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Homogenization for a class of integral functionals in spaces of probability measures

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Abstract

We study the homogenization of a class of actions with an underlying Lagrangian $\mathcal{L}$ defined on the set of absolutely continuous paths in the Wasserstein space $\mathcal{P}_p(\mathbb{R}^d)$. We introduce an appropriate topology on this set and obtain the existence of a Γ-limit of the rescaled Lagrangians. Our main goal is to provide a representation formula for these Γ-limits in terms of the effective Lagrangians. This allows us to study not only the "convexity properties" of the effective Lagrangian, but also the differentiability properties of its Legendre transform restricted to constant functions. For the case $d > 1$ we obtain partial results in terms of an effective Lagrangian defined on $L^p((0,1)^d;\mathbb{R}^d)$. Our study provides a way of computing the limit of a family of metrics on the Wasserstein space. The results of this paper can also be applied to study the homogenization of variational solutions of the one-dimensional Vlasov–Poisson system, as well as the asymptotic behavior of calibrated curves (Fathi (2003) [6], Gangbo and Tudorascu (2010) [12]). Whereas our study for the one-dimensional case covers a large class of Lagrangians, that for the higher dimensional case is concerned with special Lagrangians such as the ones obtained by regularizing the potential energy of the d-dimensional Vlasov–Poisson system.

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1. Introduction

Let us consider a continuous path defined on the set of probability measures on the phase space $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) : t \in [0, T] \to f_t \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ and consider rescaled actions defined on such paths

$$\mathbf{A}^\epsilon(f) := \frac{1}{2}\int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d} |v|^2 f_t(\mathrm{d}x, \mathrm{d}v) - \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}^d\times\mathbb{R}^d} W\left(\frac{x-y}{\epsilon}\right) f_t(\mathrm{d}x, \mathrm{d}v) f_t(\mathrm{d}y, \mathrm{d}w),$$

whose critical points satisfy a nonlinear Vlasov system. When f_t is the push forward of a probability density ϱ_t defined on the physical space $\mathbb{R}^d$ by a map of the form $\mathbf{id} \times \mathbf{v}_t$ defined on $\mathbb{R}^d \times \mathbb{R}^d$ (where $\mathbf{id}$ is the identity map), the above actions are reduced to

$$\frac{1}{2}\int_0^T \int_{\mathbb{R}^d} |\mathbf{v}_t(x)|^2 \varrho_t(\mathrm{d}x) - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W\left(\frac{x-y}{\epsilon}\right)\varrho_t(\mathrm{d}x)\varrho_t(\mathrm{d}y). \tag{1.1}$$

In this work we study the Γ-limit of a class of functionals that includes those appearing in (1.1) and their link with the homogenization of systems of PDEs of the nonlinear Vlasov type. We endow $AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$, the set of p-absolutely continuous paths on $\mathcal{P}_p(\mathbb{R}^d)$ [2], with a topology τ_w for which the sublevel sets of our actions are pre-compact (cf. Section 3). More precisely, τ_w is defined as follows: we say that $\{\sigma^n\}_n \subset AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ satisfying the condition

$$\sup_{n\in\mathbb{N}} \int_0^T \left(W_p^p(\sigma_t^n, \delta_0) + |(\sigma^n)'|_t^p\right)\mathrm{d}t < \infty$$

τ_w-converges to σ in $AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ if

$$\lim_{n\to\infty} \int_0^T W_1(\sigma_t^n, \sigma_t)\mathrm{d}t = 0.$$

Here, W_q is the L^q-Wasserstein distance for $1 \le q < \infty$ and $|\sigma'|$ is the metric derivative of $\sigma \in AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$. For such paths σ and the associated (unique) velocity $\mathbf{v}$ of minimal $\mathbf{L}^p(\sigma)$-norm [2], we define

$$\mathcal{F}^\epsilon(\sigma) = \int_0^T \mathcal{L}(D_\#^{1/\epsilon}\sigma_t, \mathbf{v}_t \circ D^\epsilon)\mathrm{d}t, \tag{1.2}$$

