $\mathbf{z}_\varepsilon(t)$ is within distance ε of ∂N. Then for $0 \le t < t_\varepsilon$

$$\begin{aligned} |(au_{x_i})_\varepsilon(\mathbf{z}_\varepsilon(t))| &= |\textstyle\int \eta_\varepsilon(\mathbf{z}_\varepsilon(t) - w)a(w)u_{x_i}(w)dw| \\ &\le C(|\mathbf{z}'_\varepsilon(t)| + \varepsilon), \end{aligned} \tag{9.7}$$

according to (9.5), where

$$\mathbf{z}'_\varepsilon(t) = (z_{1,\varepsilon}(t), \dots, z_{n-1,\varepsilon}(t)).$$

This follows since $|\mathbf{z}_\varepsilon(t) - w| \le \varepsilon$ implies $|w'| \le |\mathbf{z}'_\varepsilon(t)| + \varepsilon$. Thus (9.6), (9.7) yield the differential inequality

$$|\dot{\mathbf{z}}'_\varepsilon(t)| \le C(|\mathbf{z}'_\varepsilon(t)| + \varepsilon) \text{ for } 0 \le t < t_\varepsilon;$$

and so, since $\mathbf{z}'_\varepsilon(0) = 0$, Gronwall's inequality gives

$$|\mathbf{z}'_\varepsilon(t)| \le C\varepsilon \text{ for } 0 \le t < t_\varepsilon.$$

Consequently if $0 \le t < t_\varepsilon$

$$\begin{aligned} |(aDu)_\varepsilon(\mathbf{z}_\varepsilon(t))| &= |\textstyle\int \eta_\varepsilon((\mathbf{z}_\varepsilon(t) - w)a(w)Du(w)dw| \\ &\le C ⨍_{B(\mathbf{z}_\varepsilon(t),\varepsilon)} a\, dz \\ &\le C ⨍_{B(\hat{\mathbf{z}}_\varepsilon(t),C\varepsilon)} a\, dz, \end{aligned} \tag{9.8}$$

$\hat{\mathbf{z}}_\varepsilon(t)$ denoting the projection of $\mathbf{z}_\varepsilon(t)$ onto I. But recalling observation (6.21) we note

$$⨍_{B(\hat{\mathbf{z}}_\varepsilon(t),C\varepsilon)} a\, dz \to 0$$

uniformly for $0 \le t \le t_\varepsilon$, since $a|_I = 0$. Hence (9.8) and the ODE (8.8), (8.9) then imply

$$|\dot{\mathbf{z}}_\varepsilon(t)| \to 0 \ \ \textit{uniformly} \text{ for } 0 \le t \le t_\varepsilon.$$

In particular $t_\varepsilon = 1$ for all sufficiently small $\varepsilon > 0$. Thus in fact (9.4) is valid, since $\mathbf{s}_\varepsilon(z_0) = \mathbf{z}_\varepsilon(1) \to z_0$.

4. Subcase B: $a(z_0) > 0$.

In this situation there exists a point $c_0 \in (z_0, b_0]$ such that

$$a|_I > 0 \text{ on the interval } I = (z_0, c_0), \ a(c_0) = 0. \tag{9.9}$$

Let $l = |z_0 - c_0|$.

Our plan is to factor $\mathbf{z}_\varepsilon(t)$ by writing

$$\mathbf{z}_\varepsilon(t) = \mathbf{w}_\varepsilon(\phi_\varepsilon(t)) \qquad (0 \le t \le 1), \tag{9.10}$$

where

$$\begin{cases} \dot{\mathbf{w}}_\varepsilon(s) = \nu_\varepsilon(\mathbf{w}_\varepsilon(s)) \quad (s \ge 0) \\ \mathbf{w}_\varepsilon(0) = z_0 \end{cases} \tag{9.11}$$

and

$$\begin{cases} \dot{\phi}_\varepsilon(t) = c_\varepsilon(\phi_\varepsilon(t), t) \quad (0 \le t \le 1) \\ \phi_\varepsilon(0) = 0, \end{cases} \tag{9.12}$$

for

$$c_\varepsilon(s,t) = \frac{a_\varepsilon(\mathbf{w}_\varepsilon(s))}{t f_\varepsilon^-(\mathbf{w}_\varepsilon(s)) + (1-t) f_\varepsilon^+(\mathbf{w}_\varepsilon(s)) + \delta} \qquad (0 \le t \le 1, \ s \ge 0). \tag{9.13}$$

Remember ν_ε is defined by (8.5). Observe that if $\mathbf{w}_\varepsilon : [0,\infty) \to B(0,R)$ solves (9.11) and $\phi_\varepsilon : [0,1] \to [0,\infty)$ solves (9.12), then $\mathbf{z}_\varepsilon$ defined by (9.10) is the unique solution of the ODE (8.9). The advantage of this factorization is that the ODE (9.12) is scalar.

