A mathematically rigorous analysis of forced axisymmetric flows in the atmosphere

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This note discusses a rigorous mathematical formulation for the evolution of the Eliassen balanced vortex. It is first shown that a stable balanced vortex of finite extent can be embedded in an ambient fluid at rest, and that such a vortex exists for prescribed angular momentum and potential temperature on fluid parcels. This uses a method developed by Shutts, Booth and Norbury. This is a different way of viewing the problem from the normal methods, which analyse the stability of a prescribed vortex. The stability of the vortex depends on the presence of background rotation and on the azimuthal velocity at the boundary of the vortex being less than that in the surrounding ambient fluid. It is then shown that the evolution of this vortex under axisymmetric forcing can be written as a conservation law for a potential pseudo-density in the transformed coordinates introduced by Schubert and Hack. The stability of the vortex to non-axisymmetric perturbations is also discussed.

1. Introduction

There is a very large literature on the structure of tropical cyclones and in particular dynamical mesoscale processes which can create asymmetries on the core region. A recent review is given by Wang and Wu (2004). Despite the focus on the importance of asymmetries, however, there is still substantial interest in the evolution of the axisymmetric flow, particularly the effects of axisymmetric forcing. Two recent examples are the papers of Bell *et al.* (2012) and Huang *et al.* (2012). Dynamical theories of axisymmetric flow start from the classical Eliassen (1951) model of a balanced circular vortex. This shows how a vortex in gradient wind balance evolves slowly under axisymmetric forcing of angular momentum and potential temperature. Shutts *et al.* (1988) described this evolution as a sequence of minimum energy states. Such a model can explain a significant fraction of the observed evolution of tropical cyclones, as demonstrated for instance by Bui *et al.* (2009).

In this paper we describe a mathematically rigorous formulation of the forced axisymmetric vortex problem developed by Cullen and Sedjro (2013). This builds on the results of Fjortoft (1946) and Eliassen and Kleinschmidt (1957) that a stable balanced vortex represents a minimum energy state with respect to variations which conserve mass, angular momentum and potential temperature. In particular, this requires that the square of the angular momentum must increase with radius and the fluid be stably stratified. Thus, if the mass of fluid with a given angular momentum and potential temperature is given, the fluid can then be uniquely rearranged into a stable axisymmetric vortex. This is a different way of posing the problem from the methods

normally used in the literature, which analyse the stability of a prescribed vortex. Thus different insights can be expected.

The condition that the square of the absolute angular momentum must increase with radius implies that, if the vortex is embedded in an ambient fluid at rest, there must be background rotation. Otherwise the ambient fluid would need to have more angular momentum about the centre of the vortex than any fluid inside the vortex, which is not a realistic situation. If the ambient fluid is at rest, then in the presence of positive background rotation the angular momentum condition implies that the azimuthal velocity of a cyclonic vortex must decrease to zero or become anticyclonic at the vortex boundary, or else the boundary of the vortex would be unstable. This is consistent with the observed decrease of azimuthal velocity with radius outside the core region of tropical cyclones.

The mathematical theory implies that such a vortex can uniquely be constructed given a fluid with known angular momentum and potential temperature on fluid parcels. Shutts *et al.* (1988) demonstrated this by using a change of variables and an explicit construction assuming piecewise constant data. In order to do this, the vortex had to be confined by rigid axisymmetric boundaries. In the present paper we prove that a unique vortex with a free boundary sitting in an ambient fluid at rest exists for continuous data. This is a suitable step towards the real case where the vortex would be embedded in a time-dependent ambient flow. However, it is now necessary to show that the boundary of the vortex also represents a minimum energy config-uration. Cullen and Sedjro (2013) assume that the ambient fluid is neutrally stratified and that the potential temperature anomaly within the vortex is strictly positive. Stability of the boundary

then requires that vortex is monotonically expanding with height. However, if insufficient mass is specified for the vortex, the vortex may not extend through the full depth of the physical domain, but be confined to the upper part. This would not be physically realistic, but we show that it is not likely to occur with a reasonable choice of data. The method could in principle be extended to a case with non-zero stratification in the ambient fluid, which should also give more physically realistic results.

Note that the free boundary problem could also be obtained by solving a problem with rigid boundaries solved over a larger domain. In that case the characterisation of the free boundary established in this paper will describe the geometry of the internal interface in the fluid which represents the boundary of the vortex.

Using the methods of Craig (1991), it is then possible to show that the evolution of such a vortex under the action of axisymmetric forcing can be described by an evolution equation for a potential pseudo-density in isentropic and angular momentum coordinates. It is interesting to note that Craig, eq. (33), shows that this problem can still be written in terms of a potential vorticity. This represents the mass in suitably transformed variables. This is to be expected because of the Hamiltonian derivation of the equations. Conditions for the solution of this to be well-posed are derived. Such a model could be considered to represent the axisymmetric part of the evolution of an almost axisymmetric flow, where the forcing terms represent the integrated effects of non-axisymmetric perturbations on the basic vortex.

The physical applicability of axisymmetric vortex theory presumes that such a vortex can be stable to non-axisymmetric disturbances. In Montgomery and Shapiro (1995) this is analysed using a balanced model, and necessary conditions for instability are found that require the radial gradient of the potential vorticity to change sign on an isentropic surface. There is also a boundary contribution. This analysis is strictly only valid for small azimuthal wavenumbers as the balanced model is not valid otherwise. An alternative approach is suggested by the strong axisymmetrisation observed in tropical cyclones and analysed, for instance, by Smith and Montgomery (1995). This suggests that the end result of non-axisymmetric disturbances to a vortex would be regarded as a new axisymmetric vortex in which the potential vorticity was a rearrangement of the original potential vorticity allowing for mixing. The computations and analysis of Schubert et al. (1999) demonstrate in a two-dimensional vortex that an initial annular ring of vorticity is unstable to non-axisymmetric disturbances and then evolves towards a new axisymmetric vortex with the vorticity concentrated at the centre. Mixing is clearly visible. Thus we could expect that a stable vortex represents an extremum of the energy with respect to axisymmetric rearrangements of potential vorticity which allow for mixing. Mathematically rigorous analyses like this were carried out for quasi-geostrophic vortices in a shear flow by Burton and Nycander (1999) and for straight geostrophic flows by Cullen and Douglas (2003). In this paper we illustrate the approach for the two-dimensional case, and compare with the results of Montgomery and Shapiro.