where D^ϵ is the map of $\mathbb{R}^d$ onto itself defined by $D^\epsilon x = \epsilon x$. The notation $\mathbf{L}^p$ replaces L^p when dealing with $\mathbb{R}^d$-valued p-integrable functions. If $d = 1$, we stick with the classical notation L^p. We have denoted the push forward operator by $\# : \mathcal{P}_p(\mathbb{R}^d) \to \mathcal{P}_p(\mathbb{R}^d)$. The topology τ_w seems to be too weak to directly provide information on the $\Gamma(\tau_w)$-limit of the functionals $\mathcal{F}^\epsilon$. Our strategy is to introduce a stronger topology τ on a subset of $AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$, hoping to be able to extract enough information from the $\Gamma(\tau)$-limit of the functionals $\mathcal{F}^\epsilon$ in order to draw some conclusions on the $\Gamma(\tau_w)$-limit. The strategy completely paid off when $d = 1$, and produced partial results in the multi-dimensional case. The τ-topology that we consider is inspired from the well-known isometry between the convex cone of nondecreasing functions in $L^p(0,1)$ and the Wasserstein space $\mathcal{P}_p(\mathbb{R})$. Define $X := (0,1)^d$. We consider the set of pairs $(\sigma, \mathbf{v})$ such that $\sigma \in AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$, $\mathbf{v}$ is its minimal norm velocity and there exists $\phi \in AC^p(0,T;\mathbf{L}^p(X))$ satisfying the following properties: $\dot{\phi}_t = \mathbf{v}_t \circ \phi_t$ for almost every $t \in (0,T)$ and ϕ_t pushes ν_0 (the Lebesgue measure on the unit cube X) forward to σ_t for all $t \in [0,T]$. When these conditions are satisfied, we say that ϕ is a flow associated with $(\sigma, \mathbf{v})$. We denote by $\mathcal{S}_p(\mathbb{R}^d)$ the set of such special curves σ. When $d = 1$, $AC^p(0,T;\mathcal{P}_p(\mathbb{R}))$ and $\mathcal{S}_p(\mathbb{R})$ coincide [10]. Let $\{\sigma^n\}_n \subset \mathcal{S}_p(\mathbb{R}^d)$ and suppose $\mathbf{v}^n$ is the velocity of minimal norm for σ^n. We say that $\{\sigma^n\}_n$ τ-converges to σ in $\mathcal{S}_p(\mathbb{R}^d)$ if there exists a sequence $\{\phi\} \cup \{\phi^n\}_n \subset AC^p(0,T;\mathbf{L}^p(X))$ such that ϕ^n (and ϕ) is a flow associated with σ^n (and σ, respectively) and

$$\|\phi^n - \phi\|_{\mathbf{L}^p((0,T)\times Y)} \to 0 \quad \text{and} \quad \{\dot{\phi}^n\}_n, \{\phi^n\}_n \text{ are bounded in } \mathbf{L}^p((0,T)\times X)$$

for any $Y \Subset X$. We only consider Lagrangians $\mathcal{L}$ for which there exist real-valued functionals L defined on $\mathbf{L}^p(X) \times \mathbf{L}^p(X)$ such that $\mathcal{L}(\mu, \mathbf{v}) = L(M, \mathbf{v} \circ M)$ if $\mu \in \mathcal{P}_p(\mathbb{R}^d)$, $\mathbf{v} \in \mathbf{L}^p(\mu)$ and $M \in \mathbf{L}^p(X)$ is a map which pushes ν_0 to μ.