We introduce next the formal limits of (9.11)–(9.13) for $\varepsilon = 0$. Define

$$\mathbf{w}(s) = z_0 + s\nu \qquad (s \ge 0), \tag{9.14}$$

where, recall,

$$\nu = -Du(z_0).$$

In addition, let ϕ solve

$$\begin{cases} \dot{\phi}(t) = c(\phi(t), t) \quad (0 \le t \le 1) \\ \phi(0) = 0, \end{cases} \tag{9.15}$$

for

$$c(s,t) = \frac{a(\mathbf{w}(s))}{t f^-(\mathbf{w}(s)) + (1-t) f^+(\mathbf{w}(s)) + \delta} \qquad (0 \le s \le l, \ 0 \le t \le 1). \tag{9.16}$$

Observe that $\phi(1) < l$, since $a(c_0) = 0$. Note also

$$\mathbf{z}_\delta(t) = \mathbf{w}(\phi(t)) \ ,$$

$\mathbf{z}_\delta$ the solution of (8.20),(8.21). We intend to prove $\mathbf{w}_\varepsilon \to \mathbf{w}$ and $\phi_\varepsilon \to \phi$ as $\varepsilon \to 0$.

=750 Fig9.1

5. Lemma 9.2 *For each $\sigma > 0$ there exist positive constants C, θ such that*

$$a_\varepsilon(\mathbf{w}_\varepsilon(s)) \geq \theta > 0 \qquad (0 \leq s \leq l - \sigma) \tag{9.17}$$

and

$$\max_{0 \leq s \leq l-\sigma} |\mathbf{w}_\varepsilon(s) - \mathbf{w}(s)| \leq C\varepsilon. \tag{9.18}$$

for all sufficiently small $\varepsilon > 0$.

Proof. As usual, we may assume $z_0 = 0$, $c_0 = le_n$, $\nu = e_n$. Then

$$|\nu_\varepsilon(\mathbf{w}_\varepsilon(s)) - \nu| = |\int \frac{\eta_\varepsilon(\mathbf{w}_\varepsilon(s) - z)a(z)[Du(z) + e_n]}{\int \eta_\varepsilon(\mathbf{w}_\varepsilon(s) - z)a(z)dz} dz| \tag{9.19}$$

provided the denominator $a_\varepsilon(\mathbf{w}_\varepsilon(s)) = \int \eta_\varepsilon(\mathbf{w}_\varepsilon(s) - z)a(z)dz \neq 0$. As in (9.5) above, there exists a neighborhood N of $J = [0, (l - \sigma)e_n]$ such that

$$|Du(z) + e_n| \leq C|z'| \tag{9.20}$$

provided $z \in T \cap N$, $Du(z)$ exists, and $z = (z', z_n)$.

Let s_ε denote the minimum of $l - \sigma$ and the first time (if any) that the trajectory $\mathbf{w}_\varepsilon(s)$ is within distance ε of ∂N. Then if $0 \leq s < s_\varepsilon$ and $a_\varepsilon(\mathbf{w}_\varepsilon(s)) \neq 0$, we have

$$|\nu_\varepsilon(\mathbf{w}_\varepsilon(s)) - \nu| \leq C(|\mathbf{w}'_\varepsilon(s)| + \varepsilon), \tag{9.21}$$

where $\mathbf{w}'_\varepsilon(s) = (w_{\varepsilon,1}(s), \ldots, w_{\varepsilon,n-1}(s))$. Since $\mathbf{w}(s)$ lies on I, we have $|\mathbf{w}'_\varepsilon(s)| \leq |\mathbf{w}_\varepsilon(s) - \mathbf{w}(s)|$; and consequently

$$|\nu_\varepsilon(\mathbf{w}_\varepsilon(s)) - \nu| \leq C(|\mathbf{w}_\varepsilon(s) - \mathbf{w}(s)| + \varepsilon). \tag{9.22}$$