2. Governing equations

2.1. Physical space equations

We start with the equations for the evolution under the action of forcing of an axisymmetric vortex introduced by Eliassen (1951). The form of the equations follows Craig (1991) and describe a hydrostatic rotating Boussinesq atmosphere with the vertical coordinate z being a function of pressure. The equations are written in cylindrical polar coordinates (λ, r, z) with velocity components (u, v, w) in the coordinate directions. However, all variables are assumed to be independent of λ. The geopotential is φ. The potential temperature is written as the deviation θ' from a spatially uniform value θ_0. The angular velocity of the system rotation is Ω and the acceleration due to gravity is g. The forcing is described by terms F, S which are also independent of λ. These could represent the averaged effects of $\lambda-$dependent disturbances to the vortex, or other external forcing.

$$\begin{cases} \frac{Du}{Dt} + \frac{uv}{r} + 2\Omega v = \frac{1}{r}F(t, r, z), \\ \frac{D\theta'}{Dt} = S(t, r, z), \\ \frac{u^2}{r} + 2\Omega u = \frac{\partial \varphi}{\partial r}, \\ \frac{1}{r}\frac{\partial}{\partial r}(rv) + \frac{\partial w}{\partial z} = 0, \\ \frac{\partial \varphi}{\partial z} - g\frac{\theta'}{\theta_0} = 0. \end{cases} \quad (1)$$

The third of these equations is only valid if $u \to 0$ as $r \to 0$. Following Shutts et al. (1988), we therefore exclude a small region $r < r_0$ with $u = 0$ from the problem, and assume that equations (1) define the flow in an isolated vortex occupying a region Γ_ς defined by

$$\Gamma_\varsigma = [0, 2\pi] \times [r_0, \varsigma(t, z)] \times [0, H]. \quad (2)$$

The vortex is assumed to be sitting in an ambient fluid at rest with constant potential temperature θ_0. We assume that $\theta' > 0$, so that the potential temperature of the vortex is always greater than θ_0. These assumptions were introduced in Cullen and Sedjro (2013) to make the mathematics tractable. It is hoped that they can be relaxed in future work. The boundary conditions are

$$\begin{cases} w(t, r, 0) = w(t, r, H) = 0, \\ v(t, r_0, z) = 0, \\ \varphi(t, \varsigma(t, z), z) = 0. \end{cases} \quad (3)$$

The last of these conditions is required to make φ well-defined.

The boundary condition $v(t, r_0, z) = 0$ is physically appropriate if $u(t, r_0, z) \geq 0$, because then the assumption $u = 0$ for $r < r_0$ is consistent with stability of the vortex. Since imposing this extra condition makes the problem overdetermined, it implies a restriction on the choice of initial data which will be discussed later.

The boundary $r = \varsigma(t, z)$ is a material surface so that

$$\frac{\partial \varsigma}{\partial t} + w\frac{\partial \varsigma}{\partial z} = v. \quad (4)$$

Physically relevant solutions will require this vortex to be stable to internal axisymmetric perturbations. Similarly, we require the vortex to retain its identity, so that there is no mixing between the vortex and the surrounding fluid. This will require the boundary $r = \varsigma$ to be stable against perturbations also involving the surrounding fluid. It is then possible to justify only considering the fluid in the vortex and ignoring the motion of the ambient fluid.

2.2. Equations in new variables

In this section, we show how the equations for forced axisymmetric flows can be reformulated as a continuity equation in a set of transformed variables. A more thorough discussion can be found in Cullen and Sedjro (2013). Let φ be smooth and define $\varphi_t(r, z) = \varphi(t, r, z)$ for each fixed t. Use a similar convention for other variables. Then, following Shutts et al. (1988), define a new radial coordinate s and a potential P_t defined in terms of this coordinate by

$$P_t(s, z) = \varphi_t(r, z) + \frac{\Omega^2 r^2}{2} \quad \text{with} \quad 2s = r_0^{-2} - r^{-2}. \quad (5)$$

The domain occupied by the vortex, Γ_ς, defined in (2) written in (λ, s, z) coordinates is

$$D_\rho = [0, 2\pi] \times [0, \rho(t, z)] \times [0, H]. \quad (6)$$

We can use the third and fifth equations of (1) to show that

$$\nabla P_t = \left(u_t^2 r^2 + 2\Omega u_t r^3 + \Omega^2 r^4, g\frac{\theta_t'}{\theta_0} \right). \qquad (7)$$

The quantity $(u_t r + \Omega r^2)$ is the component of the angular momentum of the fluid in the direction of the rotation of the coordinate system. The conditions for a minimum energy state given by Fjortoft (1946) and Eliassen and Kleinschmidt (1957) are that the square of the angular momentum is monotonically increasing in r and the potential temperature is monotonically increasing in z, which means that

$$\nabla P_t \qquad \text{is invertible}, \qquad (8)$$

Shutts *et al.* (1988) show that this is equivalent to convexity of P_t. The boundary condition satisfied by φ in (3) becomes

$$P_t(\rho_t(z), z) = \frac{\Omega^2 r_0^2}{2(1 - 2r_0^2 \rho_t(z))} \quad \text{on} \quad \{\rho_t > 0\}, \qquad (9)$$

where

$$2\rho_t = r_0^{-2} - \varsigma_t^{-2}. \qquad (10)$$

If we assume that the angular momentum $(u_t r + \Omega r^2) > 0$, we can define new variables as introduced by Schubert and Hack (1983) and Shutts *et al.* (1988):

$$\begin{cases} \Omega R^2 = u_t r + \Omega r^2, \\ \Upsilon = \Omega^2 R^4, \\ Z = g\frac{\theta_t'}{\theta_0}. \end{cases} \qquad (11)$$

Then (7) becomes

$$\nabla P_t = (\Upsilon, Z). \qquad (12)$$

The monotonicity condition (8) means that Υ is monotonically increasing in r and that Z is monotonically increasing in z.

Writing the evolution equations (1) in these variables and (λ, s, z) coordinates gives

$$\begin{cases} \frac{1}{2\sqrt{\Upsilon}}\frac{D\sqrt{\Upsilon}}{Dt} = F_t(r, z), \\ \frac{DZ}{Dt} = S_t(r, z), \\ \nabla P = (\Upsilon, Z), \\ \frac{1}{e(s)}\frac{\partial}{\partial s}(e(s)\upsilon) + \frac{\partial w}{\partial z} = 0, \end{cases} \qquad (13)$$

where υ is the radial velocity in s coordinates, and the volume measure $r\mathrm{d}r\mathrm{d}z$ in (r, z) coordinates becomes $e(s)\mathrm{d}s\mathrm{d}z$ in (s, z) coordinates where

$$e(s) = r_0^4/(1 - 2sr_0^2)^2 \quad \text{for} \quad 0 \le 2r_0^2 s < 1. \qquad (14)$$

The boundary conditions become

$$\begin{cases} w(t, s, 0) = w(t, s, H) = 0, \\ \upsilon(t, 0, z) = 0, \\ \varphi(t, \rho(t, z), z) = 0. \end{cases} \qquad (15)$$

where

$$\frac{\partial \rho}{\partial t} + w\frac{\partial \rho}{\partial z} = \upsilon. \qquad (16)$$

Next introduce angular momentum and isentropic coordinates. The monotonicity condition (8) means that these will be well-defined. Define

$$\Psi_t(\Upsilon, Z) = s\Upsilon + zZ - P_t. \qquad (17)$$

This means that $\Psi_t(\Upsilon, Z)$ is the Legendre transform of $P_t(s, z)$ for each t as defined by Rockafellar (1970). Then it can be shown that Ψ_t is also convex and $\nabla \Psi_t = (s, z)$. The map from physical

coordinates (s, z) to angular momentum and potential temperature coordinates (Υ, Z) is generated by ∇P_t and its inverse is $\nabla \Psi_t$.

