A variational approach for the 2-D semi-geostrophic shallow water equations

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Abstract

Existence of weak solutions to the 3-D semi-geostrophic equations with rigid boundaries was proved by Benamou and Brenier [3], using Monge transport theory. This paper extends the results to a free surface boundary condition, which is more physically appropriate. This extension is at present for the 2-D shallow water case only. In addition, we establish stronger time regularity than was possible in [3].

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Contents

Introduction **2**

1 Introduction **2**

2 Definitions and notation **13**

3 The geostrophic energy **15**

4 Existence of Stable Solutions in Dual Variables **24**

A Background on the Wasserstein distance **34**

1 Introduction

We study the so-called *semi-geostrophic shallow water system*, a variant of the *semi-geostrophic system*. This system is used by meteorologists to model how fronts arise in large scale weather patterns. The shallow water variant provides a simpler context in which to develop the theory for the 3-D system. It is also physically important in its own right in describing the dynamics of layers of shallow fluid which do not fill the available domain and so have a free boundary. See [4] for an example. The *semi-geostrophic system* is a 3-D free boundary problem which is an approximation of the 3-D Euler equations of incompressible fluid in a rotating coordinate frame around the Ox_3-axis where the effects of rotation dominate. It was first introduced by Eliassen [9] in 1948 and rediscovered by Hoskins [13] in 1975. Hoskins showed that the *semi-geostrophic system* could be solved in particular cases by a coordinate transformation which then allowed analytic solutions to be obtained. In particular, the mechanisms for the formation of fronts in the atmosphere could be modelled analytically. Cullen and Purser showed in [5], and [6] that the equations could be given a geometrical interpretation. This interpretation allowed the equations to be used to describe a variety of phenomena in the atmosphere, such as the way fronts interact with mountains. It also appeared, in principle, that the equations could be solved for large times, without recourse to viscosity or turbulence models. This means that closed, though simplified, solutions for atmospheric behavior could be obtained. These solutions would describe aspects of the atmosphere that were controlled by large-scale behavior, and therefore highly predictable. An example is shown in [7].

Benamou and Brenier [3] showed that the existence of weak solutions of the *semi-geostrophic system* could be proved, using a rigid-wall boundary condition, thus verifying the conjectures in [6]. In this paper, we use a more physically appropriate free boundary condition. However, we simplify the full 3-D system to give the *semi-geostrophic shallow water system* by assuming that the potential temperature is constant. This yields that the pressure assumes a special form so that the original 3-dimensional equations (7) can be reduced to the 2-dimensional system (14). We denote points in the plane by $x := (x_1, x_2)$ and for that reason we write elements of $\mathbf{R}^3$ as $\bar{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$. The unknowns in the original 3-dimensional equations are:

$$\begin{cases} \bar{\mathbf{v}}_g = (v_{g1}, v_{g2}, 0) = \text{geostrophic wind velocity} \\ \bar{\mathbf{v}}_a = (v_{a1}, v_{a2}, v_{a3}) = \text{ageostrophic wind velocity} \\ p = \text{pressure} \\ \theta = \text{potential temperature,} \end{cases}$$

defined on $[0, +\infty) \times O(t)$. The domain $O(t)$ is the region occupied by the fluid at time t. Since the height of the fluid is to be determined and depends on the time t, then $O(t)$ is a time-dependent region in $\mathbf{R}^3$. We set

$$\bar{\mathbf{v}} := \bar{\mathbf{v}}_g + \bar{\mathbf{v}}_a = \text{total wind velocity.}$$

We define the convective derivatives

$$\frac{\bar{D}}{Dt} := \frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + v_3 \frac{\partial}{\partial x_3}, \qquad \frac{D}{Dt} := \frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2},$$

and the two-dimensional and three-dimensional gradients

$$D := (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}), \qquad \bar{D} := (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}).$$

Following Hoskins [13], we comment on how to derive the semi-geostrophic equations from the well-known Boussinesq equations. The Boussinesq equations describe the evolution of an incompressible fluid in a reference configuration in rotation about the $0x_3$-axis.

$$\begin{cases} (i)\frac{D\mathbf{v}}{Dt} + f\mathbf{e}_3 \times \mathbf{v} = -Dp \\ (ii)\mathbf{v}_g = \mathbf{e}_3 \times Dp/f \\ (iii)\mathrm{div}(\bar{\mathbf{v}}) = 0, \quad \frac{\bar{D}\theta}{Dt} = 0 \\ (iv)\frac{\partial p}{\partial x_3} = g\theta/\theta_o \\ (v)\frac{\bar{D}\theta}{Dt} = 0. \end{cases} \tag{1}$$

Here, f is the constant given by the Coriolis force. We currently choose to omit the boundary condition going along with these equations. Using (1) (i) we deduce that the horizontal component of the velocity is given by

$$\mathbf{v} = \frac{\mathbf{e}_3}{f} \times Dp + \frac{\mathbf{e}_3}{f} \times \frac{D\mathbf{v}}{Dt}. \tag{2}$$

Recall that according to our notation the horizontal component of an arbitrary vector $\bar{\mathbf{a}} \in \mathbf{R}^3$ is denoted by $\mathbf{a}$. We apply $\frac{D}{Dt}$ to both sides of (2) to obtain

$$\frac{D\mathbf{v}}{Dt} = \frac{D\mathbf{v}_g}{Dt} + \frac{\mathbf{e}_3}{f} \times \frac{D^2\mathbf{v}}{Dt^2}. \tag{3}$$

We combine (2) and (3) to deduce that

$$\mathbf{v} = \mathbf{v}_g + \frac{\mathbf{e}_3}{f} \times \frac{D\mathbf{v}_g}{Dt} - \frac{1}{f}\frac{D^2\mathbf{v}}{Dt^2}. \tag{4}$$

Let τ be the time scale for change in the velocity following a fluid particle. As in Hoskins [13], we assume that τ is much larger than $1/f$, and so, $\epsilon := \frac{1}{f\tau} << 1$. Set $\bar{t} = t/\tau$. We rewrite (4) in the adimensional form in time:

$$\mathbf{v} = \mathbf{v}_g + \epsilon\, \mathbf{e}_3 \times \frac{D\mathbf{v}_g}{D\bar{t}} - \epsilon^2 \frac{D^2\mathbf{v}}{D\bar{t}^2}. \tag{5}$$

Because $\epsilon << 1$ the last term in (5) is neglected. This yields new equations for $\mathbf{v}$ given by

$$\mathbf{v} = \mathbf{v}_g + \epsilon\, \mathbf{e}_3 \times \frac{D\mathbf{v}_g}{D\bar{t}} = \mathbf{v}_g + \frac{\mathbf{e}_3}{f} \times \frac{D\mathbf{v}_g}{Dt}. \tag{6}$$

We replace (i) in (1) by (6) to obtain the semigeostrophic system. More precisely, the semi-geostrophic equations are:

$$\begin{cases} (i)\frac{D\mathbf{v}_g}{Dt} + f\mathbf{e}_3 \times \mathbf{v} = -Dp, \\ (ii)\mathbf{v}_g \cdot \mathbf{e}_3 \equiv 0 \\ (iii)\mathrm{div}(\bar{\mathbf{v}}) = 0 \\ (iv)\bar{D}p = -f\mathbf{e}_3 \times \mathbf{v}_g + g\theta/\theta_o\mathbf{e}_3 \\ (v)\frac{\bar{D}\theta}{Dt} = 0. \end{cases} \tag{7}$$

in $[0, +\infty) \times O(t)$, along with the boundary condition

$$\bar{\mathbf{v}} \cdot \nu := normal\ speed\ of\ the\ boundary,$$

on $[0, +\infty) \times \partial O(t)$. (See [3], [15] and the review paper by Evans [10] when the region $O(t) \equiv O(0)$.) A derivation of (7), accessible to mathematicians, can be found in [1].

Among all solutions of (7) we are interested in those that are stable. To define the concept of stability, let us introduce the functional

$$\bar{I}[\bar{\mathbf{X}}] := f^2 \int_{O(t)} \left(\frac{|x_1 - X_1(\bar{x})|^2}{2} + \frac{|x_2 - X_2(\bar{x})|^2}{2} - x_3 X_3(\bar{x}) \right) d\bar{x}, \tag{8}$$

and the map

$$\bar{\mathbf{X}}_t(\bar{x}) := x + \bar{D}p(t, \bar{x})/f^2 = (x_1, x_2, 0) + \bar{D}p(t, \bar{x})/f^2.$$

Recall that the push-forward of $\chi_{O(t)}\mathcal{H}^3$ by the map $\bar{\mathbf{X}}_t$ is the measure $\bar{\mathbf{X}}_{t\#}(\chi_{O(t)}\mathcal{H}^3)$ defined by

$$\bar{\mathbf{X}}_{t\#}(\chi_{O(t)}\mathcal{H}^3)[B] := \mathcal{H}^3[\bar{\mathbf{X}}_t^{-1}(B)],$$

for all $B \subset \mathbf{R}^3$ Borel sets. Let $\bar{\mathcal{X}}(t)$ be the set of all Borel maps $\bar{\mathbf{X}} : O(t) \to \mathbf{R}^3$ such that

$$\bar{\mathbf{X}}_{t\#}(\chi_{O(t)}\mathcal{H}^3) = \bar{\mathbf{X}}_{\#}(\chi_{O(t)}\mathcal{H}^3).$$

In [8] it was shown that geostrophic and hydrostatic states (i.e. states which satisfy (7)) correspond to critical points of the integral (8) over the set $\bar{\mathcal{X}}(t)$. Critical points which are not minima correspond to geostrophic and hydrostatic states which are unstable to small perturbations evolving under the 3D Euler equations. The subsequent evolution of these states cannot be described by the semi-geostrophic approximation. Therefore, we seek for solutions of (7) satisfying the minimization principle

$$\bar{I}[\bar{\mathbf{X}}_t] \leq \bar{I}[\bar{\mathbf{X}}]$$

for all $\bar{\mathbf{X}} \in \bar{\mathcal{X}}(t)$. Following the terminology of [3] we refer to this principle as the *Cullen-Purser stability condition*. Observe that the *Cullen-Purser stability condition* implies that $\bar{\mathbf{X}}_t$ is the optimal transport map in the Monge-Kantorovich mass transport problem that transports $\chi_{O(t)}$ onto $\bar{\mathbf{X}}_{t\#}\chi_{O(t)}$, where optimality is measured against the cost function $c(\bar{x}-\bar{y}) = |\bar{x}-\bar{y}|^2$. Let us recall briefly what is meant by the Monge-Kantorovich mass transport problem. For details and a complete reference on the topic, we refer the reader to the recent book by Rachev and Rüschendorf [18].

Consider the cost function $c : \mathbf{R}^d \to [0, +\infty)$, given by

$$c(z) = |z|^p/p, \qquad (z \in \mathbf{R}^d),$$

where $0 < p < +\infty$. Here, $c(x-y)$ represents the cost of moving a unit mass from a point x to a point y. Denote by $\mathcal{P}(\mathbf{R}^d)$ the set of all probability Borel measures on $\mathbf{R}^d$. Assume that ρ, and α are two Borel probability density functions on $\mathbf{R}^d$ that represents mass distributions on $\mathbf{R}^d$. Let $\Gamma(\rho, \alpha)$ be the set of all Borel measures on $\mathbf{R}^d \times \mathbf{R}^d$ having ρ and α as marginals:

$$\int_B \rho(x)dx = \gamma[B \times \mathbf{R}^d] \quad \text{and} \quad \gamma[\mathbf{R}^d \times B] = \int_B \alpha(y)dy$$

for each Borel set $B \subset \mathbf{R}^d$. The p-Monge-Kantorovich problem consists in finding the cheapest way for rearranging ρ onto α, where optimality is measured against the cost c. More precisely, the problem consists in finding $\gamma_o \in \Gamma(\rho, \alpha)$, the minimizer of

$$W_p^p(\rho, \alpha) := \inf_{\gamma \in \Gamma(\rho,\alpha)} \int_{\mathbf{R}^d \times \mathbf{R}^d} c(x-y)\; d\gamma(x,y). \tag{9}$$

It is well-known that (9) admits a unique minimizer γ_o. Furthermore, there exists a unique map $\mathbf{t}$ that is the gradient of a convex function P such that γ_o is the push forward of ρ by $\mathbf{id} \times \mathbf{t}$. Consequently, $\mathbf{t}$ minimizes

$$\mathbf{r} \to \int_{\mathbf{R}^d} c(x - \mathbf{r}(x))\rho(x)dx$$

other the set of all maps that pushes ρ forward to α. (See [2], and [12]). We refer to $W_p^p(\rho, \alpha)$ as the total work required to transport ρ onto α. It is well-known that for $p \geq 1$, W_p is a metric on $\mathcal{P}(\mathbf{R}^d)$, whereas for $0 \leq p \leq 1$, it is W_p^p which is a metric on $\mathcal{P}(\mathbf{R}^d)$.

Note that the *Cullen-Purser stability condition* gives that $\bar{\mathbf{X}}_t$ is the gradient of a convex function, i.e., $\bar{x} \to |x|^2/2 + p(t, \bar{x})/f^2$ is convex. The geostrophic energy is defined to be

$$\int_{O(t)} [\frac{|\mathbf{v}_g|^2}{2} - \frac{g\theta x_3}{\theta_o}] d\bar{x}$$

which is $\bar{I}[\bar{\mathbf{X}}_t]$.

In this paper we study the shallow water model, where the fluid is within a region Ω in the (x_1, x_2)–plane but the height ρ of the surface above the reference level is unknown and can vary

$$O(t) = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid (x_1, x_2) \in \Omega, \;\; 0 \leq x_3 \leq \rho(t, x_1, x_2)\}.$$

The rigid bottom is defined by the surface $x_3 = 0$. The rotation axis of the fluid coincides with the x_3–axis in the model and the condition at the top boundary of the fluid is:

$$p(t, x_1, x_2, \rho(x_1, x_2)) = p_o, \tag{10}$$

where p_o is a constant.