In the first part of this work we study the $\Gamma(\tau)$-limit of $\mathcal{F}^\epsilon$ or, equivalently, the $\Gamma(\tau_w)$-limit when $d = 1$, without imposing any convexity assumption on $L(M, \cdot)$. Our study on $\mathcal{S} := AC^p(0,T;\mathcal{P}_p(\mathbb{R}))$ is greatly facilitated by the well-known isometry mentioned above. We considerably exploit the established fact that if $\sigma \in \mathcal{S}$ and v is the velocity of minimal norm for σ, then $\dot{\phi}_t = v_t \circ \phi_t$ for almost every $t \in (0,T)$ (cf. [10]). Here, ϕ_t is the monotone nondecreasing map that pushes ν_0 forward to σ_t. We establish, under some mild continuity assumptions on L, a representation formula for $\mathcal{F}_w$, the $\Gamma(\tau_w)$-limit of $\mathcal{F}^\epsilon$, and for $\mathcal{F}$, the $\Gamma(\tau)$-limit of $\mathcal{F}^\epsilon$, in terms of the *effective Lagrangian* $\bar{L}$ of L:

$$\mathcal{F}(\sigma) = \int_0^T \bar{L}(v_t \circ \phi_t)\mathrm{d}t = \mathcal{F}_w(\sigma). \tag{1.3}$$

We extend some of the results obtained in the one-dimensional setting to the higher dimensional one, where the Lagrangians $\mathcal{L}^\epsilon$ are defined on the "tangent bundle" of the Wasserstein space $\mathcal{P}_p(\mathbb{R}^d)$. To simplify the arguments in the second part of this manuscript, we restrict ourselves to Lagrangians assuming the special form $\mathcal{L}(\mu, \xi) = \|\xi\|_\mu^p/p + \mathcal{W}(\mu)$, where $\mathcal{W}$ is a continuous, real-valued periodic function, in a sense to be specified.

For $\mu \in \mathcal{P}_p(\mathbb{R}^d)$ and $\xi \in \mathbf{L}^p(\mu)$ it is readily seen that the expression

$$\bar{\mathcal{L}}(\mu, \xi) = \liminf_{T\to\infty} \frac{\mathcal{C}_{0,T}(\mu_0, (T\xi)_\# \mu)}{T} \tag{1.4}$$

is independent of $\mu_0 \in \mathcal{P}_p(\mathbb{R}^d)$. Here, $\mathcal{C}_{0,T}(\mu_0, (T\xi)_\#\mu)$ is the infimum of $\int_0^T \mathcal{L}(\sigma_t, \mathbf{v}_t)\mathrm{d}t$ over the set of $(\sigma, \mathbf{v})$ such that $\sigma \in AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$, $\mathbf{v}$ is a velocity for σ and $\sigma_0 = \mu_0$, $\sigma_T = (T\xi)_\#\mu$. We refer to $\bar{\mathcal{L}}$ as the *effective Lagrangian* associated with $\mathcal{L}$. Note that no topology need be mentioned in connection with the definition of $\bar{\mathcal{L}}$ and so, this functional is intrinsic.

For the strong topology τ chosen in Section 5.3, existence of the $\Gamma(\tau)$-limit $\mathcal{F}$ of $\mathcal{F}^\epsilon$ is obtained by standard arguments. We show that

$$\mathcal{F}(\sigma) = \int_0^T \bar{\mathcal{L}}(\sigma_t, \mathbf{v}_t)\mathrm{d}t = \int_0^T \bar{L}(\mathbf{v}_t \circ \phi_t)\mathrm{d}t \tag{1.5}$$

for a convex functional $\bar{L} : \mathbf{L}^p(X) \to \mathbb{R}$, which is precisely the effective Lagrangian of L. Here $\mathbf{v}$ is the velocity of minimal norm for σ and ϕ is any flow associated with $(\sigma, \mathbf{v})$. What is obvious is that $\mathcal{F}_w \le \mathcal{F}$ on $\mathcal{S}_p(\mathbb{R}^d)$. The harder and more interesting question is that of how to obtain a representation formula for $\mathcal{F}_w$. In higher dimension, only for a dense subset of $\mathcal{P}_p(\mathbb{R}^d)$ were we able to establish the representation formula

$$\mathcal{F}_w(\sigma) = \mathcal{F}(\sigma) = \int_0^T \bar{L}(\mathbf{v}_t \circ \phi_t)\mathrm{d}t. \tag{1.6}$$