We now need to recognise that these two coordinates are defined in different units, which is important when proving that the transformation can be constructed and in generating examples. To illustrate this, apply this transformation to a state of rest in hydrostatic balance, as considered in Shutts *et al.* (1988). Eq. (11) shows that this corresponds to choosing $\Upsilon = \Omega^2 r^4$, while Z can be any positive function of z. The definition of s in (5) shows that the domain $r_0 \le r < \infty$ transforms to the finite domain $0 \le s \le \frac{1}{2r_0^2}$, but the associated steady state value of Υ will be in the domain $\Omega^2 r_0^4 \le \Upsilon < \infty$. Since we wish to show that a unique vortex with prescribed angular momentum and potential temperature can be constructed, it is necessary to restrict the values of Υ to a finite range. Noting the requirement $u \ge 0$ at $r = r_0$ discussed in section 2.1, we write this range as $\Upsilon \in [\Omega^2 r_0^4, \Upsilon_m]$ for some Υ_m. We then map this into a finite physical region as in the definition (6) where the boundary ρ has to be determined as part of the solution. The definition of $e(s)$ in (14) and the definition of s in (5) show that $e(s) = r^4$. Thus a state of rest corresponds to $\Upsilon_m = \Omega^2 e(\rho)$.

2.3. Mass conservation

Since the mass in physical coordinates is represented by the volume in pressure coordinates, and the fluid occupies the region Γ_ς defined in (2), the mass in physical space can be represented as the integral of $r\chi_{\Gamma_{\varsigma_t}(r,z)}\mathrm{d}r\mathrm{d}z$, where χ_{Γ_ς} is the characteristic function of Γ_ς. Since the physical coordinate has been changed from r to s, we need to write mass conservation in terms of s using the volume measure (14). The mass then becomes the integral of

$$e(s)\chi_{D_{\rho_t}}\mathrm{d}s\mathrm{d}z, \qquad (18)$$

where $\chi_{D_{\rho_t}}$ is the characteristic function of D_{ρ_t}, the domain occupied by the fluid in (s, z) coordinates.

Next define the mass density in (Υ, Z) coordinates as σ_t, where

$$\sigma_t \mathrm{d}\Upsilon \mathrm{d}Z = e(s)\mathrm{d}s\mathrm{d}z, \qquad (19)$$

Since $(\Upsilon, Z) = \nabla P_t(s, z)$, we have using (19)

$$\sigma_t \frac{\partial(\Upsilon, Z)}{\partial(s, z)} = \sigma_t \det(\partial^2 P_t) = e(s), \qquad (20)$$

where $\partial^2 P_t$ is the Hessian matrix of P_t.

Mathematically we represent mass densities as probability measures, since these take non-negative values and have a prescribed integral equal to the total mass. It is then convenient to write (20) in the form

$$\sigma_t = \nabla P_t \# e(s)\chi_{D_{\rho_t}} \qquad (21)$$

\# indicates the push forward of the measure $e(s)\chi_{D_{\rho_t}}$ in physical space by the map ∇P_t which takes physical space to (Υ, Z) space. This concept is defined by Villani (2003).

As ∇P_t is invertible with inverse $\nabla \Psi_t$, σ_t is equivalently defined by

$$e(\partial_\Upsilon \Psi)\det(\partial^2 \Psi) = \sigma_t \qquad \nabla \Psi(\Delta) = D_{h_t}, \qquad (22)$$

where $s = \partial_\Upsilon \Psi = \partial \Psi/\partial \Upsilon$ and Δ is the region of $\mathbb{R}_+^2$ where σ_t is non-zero. (22) can be written as

$$\nabla \Psi_t \# \sigma_t = e(s)\chi_{D_{\rho_t}}. \qquad (23)$$

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Conservation of mass then requires that

$$\begin{cases} \frac{\partial \sigma}{\partial t} + \nabla \cdot (\sigma V_t) = 0 & (0, T) \times \mathbb{R}^2 \\ \sigma|_{t=0} = \sigma_0. \end{cases} \tag{24}$$

where V_t is the velocity in (Υ, Z) coordinates, so that

$$V_t = \left(\frac{\mathrm{D}\Upsilon}{\mathrm{D}t}, \frac{\mathrm{D}Z}{\mathrm{D}t} \right). \tag{25}$$

Equations (1) give, using the definitions (11),

$$\begin{cases} \frac{1}{2\sqrt{\Upsilon}} \frac{\mathrm{D}\sqrt{\Upsilon}}{\mathrm{D}t} = F_t(r, z), \\ \frac{\mathrm{D}Z}{\mathrm{D}t} = S_t(r, z). \end{cases} \tag{26}$$

Writing F_t and S_t as functions of (Υ, Z), using the relation $\nabla \Psi_t = (s, z)$ and the definition of s from (5) gives

$$V_t = \left(2\sqrt{\Upsilon} F_t \left(\frac{r_0}{\sqrt{1 - 2r_0^2 \frac{\partial \Psi}{\partial \Upsilon}}}, \frac{\partial \Psi}{\partial Z} \right), \right. \tag{27}$$
$$\left. \frac{g}{\theta_0} S_t \left(\frac{r_0}{\sqrt{1 - 2r_0^2 \frac{\partial \Psi}{\partial \Upsilon}}}, \frac{\partial \Psi}{\partial Z} \right) \right).$$

Solving (26) requires calculating Ψ_t given σ_t. If this can be done, then (23) determines ρ_t and (17) is used to calculate P_t. Equation (7) is then used to calculate u and θ', giving a complete solution for the vortex. The radial and vertical velocities (v, w) can be determined by transforming V_t as defined in (25) into (s, z) coordinates using the map $\nabla \Psi_t$, allowing for the time dependence of Ψ_t.

The result will correspond to a solution of (13) if Ψ_t is convex, since then the change of coordinates is valid. The first two equations of (13) correspond to the definition of V_t (27), the third equation corresponds to equation (17), and the fourth equation corresponds to equation (24). The boundary equation (16) corresponds to (23) which determines ρ_t.

In the next section, we show that Ψ_t can be uniquely calculated for a given σ_t by minimising the energy subject to mass conservation. We show that the resulting Ψ_t is convex. This will then generate a solution of (26) for the reasons above.

3. Solution of the free boundary problem

In this paper we show rigorously that the energy can be minimised under rearrangements of the fluid conserving angular momentum, represented by Υ, and potential temperature Z. This is equivalent to specifying the mass of the fluid in angular momentum and isentropic coordinates and then minimising the energy with respect from maps from (Υ, Z) to physical coordinates (s, z) which conserve mass. In order to confirm the conditions found by Fjortoft (1946) and Eliassen and Kleinschmidt (1957), we need to show that this minimising state is characterised by a map $\nabla \Psi_t$ with Ψ_t convex.