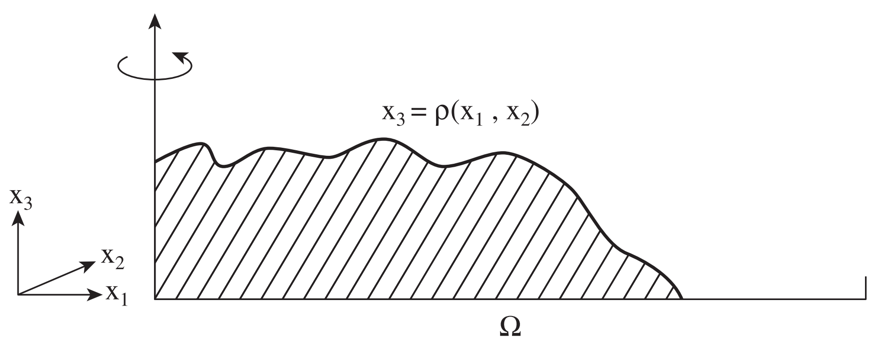

Figure 1: A typical cross-section of the solution of the shallow water model illustrating the notation in the text, and showing that the water need not fill the whole of Ω.

The semi-geostrophic equations are only a valid approximation to the Euler equations if the Rossby number U/fL is small (where U, L are velocity and length scales). In the shallow water model we additionally assume that θ is uniform and so, the following approximation can be made: the pressure is of the form

$$p(t, x_1, x_2, x_3) = gx_3 + A(t, x_1, x_2). \tag{11}$$

(See [17]) where g is the Newton's constant. Due to the orientation of the vertical axis, g is negative. In what follow, we set in the sequel that $g = -1$.

Combining (11) and (10) we deduce that

$$p = (\rho - x_3) + p_o. \tag{12}$$

Note that (12) implies that the horizontal pressure gradient is independent of x_3 so that the horizontal accelerations must be independent of x_3. It is therefore consistent to assume that

$$\text{the horizontal velocities remain } x_3 \text{ independent} \tag{13}$$

if they are so initially. In the case (12) and (13) hold, (7) becomes a 2-dimensional system called the *semi-geostrophic shallow water* equations. Given ρ^o find $\mathbf{v} := (\mathbf{v}^1, \mathbf{v}^2)$, and ρ defined on $[0, +\infty) \times \Omega$ such that

$$\begin{cases} (i)\ \frac{\partial}{\partial t}\mathbf{v}_g + D\mathbf{v}_g \cdot \mathbf{v} + fJ\mathbf{v} = -D\rho \qquad \text{in} \quad [0, +\infty) \times \Omega \\ \\ (ii)\ \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \qquad (iii)\ \mathbf{v}_g = \frac{J}{f} D\rho \qquad \text{in} \quad [0, +\infty) \times \Omega \\ \\ (vi)\ \mathbf{v} \cdot \mathbf{n} = 0 \qquad \text{in} \quad [0, +\infty) \times \partial\Omega, \quad \rho(0, \cdot) = \rho^o \quad \text{in} \quad \Omega, \end{cases} \tag{14}$$

where $\mathbf{n}$ denotes the outward unit normal to $\partial\Omega$. The third component of the velocity v_3 can be recovered by using that $\operatorname{div}(\bar{\mathbf{v}}) = 0$ and that the expression $\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}$ does not depend on x_3. We have

$$v_3 = -x_3\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}\right).$$

To obtain (14) (i) we have used (7) (i) and the relation between ρ and p given in (12). Now, (14) (ii) expresses the fact that on

$$\{(x_1, x_2, x_3) \mid x_3 = \rho(t, x_1, x_2)\},$$

which is the top of the fluid, the normal velocity $\mathbf{v} \cdot \nu = v_3$ coincides with the convective derivative of ρ.

The interpretation of the reduction of the 3–D system to the 2–D system is that in the shallow water model the 3–D fluid moves as a set of columns oriented parallel to the x_3–axis. During the stretching or contraction of each column the relative position of a fluid element in the column is unchanged. The position at time t of the fluid element that was initially at (x_1, x_2, x_3) is completely determined by x_3, the height of the fluid at (x_1, x_2) at time 0 and t.

Following Cullen & Purser ([5] and [6]) we introduce the generalized geopotential function, defined on $[0, +\infty) \times \Omega$ by

$$P(t, x) = |x|^2/2 + \rho(t, x)/f^2.$$

Define

$$\mathbf{X}_t(x) := x - \frac{J}{f}\mathbf{v}_g(\mathbf{t}, x) = DP(t, x),$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let $\mathcal{P}^{ac}(\Omega)$ be the set of all probability density functions defined on Ω. In $2D$, *Cullen-Purser stability condition* means that at each time t, the pair $(\rho(t, \cdot), \mathbf{X}_t)$ minimizes

$$I[\eta, \mathbf{X}] := 1/2 \int_\Omega (f^2 |x - \mathbf{X}(x)|^2 \eta(x) + \eta^2(x)) dx. \tag{15}$$

The minimization is performed over the set of all pairs $(\eta, \mathbf{X})$ satisfying $\eta \in \mathcal{P}^{ac}(\Omega)$, and $\mathbf{X}_\# \eta = \alpha(t, \cdot)$, where

$$\alpha(t, \cdot) := \mathbf{X}_{t\#} \rho(t, \cdot). \tag{16}$$

The $2D$ geostrophic energy up to a multiplicative constant is

$$\mathcal{E}[\rho] = \frac{1}{2} \int_\Omega (|D\rho|^2 \rho + f^2 \rho^2) dx. \tag{17}$$

In [13], $\rho(t, DP^*(t, y))/\alpha(t, y)$ is interpreted as the geostrophic vorticity. Observe that as in the $3D$ case, $\mathbf{X}_t$ is the optimal map of the Monge-Kantorovich problem which consists in rearranging $\rho(t, \cdot)$ onto $\alpha(t, \cdot)$, where optimality is measured against the cost function $c(x - y) = |x - y|^2$. Since $\mathbf{X}_t = DP(t, \cdot)$ we deduce that $P(t, \cdot)$ is convex. Let W_2 be the Wasserstein distance introduced in Appendix A, and let $\mathcal{P}^{ac}(\mathbf{R}^2)$ be the set of all Borel probability density functions on $\mathbf{R}^2$. Note that if $\bar{\alpha} \in \mathcal{P}^{ac}(\mathbf{R}^2)$ then by Lemmas A.4, and A.5, the map

$$\eta \to f^2 W_2^2(\bar{\alpha}, \eta) + 1/2 \int_\Omega \eta^2(x) dx$$

is strictly convex and lower semicontinuous on $\mathcal{P}^{ac}(\Omega)$. Hence, it admits a unique minimizer $\mathcal{L}[\bar{\alpha}]$ in $\mathcal{P}^{ac}(\Omega)$.

Definition 1.1 *We define $\mathcal{L}$ to be the functional that maps $\bar\alpha \in \mathcal{P}^{ac}(\mathbf{R}^2)$ to $\mathcal{L}[\bar\alpha]$, the unique minimizer over $\mathcal{P}^{ac}(\Omega)$ of*

$$\eta \to f^2 W_2^2(\bar\alpha, \eta) + 1/2 \int_\Omega \eta^2 dx.$$

Observe that

$$\mathcal{L}[\alpha(t,\cdot)] := \rho(t,\cdot). \tag{18}$$

As shown by (26) the velocity $\mathbf{v}$ can be expressed in terms of

$$DP, \quad \frac{\partial}{\partial t} DP, \quad D^2 P^*.$$

One of the difficulties encountered when trying to solve (14) is that $\mathbf{v}$ may not be smooth enough. In other words, either functions DP, $\frac{\partial}{\partial t} DP$, or $D^2 P^*$ may not be smooth enough. In fact, we could not even prove that $\mathbf{v}$ is locally integrable.

Let us fix a time interval $[0,T]$ on which we study (14). Following Hoskins [13], we substitute (14) by the system (25), which turns out to be easier to handle for reasons which will soon be apparent. Formally, (14) and (25) are equivalent in the following sense. Assume that $\mathbf{v}$ is smooth and write (14) using Lagrangian coordinates. More precisely, introduce the flow

$$\begin{cases} \frac{\partial Z}{\partial t}(t,x) = \mathbf{v}(t, Z(t,x)) & x \in \Omega, t \in [0,T] \\ Z(0,x) = x & x \in \Omega. \end{cases} \tag{19}$$

Combining (i) and (iii) in (14), we have that

$$\frac{\partial}{\partial t}[DP(t, Z(t,x))] = fJ[DP(t,Z(t,x)) - Z(t,x)]. \tag{20}$$

In view of (20), it is natural to introduce the velocity that produces the flow in (20). We use the change of variables $y = DP(t,x)$ to define the so-called *geostrophic velocity* in *dual variables*

$$\mathbf{w}(t,y) := \mathbf{v}_g(t, DP^*(t,y)) = fJ[y - DP^*(t,y)], \tag{21}$$

and the flow

$$M(t,y) := DP(t, Z(t, DP^*(0,y))).$$

Now, in these new variables, (20) reads

$$\frac{\partial}{\partial t} M(t,y) = \mathbf{w}(t, M(t,y)). \tag{22}$$

The density corresponding to ρ in the new variables is

$$\alpha := DP(t,\cdot)_{\#}\rho(t,\cdot). \tag{23}$$

In the light of (14) (ii) and the definition of Z we have that

$$Z(t,\cdot)_{\#}\rho(0,\cdot) = \rho(t,\cdot).$$

This, combined with (23) implies that

$$\mathbf{M}(t,\cdot)_{\#}\alpha(0,\cdot) = \alpha(t,\cdot). \tag{24}$$

Combining (21), (23) and (24), we deduce that the so-called *semigeostrophic system in dual variables* can be written as:

$$\begin{cases} (i) \quad \frac{\partial \alpha}{\partial t} + \mathrm{d}iv\,(\alpha \mathbf{w}) = 0 \qquad \text{in the weak sense in} \quad [0,T]\times \mathbf{R}^2 \\ (ii) \quad \mathbf{w}(t,y) := fJ(y - DP^*(t,y)), \qquad \text{in} \quad [0,T]\times \mathbf{R}^2 \\ (iii) \quad P(t,x) := |x|^2/2 + \rho(t,x)/f^2, \qquad \text{in} \quad [0,T]\times\Omega \\ (iv) \quad \alpha(t,\cdot) := DP(t,\cdot)_{\#}\rho, \quad t\in[0,T] \\ (v) \quad \alpha(0,\cdot) = \alpha^o \quad \text{in} \quad \mathbf{R}^2. \end{cases} \tag{25}$$

We have formally shown that (14) implies (25). We next comment on properties of the *geostrophic velocity* in *dual variables*. Because $P^*(t,\cdot)$ is convex, we have that $\mathbf{w}(t,\cdot)$ is locally of bounded variations. In fact, only the restriction of $\mathbf{w}(t,\cdot)$ to $spt(\alpha(t,\cdot))$ is relevant in our study. Assume that at time $t=0$ we have that

$$spt(\alpha(0,\cdot)) \subset B_S.$$

Here B_S is the open ball of center 0 and radius S. We show that $spt(\alpha(t,\cdot))$ is contained in a ball whose radius evolves in time with a speed less than or equal to Sf. By symmetry (25) (iv) reads off

$$DP^*(t,\cdot)_{\#}\alpha(t,\cdot) = \rho(t,\cdot).$$

Since $spt(\rho(t,\cdot)) \subset \bar{\Omega}$, we may then assume without loss of generality that $DP^*(t,\cdot)$ maps $\mathbf{R}^2$ into the convex hull of $\bar{\Omega}$. Therefore, the restriction of $\mathbf{w}(t,\cdot)$ to $spt(\alpha(t,\cdot))$ is of class L^∞. These properties of $\mathbf{w}$ are exploited in the present work.

It remains to formally show that (25) implies (14). Assume that ρ, P, α, and $\mathbf{w}$ satisfy (25). Set

$$\mathbf{v}(t,x) := D^2P^*(t,DP(t,x))\Big(\mathbf{v}_g + \frac{J}{f}\frac{\partial \mathbf{v}_g}{\partial t}\Big), \quad \text{and} \quad \mathbf{v}_g := \frac{J}{f}D\rho. \tag{26}$$

Straightforward calculations show that ρ, P, $\mathbf{v}$, and $\mathbf{v}_g$ satisfy (14).

The aim of this paper is to show that (25) has a *stable solution* (α, ρ). At the present time we do not know whether or not *stable solutions* (α, ρ) in (25) are unique. We hope to address this issue in the future. We hereafter summarize our main result:

Theorem 1.2 *Assume that* $T > 0$, *that* $1 < r < +\infty$, *and that* $\Omega \subset \mathbf{R}^2$ *is open and connected. Assume that* $\alpha^o \in L^r(\mathbf{R}^2)$, $\rho^o \in L^1(\Omega)$ *are two probability density functions. Assume that* $spt(\alpha^o), \bar{\Omega} \subset B_S$. *Here,* B_S *is the ball of center* 0, *and radius* S. *Set* $P^o(x) := |x|^2/2 + \rho^o(x)/f^2$, *and assume that* P^o *can be extended into a convex function defined on* $\mathbf{R}^2$. *Assume that* (α^o, ρ^o) *satisfies the compatibility condition* $\alpha^o = DP^o_{\#}\rho^o$. *Then,*

(i) the system (25) has a stable solution (α, ρ).