Now

$$\begin{aligned} \mathbf{w}_\varepsilon(s) &= z_0 + \textstyle\int_0^s \nu_\varepsilon(\mathbf{w}_\varepsilon(t))dt \\ \mathbf{w}(s) &= z_0 + s\nu \ ; \end{aligned}$$

whence

$$\begin{aligned} |\mathbf{w}_\varepsilon(s) - \mathbf{w}(s)| &\leq \textstyle\int_0^s |\nu_\varepsilon(\mathbf{w}_\varepsilon(t)) - \nu| dt \\ &\leq C \textstyle\int_0^s |\mathbf{w}_\varepsilon(t) - \mathbf{w}(t)| dt + C\varepsilon, \end{aligned}$$

according to (9.22). Therefore Gronwall's inequality implies

$$|\mathbf{w}_\varepsilon(s) - \mathbf{w}(s)| \leq C\varepsilon, \tag{9.23}$$

provided $0 \leq s < s_\varepsilon$ and $a_\varepsilon(\mathbf{w}_\varepsilon(t)) > 0$ for $0 \leq t \leq s$.

6. Next we estimate $a_\varepsilon(\mathbf{w}_\varepsilon(t))$ from below.

Recall that mollifier function η introduced in Section 8 satisfies $\eta > 0$ on $B(0,1/2)$: see (8.1). Thus for some constant $C > 0$,

$$\begin{aligned} a_\varepsilon(\mathbf{w}_\varepsilon(t)) &\geq C \fint_{B(\mathbf{w}_\varepsilon(t),\frac{\varepsilon}{2})} a\, dz \\ &\geq \frac{C}{\varepsilon} \int_{\mathbf{w}_{\varepsilon,n}(t)-\gamma\varepsilon}^{\mathbf{w}_{\varepsilon,n}(t)+\gamma\varepsilon} \Big(\fint_{B'(\mathbf{w}'_\varepsilon(t),\gamma\varepsilon)} a\, dz'\Big) dz_n \end{aligned} \tag{9.24}$$

for some constant $1 > \gamma > 0$, depending only on dimension. (Here and afterwards, remember our notational convention that a superscript $'$ means a point, ball, etc. in $\mathbb{R}^{n-1}$.)

Set $D = C + 1$, C the constant from (9.23). Then there exists a small constant $\delta > 0$ and finitely many points $\{w'_k\}_{k=1}^M \subset B'(0,D)$ such that if B' is any ball with center in $B'(0,D)$ and of radius γ, then

$$B' \supset B'(w'_k, \delta) \text{ for some index } k \in \{1,\dots,M\}. \tag{9.25}$$

For each such point w'_k, select a function ϕ^k satisfying the conditions

$$\begin{cases} \phi^k : \mathbb{R}^{n-1} \to \mathbb{R} \text{ is smooth, } \phi^k \geq 0, \\ \int_{\mathbb{R}^{n-1}} \phi^k dz' = 1, \ \operatorname{supp}(\phi^k) \subset B'(w'_k,\delta) \qquad (k = 1,\dots M). \end{cases} \tag{9.26}$$

Let $\phi^k_\varepsilon(z') = \frac{1}{\varepsilon^{n-1}} \phi^k(\frac{z'}{\varepsilon})$. Then (9.24), (9.25) imply for each t that

$$\begin{aligned} a_\varepsilon(\mathbf{w}_\varepsilon(t)) &\geq \frac{C}{\varepsilon} \int_{\mathbf{w}_{\varepsilon,n}(t)-\gamma\varepsilon}^{\mathbf{w}_{\varepsilon,n}(t)+\gamma\varepsilon} \fint_{B'(\varepsilon \mathbf{w}'_k,\varepsilon\delta)} a\, dz'\, dz_n \\ &\geq \frac{C}{\varepsilon} \int_{\mathbf{w}_{\varepsilon,n}(t)-\gamma\varepsilon}^{\mathbf{w}_{\varepsilon,n}(t)+\gamma\varepsilon} \int_{\mathbb{R}^{n-1}} a\ \phi^k_\varepsilon\, dz'\, dz_n \end{aligned} \tag{9.27}$$

for some index $k \in \{1,\dots,M\}$.