3.1. Energy minimisation

Suppose that the mass in (Υ, Z) is σ_t, which is a probability measure with support $\Delta \subset \mathbb{R}^2_+$. We assume additionally that Δ is compact (i.e. closed and bounded). We now show that finding the map from Δ to D_ρ which conserves mass while minimising the energy corresponds to finding an invertible map $\nabla \Psi_t$, together with a function $\rho_t(z)$ which defines the region D_ρ in physical space occupied by the vortex. This corresponds to the energy minimisation problem solved by Shutts *et al.* (1988), but without making the assumption of piecewise constant data and using a

free vortex boundary rather than a rigid boundary. A proof that the energy minimiser is unique will show that a unique stable axisymmetric vortex with a free boundary with suitably prescribed angular momentum and potential temperature on fluid parcels can be embedded in an ambient isentropic fluid at rest.

The total energy to be minimised is

$$E = \frac{1}{2\pi} \int_{D_\rho} \left(\frac{1}{2} u_t^2 - \frac{g\theta_t' z}{\theta_0} \right) e(s) \mathrm{d}\lambda \mathrm{d}s \mathrm{d}z. \tag{28}$$

The energy density $\frac{1}{2} u_t^2 - \frac{g\theta_t' z}{\theta_0}$ can be rewritten using (11) as

$$\frac{r_0^2 \Omega^2}{2(1 - 2r_0^2 s)} - s\Upsilon - zZ + \frac{\Upsilon}{2r_0^2} - \Omega\sqrt{\Upsilon}. \tag{29}$$

We wish to minimise (28) over maps from Δ to D_ρ for a given σ_t. This type of constraint means that the problem is an optimal transport problem, as described in Villani (2003). In order to solve it, it is necessary to solve a relaxed problem called the Monge-Kantorovich problem. The relaxed problem does not require individual points in Δ to be mapped to individual points in D_ρ, but only requires the mass in subsets of Δ to be mapped to well-defined subsets of D_ρ. Thus instead of seeking a map from Δ to D_ρ, and thus finding a Ψ_t such that $(s, z) = \nabla \Psi_t$, we seek a joint probability measure γ with marginals σ_t and $e(s)\chi_{D_{\rho_t}}$, where $\chi_{D_{\rho_t}}$ is the characteristic function of D_{ρ_t}. This means that for any functions $\phi(s, z), \psi(\Upsilon, Z)$,

$$\int_{\Delta \times D_{\rho_t}} (\psi(\Upsilon, Z) + \phi(s, z)) \mathrm{d}\gamma = \tag{30}$$
$$\int_\Delta \psi \sigma_t \mathrm{d}\Upsilon \mathrm{d}Z + \int_{D_{\rho_t}} \phi e(s) \mathrm{d}s \mathrm{d}z.$$

This procedure is described in Villani (2003).

In general we write such a joint probability measure as

$$\gamma \in G(\sigma_t, e(s)\chi_{D_{\rho_t}}), \tag{31}$$

where G is the set of joint probability measures on $\mathbb{R}^2_+ \times \mathbb{R}^2_+$. In the special case where this corresponds to a map from Δ to D_ρ, (31) reduces to

$$\gamma = \sigma_t \delta[(s, z) = \nabla \Psi_t], \tag{32}$$

where δ denotes the Dirac delta function.

We now have to minimise (28) over choices of γ as in (31) and choices of ρ_t which define the vortex domain D_{ρ_t}.

Write (28) using (29) in the form

$$I(\rho, \gamma) := \int_{D_\rho \times \Delta} \left(-s\Upsilon - zZ + \frac{\Upsilon}{2r_0^2} \right. \tag{33}$$
$$\left. -\Omega\sqrt{\Upsilon} + \frac{r_0^2 \Omega^2}{2(1 - 2r_0^2 s)} \right) \mathrm{d}\gamma$$

We first assume a given value $\tilde{\rho}$ of ρ, and minimise (33) over choices of γ. The result can be written as

$$\bar{I}[\sigma_t](\tilde{\rho}) := \inf_{[\gamma \in G(\sigma_t, e(s)\chi_{D_{\tilde{\rho}}})]} \left\{ \int_{\Delta \times D_{\tilde{\rho}}} (-s\Upsilon - zZ) \mathrm{d}\gamma \right\} +$$
$$\int_{D_{\tilde{\rho}}} \left(\frac{\Omega^2 r_0^2}{2(1 - 2sr_0^2)} \right) e(s) \mathrm{d}s \mathrm{d}z + \tag{34}$$
$$\int_{\mathbb{R}^2} \left(\frac{\Upsilon}{2r_0^2} - \Omega\sqrt{\Upsilon} \right) \sigma_t \mathrm{d}\Upsilon \mathrm{d}Z.$$

The second term in (34) depends only on the choice of $D_{\tilde{\rho}}$ and the third term depends only on σ_t.

In order to prove that the first term in (34) can be uniquely minimised, we write it in a dual formulation, as discussed in Villani (2003). It can be proved that

$$\int_{\mathbb{R}^2} -\tilde{\Psi}\sigma \mathrm{d}\Upsilon \mathrm{d}Z + \int_0^H \int_0^{\tilde{\rho}(z)} -\tilde{P}(s,z)e(s)\mathrm{d}s\mathrm{d}z, \qquad (35)$$

can be uniquely maximised over the set of continuous functions $\tilde{\Psi}, \tilde{P}$ such that

$$\tilde{P}(s,z) + \tilde{\Psi}(\Upsilon, Z) \geq s\Upsilon + zZ \text{ for all} \qquad (36)$$
$$((s,z),(\Upsilon,Z)) \in D_{\tilde{\rho}} \times \mathbb{R}_+^2.$$

At the solution the inequality in (36) becomes an equality and the value is the value of $s\Upsilon + zZ$ to be used in the first term of (34), so that $\bar{I}[\sigma_t](\tilde{\rho})$ is defined for a given $\tilde{\rho}$.

The complete minimization problem can then be written as the calculation of

$$H_*(\sigma_t) = \inf_{\tilde{h}} \bar{I}[\sigma_t](\tilde{h}). \qquad (37)$$

Using (34) and (35), we can write

$$H_*(\sigma_t) = \sup\left(\int_{\mathbb{R}^2} \left(\frac{\Upsilon}{2r_0^2} - \Omega\sqrt{\Upsilon} - \tilde{\Psi} \right) \sigma \mathrm{d}\Upsilon \mathrm{d}Z + \qquad (38) \right.$$
$$\left. \inf_{\tilde{\rho} \in \mathcal{H}_I} \int_0^H \int_0^{\tilde{\rho}(z)} \left(\frac{\Omega^2 r_0^2}{2(1-2sr_0^2)} - \tilde{P}(s,z) \right) e(s)\mathrm{d}s\mathrm{d}z \right).$$

Here, $\mathcal{H}_I$ consists of all measurable functions $\tilde{\rho} : [0,H] \longmapsto [0,1/(2r_0^2)]$. The supremum in (38) is taken over the same set of continuous functions $\tilde{\Psi}, \tilde{P}$ as is defined in (36).