(ii)

$$W_1(\alpha(s_2, \cdot), \alpha(s_1, \cdot)) \leq C_T|s_1 - s_2|$$

for all $s_1, s_2 \in [0, T]$; *here,* $C_T := Sf(2 + fT)||\alpha^o||_{L^1(\mathbf{R}^2)}$.

(iii) If $\mathbf{w}$ *is the semigeostrophic velocity in (25), then*

$$||\mathbf{w}(t, \cdot) - fJy||_{L^\infty(spt(\alpha(t,\cdot))} \leq fS, \quad ||\mathbf{w}(t, \cdot) - fJy||_{BV(spt(\alpha(t,\cdot))} \leq fCSR(R+1),$$

for all $0 \leq t \leq T$; *here,* $R := S(1 + fT)$.

(iv) $\rho \in C([0, T]; W^{1,s}(\Omega))$ *for all* $1 \leq s < +\infty$, *and* $\rho \in L^\infty((0, T); W^{1,\infty}(\Omega))$.

Throughout all this study the stability lemma proved in Lemma 3.6 plays an important role. It asserts the following. If the sequence $\{\alpha_j\}_{j=1}^\infty \subset \mathcal{P}^{ac}(B)$ converges weakly to α as j tends to $+\infty$, then the sequence $\{\mathcal{L}[\alpha_j]\}_{j=1}^\infty$ converges uniformly to $\mathcal{L}[\alpha]$ in Ω as j tends to $+\infty$. A consequence of the stability lemma is that since $t \to \alpha(t, \cdot)$ is Lipschitz continuous with respect to the W_1-distance we have that $\rho(\cdot, x) := \mathcal{L}[\alpha](\cdot, x) \in C[0, T]$. This implies a time regularity of the pressure

$$p(t, x_1, x_2, x_3) = \rho(t, x) - x_3 + p_o.$$

This time regularity is stronger than the regularity proved by Benamou and Brenier [3] for the 3-D semi-geostrophic equations. Our solutions may require the height of the fluid ρ to vanish in part of the domain Ω, and can thus describe the free boundary case. This situation does not arise in the 3-D problem solved in [3] because of their choice of boundary conditions. Future work will study the 3-D problem with a free boundary.

In a work in progress, N. Georgy shows that the energy $\mathcal{E}[\rho(t,\cdot)]$ defined in (17) is conserved in time for the solutions constructed in the present work. His proof is similar to the one in [15].

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2 Definitions and notation

Definition 2.1 *Suppose $\mathbf{t} : \mathbf{R}^d \to \mathbf{R}^m$ is a Borel map.*

(i) If μ is a Borel probability measure on $\mathbf{R}^d$, then $\mathbf{t}_\#\mu$ is the Borel probability measure on $\mathbf{R}^m$ defined by

$$\mathbf{t}_\#\mu[B] := \mu[\mathbf{t}^{-1}(B)]$$

for all $B \subset \mathbf{R}^m$ Borel. We say that $\mathbf{t}_\#\mu$ is the push-forward of μ by $\mathbf{t}$, or $\mathbf{t}$ pushes μ forward to $\mathbf{t}_\#\mu$.

(ii) Assume that ρ is a Borel probability function on $\mathbf{R}^d$, and that

$$\int_{\mathbf{t}^{-1}(N)} \rho(x)dx = 0$$

whenever $N \subset \mathbf{R}^m$ satisfies $\mathcal{H}^2[N] = 0$. We define $\mathbf{t}_\#\rho$ to be the unique Borel probability function α on $\mathbf{R}^m$ such that $\int_{\mathbf{t}^{-1}[B]} \rho(x)dx = \int_B \alpha(y)dy$ for all $B \subset \mathbf{R}^m$. We say that $\mathbf{t}_\#\rho$ is the push-forward of ρ by $\mathbf{t}$. We sometimes write

$$\alpha(\mathbf{t}(x)) det D\mathbf{t}(x) = \rho(x) \qquad in\, \Omega$$

in the weak sense, where $\alpha := \mathbf{t}_\#\rho$.

Definition 2.2 *Assume that $T > 0$, that $\alpha \in L^1((0,T) \times \mathbf{R}^2)$, and that $\mathbf{k} \in L^1((0,T) \times \mathbf{R}^2)$. Assume that $\alpha^o \in L^1(\mathbf{R}^2)$. We say that*

$$\frac{\partial \alpha}{\partial t} + div\, (\mathbf{k}) = 0, \qquad \alpha(t,\cdot) = \alpha^o, \tag{27}$$

in the weak sense, in $[0,T) \times \mathbf{R}^2$*, if*

$$\int_{\mathbf{R}^2} \alpha^o \varphi dy + \int_0^T dt \int_{\mathbf{R}^2} \alpha \frac{\partial \varphi}{\partial t} + \mathbf{k} \cdot D\varphi dy = 0$$

for all $\varphi \in C_o^1([0,T) \times \mathbf{R}^2)$.

Definition 2.3 *We say that* (α, ρ) *is a stable solution of (25) if* (α, ρ) *satisfies (25) and the function* $P(t, \cdot)$ *in (25) (iii) can be extended into a convex function on* $conv(\Omega)$.

Notation

For the convenience of the reader we collect together some of the notation introduced throughout the text.

• $\mathcal{H}^d$ denotes d-dimensional Hausdorff measure on the Borel σ-algebra of sets.

• If $A \subset \mathbf{R}^d$, $\mathcal{P}_{ac}(A)$ is the set of all Borel probability density functions on A.

• If $\Omega \subset \mathbf{R}^2$ then $\overline{\Omega}$ denotes the closure, $\Omega^c := \mathbf{R}^2 \setminus \Omega$ the complement, and $conv(\Omega)$ the convex hull of Ω, meaning the smallest convex set containing Ω.

• If $A \subset \mathbf{R}^2$ we denote by χ_A the characteristic function of A.

• If $P : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ is not identically $+\infty$, the Legendre-Fenchel transform of P is the convex, lower semicontinuous function $P^* : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ defined by

$$P^*(y) := \sup_{x \in \mathbf{R}^2} \{x \cdot y - P(x)\}.$$

Hence P^{**} is the greatest lower semicontinuous convex function dominated by P. If P is a Lipschitz function we denote by $Lip(P)$ the smallest constant R such that $|P(x) - P(y)| \leq R|x - y|$ for all x, y.

• The set where P is finite is denoted by $dom(P) \subset \mathbf{R}^2$, and the set where P is differentiable is denoted by $dom(DP) \subset \mathbf{R}^2$.

• The subdifferential of a convex function $P : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ is the set $\partial P \subset \mathbf{R}^2 \times \mathbf{R}^2$ consisting of all (x, y) satisfying

$$P(\mathbf{z}) - P(x) \geq y \cdot (\mathbf{z} - x), \qquad (\forall \mathbf{z} \in \mathbf{R}^2).$$

If $(x, y) \in \partial P$ we may also write $y \in \partial P(x)$ and for $A \subset \mathbf{R}^2$ we define $\partial P(A)$ to be $\{y \in \mathbf{R}^2 \mid \exists x \in \mathbf{R}^2, (x, y) \in \partial P\}$. Recall $x \in \partial P^*(y)$ whenever $y \in \partial P(x)$, while the converse also holds true if P is convex lower semicontinuous.

• The support of a nonnegative function ρ defined on $\mathbf{R}^2$ is $\mathrm{spt}\,\rho$, the intersection of all closed set $K \subset \mathbf{R}^2$ such that $\int_{K^c} \rho(x)dx = 0$.

• We denote the identity map $\mathbf{id}(x) = x$ by $\mathbf{id}$.

• If ρ and α are two Borel probability density functions on $\mathbf{R}^2$, $\Gamma(\rho, \alpha)$ stands for the set of all Borel measures on $\mathbf{R}^2 \times \mathbf{R}^2$ having $\rho d\mathcal{H}^2$ and $\alpha d\mathcal{H}^2$ as their marginals: $\int_B \rho(x)dx = \gamma[B \times \mathbf{R}^2]$ and $\gamma[\mathbf{R}^2 \times B] = \int_B \alpha(y)dy$ for all Borel sets $B \subset \mathbf{R}^2$.

• J is the rotation matrix of angle $\pi/2$:

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

3 The geostrophic energy

Throughout this section we assume that B_S, and B are open balls in $\mathbf{R}^2$ centered at the origin. The radius of B_S is S. We assume that α is a probability density function of support $\bar{\Lambda}$ and that $\bar{\Omega} \subset B_S$, $\bar{\Lambda} \subset B$.

Consider $(\bar{\rho}(t, \cdot), \bar{\alpha}(t, \cdot))$ a solution of (25) at time t. Then Proposition 3.4 shows that at time t the geostrophic energy defined in (17) is the infimum of the functional

$$(\eta, \mathbf{X}) \to E[\eta, \mathbf{X}] := 1/2 \int_\Omega (f^2 |x - \mathbf{X}(x)|^2 \eta(x) + \eta^2(x))dx$$

over the set of all pairs $(\eta, \mathbf{X})$ such that $\eta \in \mathcal{P}^{ac}(\Omega)$ and $\mathbf{X}_\# \eta = \alpha(t, \cdot)$.

The aim of this section is to characterize minimizers of E. More precisely, assume that we are given a probability density function α whose support is bounded. We give a necessary and sufficient condition for $(\rho, \mathbf{X}_o)$ to minimize E over the set of all pairs $(\eta, \mathbf{X})$ such that $\eta \in \mathcal{P}^{ac}(\Omega)$ and $\mathbf{X}_\# \eta = \alpha$.

Remark 3.1 *We show in Lemma A.4 that* $\eta \to W_2^2(\alpha, \eta)$ *is weakly lower semicontinuous on* $\mathcal{P}^{ac}(\Omega) \subset L^1(\Omega)$, *thus* I *is weakly lower semicontinuous on* $\mathcal{P}^{ac}(\Omega) \subset L^1(\Omega)$ *as the sum of two weakly lower semicontinuous functionals. The functional* $\eta \to \int_\Omega \eta^2(x)dx$ *is strictly convex on* $\mathcal{P}^{ac}(\Omega)$ *and so, by Lemma A.5*

$$I : \eta \to f^2 W_2^2(\alpha, \eta) + \int_\Omega \eta^2/2(x)dx$$

is also strictly convex on $\mathcal{P}^{ac}(\Omega)$. *Consequently,* I *admits a unique minimizer over any subset of* $\mathcal{P}^{ac}(\Omega)$ *which is precompact for the weak* L^1*-topology.*

The following Lemma was proved by Otto in [16] when $\Lambda = \Omega$.

Lemma 3.2 *Assume that $\Omega \subset \Lambda$, $0 < \delta_1 < \delta < 1$ are two real numbers, and that α satisfies $\delta \leq \alpha \leq 1/\delta$ on Λ. Then I admits a unique minimizer ρ over the set $\{\eta \in \mathcal{P}^{ac}(\Omega) \ : \ \delta_1 \leq \eta \leq 1/\delta_1 \text{ on } \Omega\}$. Furthermore, $\mathcal{H}^2(E) = 0$ where $E := \{x \in \Omega \ : \ \rho(x) < \delta\}$.*

Proof: 1. Existence and uniqueness of a minimizer ρ of I over the set $\{\eta \in \mathcal{P}^{ac}(\Omega) \ : \ \delta_1 \leq \eta \leq 1/\delta_1 \text{ on } \Omega\}$ follows from Remark 3.1.
2. We next prove that $\mathcal{H}^2(E) = 0$. Assume on the contrary that $\mathcal{H}^2(E) > 0$. Proposition A.2 gives existence of a convex $\psi : \mathbf{R}^2 \to \mathbf{R}$ such that $\mathbf{t} := D\psi$ is the optimal map that pushes α forward to ρ, where optimality is measured against the cost function $c(z) = |z|^2/2$. We have

$$W_2^2(\alpha, \rho) = 1/2 \int_\Lambda |x - \mathbf{t}(x)|^2 \alpha(x) dx. \tag{28}$$

Note that $\mathcal{H}^2[E \cap \ \mathbf{t}^{-1}(E^c)] = 0$ implies

$$\delta \mathcal{H}^2[E] \leq \mu[E] = \mu[E \cap \ \mathbf{t}^{-1}(E)] \leq \mu[\mathbf{t}^{-1}(E)] = \nu[E] < \delta \mathcal{H}^2[E],$$

which yields a contradiction and so,

$$\mathcal{H}^2[E \cap \ \mathbf{t}^{-1}(E^c)] > 0. \tag{29}$$

3. We next introduce density functions

$$e_o := \chi_{E \cap \, \mathbf{t}^{-1}(E^c)} \cdot \alpha,$$

$$e_1 := \mathbf{t}_\# e_o = \chi_{E^c \cap \, \mathbf{t}(E)} \cdot \rho,$$

$$\rho_\epsilon := \rho + \epsilon(e_o - e_1).$$

Clearly, $\rho_\epsilon \in \mathcal{P}^{ac}(\Omega)$. Note that for $\epsilon \in (0, \delta)$, the function $\alpha - \epsilon e_o$ is nonnegative. We define on $\mathbf{R}^2 \times \mathbf{R}^2$ the measure γ_ϵ given

$$\gamma_\epsilon := (\mathbf{id} \times \mathbf{t})_\# (\alpha - \epsilon e_o) + \epsilon (\mathbf{id} \times \mathbf{id})_\# e_o.$$

Observe that γ_ϵ has α and ρ_ϵ as its marginals and so, using (28) we have

$$\begin{aligned} 2W_2^2(\alpha, \rho_\epsilon) &\leq \int_{\mathbf{R}^2 \times \mathbf{R}^2} |x - y|^2 d\gamma_\epsilon(x, y) \\ &= \int_\Lambda |x - \mathbf{t}(x)|^2 \alpha(x) dx - \epsilon \int_\Lambda |x - \mathbf{t}(x)|^2 e_o(x) dx \\ &= 2W_2^2(\alpha, \rho) - \epsilon \int_{E \cap \, \mathbf{t}^{-1}(E^c)} |x - \mathbf{t}(x)|^2 \alpha(x) dx. \end{aligned} \tag{30}$$