However for each k,

$$\begin{cases} a^k_\varepsilon(s) = \frac{1}{2\gamma\varepsilon} \int_{s-\gamma\varepsilon}^{s+\gamma\varepsilon} \int_{\mathbb{R}^{n-1}} a\ \phi^k_\varepsilon\, dz'\, dz_n \\ \to a(s) = a(se_n) \text{ as } \varepsilon \to 0, \quad \textit{uniformly} \text{ for } 0 \leq s \leq l - \sigma. \end{cases} \tag{9.28}$$

This follows from observation (6.21) after the proof of Proposition 6.1. Since $\min_{0 \leq s \leq l-\sigma} a(se_n) > 0$, we deduce from (9.27), (9.28) that for some constant $\theta > 0$

$$\mathbf{a}_\varepsilon(\mathbf{w}_\varepsilon(t)) \geq \theta > 0$$

provided $0 \leq t \leq s_\varepsilon$ and ε is small enough.

Consequently the estimate (9.23) forces $s_\varepsilon = l - \sigma$ if $\varepsilon > 0$ is small, and so Lemma 9.2 is proved. □

7. Lemma 9.3 *We have*

$$\phi_\varepsilon \to \phi \quad \textit{uniformly on } [0,1] \tag{9.29}$$

as $\varepsilon \to 0$.

Proof. We first note that for each $0 \le t \le 1$ we have

$$c_\varepsilon(s,t) \to c(s,t) \text{ for a.e. } 0 \le s \le l. \tag{9.30}$$

To prove this, observe as before that if $0 \le s \le l - \sigma$

$$\begin{aligned} |a_\varepsilon(\mathbf{w}_\varepsilon(s)) - a(\mathbf{w}(s))| &\le \int \eta_\varepsilon(z - \mathbf{w}_\varepsilon(s))|a(z) - a(\mathbf{w}(s))|dz \\ &\le C ⨍_{B(\mathbf{w}_\varepsilon(s),\varepsilon)} |a(z) - a(\mathbf{w}(s))|dz. \end{aligned}$$

Since Lemma 9.2 asserts $|\mathbf{w}_\varepsilon(s) - \mathbf{w}(s)| \le C\varepsilon$, we have

$$|a_\varepsilon(\mathbf{w}_\varepsilon(s)) - a(\mathbf{w}(s))| \le C ⨍_{B(\mathbf{w}(s),C\varepsilon)} |a(z) - a(\mathbf{w}(s))|dz.$$

The expression on the right hand side goes to zero for a.e. $0 \le s \le l - \sigma$, owing to (8.17). This is true for each $\sigma > 0$. Thus

$$a_\varepsilon(\mathbf{w}_\varepsilon(s)) \to a(\mathbf{w}(s)) \text{ for a.e. } 0 \le s \le l. \tag{9.31}$$

Since $f^\pm_\varepsilon \to f^\pm$ uniformly, we have established (9.30).

We now make use of (9.30) to prove (9.29). Our argument uses very strongly the fact (9.17) that a_ε, a are positive, bounded away from zero.

Now (9.12) says

$$\frac{\dot\phi_\varepsilon(t)}{c_\varepsilon(\phi_\varepsilon(t),t)} = 1 \qquad (0 \le t \le 1).$$

Consequently

$$\frac{d}{dt}\left(\int_0^{\phi_\varepsilon(t)} \frac{ds}{c_\varepsilon(s,t)}\right) = 1 + \int_0^{\phi_\varepsilon(t)} \frac{f^-_\varepsilon(\mathbf{w}_\varepsilon(s)) - f^+_\varepsilon(\mathbf{w}_\varepsilon(s))}{a_\varepsilon(\mathbf{w}_\varepsilon(s))}\, ds\ . \tag{9.32}$$

Since the functions $\{\phi_\varepsilon\}_{\varepsilon>0}$ are uniformly Lipschitz, we may extract a subsequence $\varepsilon_j \to 0$ such that

$$\phi_{\varepsilon_j} \to \psi \text{ uniformly on } [0,1], \tag{9.33}$$

for some Lipschitz function ψ.

Note $f_\varepsilon^\pm \to f^\pm$ uniformly, and $\mathbf{w}_\varepsilon(s) \to \mathbf{w}(s)$ uniformly on $[0, l-\sigma]$ for each $\sigma > 0$, according to (9.18). Furthermore for each fixed $0 \le t \le 1$

$$\frac{1}{c_\varepsilon(s,t)} \to \frac{1}{c(s,t)}\,, \qquad a_\varepsilon(\mathbf{w}_\varepsilon(s)) \to a(\mathbf{w}(s)) \text{ for a.e. } 0 \le s \le l,$$

according to (9.17), (9.30), (9.31). Hence for any $t > 0$ such that $\phi_\varepsilon(t) \le l - \sigma$ for all small $\varepsilon > 0$, we may pass to limits in (9.32) in the sense of distributions:

$$\frac{d}{dt}\left(\int_0^{\psi(t)} \frac{ds}{c(s,t)}\right) = 1 + \int_0^{\psi(t)} \frac{f^-(\mathbf{w}(s)) - f^+(\mathbf{w}(s))}{a(\mathbf{w}(s))} ds.$$