It is proved in Cullen and Sedjro (2013) that there is a unique minimiser in (37). The proof is carried out by solving the maximisation problem (38). Write ρ_t for the minimiser in (37) for a given σ_t and Ψ_t, P_t for the functions that maximise (38). It is proved that the maximiser corresponds to equality in (36), so that Ψ_t, P_t are Legendre transforms of each other as required at the end of section 2.2. It is proved that Ψ_t satisfies (22) together with (9). It is also proved that Ψ_t, P_t are convex and so $\nabla_{s,z} P_t$ is invertible almost everywhere. Provided that σ_t is bounded, so that a finite amount of mass is not associated with a single value of (Υ, Z), the solutions satisfy (21) and the inverse relation (23). This means that the mass of fluid specified in (Υ, Z) coordinates completely fills the physical domain D_{ρ_t}. However, ρ_t may be zero for some values of $z \in [0,H]$. It is proved that ρ_t is monotonically increasing in z if the region Δ in which the mass is specified is contained in $Z > 0$. This corresponds to the original assumption $\theta' > 0$ in (1). In the original coordinates (r,z) this means that the boundary $r = \varsigma(z)$ is monotonically increasing in z. The solution provides a rigorous solution of the two-dimensional elliptic problem solved by Craig (1991), eq. (64) though Craig did not use the same boundary conditions.

3.2. Properties of the solution

The mathematical problem that is solved in the preceding subsections shows that, given the mass of fluid with given bounded values of angular momentum and potential temperature, the fluid can be uniquely arranged to give an axisymmetric vortex in an unbounded ambient fluid at rest. However, the vortex may not fill the depth of the domain.

It is simplest to study the implications of the results using original physical coordinate r. First consider the implications of the result at the boundary $r = \varsigma_t$ of the physical domain Γ_{ς_t}. The discussion above shows that ς_t is monotonically increasing in z.

Outside the physical domain $\varphi_t = 0$, so that (5) implies using (11) that $\nabla P_t = (\Omega^2 r^4, 0)$. In general $\nabla P_t = (\Upsilon, Z) = ((u_t r + \Omega r^2)^2, g\theta_t'/\theta_0)$. The assumption that the support of σ_t, $\Delta_t \subset \mathbb{R}_+^2$ is compact means that Υ is less than some Υ_m for all points in Δ_t. Thus the vortex in physical space is bounded by $r = r_m$ where $\Omega^2 r_m^4 = \Upsilon_m$. Since θ_t' has been chosen to be positive, the invertibility of ∇P_t at $r = \varsigma_t$ implies that ς is monotonically nondecreasing in z. It also implies that $\varphi_t > 0$ within Γ_{ς_t}. Thus $\partial\varphi_t/\partial r$ will be negative as r approaches ς_t, which implies that $u_t < 0$. Thus the flow must become anticyclonic at the vortex boundary. This is an artificial restriction resulting from embedding the vortex in an ambient fluid at rest. More realistically, as discussed in the Introduction, u_t would be less cyclonic at the boundary than the ambient flow.

We first illustrate the nature of a stable vortex with a free boundary. At $z = 0$ we construct a vortex with radius $\varsigma(0) = r_1$. Using (11), we choose $\Upsilon = \Upsilon_1$ throughout the vortex, where $\Upsilon_1 = \Omega^2 r_1^4$. Then

$$u = \frac{1}{r}\Omega(r_1^2 - r^2). \qquad (39)$$

Eq. (1) gives

$$\frac{\partial\varphi}{\partial r} = \frac{\Omega^2}{r^3}(r_1^4 - r^4). \qquad (40)$$

Integrating this and using the boundary condition (3) gives

$$\varphi(r,0) = \Omega^2\left(r_1^2 - \frac{1}{2}\left(\frac{r_1^4}{r^2} + r^2\right)\right). \qquad (41)$$

Now extend this to a three-dimensional vortex by setting $Z = g\theta'/\theta_0 = g\hat{\theta}'z/H$, where $\hat{\theta}'$ is a constant so that the static stability is constant. Then

$$\varphi(r,z) = \varphi(r,0) + \frac{1}{2}g\hat{\theta}'z^2/H : \quad \text{for} \quad r \leq r_1. \qquad (42)$$

Using (41), (42) and the boundary condition $\varphi = 0$ at $r = \varsigma(z)$ gives

$$\Omega^2\left(r_1^2 - \frac{1}{2}\left(\frac{r_1^4}{\varsigma(z)^2} + \varsigma(z)^2\right)\right) + \frac{g\hat{\theta}'}{2H}z^2 = 0, \qquad (43)$$

which determines $\varsigma(z)$.

This solution is illustrated in Fig. 1 which shows plots of u against r and ς against z. r_1 is chosen to be 100km rather than a more representative choice for the overall radius of a tropical cyclone of 1000km. The solution at $z = 0$ shows azimuthal winds of about 50ms^{-1}. If r_1 is chosen to be 1000km, then the maximum azimuthal wind is about 5000ms^{-1}, indicating that it is not necessary for the angular momentum from the outer part of the vortex to be brought into the centre in order to explain observed tropical cyclone intensities. (39) shows that u will become increasingly negative in the outer part of the vortex $r > r_1$ as z increases. This is an artifact of the simple data used to construct the illustration.

We now illustrate the solution procedure based on the construction of a vortex where the mass σ_t is given as a function of Υ and Z. We thus have to find $\varsigma_t(z)$. The total mass of the fluid has to be the same in both (Υ, Z) and physical (r, z) coordinates. Since the vortex fills D_{ς_t}, this means that

$$\int_{\Delta_t} \sigma_t \mathrm{d}\Upsilon \mathrm{d}Z = \int_{D_{\varsigma_t}} r\mathrm{d}r\mathrm{d}z. \qquad (44)$$

However, ς_t may be zero for some values of z.

Next consider the nature of ς. The condition $\varphi_t = 0$ applied at $r = \varsigma_t(z)$ implies

$$\frac{\partial\varphi_t}{\partial r}\frac{\partial\varsigma_t}{\partial z} + \frac{\partial\varphi_t}{\partial z} = 0. \qquad (45)$$
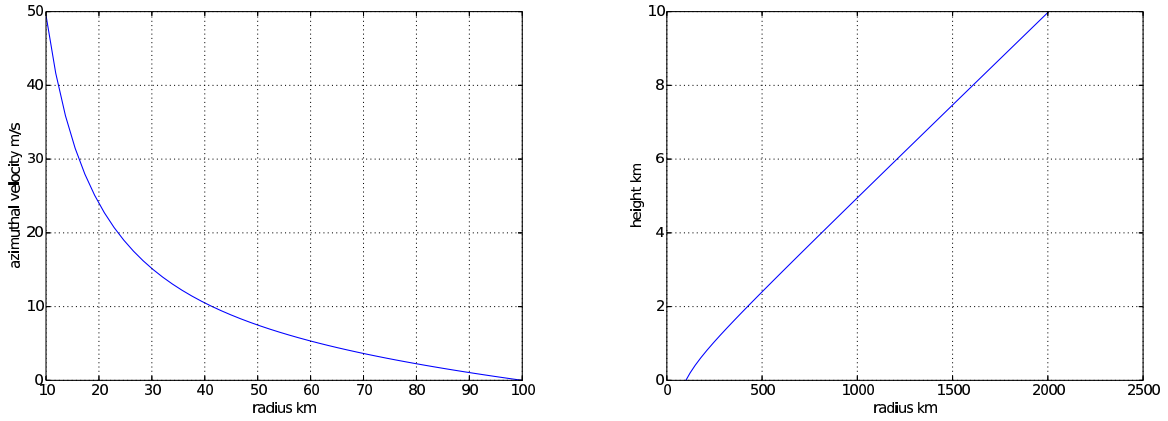
Figure 1. Left: Azimuthal velocity profile (ms^{-1} at $z = 0$ given by Eq. 39 with $\Omega = 0.5$E-4 s^{-1}, $r_0 = 10$km, $r_1 = 100$km. Right: Vertical profile of ς (km) given by Eq. 43 with additionally g=10 ms^{-2}, H=10 km