The convexity of $t \to t^2$ and the fact that both ρ and ρ_ϵ are probability density functions yield

$$1/2 \int_\Omega (\rho^2 - \rho_\epsilon^2) dx \geq \int_\Omega \rho_\epsilon (\rho - \rho_\epsilon) dx = \int_\Omega (\rho_\epsilon - \delta)(\rho - \rho_\epsilon) dx. \tag{31}$$

Simple computations show that

$$\int_\Omega (\rho_\epsilon - \delta)(\rho - \rho_\epsilon) dx = \epsilon \int_E (\delta - \rho) e_o dx + \epsilon \int_{E^c} (\rho - \delta) e_1 dx - \epsilon^2 \int_\Omega (e_o - e_1)^2 dx. \tag{32}$$

But by definition of the set E, the first and second terms in (32) are nonnegative and so,

$$\int_\Omega (\rho_\epsilon - \delta)(\rho - \rho_\epsilon) dx \geq -\epsilon^2 \int_\Omega (e_o - e_1)^2. \tag{33}$$

Next we combine (30), (31) and (33) to obtain that

$$I[\rho_\epsilon] - I[\rho] \leq -c\epsilon/2 \int_{E \cap \, \mathbf{t}^{-1}(E^c)} |x - \mathbf{t}(x)|^2 \alpha(x) dx + \epsilon^2 \int_\Omega (e_o - e_1)^2 dx. \tag{34}$$

In the light of (29), the factor of ϵ in (34) is positive and so,

$$I[\rho_\epsilon] < I[\rho], \tag{35}$$

for $\epsilon > 0$ small enough. Since $\delta_1 \leq \rho \leq 1/\delta_1$ (35) is at a variance with the fact that ρ minimizes I over the sets $\mathcal{P}^{ac}(\Omega) \cap \{\eta \ : \ \delta_1 \leq \eta \leq 1/\delta_1\}$. Consequently, $\mathcal{H}^2[E] = 0$. QED.

Lemma 3.3 (Euler-Lagrange equations $I'(\rho) = 0$ **)** *Assume that $\Omega \subset \Lambda$, and that for a positive number $\delta \in (0,1)$, we have that $\delta \leq \alpha \leq 1/\delta$ on Λ. Then I admits a minimizer ρ over $\mathcal{P}^{ac}(\Omega)$ (which in turn is unique since I is strictly convex). Furthermore the following hold:*

(i) $\rho \geq \delta$ on Ω.

(ii) If in addition it is assumed that Ω is connected, then

$$P(x) := |x|^2/2 + \rho(x)/f^2 \qquad (x \in \Omega)$$

has a convex extension to B_S (still denoted by P) and $dom(P) = \bar{B}_S$, $\partial P(B_S) \subset \bar{B}$, and $DP_\# \rho = \alpha$.

(iii) $||P||_{W^{1,\infty}(B_S)}$, $||\rho||_{W^{1,\infty}(\Omega)}$ and $||D\rho||_{BV(\Omega)}$ are bounded by a constant that depends only on S, B and f.

Proof: 1. By Lemma 3.2, I admits a unique minimizer ρ_n over $\mathcal{P}^{ac}(\Omega)_n := \{\eta \in \mathcal{P}^{ac}(\Omega) \ : \ 1/n \leq \eta \leq n\}$. That minimizer satisfies

$$\delta \leq \rho_n \quad \text{on } \Omega. \tag{36}$$

For n large enough the nonnegative function $\frac{\chi_\Omega}{\mathcal{H}^2(\Omega)}$ belongs to $\mathcal{P}^{ac}(\Omega)_n$ and so,

$$\int_\Omega \frac{\rho_n^2}{2} dx \leq I[\rho_n] \leq I[\frac{\chi_\Omega}{\mathcal{H}^2(\Omega)}] < \infty.$$

We deduce that $\{\rho_n\}_{n=1}^\infty$ is weakly precompact in $L^2(\Omega)$. So, there exist an increasing sequence $\{n_i\}_{i=1}^\infty \subset \mathbf{N}$ and a function $\rho \in L^2(\Omega)$ such that $\{\rho_{n_i}\}_{i=1}^\infty$ converges weakly to ρ in $L^2(\Omega)$ as i tends to $+\infty$. By (36) we have

$$\delta \leq \rho. \tag{37}$$

2. We claim that ρ minimizes I over $\mathcal{P}^{ac}(\Omega)$.

Proof : Let $\eta \in \mathcal{P}^{ac}(\Omega) \cap L^2(\Omega)$. Choose, $\eta_{n_i} \in \mathcal{P}^{ac}(\Omega)_{n_i}$, $(i = 1, 2, \cdots)$ such that $\{\eta_{n_i}\}_{i=1}^\infty$ converges to η strongly in $L^2(\Omega)$ as i tends to $+\infty$. We have

$$f^2 W_2^2(\alpha, \rho_{n_i}) + \int_\Omega \rho_{n_i}^2/2dx \leq f^2 W_2^2(\alpha, \eta_{n_i}) + \int_\Omega \eta_{n_i}^2/2(x)dx. \tag{38}$$

Lemma A.4 says that

$$\bar\eta \to W_2^2(\alpha, \bar\eta)$$

is weakly lower semicontinuous on $\mathcal{P}^{ac}(\Omega) \cap L^2(\Omega)$. Hence, I is weakly lower semicontinuous on $\mathcal{P}^{ac}(\Omega) \cap L^2(\Omega)$. So, letting n_i goes to ∞ in (38) we deduce that

$$I[\rho] \leq I[\eta].$$

Since $\eta \in \mathcal{P}^{ac}(\Omega) \cap L^2(\Omega)$ is arbitrary we conclude that ρ minimizes I over $\mathcal{P}^{ac}(\Omega)$.

3. We next write Euler-Lagrange equations for ρ. Let $\xi \in C_o^\infty(\Omega)$. Consider the one-parameter family $\{\Psi(\tau, \cdot)\}$ of diffeomorphisms given by

$$\begin{cases} \frac{\partial \Psi}{\partial \tau}(\tau, x) &= \xi(\Psi(\tau, x)) \\ \Psi(0, x) &= x. \end{cases} \tag{39}$$

Let ρ_τ be the push forward of ρ through $\Psi(\tau, \cdot)$ i.e.

$$\rho_\tau(\Psi(\tau, x)) det D\Psi(\tau, x) = \rho(x). \tag{40}$$

The smoothness of ξ and (39) yield that

$$\sup_{x\in\Omega} |\Psi(\tau, x) - x - \tau\xi(x)| = o(\tau), \tag{41}$$

and

$$\sup_{x\in\Omega} |detD\Psi(\tau, x) - 1 - \tau \text{d}iv\, \xi(x)| = o(\tau). \tag{42}$$

We use (40) and (42) to obtain that

$$\begin{aligned} \lim_{\tau\to 0^+} \int_\Omega \frac{\rho_\tau^2(x) - \rho^2(x)}{\tau} dx &= \lim_{\tau\to 0^+} \int_\Omega \rho^2(x)\Big(\frac{1}{detD\Psi(\tau, x)} - 1\Big)dx \\ &= -\int_\Omega \rho^2(x) \text{d}iv\, \xi(x) dx. \end{aligned} \tag{43}$$

Proposition A.2 guarantees existence of an optimal map that pushes ρ forward to α, where optimality is measured against the cost function $c(z) = |z|^2$. This map is the gradient of a convex function P. By symmetry DP^* pushes α forward to ρ, and

$$W_2^2(\alpha, \rho) = 1/2 \int_\Lambda |y - DP^*(y)|^2 \alpha(y) dy. \tag{44}$$

Also, we may assume that

$$dom(P) = \bar{B}_S, \;\; dom(P^*) = \mathbf{R}^2, \quad \partial P(B_S) \subset \bar{B}, \;\; \partial P^*(\mathbf{R}^2) \subset B_S. \tag{45}$$

We can readily check that $y \to \Psi(\tau, DP^*(y))$ pushes α forward to ρ_τ and deduce that

$$W_2^2(\alpha, \rho(\tau, \cdot)) \le 1/2 \int_\Lambda |y - \Psi(\tau, DP^*(y))|^2 \alpha(y) dy. \tag{46}$$

We combine (41), (44) and (46) to deduce that

$$\begin{aligned} &\limsup_{\tau\to 0^+} \frac{W_2^2(\alpha, \rho_\tau) - W_2^2(\alpha, \rho)}{\tau} \\ \le\ & \limsup_{\tau\to 0^+} \int_\Lambda (y - DP^*(y)) \cdot \frac{DP^*(y) - \Psi(\tau, DP^*(y))}{\tau} \alpha(y) dy \\ =\ & \int_\Lambda (DP^*(y) - y) \cdot \xi(DP^*(y))) \alpha(y) dy. \end{aligned} \tag{47}$$

Since ρ minimizes I over $\mathcal{P}^{ac}(\Omega)$, (43) and (47) imply

$$\begin{aligned} 0 &\le \lim_{\tau\to 0^+} \frac{I[\rho_\tau] - I[\rho]}{\tau} \\ &\le \int_\Lambda f^2 (DP^*(y) - y) \cdot \xi(DP^*(y)) \alpha(y) dy - 1/2 \int_\Omega \rho_\infty^2 \text{d}iv\, \xi dx. \end{aligned} \tag{48}$$

Because ξ is arbitrary in (48) we deduce that in fact

$$f^2 \int_\Lambda (DP^*(y) - y) \cdot \xi(DP^*(y))\alpha(y)dy - 1/2 \int_\Omega \rho^2 \mathrm{d}iv\, \xi dx = 0. \qquad (49)$$

We exploit in (49) the fact that DP pushes ρ forward to α to obtain that

$$f^2 \int_\Omega (x - DP(x)) \cdot \xi(x))\rho(x)dx - 1/2 \int_\Omega \rho^2(x)\mathrm{d}iv\, \xi(x)dx = 0, \qquad (50)$$

for all $\xi \in C_o^\infty(\Omega)$. By (45), the subgradient of P being contained in $\bar{B}$ we conclude that $P \in W^{1,\infty}(\Omega)$ with $Lip(P) \leq diam(B)/2$. This, together with (50) and the fact that $\rho \in L^2(\Omega)$ implies that

$$1/2D(\rho^2) = f^2(DP - \mathbf{id})\rho. \qquad (51)$$

Because (37) gives that $\rho > 0$, we can divide both sides of (51) by ρ to obtain that

$$D\rho = f^2(DP - \mathbf{id}). \qquad (52)$$

So, $\rho \in W^{1,\infty}(\Omega)$. Since Ω is connected and open, (52) yields that

$$\rho(x) = f^2(P(x) - |x|^2/2)$$

up to an additive constant which we set to be 0 since P is determined up to an additive constant. Also,

$$\int_\Omega |P|dx = 1/2 \int_\Omega |x|^2 + 1/f^2 \leq 1/2 \int_{B_S} |x|^2 + 1/f^2$$

is bounded by a constant that depends only on S and f. So, $||\rho||_{W^{1,\infty}(\Omega)}$, and $||P||_{W^{1,\infty}(B_S)}$ are bounded by a constant that depends only on S, B and f. Since in addition P is convex, and $\Omega \subset\subset B_S$, we use $||P||_{W^{1,\infty}(B_S)}$ to control $||DP||_{BV(\Omega)}$. (See [11]). We find that $||DP||_{BV(\Omega)}$ is bounded by a constant that depends only on S, B and f.

Proposition 3.4 (Characterisation of minimizers) *Suppose that Ω is connected, that $\alpha \in \mathcal{P}^{ac}(B)$, and that $\rho \in \mathcal{P}^{ac}(\Omega)$. Then the following are equivalent:*

(i) ρ is a minimizer of I over $\mathcal{P}^{ac}(\Omega)$.

(ii) The function $P : x \to |x|^2/2 + \rho(x)/f^2$ can be extended to $\bar{B}_S$ into a convex function such that $DP_\# \rho = \alpha$, and $P \equiv +\infty$ on the complement of $\bar{B}_S$. Consequently, $\partial P^*(\mathbf{R}^2) \subset \bar{B}_S$.

In either case $||\rho||_{W^{1,\infty}(\Omega)}$, $||DP||_{BV(\Omega)}$, and $||P||_{W^{1,\infty}(B_S)}$ are bounded by a constant that depends only on S, B and f.