For a.e. t therefore

$$\frac{\dot{\psi}(t)}{c(\psi(t),t)} = 1.$$

Thus ψ solves the ODE (9.15). Since the transport density a restricted to R_{z_0} is locally Lipschitz (condition (8.19)), the solution is unique.

Thus $\psi(t) = \phi(t)$ for those t such that $\phi_\varepsilon(t) \le l - \sigma$ if ε is sufficiently small, $\sigma > 0$. Since $\phi(1) < l$, in fact $\psi = \phi$ on $[0,1]$. Thus the limit (9.29) is valid. □

8. Conclusion of the proof of Theorem 9.1. According to Lemmas 9.2, 9.3 above

$$\mathbf{s}_{\varepsilon,\delta}(z_0) = \mathbf{z}_\varepsilon(1) = \phi_\varepsilon(\mathbf{w}_\varepsilon(1)) \to \phi(\mathbf{w}(1)) = \mathbf{z}_\delta(1) = \mathbf{s}_\delta(z_0).$$

This at last verifies assertion (9.1) for a.e. z_0. Using the Dominated Convergence Theorem, we pass to limits in (8.12) as $\varepsilon \to 0$, and thereby derive (9.2). □

10 Optimality

It remains to pass to limits as $\delta \to 0$. As noted in (8.26) the limit

$$\lim_{\delta \to 0} \mathbf{s}_\delta(z_0) = \mathbf{s}(z_0) \tag{10.1}$$

exists for a.e. $z_0 \in B(0,R)$. We finally verify that the mapping $\mathbf{s}$, defined thusly for a.e. z_0, solves the Monge–Kantorovich mass transfer problem.

Theorem 10.1 (i) *The mapping* $\mathbf{s}$ *satisfies*

$$\int_X h(\mathbf{s}(x)) f^+(x)\, dx = \int_Y h(y) f^-(y)\, dy \tag{10.2}$$

for each function $h \in C(B(0,R))$.

(ii) *Furthermore,*

$$\int_X |x - \mathbf{s}(x)| f^+(x)\ dx = \min_{\mathbf{r} \in \mathcal{A}} \int_X |x - \mathbf{r}(x)| f^+(x)\ dx. \tag{10.3}$$

Here

$$\mathcal{A} = \{\mathbf{r} : X \to Y \mid \begin{array}{l} \mathbf{r} \ \ \textit{is measurable, and} \\ \int_X h(\mathbf{r}(x)) f^+(x)\ dx = \int_Y h(y) f^-(y)\ dy \\ \textit{for all} \ \ h \in C(B(0,R))\} \end{array}$$

Proof. 1. The identity (10.2) follows from (10.1) and (9.2) as $\delta \to 0$, since $\mathbf{s}_\delta \to \mathbf{s}$ a.e.

2. To verify (10.3) we recall from the construction that for a.e. $x_0 \in X$ the point $y_0 = \mathbf{s}(x_0)$ lies on the transport ray R_{x_0}. Thus

$$u(x) - u(\mathbf{s}(x)) = |x - \mathbf{s}(x)| \qquad (\text{a.e. } x \in X).$$

Consequently:

$$\begin{aligned} \int_X |x - \mathbf{s}(x)| f^+(x)\ dx &= \int_X [u(x) - u(\mathbf{s}(x))] f^+(x)\ dx \\ &= \int_X u(x) f^+(x) dx - \int_Y u(y) f^-(y)\ dy \text{ by (10.2)} \\ &= \int_{B(0,R)} u(f^+ - f^-)\ dz \\ &= \max\nolimits_{|Dw| \le 1} \int_{B(0,R)} wf\ dz \quad \text{(Theorem 2.1)} \\ &= \min\nolimits_{q \in \mathcal{M}} \int_{B(0,R)} \int_{B(0,R)} |x - y|\ dq \\ &\le \inf\nolimits_{\mathbf{r} \in \mathcal{A}} \int_X |x - \mathbf{r}(x)| f^+(x)\ dx, \end{aligned}$$

the last equality holding by Kantorovich's principle, as explained in §1. Since $\mathbf{s} \in \mathcal{A}$, the optimality condition (10.3) is proved. □