Using (1) then gives

$$\left(\frac{u_t^2}{\varsigma_t} + 2\Omega u_t\right)\frac{\partial \varsigma_t}{\partial z} + g\frac{\theta_t'}{\theta_0} = 0.$$

This can be rewritten as

$$\left(\frac{1}{\varsigma_t^3}\Upsilon - \Omega^2\varsigma_t\right)\frac{\partial \varsigma_t}{\partial z} + Z = 0.$$

At a state of rest, $\Upsilon = \Omega^2 r^4$ so at $r = \varsigma_t$ write $\Upsilon' = \Upsilon - \Omega^2\varsigma_t^4$. Then

$$\frac{\partial}{\partial z}\varsigma_t^{-2} = 2\frac{Z}{\Upsilon'}. \tag{46}$$

If ς_t increases with z then Υ' has to be negative as expected from the condition $u_t < 0$ at $r = \varsigma_t$ as found above.

To illustrate the solution, choose $Z = g\hat{\theta}'z/H$ and Υ' equal to a negative constant, and then

$$\varsigma_t^{-2} = \varsigma_t^{-2}(0) + g\hat{\theta}'z^2/(\Upsilon'H). \tag{47}$$

Since ς_t has to be positive for $0 \leq z \leq H$, (47) shows that Υ' has to be chosen to be sufficiently negative. Then σ_t satisfies

$$\int_{\Delta_t}\sigma_t \mathrm{d}\Upsilon \mathrm{d}Z = \frac{1}{2}\int_0^H \left(\frac{\Upsilon'H}{\Upsilon'H\varsigma_t^{-2}(0) + gz^2}\right)\mathrm{d}z - \frac{1}{2}r_0^2 H. \tag{48}$$

Noting that $\varsigma_t(0) \geq r_0$ and $\Upsilon' < 0$, $\varsigma_t(0)$ can be found to ensure positivity of (48) if $-gH > \Upsilon'r_0^{-2}$. The right hand side of (48) can then be made arbitrarily large by letting $\varsigma_t(0)$ approach $\sqrt{-gH/\Upsilon'}$. Thus the $\varsigma_t(0)$ can be chosen to fit any desired value of the total mass on the left hand side provided that

$$\int_{\Delta_t}\sigma_t \mathrm{d}\Upsilon \mathrm{d}Z \geq \frac{1}{2}\int_0^H \left(\frac{\Upsilon'H}{\Upsilon'Hr_0^{-2} + gz^2}\right)\mathrm{d}z. \tag{49}$$

If the total mass is smaller than the expression on the right hand side of (49), then we set

$$\varsigma_t = r_0: \quad z < z_1, \tag{50}$$

$$\varsigma_t = \sqrt{-\frac{\Upsilon'H}{\Upsilon'Hr_0^{-2} + g(z^2 - z_1^2)}}: \quad z \geq z_1,$$

which is consistent with (46).

The effect of this is illustrated in Fig. 2. Choosing $\hat{\theta}' = 0.1$, which implies a potential temperature excess of 30K at the top of the troposphere, and Υ'=-2.5E12, which implies a 5% angular

momentum deficit compared to the rest state value at a radius of about 200km, means that the rate of increase of vortex diameter with height is very sensitive to the radius. The situation where the vortex does not reach the ground is illustrated. For a more realistic choice of 100km for the bottom level radius a much larger value of the angular momentum deficit would be required to allow (49) to be solved. If a value corresponding to a 5% deficit at a 500km radius is used, the radius at the top of the troposphere becomes 330km. The vertical profile of ς shown in Fig. 2 is quite different from that in the example of Fig. 1 because the assumption of uniform Υ is replaced by the assumption that the angular momentum deficit Υ' is uniform with height.

These examples are limited by the need to choose a uniform angular momentum or angular momentum deficit in order to allow analytic solution. This is not very realistic. In order to simulate a realistic hurricane-like vortex which extends through the vertical domain, it is necessary to choose a sufficiently large total σ_t, as illustrated above, and to choose σ_t small for small Υ so that large values of the angular momentum Υ are mapped onto small values of r.

In order to understand the implications further, consider the two-dimensional case where there is no variation in z. The variational problem solved by Cullen and Sedjro becomes highly degenerate in this case as we illustrate. The angular momentum is given by $\sqrt{\Upsilon}$. A stable vortex is given by choosing s to be a monotonically increasing function of Υ such that $e(s)\partial s/\partial \Upsilon = \sigma_t$. Suppose the maximum angular momentum of the fluid specified to be in the vortex is $\sqrt{\Upsilon_m}$. The boundary of the vortex is given by $r = \varsigma_t$, and so the area of the vortex will be given by $\frac{1}{2}\varsigma_t^2$. This must be equal to the integral of σ_t over $\Delta_t = [0, \Upsilon_m]$. The angular momentum of the ambient fluid is Ωr^2 for $r \geq \varsigma_t$. Applying the condition that the angular momentum increases with r at the vortex boundary $r = \varsigma_t$ implies that $\Upsilon_m \leq \Omega^2\varsigma_t^4$, so that $\varsigma_t \geq \Omega^{-\frac{1}{2}}\Upsilon_m^{\frac{1}{4}}$. If the specified total mass is too small, then ς_t will be smaller than this. Sufficient additional mass from the ambient fluid has then to be incorporated into the vortex so that $\varsigma_t = \Omega^{-\frac{1}{2}}\Upsilon_m^{\frac{1}{4}}$. This imposes a restriction on σ_t which is not present in the three-dimensional problem.