Proof: 1. We first show that (i) implies (ii). Assume that ρ is a minimizer of I over $\mathcal{P}^{ac}(\Omega)$. Choose, $\{\alpha_n\}_{n=1}^{\infty} \subset \mathcal{P}^{ac}(B)$ such that

$$1/n \leq \alpha_n \leq n \qquad \text{on } B, \tag{53}$$

for n large enough, and

$$||\alpha_n - \alpha||_{L^1(B)} \to 0 \quad \text{as} \quad n \to +\infty. \tag{54}$$

By Lemma 3.3 the unique minimizer ρ_n of

$$I_n : \eta \to f^2 W_2^2(\alpha_n, \eta) + \int_\Omega \frac{\eta^2}{2} dx$$

is such that $P_n : x \to |x|^2/2 + \rho_n(x)/f^2$ can be extended into a convex function defined on $\bar{B}_S$. This function satisfies

$$DP_{n\#}\rho_n = \alpha_n, \tag{55}$$

and

$$||DP_n||_{BV(\Omega)},\ ||P_n||_{W^{1,\infty}(B_S)} \leq c_1, \tag{56}$$

where c_1 is a constant that depends only on S, B and f. By (56) there exists a sequence $\{n_i\}_{i=1}^{\infty} \subset \mathbf{N}$ such that as i tends to $+\infty$, $\{\rho_{n_i}\}_{i=1}^{\infty}$ converges uniformly in Ω to some $\bar{\rho}$, $\{P_{n_i}\}_{i=1}^{\infty}$ converges uniformly in $\bar{B}_S$ to some P. Furthermore,

$$P(x) = |x|^2/2 + \bar{\rho}(x)/f^2, \tag{57}$$

in Ω, and

$$||DP||_{BV(\Omega)},\ ||P||_{W^{1,\infty}(B_S)} \leq c_1. \tag{58}$$

We have that

$$I_n[\rho_n] \leq I_n[\eta]$$

for all $\eta \in \mathcal{P}^{ac}(\Omega)$. This, together with (54) and the fact that by Lemma A.4 W_2 is weakly continuous on $\mathcal{P}^{ac}(\Omega) \times \mathcal{P}^{ac}(B)$ shows that $\bar{\rho}$ minimizes I over $\mathcal{P}^{ac}(\Omega)$. Remark 3.1 asserts uniqueness of minimizer of I over $\mathcal{P}^{ac}(\Omega)$ and so, $\bar{\rho} = \rho$. Combining the fact that $\bar{\rho} = \rho$ with (57) we conclude that

$$P(x) = |x|^2/2 + \rho(x)/f^2$$

in Ω. We can now use (55) and Lemma A.3 to conclude that $DP_{\#}\rho = \alpha$. Note that we may extend P to the complement of $\bar{B}_S$ by setting

$$P(x) = +\infty,$$

for $x \notin \bar{B}_S$. This proves (ii). In the light of (58) we have that $||\rho||_{W^{1,\infty}(\Omega)}$, and $||DP||_{BV(\Omega)}$ are bounded by a constant that depends only on S, B and f.

2. Conversely, assume that $P : x \to |x|^2/2 + \rho(x)/f^2$ can be extended from Ω to $\bar{B}_S$ into a convex function such that $DP_{\#}\rho = \alpha$. Let $\eta \in \mathcal{P}^{ac}(\Omega)$ and let $\mathbf{r} : \Lambda \to \Omega$ be a Borel map that pushes α forward to η. The convexity of P implies

$$\int_\Lambda [P(\mathbf{r}(y)) - P(DP^*(y))]\alpha(y)dy \geq \int_\Lambda y \cdot (\mathbf{r}(y) - DP^*(y))\alpha(y)dy.$$

We use the fact that, by symmetry, DP^* is the unique optimal map that pushes α forward to ρ, where optimality is measured against the cost function $c(z) = |z|^2$. We rewrite the above inequality as

$$\begin{aligned} 2\int_\Omega P(x)\eta(x)dx - 2\int_\Omega P(x)\rho(x)dx \;\; &\geq \;\; \int_\Lambda |y - DP^*(y)|^2\alpha(y)dy \\ &- \;\; \int_\Lambda |y - \mathbf{r}(y)|^2\alpha(y)dy \\ &+ \;\; \int_\Lambda (|\mathbf{r}(y)|^2 - |DP^*(y)|^2)\alpha(y)dy. \end{aligned} \tag{59}$$

Observe that (59) is equivalent to

$$\begin{aligned} f^2/2\int_\Lambda |y - \mathbf{r}(y)|^2\alpha(y)dy \;\; &\geq \;\; f^2/2\int_\Lambda |y - DP^*(y)|^2\alpha(y)dy \\ &+ \;\; \int_\Omega \rho(x)(\rho(x) - \eta(x))dx, \end{aligned}$$

which can be read as

$$E[\eta, \mathbf{r}] \geq E[\rho, DP^*] + 1/2\int_\Omega (\rho(x) - \eta(x))^2 dx. \tag{60}$$

By (60), we have

$$E[\eta, \mathbf{r}] \geq E[\rho, DP^*], \tag{61}$$

and equality holds if and only if $\mathbf{r} = DP^*$ and $\eta = \rho$. Note that

$$I[\eta] = \inf_{\mathbf{r}} E[\eta, \mathbf{r}],$$

where the infimum is performed over the set of all Borel maps $\mathbf{r} : \Lambda \to \Omega$ that push α forward to η. So, using (61) we conclude that $I[\eta] \geq I[\rho]$. QED.

Remark 3.5 *Let $\mathcal{L}$ be the functional introduced in Definition 1.1. Assume that $\alpha \in \mathcal{P}^{ac}(\mathbf{R}^2)$ has its support contained in B, and let $\rho := \mathcal{L}[\alpha]$. Then, by Proposition 3.4, $\rho \in W^{1,\infty}(\Omega)$, and $||\rho||_{W^{1,\infty}(\Omega)}$, $||P||_{W^{1,\infty}(B_S)}$ are bounded by a constant with depends only on S, B, and f. Furthermore, we may assume that $Lip(P^*) \leq S$. Here, $x \to |x|^2/2 + \rho(x)/f^2$ has a convex extension on $\bar{B}_S$ we denote by P.*

Lemma 3.6 (Stability: strong compactness of $\alpha \to \rho$) *Suppose that $\{\alpha_j\}_{j=1}^{\infty} \subset L^1(B) \cap \mathcal{P}^{ac}(B)$ converges weakly in $L^1(B)$ to α as j tends to $+\infty$. Then, $\{\mathcal{L}[\alpha_j]\}_{j=1}^{\infty}$ converges to $\mathcal{L}[\alpha]$ in $W^{1,s}(\Omega)$, for all $1 \leq s < +\infty$. Furthermore, $\{\mathcal{L}[\alpha_j]\}_{j=1}^{\infty}$ converges uniformly to $\mathcal{L}[\alpha]$ on Ω.*

Proof: Set

$$\rho_j := \mathcal{L}[\alpha_j], \quad P_j(x) = |x|/2 + \rho_j(x)/f^2, \quad \rho := \mathcal{L}[\alpha].$$

By Proposition 3.4, P_j can be extended on $\bar{B}_S$ into a convex function we still denote P_j, such that

$$||P_j||_{W^{1,\infty}(B_S)}, \ \ ||DP_j||_{BV(\Omega)}, \ \text{and} \ ||\rho_j||_{W^{1,\infty}(\Omega)} \leq c_1,$$

where, c_1 is a constant that depends only on S, B, and f. Furthermore,

$$DP_{j\#}\rho_j = \alpha_j. \tag{62}$$

Let $\{P_{j_k}\}_{k=1}^{\infty}$ be an arbitrary subsequence of $\{P_j\}_{j=1}^{\infty}$. We make use of the fact that $\{P_{j_k}\}_{k=1}^{\infty}$ a subsequence of convex functions that is bounded in $W^{1,\infty}(B_S)$. Hence, we may extract from $\{P_{j_k}\}_{k=1}^{\infty}$ a subsequence $\{P_{j_{k_l}}\}_{l=1}^{\infty}$ which converges strongly in $W^{1,s}(B_S)$ to a convex function P, as l tends to $+\infty$. We may assume that $\{P_{j_{k_l}}\}_{l=1}^{\infty}$ converges uniformly to P on B_S, as l tends to $+\infty$. Using (62) and Lemma A.3 we conclude that

$$DP_{\#}\rho = \alpha, \tag{63}$$

where

$$\rho(x) = f^2(P(x) - |x|^2/2). \tag{64}$$

In the light of Proposition 3.4, (63) and (64) we deduce that $\rho = \mathcal{L}[\alpha]$ in Ω. Let us summarize a byproduct of what we have proved. If $\{\rho_{j_k}\}_{k=1}^{\infty}$ is any subsequence of $\{\rho_j\}_{j=1}^{\infty}$, then we can extract a subsequence $\{\rho_{j_k}\}_{k=1}^{\infty}$ which converges to $\mathcal{L}[\alpha]$ in $W^{1,s}(\Omega)$ and in $C(\bar{\Omega})$. The limit $\mathcal{L}[\alpha]$ being independent of the subsequence $\{\rho_{j_k}\}_{k=1}^{\infty}$, we conclude the proof of Lemma 3.6. QED.

4 Existence of Stable Solutions in Dual Variables

Throughout this section we assume that $\Omega \subset \mathbf{R}^2$ is an open, bounded, connected set, and $\bar{\Omega} \subset B_S$. Here, B_S is the open ball of center 0, and radius S. We assume that $\rho^o \in \mathcal{P}^{ac}(\Omega)$, and that

$$x \to P^o(x) := |x|^2/2 + \rho^o(x)/f^2$$

can be extended into a convex function on B_S. We assume that $\alpha^o \geq 0$ is a probability density function, $spt(\alpha^o) \subset B_S$, and that the compatibility condition

$$DP^o_{\#}\rho^o = \alpha^o$$

is satisfied. Here, $f > 0$ is a constant occuring as the Coriolis force.

The aim of this section is to prove that (25) admits a stable solution (α, ρ) for all times t in $(0, +\infty)$. Let $T > 0$ be an arbitrary integer and denote by B the ball of center 0 and radius

$$R := S(2 + fT).$$

The first part of this section consists in discretizing (25) in time to construct approximate solutions. Fix a time step size $h > 0$ such that $n_h := T/h$ is an integer. Assume that the density functions α^o_h, and ρ^o_h are given. For k integer, we shall inductively determine α^k_h, ρ^k_h, approximate solutions of (25) in the time interval $[kh, (k+1)h)$. While discretizing (25) we need to smooth out functions. Let us introduce the standard mollifiers $j_h : \mathbf{R}^2 \to [0, +\infty)$ defined by

$$j_h(y) := \frac{1}{h^2} j(\frac{y}{h}).$$

Here $j : \mathbf{R}^2 \to [0, +\infty)$ is of class C^∞, is symmetric, has its support equal to the closed ball of center 0 and radius 1, and $\int_{\mathbf{R}^2} j(y)dy = 1$. Set

$$\alpha^o_h := j_h * \alpha^o.$$

Given α^k_h, we use the functional $\mathcal{L}$ of Definition 1.1 to define on Ω the functions

$$\begin{cases} \rho^k_h & := & \mathcal{L}[\alpha^k_h] \\ P^k_h(x) & := & |x|^2/2 + \rho^k_h(x)/f^2. \end{cases} \tag{65}$$

We define on $\mathbf{R}^2$ the functions

$$\begin{cases} Q_h^k & := & j_h * (P_h^k)^* \\ \mathbf{w}_h^k(y) & := & fJ(y - D(P_h^k)^*(y)) \\ \mathbf{u}_h^k(y) & := & fJ(y - DQ_h^k(y)) \end{cases} \tag{66}$$

By Remark 3.5 we may assume that

$$Lip(P_h^k)^* \leq S, \tag{67}$$

and so,

$$||DQ_h^k||_{L^\infty(\mathbf{R}^2)} \leq S. \tag{68}$$

Step 1. Use $\mathbf{u}_h^k$ and the transport equation to determine α_h^{k+1} by solving

$$\begin{cases} \frac{\partial \alpha_h}{\partial t} + \mathrm{d}iv\, [\alpha_h \mathbf{u}_h^k] = 0 & \text{in} & [kh, (k+1)h] \times \mathbf{R}^2 \\ \alpha_h(kh, y) = \alpha_h^k(y) & \text{in} & \mathbf{R}^2, \end{cases} \tag{69}$$

in the weak sense. We define

$$\alpha_h^{k+1}(y) := \alpha_h((k+1)h, y).$$

Lemma 4.1 shows that

$$spt(\alpha_h^{k+1}) \subset B_{S_{k+1}}.$$

Step 2. Inductively, we use α_h^{k+1}, (65), and (66) to define ρ_h^{k+1}, P_h^{k+1}, Q_h^{k+1}, $\mathbf{w}_h^{k+1}$, and $\mathbf{u}_h^{k+1}$.

Step 3. We introduce the following functions that depend on the time and space variables. Note first of all that α_h is well-defined over $[0, T] \times \mathbf{R}^2$. As in (65) we define

$$\begin{cases} \rho_h(t, \cdot) & := & \mathcal{L}[\alpha_h(t, \cdot)] \\ P_h(t, x) & := & |x|^2/2 + \rho_h(t, x)/f^2, \end{cases} \tag{70}$$

for all $t \in [0, T]$ and all $x \in \Omega$. We define

$$\begin{cases} Q_h(t, \cdot) & := & j_h * (P_h)^*(t, \cdot) \\ \mathbf{w}_h(t, y) & := & fJ(y - D(P_h)^*(t, y)) \\ \mathbf{u}_h(t, y) & := & fJ(y - DQ_h(t, y)), \end{cases} \tag{71}$$

for all $t \in [0, T]$ and all $y \in \mathbf{R}$. Similarly, we define the following functions which are stepwise constant in the time variable.

$$\begin{cases} \bar{\rho}_h(t,x) := \rho_h^k(x) = \rho_h(kh,x) \\ \bar{P}_h(t,x) := P_h^k(x) = P_h(kh,x), \end{cases} \tag{72}$$

for $t \in [kh,(k+1)h)$ and $x \in \Omega$. We define

$$\begin{cases} \bar{\alpha}_h(t,y) := \alpha_h^k(y) = \alpha_h(kh,y) \\ \bar{\mathbf{w}}_h(t,y) := \mathbf{w}_h^k(y) = \mathbf{w}_h(kh,y) \\ \bar{\mathbf{u}}_h(t,y) := \mathbf{u}_h^k(y) = \mathbf{u}_h(kh,y), \end{cases} \tag{73}$$

for $t \in [kh,(k+1)h)$ and $y \in \mathbf{R}^2$. We next start listing useful properties of the functions defined above.