Remark. So in particular

$$\int_T a\ dz = \int_{B(0,R)} u(f^+ - f^-) dz = \min_{\mathbf{r} \in \mathcal{A}} \int_X |x - \mathbf{r}(x)| f^+(x) dx. \tag{10.4}$$

The integral of the transport density a equals the least cost for the mass transfer. □

11 Appendix: Approximating semiconvex and semiconcave functions by C^2 functions

Consider first a function $u : B(0,R) \to \mathbb{R}$ and suppose u is convex. Then we can write (cf. [E-G, §6.3])

$$[D^2 u] = \begin{pmatrix} \mu^{11} & \dots & \mu^{1n} \\ & \ddots & \\ \mu^{n1} & \dots & \mu^{nn} \end{pmatrix},$$

where μ^{ij} is a (signed) Radon measure satisfying

$$\int_{B(0,R)} u\, \phi_{x_i x_j}\, dx = \int_{B(0,R)} \phi\, d\mu^{ij} \qquad (1 \le i,j \le n)$$

for each $\phi \in C^2(B(0,R))$ with compact support. Using Lebesgue's Decomposition Theorem for measures we write

$$\mu^{ij} = \mu^{ij}_{ac} + \mu^{ij}_{s} \qquad (1 \le i,j \le n),$$

where μ^{ij}_{ac} is absolutely continuous and μ^{ij}_{s} is singular with respect to n-dimensional Lebesgue measure. Thus

$$d[D^2u] = d[D^2u]_s + [D^2u]_{ac}dx,$$

where

$$[D^2u]_{ac} = \begin{pmatrix} u_{11} & \cdots & u_{1n} \\ & \ddots & \\ u_{n1} & & u_{nn} \end{pmatrix}$$

and

$$d\mu^{ij}_{ac} = u_{ij}dx.$$

Alexandrov's Theorem (see, e.g., [E-G, §6.4]) asserts that u is twice differentiable for a.e. $x_0 \in B(0,R)$. More precisely for a.e. x,

$$u(y) - [u(x) + Du(x)\cdot(y-x) + \tfrac{1}{2}(y-x)^T[D^2u]_{ac}(x)(y-x)] = o(|y-x|^2) \text{ as } y \to x. \tag{11.1}$$

Proposition A.1 *For each $\varepsilon > 0$ there exists a C^2 function $\tilde{u} : B(0,R) \to \mathbb{R}$ such that*

$$\tilde{u} = u,\ D\tilde{u} = Du,\ D^2\tilde{u} = [D^2u]_{ac},$$

except for a measurable set of Lebesgue measure less than ε.

Proof. 1. For $k = 1, \dots$ set

$$E_k = \{x \in B(0,R) \mid \sup_{y \in B(x,r)} |u(y) - u(x) - Du(x)\cdot(y-x)| \le kr^2 \text{ for } 0 < r < \frac{1}{k} \text{ and } |u(x)|, |Du(x)| \le k\}.$$

Then $|B(0,R) - \cup_{k=1}^{\infty} E_k| = 0$, according to (11.1).

Thus there exists k_0 such that

$$|B(0,R) - E_{k_0}| \le \varepsilon, \tag{11.2}$$

and so there exists a compact set $A \subset E_{k_0}$ such that

$$|E_{k_0} - A| \le \varepsilon. \tag{11.3}$$

Then in the terminology of Ziemer [Z, §3.5.4] we see

$$\|u\|_{T^{\infty,2}(x)} \le M$$

for some constant M and each $x \in A$. Thus [Z, Theorem 3, 6.2] implies that there exists a $C^{1,1}$ function $\bar{u}$ defined on $B(0,R)$ such that

$$u = \bar{u} \text{ on } A.$$

Finally there exists a C^2 function $\tilde{u}$ which agrees with $\bar{u}$ except for a set of measure less than ε. This last statement follows from [Z, Theorem 3.10.5]. □

A function u is called *semiconvex* if $u + \frac{C}{2}|x|^2$ is convex for some constant C, and u is *semiconcave* if $-u + \frac{C}{2}|x|^2$ is convex for some C. Applying Proposition A.1 to $\pm u + \frac{C}{2}|x|^2$ we deduce the same result for semiconvex and semiconcave functions.

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