4. Solution of the evolution equations

4.1. Theoretical results

We now solve equations (24), (27) and (23) in time, given an initial mass distribution σ_0 in angular momentum and isentropic coordinates. In order to solve (24) we have to calculate V_t from σ_t at each t as described in section 2.2.
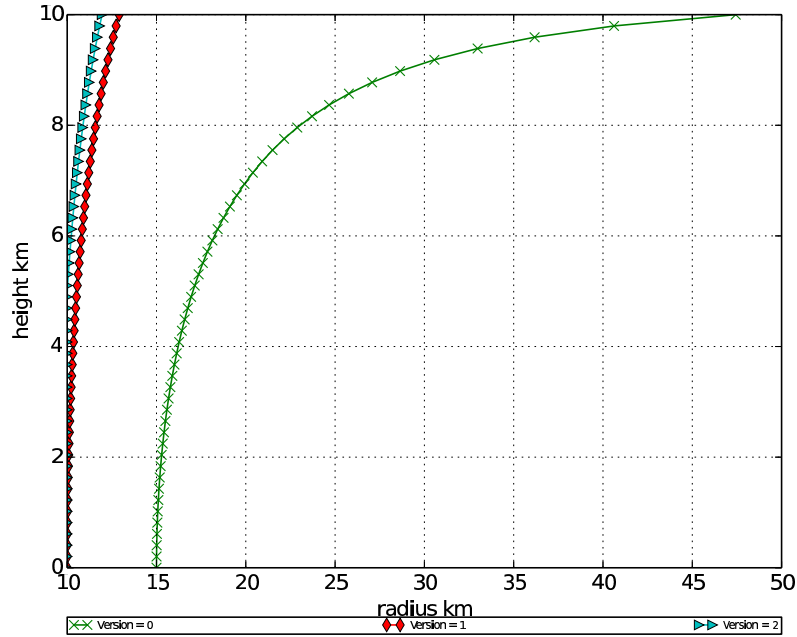
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Figure 2. Vertical profiles of vortex radius $\varsigma(z)$ (km) given by Eq. 49 for various choices of total mass, setting $\Omega = 0.5E-4\ s^{-1}$, $\hat{\theta}' = 0.1$ and $\Upsilon' = $-2.5E12m^{4}s^{-2}.

As noted in section 2.1, physical applicability requires choosing σ_0 so that $u \geq 0$ at $r = r_0$. This will be assured if the domain Δ_0 where σ_0 is non-zero satisfies

$$\Delta_0 = [(\Upsilon, Z) : \Omega^2 r_0^4 \leq \Upsilon \leq \Upsilon_m < \infty, 0 \leq Z \leq Z_m < \infty], \tag{51}$$

for some Υ_m, Z_m.

In Cullen and Sedjro (2013), equation (24) is solved for particular classes of angular momentum forcing F and thermal forcing S. The main difficulty is that the forcing terms are specified as functions of physical space but, as shown in (27), have to be applied as functions of angular momentum and potential temperature, Υ and Z. Since the mapping from (Υ, Z) to (r, z) is unknown until the problem has been solved, the results are highly non-trivial. However, some types of physical forcing, for instance those resulting from air-sea interaction, are naturally specified in physical coordinates. The lack of prior knowledge of the mapping leads to some severe restrictions on what can be proved. These come from the requirements that Υ is an inherently positive quantity and that Z has been assumed to be positive to distinguish it from the ambient fluid. It is likely that the restriction on Z could be relaxed, though the proof of the results in section 3.1 would have to be modified, and the nature of $\varsigma(z)$ would change.

Some types of forcing, such as latent heat release, could be more naturally imposed as a function of Υ and Z. In that case it is quite easy to solve (24) because V_t as defined in (25) will be known explicitly from (27).

Equation (24) is solved under two different sets of assumptions. In the first case, $\sigma(0, \cdot)$ is assumed to be bounded, so that the results of section 3.1 mean that (23) holds, as well as (21). Essentially this means that the transformation between physical coordinates (s, z) and angular momentum and potential temperature coordinates (Υ, Z) is invertible. If $\sigma(0, \cdot)$ contains Dirac masses, then there will be values of (Υ, Z) where a single value of angular momentum and potential temperature is mapped to a finite region of physical space. Cullen and Sedjro prove that if

1. $0 \leq F, \frac{g}{\theta_0} S \leq M$ for some positive constant M,
2. $\frac{\partial F}{\partial z} = \frac{\partial S}{\partial r} = 0$,

3. $\frac{\partial F}{\partial r}, \quad \frac{\partial S}{\partial z} > 0$,

then σ remains bounded as it evolves in time under the action of the forcing terms F and S.

These restrictions are very artificial, for instance they require that diabatic heating increases with height which is the reverse of the usual case. If these restrictions are not observed, it is possible that Dirac masses will be created in σ, corresponding to well-mixed layers of uniform potential temperature and absolute angular momentum. This is definitely possible in the real system and is illustrated in the computations of Shutts *et al.* (1988).. If this happens, then Cullen and Sedjro prove that (24) can still be solved, and the solutions will respect (21), if F and S are continuous, bounded and non-negative. However, the map from (Υ, Z) to physical space will be multivalued. The artificial restrictions that $F, S \geq 0$ are required because Υ is an inherently positive quantity and Z is assumed to be positive as discussed above.

In the results of Shutts *et al.* (1988), which showed an eyewall discontinuity and a strong low-level vortex, they started by defining a state of rest in hydrostatic balance in a fixed domain $D_{\varsigma_t} = [r_0, \varsigma_t] \times [0, H]$. This corresponds to choosing $\sigma_t = \frac{1}{4\Omega\sqrt{\Upsilon}}$ over a domain $\Delta_0 = [\Omega^2 r_0^4, \Upsilon_m] \times [0, H]$. This was represented discretely by assigning masses σ_i to points $(\Upsilon_i(0), Z_i(0))$ in Δ_0. The results illustrated in their paper were obtained by adding a single time increment S to Z_i at points i with small values of r and z, so that $Z_i(t) = Z_i(0) + S$. This gives a discrete solution of (24) with $F = 0$ and with $S > 0$ for small values of r and z. This choice is consistent with the second set of assumptions used by Cullen and Sedjro, though the smoothness assumption is not relevant for a discrete scheme. Mapping the solution at t back to physical space produced a low level vortex bounded by an eyewall discontinuity, and an upper level anticyclonic lens representing the convective outflow. A number of experiments were carried out with choices of region with different aspect ratios where S was non-zero, and also with regions where $S < 0$ to represent evaporative cooling. In the latter case, the values of S were chosen to ensure that Z_i remained positive which is required for the results of Cullen and Sedjro to be valid.

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4.2. Stability of the solutions

The requirement that absolute angular momentum increases with radius ensures that a vortex whose evolution is described by (24) is stable to axisymmetric perturbations. In order for the evolution to be physically relevant, it is also necessary for the solutions to be stable to non-axisymmetric disturbances. We therefore consider what distributions of σ_t as a function of Υ and Z correspond to vortices which are stable to non-axisymmetric disturbances. The study of Schubert *et al.* (1999) shows that the evolution of a barotropic vortex in a non-divergent rotating fluid from initial data consisting of an annulus of vorticity with a small azimuthal perturbation goes towards a vortex concentrated at the origin. However, conservation of energy and angular momentum require that some of the initial vorticity is left behind as filamentary structures outside the core vortex. This evolution was discussed in terms of either a minimum enstrophy hypothesis or a maximum entropy hypothesis.