Lemma 4.1 *Assume that $r \in [1,+\infty]$, and that $||\alpha_h^o||_{L^r(\mathbf{R}^2)} < +\infty$. Assume that $spt(\alpha_h^k) \subset B_{S_k}$, where $S_k := S(1+fhk)$, and that $||\alpha_h^k||_{L^r(\mathbf{R}^2)} = ||\alpha_h^o||_{L^r(\mathbf{R}^2)}$. Let α_h be the unique solution of (69) over $[kh,(k+1)h] \times \mathbf{R}^2$, and suppose that $kh < T$. Then we have that*

$$||\alpha_h(t,\cdot)||_{L^r(\mathbf{R}^2)} = ||\alpha_h^o||_{L^r(\mathbf{R}^2)}, \quad (t \in [kh,(k+1)h]). \tag{74}$$

$$spt(\alpha_h(t,\cdot)) \subset B_{S_{k+1}}, \quad (t \in [kh,(k+1)h]). \tag{75}$$

$$W_1(\alpha_h(s_2,\cdot),\alpha_h(s_1,\cdot)) \leq C_T|s_2 - s_1|, \quad (s_1,s_2 \in [kh,(k+1)h]), \tag{76}$$

where $C_T := S\,f(2+fT)||\alpha^o||_{L^1(\mathbf{R}^2)}$.

Proof: We introduce the characteristic system associated with (69) by defining the flow N satisfying

$$\begin{cases} \frac{\partial N}{\partial t}(t,y) = \mathbf{u}_h^k(N(t,y)) & y \in \mathbf{R}^2 \\ N(kh,y) = y & y \in \mathbf{R}^2. \end{cases} \tag{77}$$

Since $\mathbf{u}_h^k$ is divergence free and belongs to $C^\infty(\mathbf{R}^2)$, the classical theory of ODEs tells us that $N \in C^\infty([kh,(k+1)h] \times \mathbf{R}^2)$, $N(t,\cdot)$ is a diffeomorphism and

$$detDN(t,\cdot) \equiv 1, \tag{78}$$

for $t \in [kh,(k+1)h]$. It is well-known that the unique solution α_h of (69) over $[kh,(k+1)h] \times \mathbf{R}^2$ is given by

$$\alpha_h(t,N(t,y)) \equiv \alpha_h^k(kh,y), \tag{79}$$

for $(t, y) \in [kh, (k+1)h] \times \mathbf{R}^2$. Using the definition of $\mathbf{u}_h^k$, (68), (77), we have

$$\frac{\partial |N|}{\partial t} = \frac{\partial N}{\partial t} \cdot \frac{N}{|N|} = -fJDQ_h^k(N) \cdot \frac{N}{|N|} \leq Sf,$$

for all $t \in [kh, (k+1)h]$. So,

$$|N(t, y)| \leq |y| + fS(t - kh), \tag{80}$$

for all $t \in [kh, (k+1)h]$. Using that $spt(\alpha_h^k) \subset B_{S_k}$, (79) and (80) we conclude that

$$spt(\alpha_h(t, \cdot)) \subset B_{S_{k+1}}.$$

for all $t \in [kh, (k+1)h]$. This proves (75). We obtain (74) as a direct consequence of (78), (79), and the fact that $||\alpha_h^k||_{L^r(\mathbf{R}^2)} = ||\alpha_h^o||_{L^r(\mathbf{R}^2)}$.

Now, assume that $kh \leq s_1 < s_2 \leq (k+1)h$. From (68), and (75) we have

$$\alpha_h(t, y)|\mathbf{u}_h^k(y)| \leq S\, f(2 + fT)|\alpha_h(t, y)|, \tag{81}$$

for all $t \in [kh, (k+1)h]$. Let $\phi \in C_o^\infty(\mathbf{R}^2)$. We have

$$\int_{\mathbf{R}^2} \phi(y)(\alpha_h(s_2, y) - \alpha_h(s_1, y))dy = \int_{\mathbf{R}^2} \int_{s_1}^{s_2} \phi(y) \frac{\partial \alpha_h}{\partial s}(s, y)dyds.$$

This, together with (69) yields

$$\int_{\mathbf{R}^2} \phi(y)(\alpha_h(s_2, y) - \alpha_h(s_1, y))dy = -\int_{\mathbf{R}^2} \int_{s_1}^{s_2} \phi(y) div(\alpha_h(s, y)\mathbf{u}_h^k(y))dyds. \tag{82}$$

We use (74), (81) and integrate by parts the right handside of (82) to obtain that

$$\int_{\mathbf{R}^2} \phi(y)(\alpha_h(s_2, y) - \alpha_h(s_1, y))dy \leq C_T Lip(\phi)|s_2 - s_1|. \tag{83}$$

In the light of (83), and the dual representation of $W_1(\alpha_h(s_2, \cdot), \alpha_h(s_1, \cdot))$ given by Proposition A.1, we deduce (76). QED.

Corollary 4.2 *Assume that* $1 \leq r \leq +\infty$, *and that* $\alpha^o \in \mathcal{P}^{ac}(B_S) \cap L^r(B_S)$, *has its support strictly included in* B_S. *Assume that* $h > 0$ *is small enough, say* $0 < h < \bar{h}$, *and set* $C_T := f(2 + fT)||\alpha^o||_{L^1(B_S)}$. *Let* B_R *be the open ball of center* 0, *and radius* $R := S(1 + fT)$. *Then,*

$$\begin{cases} \frac{\partial \alpha_h}{\partial t} + div\, [\alpha_h \bar{\mathbf{u}}_h] = 0 & in \quad [0, T] \times \mathbf{R}^2 \\ \alpha_h(0, y) = \alpha_h^o(y) & in \quad \mathbf{R}^2. \end{cases} \tag{84}$$

We have that

$$||\alpha_h(t,\cdot)||_{L^r(\mathbf{R}^2)} = ||\alpha_h^o||_{L^r(\mathbf{R}^2)}, \tag{85}$$

for all $t \in [0,T]$. *Also,*

$$DP_h(t,\cdot)_{\#}\rho_h(t,\cdot) = \alpha_h(t,\cdot), \quad D\bar{P}_h(t,\cdot)_{\#}\bar{\rho}_h(t,\cdot) = \bar{\alpha}_h(t,\cdot), \tag{86}$$

$$||DP_h^*(t,\cdot)||_{L^\infty}, \quad ||DQ_h(t,\cdot)||_{L^\infty} \leq S, \quad ||P_h^*(t,\cdot) - Q_h(t,\cdot)||_{L^\infty} \leq hS, \tag{87}$$

for all $t \in [0,T]$. *We have that*

$$spt(\alpha_h(t,\cdot)) \subset B_R, \tag{88}$$

for all $t \in [0,T]$. *Hence, there exists a constant* c_1 *depending only on* S, B_R, *and* f, *such that*

$$||\rho_h(t,\cdot)||_{W^{1,\infty}(\Omega)} \leq c_1, \tag{89}$$

for all $t \in [0,T]$. *Furthermore,*

$$W_1(\alpha_h(s_2,\cdot), \alpha_h(s_1,\cdot)) \leq C_T|s_2 - s_1|, \tag{90}$$

for all $s_1, s_2 \in [0,T]$.

Proof: We obtain (84) as a direct consequence of (69). Lemma 4.1 yields (85). Proposition 3.4, and (70) imply (86) and the first two inequalities in (87). Because $j_h * P_h^*(t,\cdot) = Q_h(t,\cdot)$, we obtain the second inequality in (87). We use (75) to deduce (88).

Recall that by (70) we have $\rho_h(t,\cdot) := \mathcal{L}[\alpha_h(t,\cdot)]$. Using (88), and Remark 3.5, we conclude (89). Since W_1 is a distance, we use Lemma 4.1 to obtain (90). QED.

We next state a general result in partial differential equations which is needed in this work.

If $1 < r < +\infty$, $r' := r/(r-1)$, $\alpha \in L^r((0,T) \times B)$, and $\varphi \in W^{1,r'}(B)$, we define

$$< \alpha : \varphi > (t) := \int_B \alpha(t,y)\varphi(y)dy.$$

Lemma 4.3 *Suppose that* $1 < r < +\infty$, *that* $T > 0$ *is an integer, and that* $B \subset \mathbf{R}^2$ *is an open bounded ball. Suppose that* $\{\alpha_h\}_{h>0} \subset W^1((0,T) \times \bar{B})$, *and that* $\{\mathbf{k}_h\}_{h>0} \subset L^r((0,T) \times B)$ *satisfy, in the weak sense,*

$$\frac{\partial \alpha_h}{\partial t} + div\ (\mathbf{k}_h) = 0, \tag{91}$$

in $[0,T] \times \mathbf{R}^2$*, using the convention that* α_h *and* $\mathbf{k}_h$ *are defined to be* 0 *outside* $\bar{B}$*. Suppose that there exists a constant* $C > 0$ *such that*

$$\sup_{t\in[0,T],h>0} ||\alpha_h(t,\cdot)||_{L^r(B)}, \qquad \sup_{t\in[0,T],h>0} ||\mathbf{k}_h(t,\cdot)||_{L^r(B)} \leq C. \tag{92}$$

Then, there exists a sequence $\{h_j\}_{j=1}^{\infty} \subset (0,1)$ *converging to* 0 *as* j *tends to* $+\infty$*, there exist* $\alpha \in L^r((0,T)\times B)$*, and a map* $\mathbf{k} \in L^r((0,T)\times B)$ *such that the following hold.*

(i) $\{\alpha_{h_j}\}_{j=1}^{\infty}$ *converges weakly to* α *in* $L^r((0,T)\times B)$ *,* $\{\mathbf{k}_{h_j}\}_{j=1}^{\infty}$ *converges weakly to* $\mathbf{k}$ *in* $L^r((0,T)\times B)$*, as* j *tends to* $+\infty$*.*

(ii) $\{\alpha_{h_j}(t,\cdot)\}_{j=1}^{\infty}$ *converges weakly to* $\alpha(t,\cdot)$ *in* $L^r(B)$ *for every* $t \in [0,T]$*, as* j *tends to* $+\infty$*.*

Proof: 1. The existence of a sequence $\{h_j\}_{j=1}^{\infty} \subset (0,1)$ converging to 0, as j tends to $+\infty$, and the existence of a function $\alpha \in L^r((0,T)\times B)$, a map $\mathbf{k} \in L^r((0,T)\times B)$ such that $\{\alpha_{h_j}\}_{j=1}^{\infty}$ converges weakly to α in $L^r((0,T)\times B)$, $\{\mathbf{k}_{h_j}\}_{j=1}^{\infty}$ converges weakly to $\mathbf{k}$ in $L^r((0,T)\times B)$ as j tends to $+\infty$ is straightforward to obtain. This concludes the proof of (i).

One can readily check that (92) implies that

$$\sup_{t\in[0,T]} ||\alpha(t,\cdot)||_{L^r(B)} \leq C. \tag{93}$$

Combining (91) and (92) we deduce that

$$|\frac{\partial}{\partial t}\int_B \alpha_h \varphi dy| = |\int_B \mathbf{k}_h \cdot D\varphi dy| \leq C||D\varphi||_{L^{r'}(B)}, \tag{94}$$

in $[0,T]$, and for all $\varphi \in W^{1,r'}(B)$. Also,

$$|\int_B \alpha_h \varphi dy| \leq C||\varphi||_{L^{r'}(B)}, \tag{95}$$

in $[0,T]$ and for all $\varphi \in L^{r'}(B)$.

2. Let $\{\psi_k\}_{k=1}^{\infty} \subset C^1(\bar{B})$ be a dense subset of $L^{r'}(B)$ and $W^{1,r'}(B)$. In the light of (94) and (95) we find that for each $k \in \mathbf{N}$ the sequence of real functions $\{< \alpha_{h_j} : \psi_k >\}_{j=1}^{\infty}$ is bounded in $W^{1,\infty}(0,T)$ and that

$$Lip(< \alpha_{h_j} : \psi_k >) \leq C||D\psi_k||_{L^{r'}(B)}. \tag{96}$$

Consequently, we may extract from $\{h_j\}_{j=1}^{\infty}$ a subsequence we still label $\{h_j\}_{j=1}^{\infty}$ such that for each $k \in \mathbf{N}$

$$< \alpha_{h_j} : \psi_k > \quad \text{converges uniformly to } a(\psi_k) \ \text{ in } C[0,T], \tag{97}$$

$$< \alpha_{h_j} : \psi_k > \quad \text{converges weak } * \text{ to } a(\psi_k) \ \text{ in } W^{1,\infty}(0,T), \tag{98}$$

as j tends to $+\infty$, for some $a(\psi_k) \in W^{1,\infty}(0,T)$. Now, using (95), (97) and the fact that $\{\psi_k\}_{k=1}^{\infty}$ is dense in $L^{r'}(B)$ we deduce that for each $\varphi \in L^{r'}(B)$,

$$\{< \alpha_{h_j} : \varphi >\}_{j=1}^{\infty}$$

is a Cauchy sequence in $C[0,T]$. Thus, there exists $a(\varphi) \in C[0,T]$ such that

$$< \alpha_{h_j} : \varphi > \ \text{converges uniformly to } a(\varphi) \ \text{ in } C[0,T], \tag{99}$$

as j tends to $+\infty$. By (95)

$$|a(\varphi)|_{L^{\infty}(0,T)} \leq C||\varphi||_{L^{r'}(B)}. \tag{100}$$

Combining (99) and (100) we deduce that for each $t \in [0,T]$ there exists $\beta(t,\cdot) \in L^r(B)$ such that

$$a(\varphi)(t) = \int_B \beta(t,y)\varphi(y)dy, \tag{101}$$

and

$$||\beta(t,\cdot)||_{L^r(B)} \leq C. \tag{102}$$

Using the fact that $\{\alpha_{h_j}\}_{j=1}^{\infty}$ converges weakly to α in $L^r((0,T)\times B)$ as j tends to $+\infty$, and using (95), and (99) we deduce that for each $\omega \in C[0,T]$ and each $k \in \mathbf{N}$ we have

$$\int_0^T \omega(t)dt \int_B \beta(t,y)\psi_k(y)dy = \int_0^T \omega(t)dt \int_B \alpha(t,y)\psi_k(y)dy. \tag{103}$$

In the light of (103) there exists a set $N \subset [0,T]$ such that $\mathcal{H}^1(N) = 0$ and

$$\int_B \beta(t,y)\psi_k(y)dy = \int_B \alpha(t,y)\psi_k(y)dy, \tag{104}$$

for all $t \in [0,T] \setminus N$. Since $\{\psi_k\}_{k=1}^{\infty}$ is dense in $L^{r'}(B)$, using (93), (102) and (104) we have that

$$\int_B \beta(t,y)\varphi(y)dy = \int_B \alpha(t,y)\varphi(y)dy, \tag{105}$$

for all $t \in [0,T] \setminus N$, and all $\varphi \in L^{r'}(B)$. Define the function

$$\bar{\alpha}(t,y) := \begin{cases} \alpha(t,y) & \text{if} \quad t \in [0,T] \setminus N,\ y \in B \\ \beta(t,y) & \text{if} \quad t \in N,\ y \times B. \end{cases} \tag{106}$$

Clearly, α and $\bar{\alpha}$ coincide $\mathcal{H}^3$ a.e. in $[0,T] \times B$, and by (101), (105) we have that

$$a(\varphi)(t) = \int_B \bar{\alpha}(t,y)\varphi(y)dy, \tag{107}$$

for every $t \in [0,T]$ and every $\varphi \in L^{r'}(B)$. From (106), and (107) we assume that

$$\alpha \equiv \bar{\alpha} \equiv \beta, \tag{108}$$

on $[0,T] \times B$.

Combining (99), (101), and (108) we conclude the proof of (ii). QED.

Theorem 4.4 (Main existence result) *Assume that $1 < r < +\infty$, and that $\alpha^o \in \mathcal{P}^{ac}(B_S) \cap L^r(B_S)$ has its support strictly contained in B_S. Define $C_T := Sf(2+fT)||\alpha^o||_{L^1(\mathbf{R}^2)}$, and let B_R be the ball of center 0, and radius $R := S(1+fT)$. Then there exist two functions $\rho \in L^\infty((0,T);W^{1,\infty}(\Omega))$, $\alpha \in L^\infty((0,T);L^r(\mathbf{R}^2))$, such that (α,ρ) is a stable solution of (25). In addition, the following hold.*

(i) $\alpha(t,\cdot) \in \mathcal{P}^{ac}(B_R)$, and $||\alpha(t,\cdot)||_{L^r(B_R)} \le ||\alpha^o||_{L^r(B_R)}$.

(ii) $spt(\alpha(t,\cdot) \subset \bar{B}_R$.

(iii)

$$W_1(\alpha(s_2,\cdot),\alpha(s_1,\cdot)) \le C_T|s_1 - s_2|, \tag{109}$$

for all $s_1, s_2 \in [0,T]$.

(iv) There exists a universal constant C such that $\mathbf{w}$, the geostrophic velocity in dual variables satisfies

$$||\mathbf{w}(t,\cdot) - fJy||_{L^\infty(B_R)} \le fS, \quad ||\mathbf{w}(t,\cdot) - fJy||_{BV(B_R)} \le fCSR(R+1).$$

(v) $\rho(t,\cdot) \in \mathcal{P}^{ac}(\Omega)$, for all $t \in [0,T]$. Furthermore, $\rho \in L^\infty((0,T);W^{1,\infty}(\Omega))$, and $\rho \in C([0,T];W^{1,s}(\Omega))$ for each $1 \le s < +\infty$.

Proof: 1. Let α_h be as in (69), let P_h, ρ_h be as in (70), let Q_h be as in (71), and let $\bar{\mathbf{u}}_h$ be as in (73). Recall that Corollary 4.2 gives that

$$||\alpha_h(t,\cdot)||_{L^r(\mathbf{R}^2)} \le ||\alpha^o||_{L^r(\mathbf{R}^2)}, \tag{110}$$

and

$$||DP_h^*(t,\cdot)||_{L^\infty(B_R)}, \quad ||DQ_h(t,\cdot)||_{L^\infty(B_R)} \le S, \tag{111}$$

for all $t \in [0,T]$ and all $h > 0$. So, in the light of Lemma 4.3, we conclude that there exists a sequence $\{h_j\}_{j=1}^\infty \subset (0,+\infty)$ converging to 0, there exists $\alpha \in L^r((0,T)\times B_R)$, and there exists $\mathbf{k} \in L^r((0,T)\times B_R)$ such that the following hold: as j tends to $+\infty$, we have that

$$\alpha_{h_j} \rightharpoonup \alpha \quad \text{in } L^r((0,T)\times \mathbf{R}^2), \quad \alpha_{h_j}\bar{\mathbf{u}}_{h_j} \rightharpoonup \mathbf{k} \ \text{ in } L^r((0,T)\times \mathbf{R}^2) \tag{112}$$

and

$$\alpha_{h_j}(t,\cdot) \rightharpoonup \alpha(t,\cdot) \quad \text{in } L^r(\mathbf{R}^2), \tag{113}$$

for each $t \in [0,T]$. Thanks to (110), (113), and the fact that the L^r-norm is weakly lower semicontinuous, we conclude the proof of (i). Using (88) in Corollary 4.2, and (113) we deduce (ii).

Observe that by Corollary 4.2

$$W_1(\alpha_{h_j}(s_2,\cdot), \alpha_{h_j}(s_1,\cdot)) \le C_T|s_2 - s_1|, \tag{114}$$

for $s_1, s_2 \in [0,T]$. Letting i tends to $+\infty$, using (113), (114), and that by Lemma A.4, W_1 is continuous for the weak $*$ topology, we concludes the proof of (iii).

2. Set

$$\rho(t,\cdot) := \mathcal{L}(\alpha(t,\cdot)), \quad P(t,x) := |x|^2/2 + \rho(t,x)/f^2. \tag{115}$$

Recall that (ii) gives that $\text{spt}\,(\alpha(t,\cdot)) \subset \bar{B}_S$, and by assumption $\bar{\Omega} \subset B_S$. So, the definition of $\mathcal{L}$, and Proposition 3.4 yield that the function $x \to P(t,x) := |x|^2/2 + \rho(t,x)/f^2$ can be extended into a convex function on B_S, such that

$$DP(t,\cdot)_\# \rho(t,\cdot) = \alpha(t,\cdot).$$

Also, we may assume without loss of generality that

$$\partial P^*(t,\cdot)(\mathbf{R}^2) \subset \bar{B}_S, \tag{116}$$

for all $t \in [0,T]$.

We claim that as j tends to $+\infty$, $\{\bar{\alpha}_{h_j}(t,\cdot)\}_{j=1}^\infty$ converges weakly in $L^2(\mathbf{R}^2)$ to $\alpha(t,\cdot)$, for each $t \in [0,T]$. Indeed, Using (114), and that $\bar{\alpha}_{h_j}(kh,\cdot) = \alpha_{h_j}(kh,\cdot)$ for all k, we conclude that

$$W_1(\alpha_{h_j}(t,\cdot), \bar{\alpha}_{h_j}(t,\cdot)) \le C_T h_j. \tag{117}$$

for each $t \in [0, T]$. Combining (113), and (117) we conclude that

$$\lim_{j \to +\infty} W_1(\bar{\alpha}_{h_j}(t, \cdot), \alpha(t, \cdot)) = 0. \tag{118}$$

But, (118) reads off $\{\bar{\alpha}_{h_j}(t, \cdot)\}_{j=1}^{\infty}$ converges weakly to $\alpha(t, \cdot)$ in $L^r(\mathbf{R}^2)$, for each $t \in [0, T]$. (See Rachev and Rüschendorf [18]). We use the stability result in Lemma 3.6 to deduce that

$$\bar{\rho}_{h_j}(t, \cdot) \to \rho(t, \cdot), \quad \bar{P}_{h_j}(t, \cdot) \to P(t, \cdot) \text{ in } C(\bar{\Omega}), \tag{119}$$

for each $t \in [0, T]$. Recall that $dom(P_h(t, \cdot)) = \bar{B}_S$, and so, by (119) we have that

$$\bar{P}^*_{h_j}(t, \cdot) \to P^*(t, \cdot) \text{ in } C(\bar{B}_R), \tag{120}$$

for each $t \in [0, T]$. Combining (111), (120), and using the fact that $\bar{P}^*_{h_j}(t, \cdot)$ is convex, we deduce that

$$||\bar{\mathbf{u}}_{h_j}(t, \cdot)||_{L^\infty(B_R)} \leq 2fR, \tag{121}$$

and

$$\bar{\mathbf{u}}_{h_j}(t, \cdot) \to fJ(\mathbf{id} - DP^*(t, \cdot)), \tag{122}$$

almost everywhere in B_R, and for each $t \in [0, T]$. Define

$$\mathbf{w} := fJ(\mathbf{id} - DP^*) \tag{123}$$

By (112), (121), and (122) we have that

$$\alpha_{h_j} \bar{\mathbf{u}}_{h_j} \rightharpoonup \alpha \mathbf{w} \quad \text{in } L^r((0, T) \times \mathbf{R}^2). \tag{124}$$

Since $\{\alpha_h^o\}_{h>0}$ converges to α^o in $L^r(\mathbf{R}^2)$ as h tends to 0, combining (84), (112), and (124) we deduce that

$$\begin{cases} \frac{\partial \alpha}{\partial t} + \mathrm{d}iv\, [\alpha \mathbf{w}] = 0 & \text{in} \quad [0, T] \times \mathbf{R}^2 \\ \alpha(0, \cdot) = \alpha^o & \text{in} \quad \mathbf{R}^2, \end{cases} \tag{125}$$

in the weak sense. Thanks to (115), (123), (125) we conclude that (α, ρ) is a stable solution of (25).

3. Since P^* is convex, $D^2 P^*$ is a Radon measure. Hence, the totale variation of $D^2 P^*$ on B_R is bounded by a multiple of the totale variation of $\triangle P^*$ on B_R. But, Green's formula gives that

$$\int_{B_R} \triangle P^*(y) dy = \int_{\partial B_R} DP^*(y) \cdot \frac{y}{||y||} dy.$$

Since in addition by (116), $||DP^*|| \leq S$, and $\mathbf{w} - fJy = -fJDP^*$ we deduce that

$$||\mathbf{w}(t,\cdot) - fJy||_{L^\infty(B_R)} \leq fS, \quad ||\mathbf{w}(t,\cdot) - fJy||_{BV(B_R)} \leq fCSR(R+1).$$

Here, C is a universal constant. (See [11]).

4. In Corollary 4.2, (89) gives that $\rho \in L^\infty((0,T); W^{1,\infty}(\Omega))$. We use (109), the stability Lemma given in 3.6, and the fact that $\rho(t,\cdot) = \mathcal{L}[\alpha(t,\cdot)]$ to obtain that $\rho \in C([0,T]; W^{1,s}(\Omega))$ for each $1 \leq s < +\infty$. Also, $\rho(t,\cdot) \in \mathcal{P}^{ac}(\Omega)$, for all $t \in [0,T]$. This concludes the proof of Theorem 4.4. QED.

A Background on the Wasserstein distance

Throughout this section we assume that $\Omega \subset\subset \bar{B}_S$, $\Lambda \subset\subset \bar{B}$ are two open sets, where B_S, and B are open balls in $\mathbf{R}^2$. We identify $\rho \in \mathcal{P}^{ac}(\Omega)$ with its restriction to Ω.

Assume that ρ, that α are two Borel probability density functions on $\mathbf{R}^2$, and that $c(z) = |z|^p/p$, for $z \in \mathbf{R}^2$. Recall that if $1 \leq p < +\infty$, the quantity

$$W_p^p(\rho,\alpha) := \inf_{\gamma \in \Gamma(\rho,\alpha)} \int_{\mathbf{R}^2 \times \mathbf{R}^2} c(x-y)\, d\gamma(x,y), \tag{126}$$

was introduced in Section 1 as the p-Monge-Kantorovich distance to the power p. We sometimes refer to the 2-Monge-Kantorovich distance as the Wasserstein distance. We recall results on the Monge-Kantorovich mass transport theory in special cases that are relevant in this work.

Proposition A.1 (Duality) *Assume that $0 < p < +\infty$, that $spt(\rho) \subset \bar{B}_S$, and that $spt(\alpha) \subset \bar{B}$. Then we have*

(i)

$$W_p^p(\rho,\alpha) := \sup_{u,v}\{\int_{\mathbf{R}^2} \rho(x)u(x)dx + \int_{\mathbf{R}^2} \alpha(y)v(y)dy\}, \tag{127}$$

where the supremum is performed over the set of all pairs (u,v) such that $u : B_S \to \mathbf{R}$, $v : B \to \mathbf{R}$, and $u(x) + v(y) \leq c(x-y)$ for all $x \in \bar{B}_S$, and all $y \in \bar{B}$.

(ii) If in addition $B = B_S$, and $0 < p \leq 1$ then

$$W_p^p(\rho,\alpha) := \sup_u\{\int_{\mathbf{R}^2} (\rho(x) - \alpha(x))u(x)dx\}. \tag{128}$$

where the supremum is performed over the set of all $u : B \to \mathbf{R}$ such that $u(x) - u(y) \leq c(x-y)$ for all $x, y \in \bar{B}$.

Proof: The proof can be found in [14]. We also refer the reader to [12]. QED.