Burton and Nycander (1999) analysed the nonlinear stability of a three-dimensional quasi-geostrophic vortex to perturbations which rearranged and/or mixed the potential vorticity. The inclusion of mixing is necessary to make the extremisation problem well posed. In the case where there is no background shear flow, the stable states are axisymmetric and represent maximisers of the energy. If an axisymmetric vortex is stable to axisymmetric perturbations, but unstable to non-axisymmetric perturbations, it is expected that non-axisymmetric perturbations will grow, equilibrate and the solution will evolve to a new axisymmetric state with the vorticity mixed, as discused by Smith and Montgomery (1995) and illustrated in the computations of Schubert *et al.*. Thus, in particular, a stable state can be characterised as an extremum of the energy with respect to axisymmetric perturbations which rearrange and/or mix vorticity and conserve total angular momentum. Similar results were obtained by Cullen and Douglas (2003) for straight geostrophic flows.

We now show how our analysis is equivalent to Burton and Nycander's in the barotropic case. We assume that the vortex is coherent, so that it occupies a region $0 \leq r \leq \varsigma$ where $\int_\Delta \sigma \mathrm{d}\Upsilon = \frac{1}{2}\varsigma^2$. The angular momentum conservation requirement is then that $\int_0^\varsigma \sqrt{\Upsilon} r \mathrm{d}r$ is conserved. The absolute vorticity, which is assumed positive, is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(ur + \Omega r^2\right) = \frac{1}{r}\frac{\partial \sqrt{\Upsilon}}{\partial r}$$

Rearranging or mixing absolute vorticity in an axisymmetric way thus corresponds to mixing $\frac{1}{r}\frac{\partial \sqrt{\Upsilon}}{\partial r}$ allowing for mass weighting. Thus if we mix two annular rings of vorticity at radii r_1 and r_2, both with width δr, we will obtain a single ring with vorticity

$$\frac{1}{(r_1 + r_2)\delta r}\left[\sqrt{\Upsilon}\right]_{r_1 - \frac{1}{2}\delta r}^{r_2 + \frac{1}{2}\delta r}. \tag{52}$$

Since $\sqrt{\Upsilon}$ has to be monotonically increasing in r, the maximum and minimum values of $\sqrt{\Upsilon}$ will not be changed by the mixing. However, imposing conservation of the total angular momentum will then require both to be changed by the same amount. Complete mixing will give a profile where $\partial/\partial r(\sqrt{\Upsilon})$ is a constant. If we allow the mixing to include the region $r \leq r_0$, where $u = 0$ as assumed in section 2.1, then we will have $\Upsilon = 0$ at $r = 0$ so that $\sqrt{\Upsilon}$ is proportional to r, giving solid body rotation. However, this will not in general be compatible with conservation of total angular momentum.

A maximum energy state corresponding to a cyclonic vortex is obtained by rearranging the vorticity to be axisymmetric and monotonically decreasing, which allows the largest values of u to

be obtained over the maximum region, given $u = 0$ at $r = 0$. This is the solution found by Burton and Nycander (1999). It is like the end state of the simulations in Schubert *et al.* (1999), though this state could not be reached because the energy would have been larger than the initial energy. The initial data used by Schubert *et al.* and the vortex analysed by Smyth and McWilliams (1998) do not satisfy this monotonicity condition. This is consistent with the instability to non-axisymmetric disturbances found by Smyth and McWilliams. The stability condition derived by Montgomery and Shapiro (1995) for three-dimensional vortices using linear theory reduces to the requirement that $\frac{\partial \sqrt{\Upsilon}}{r\partial r}$ is either monotonically increasing or decreasing in r, which is consistent with Burton and Nycander's result.

In our case, the evolution is given by equation (24) which states that the mass distribution σ evolves as a function of the square of the absolute angular momentum Υ. We can write σ as

$$\sigma = r\frac{\partial r}{\partial \Upsilon} = \frac{1}{2}\frac{\partial r^2}{\partial \Upsilon} \tag{53}$$

which follows from (18) and (21). Mixing σ as a function of Υ thus implies that r^2 is a linear function of Υ. If $\Upsilon = 0$ at $r = 0$, then r^2 will be proportional to Υ, which implies solid body rotation as found above. However, imposing angular momentum conservation will in general lead to a state with $u \neq 0$ ar $r = r_0$.

Schubert *et al.* (1999)'s results do not suggest why there is an 'eye' with small relative vorticity at the centre of the vortex, which would correspond to a vortex which was unstable to non-axisymmetric perturbations. An eye was found in the three-dimensional vortex derived by Shutts *et al.* (1988). The analysis of Montgomery and Shapiro (1995) suggests that stability of such a vortex requires the potential vorticity to vary monotonically with r on each isentropic surface. It would be desirable to extend the nonlinear stability analysis discussed above to this case using potential vorticity instead of vorticity. However, there is now the additional difficulty that the analysis would have to be carried out with the full Euler equations, whose solutions cannot be completely described by the potential vorticity. In particular, the stability of an energy maximiser with respect to rearrangements of potential vorticity could be compromised by the radiation of energy in inertio-gravity waves. This possibility is discussed by Schecter and Montgomery (2006), who derive conditions under which such radiation is inhibited. This requires regimes where there is a timescale separation between the evolution of perturbations to the vortex and the frequency of inertio-gravity and acoustic waves. Burton and Nycander (1999) state that their quasi-geostrophic analysis is only valid in regimes where the quasi-geostrophic equations remain accurate for an extended time, so that gravity wave radiation is small.

5. Physical implications

The results stated in section 3.1 mean that we can uniquely find an axisymmetric vortex where the angular momentum and potential temperature are given on fluid parcels that is embedded in an ambient fluid at rest with specified potential temperature. The proof does not require any more physics than the original results of Fjortoft, Eliassen and Kleinschmidt. The stability condition requires the azimuthal velocity at the vortex boundary to be anticyclonic, though the bulk of the vortex can, of course, be cyclonic. This boundary restriction is an artefact of blending two different simple solutions together. It is likely that this blending procedure could be used to insert a vortex into a less trivial large-scale flow, stability of the boundary of the vortex would require a cyclonic discontinuity at the boundary.

It is then shown that the evolution of this vortex under axisymmetric forcing is also well-posed, including cases where

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the forcing creates well-mixed regions within the vortex. This will certainly happen in the presence of convection, which is a key mechanism. If the axisymmetric forcing is regarded as representing the effect of non-axisymmetric features on the axisymmetric flow, this supports the ideas that non-axisymmetric forcing is important for maintaining the observed axisymmetric structure. The artificial restrictions in the proof could be removed if the forcing were specified as a function of angular momentum and potential temperature, which is probably more realistic than assuming that it is specified as a function of the physical coordinates.

Before considering non-axisymmetric dynamics, it is worth considering whether axisymmetric solutions are stable to non-axisymmetric disturbances. We demonstrated for a two-dimensional vortex that this would be true under a condition that the vortex was a maximum energy rearrangement of the absolute vorticity, as in the case studied by Burton and Nycander (1999). This also showed that a two-dimensional vortex with a central eye would not be stable to non-axisymmetric disturbances, as found by Schubert *et al.* (1999). It is likely that three-dimensional effects are critical for the stability of observed tropical cyclone vortices.

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