Proposition A.2 *Assume that $p = 2$, that $spt(\rho) \subset \bar{B}_S$, and that $spt(\alpha) \subset \bar{B}$. Then*

(i) (126) admits a minimizer. Furthermore, $\gamma_o \in \Gamma(\rho, \alpha)$ is a minimizer of (126) if and only if there exists a convex function $\psi_o : \mathbf{R}^2 \to \mathbf{R}$ such that $dom(\psi_o) = \bar{B}_S$, $\partial\psi_o(B_S) \subset \bar{B}$, $dom(\psi_o^*) = \mathbf{R}^2$, $\partial\psi_o^*(\mathbf{R}^2) \subset \bar{B}$, and

$$\int_{\mathbf{R}^2\times\mathbf{R}^2} x \cdot y \, d\gamma_o(x, y) = \int_{\mathbf{R}^2} \psi_o \rho(x) dx + \int_{\mathbf{R}^2} \psi_o^* \alpha(y) dy.$$

In that case,

$$W_2^2(\rho, \alpha) = \int_{\mathbf{R}^2} |x|^2/2(\rho(x) + \alpha(x))dx - \int_{\mathbf{R}^2\times\mathbf{R}^2} x \cdot y \, d\gamma_o(x, y). \tag{129}$$

(ii) $D\psi_o(x)$ exists for ρ-almost every $x \in \mathbf{R}^2$, and $D\psi_o$ is the unique Borel map minimizing $\int_{\mathbf{R}^2}(|x - \mathbf{r}(x)|^2)/2\rho(x)dx$ over the set of all Borel maps $\mathbf{r} : \mathbf{R}^2 \to \mathbf{R}^2$ that push ρ forward to α.

Proof: Since $\Gamma(\rho, \alpha)$ is a set of probability measures, closed under weak $*$ convergence, existence of a minimizer in (129) is a direct consequence of the Banach-Alaoglu theorem. We refer the reader to [12] for a detailed proof of the Proposition. QED.

Lemma A.3 *Let $\phi_n : B \to \mathbf{R}$ $(n = 1, 2, \cdots)$, be a collection of convex functions satisfying $\int_B |\phi_n| dx \leq R_1$, $Lip(\phi_n) \leq R_1$ for some constant $R_1 > 0$. Assume that as n tends to $+\infty$, $\{\alpha_n\}_{n=1}^\infty \subset \mathcal{P}^{ac}(B_S)$ converges weakly in $L^1(B)$ to $\alpha \in \mathcal{P}^{ac}(B_S)$, and that $\{\rho_n\}_{n=1}^\infty \subset \mathcal{P}^{ac}(B_S)$ converges weakly in $L^1(B)$ to $\rho \in \mathcal{P}^{ac}(B_S)$. Suppose that $D\phi_{n\#}\alpha_n = \rho_n$. For $F \in C(\mathbf{R}^2 \times \mathbf{R}^2)$, define $F_{\phi_n}(x) := F(x, D\phi_n(x))$.*

(i) Then there exists a convex function $\phi : B \to \mathbf{R}$ such that $\{\alpha_n \cdot F_{\phi_n}\}_{n=1}^\infty$ converges weakly in $L^1(B)$ to $\alpha_F \cdot F_\phi$ as n tends to $+\infty$. Here, $F_\phi(x) := F(x, D\phi(x))$.

(ii) Furthermore, $D\phi_\#\alpha = \rho$.

Proof: The set $\mathcal{S} := \{\phi : \bar{B} \to \mathbf{R} \; : \; \int_B |\phi(x)| dx \leq R_1, Lip(\phi) \leq R\}$ is equicontinuous on $\bar{B}$, and so, by Ascoli-Arzela theorem it is precompact in $C(\bar{B})$. Let $\{n_i\}_{i=1}^\infty \subset \mathbf{N}$ be any arbitrary sequence. We may extract a subsequence we still label $\{n_i\}_{i=1}^{\infty}$ such that as i tends to $+\infty$, $\{\phi_{n_i}\}_{i=1}^{\infty}$ converges uniformly on B to a convex function $\phi \in \mathcal{S}$, and $\{D\phi_{n_i}\}_{i=1}^{\infty}$ converges $\mathcal{H}^2$-almost everywhere to $D\phi$ on B. (See [19]). Thus, $\{F_{\phi_{n_i}}\}_{i=1}^{\infty}$ converges $\mathcal{H}^2$-almost everywhere to F_ϕ on B as i tends to $+\infty$. Since B is a bounded set, and the sequence $\{F_{\phi_{n_i}}\}_{i=1}^{\infty}$ is bounded in $L^\infty(B)$ we deduce that $\{\alpha_{n_i} \cdot F_{\phi_{n_i}}\}_{i=1}^{\infty}$ converges weakly in $L^1(B)$ to $\alpha \cdot F_\phi$. In particular for $g \in C(B)$ we have

$$\begin{aligned} \int_B g(D\phi(x))\alpha(x)dx &= \lim_{i\to+\infty} \int_\Lambda g(D\phi_{n_i}(x))\alpha_{n_i}(x)dx \\ &= \lim_{i\to+\infty} \int_\Omega g(y)\rho_{n_i}(y)dy \\ &= \int_\Omega g(y)\rho(y)dy, \end{aligned}$$

thus, $D\phi$ pushes α forward to ρ.

Note that in the light of Proposition A.2, $D\phi$ is uniquely determined by ρ, and α. Since $\{n_i\}_{i=1}^{\infty}$ is an arbitrary subsequence of $\mathbf{N}$ we deduce that $\{\alpha_n \cdot F_{\phi_n}\}_{n=1}^{\infty}$ converges weakly in $L^1(B)$ to $\alpha_F \cdot F_\phi$ as n tends to $+\infty$. QED.

Lemma A.4 (Weak $*$ semicontinuity of W_1 and W_2.) *Assume that $\{\rho_n\}_{n=1}^{\infty}$, and $\{\alpha_n\}_{n=1}^{\infty}$ are two sequences of probability Borel density functions on $\mathbf{R}^2$, that $\operatorname{spt}(\rho_n) \subset \bar{\Omega}$, and that $\operatorname{spt}(\alpha_n) \subset \bar{B}$. Assume that $\{\rho_n\}_{n=1}^{\infty}$ converges weak $*$ to ρ, and that $\{\alpha_n\}_{n=1}^{\infty}$ converges weak $*$ to α in the sense that*

$$\lim_{n\to+\infty} \int_{\mathbf{R}^2} g(x)\rho_n(x)dx = \int_{\mathbf{R}^2} g(x)\rho(x)dx,$$

for all $g \in C_o(\mathbf{R}^2)$. Then the sequence of real numbers $\{W_2(\rho_n, \alpha_n)\}_{n=1}^{\infty}$ converges to $W_2(\rho, \alpha)$, as n tends to $+\infty$. Similarly, $\{W_1(\rho_n, \alpha_n)\}_{n=1}^{\infty}$ converges to $W_1(\rho, \alpha)$, as n tends to $+\infty$.

Proof: We first prove that $\{W_2(\rho_n, \alpha_n)\}_{n=1}^{\infty}$ converges to $W_2(\rho, \alpha)$. It suffices to show that from any subsequence $\{W_2(\rho_{n_i}, \alpha_{n_i})\}_{i=1}^{\infty}$ of $\{W_2(\rho_n, \alpha_n)\}_{n=1}^{\infty}$ we may extract a subsequence $\{W_2(\rho_{n_{i_k}}, \alpha_{n_{i_k}})\}_{k=1}^{\infty}$ which converges to $W_2(\rho, \alpha)$. By Proposition A.2 there exists a measure γ_n on $\mathbf{R}^2 \times \mathbf{R}^2$ that has ρ_n and α_n as its marginals, and such that

$$W_2^2(\rho_n, \alpha_n) := \int_{\mathbf{R}^2\times\mathbf{R}^2} (|x-y|^2)/2 \; d\gamma_n(x,y).$$

Furthermore, there exists convex functions $\psi_n : \bar{B}_S \to \mathbf{R}$ such that $\partial\psi_n(B_S) \subset \bar{B}$, $\partial\psi_n^*(\mathbf{R}^2) \subset \bar{B}_S$, and

$$\int_{\mathbf{R}^2\times\mathbf{R}^2} x \cdot y \; d\gamma_n(x,y) = \int_{\mathbf{R}^2} \psi_o \rho_n(x)dx + \int_{\mathbf{R}^2} \psi_o^* \alpha_n(y)dy. \qquad (130)$$

Hence,

$$W_2^2(\rho_n, \alpha_n) = \int_{\mathbf{R}^2} |x|^2/2(\rho_n(x) + \alpha_n(x)) - \int_{\mathbf{R}^2\times\mathbf{R}^2} x \cdot y \; d\gamma_n(x,y). \tag{131}$$

We assume without loss of generality that $\psi_n(0) = 0$. Clearly, $\{\psi_n\}_{n=1}^\infty$ is equicontinuous on $\bar{B}_S$, and so, from any subsequence $\{\psi_{n_i}\}_{i=1}^\infty$ of $\{\psi_n\}_{n=1}^\infty$ we may extract another subsequence we still label $\{\psi_{n_i}\}_{i=1}^\infty$ such that $\{\psi_{n_i}\}_{i=1}^\infty$ converges uniformly to a convex function $\psi : \bar{B}_S \to \mathbf{R}$, on $\bar{B}_S$ as i goes to $+\infty$. One can readily check that $\{\psi_{n_i}^*\}_{i=1}^\infty$ converges uniformly to ψ^* on $\bar{B}$ as i goes to $+\infty$. Since $\{\gamma_{n_i}\}_{i=1}^\infty$ is a sequence of probability measures on $\mathbf{R}^2 \times \mathbf{R}^2$ whose supports are in $\bar{B} \times \bar{B}$ we may as well assume that $\{\gamma_{n_i}\}_{i=1}^\infty$ converges weak $*$ to some γ which is necessarily in $\Gamma(\rho, \alpha)$. We substitute n by n_i in (130), and let n_i go to $+\infty$ to deduce that

$$\int_{\mathbf{R}^2\times\mathbf{R}^2} x \cdot y \; d\gamma(x,y) = \int_{\mathbf{R}^2} \psi_o \rho(x) dx + \int_{\mathbf{R}^2} \psi_o^* \alpha(y) dy, \tag{132}$$

and so, in the light of Proposition A.2

$$W_2^2(\rho, \alpha) = \int_{\mathbf{R}^2} |x|^2/2(\rho(x) + \alpha(x)) dx - \int_{\mathbf{R}^2\times\mathbf{R}^2} x \cdot y \; d\gamma(x,y). \tag{133}$$

Using (131), (132), and (133) we deduce that

$$\begin{aligned}
\lim_{i\to+\infty} W_2^2(\rho_{n_i}, \alpha_{n_i}) &= \lim_{i\to+\infty} \int_{\mathbf{R}^2} |x|^2/2(\rho_{n_i}(x) + \alpha_{n_i})(x)) dx - \int_{\mathbf{R}^2\times\mathbf{R}^2} x \cdot y \; d\gamma_{n_i}(x,y) \\
&= \int_{\mathbf{R}^2} |x|^2/2(\rho(x) + \alpha(x)) dx - \int_{\mathbf{R}^2\times\mathbf{R}^2} x \cdot y \; d\gamma(x,y) \\
&= W_2^2(\rho, \alpha).
\end{aligned}$$

Since $\{W_2(\rho_{n_i}, \alpha_{n_i})\}_{i=1}^\infty$ is an arbitrary subsequence of $\{W_2(\rho_n, \alpha_n)\}_{n=1}^\infty$ we conclude that the sequence $\{W_2(\rho_n, \alpha_n)\}_{n=1}^\infty$ converges to $W_2(\rho, \alpha)$, as n tends to $+\infty$.

The proof of the convergence of $\{W_1(\rho_n, \alpha_n)\}_{n=1}^\infty$ to $W_1(\rho, \alpha)$ is similar to the above proof. QED.

Lemma A.5 *Let $\alpha \in \mathcal{P}(\mathbf{R}^2)$. Then the map $\rho \to W_2^2(\rho, \alpha)$ is convex over the set $\mathcal{P}(\mathbf{R}^2)$.*

Proof: Let ρ_o, ρ_1 be two Borel probability measures on $\mathbf{R}^2$, and let $t \in (0,1)$. If $\gamma_o \in \Gamma(\rho_o, \alpha) := \Gamma_o$ and $\gamma_1 \in \Gamma(\rho_1, \alpha) := \Gamma_1$ then $\gamma_t := (1-t)\gamma_o + t\gamma_1 \in \Gamma(\rho_t, \alpha)$, where $\rho_t = (1-t)\rho_o + t\rho_1$. Consequently,

$$\begin{aligned}
& (1-t)W_2^2(\rho_o, \alpha) + tW_2^2(\rho_1, \alpha) \\
= & \inf_{\gamma_o \in \Gamma_o} \int_{\mathbf{R}^2 \times \mathbf{R}^2} (1-t)(|x-y|^2/2) d\gamma_o(x,y) + \inf_{\gamma_1 \in \Gamma_1} \int_{\mathbf{R}^2 \times \mathbf{R}^2} t(|x-y|^2/2) d\gamma_1(x,y) \\
\geq & \inf_{\gamma_o, \gamma_1} \{ \int_{\mathbf{R}^2 \times \mathbf{R}^2} |x-y|^2/2 d\gamma_t(x,y) \; : \; \gamma_o \in \Gamma_o, \gamma_1 \in \Gamma_1 \} \\
\geq & \inf_{\gamma} \{ \int_{\mathbf{R}^2 \times \mathbf{R}^2} |x-y|^2/2 d\gamma(x,y) \; : \; \gamma \in \Gamma(\mu, \nu_t) \} \\
= & W_2^2(\rho_t, \alpha).
\end{aligned}$$

This concludes the proof of the Lemma. QED.

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