More precisely, we have proved that (1.6) holds if the path σ is an average of finitely many Dirac masses and is piecewise a geodesic in $\mathcal{P}_p(\mathbb{R}^d)$. We also establish this representation formula for σ of the form $\sigma_t = (t\xi)_\# \nu_0$ where $\xi \in \mathbf{L}^p(X)$. It remains unclear whether one can extend that result to paths of the form $\sigma_t = (\mathbf{id} + t\xi)_\# \nu_0$. This seems to be the obstruction to extending the representation formula to arbitrary elements of $AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$.

There is a vast literature in the finite dimensional setting, on the Γ-limit theory of functionals of the form $\int_0^T l(\mathbf{r}/\epsilon, \dot{\mathbf{r}})\mathrm{d}t$ where $\mathbf{r} : \mathbb{R} \to \mathbb{R}^d$ and so, the Lagrangian l corresponds to a system which has finitely many particles. One can view our study as an extension of the previous ones to encompass systems of infinitely many particles.

When $\mathcal{L}$ is appropriately chosen, critical points of $\mathcal{F}^\epsilon$ satisfy some interesting PDEs. For instance, let $|\cdot|_{\mathbb{T}^1}$ be the periodic metric on the one-dimensional torus $\mathbb{T}^1$ and set

$$\mathcal{L}(\mu, w) = \frac{1}{2}\|w\|_\mu^p - \frac{1}{4}\int_{\mathbb{R}^2} (|x-y|_{\mathbb{T}^1}^2 - |x-y|_{\mathbb{T}^1})\mu(\mathrm{d}x)\mu(\mathrm{d}y) \quad (\mu, w) \in \mathcal{T}\mathcal{P}_p(\mathbb{R}).$$

This is the Lagrangian for the one-dimensional Vlasov–Poisson system. Let $\sigma^{\epsilon *}$ be a critical point of $\mathcal{F}^\epsilon$ and let $v^{\epsilon *}$ be the velocity of minimal norm for $\sigma^{\epsilon *}$. The path $\sigma^{\epsilon,*}$ induces a path g^ϵ on $\mathcal{P}_p(\mathbb{R} \times \mathbb{R})$, which itself induces a path f^ϵ on $\mathcal{P}_p(\epsilon\mathbb{T}^1 \times \mathbb{R})$ as follows:

$$g_t^\epsilon = (\mathbf{id} \times v_t^{\epsilon *})_\# \sigma^{\epsilon,*}, \qquad f_t^\epsilon = \sum_{k \in \mathbb{Z}} g_t^\epsilon(x + \epsilon k, v).$$

According to [11], f^ϵ is a solution of the Vlasov–Poisson system, ϵ-periodic in the space variable:

$$\begin{cases} \partial_t f_t^\epsilon + v\partial_x f_t^\epsilon = \dfrac{1}{\epsilon}\partial_x P_t^\epsilon \partial_v f_t^\epsilon \\ 1 - \epsilon\partial_{xx} P_t = \epsilon\rho_t^\epsilon \\ \rho_t^\epsilon = \displaystyle\int_{\mathbb{R}} f^\epsilon(\cdot, \mathrm{d}v). \end{cases} \tag{1.7}$$

For variational solutions of (1.7), using standard arguments of the Γ-limit theory we have been able to characterize the points of accumulation of $\{f^\epsilon\}_\epsilon$.

Now let us consider $\mathcal{L}$ such that, given $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R})$, the infimum of $\mathcal{F}^\epsilon(\sigma)$ over the set of paths σ with endpoints μ_0 and μ_1 is a number $(\mathrm{dist}^\epsilon(\mu_0, \mu_1))^2$ such that dist^ϵ is a metric. As $AC^2(0, T; \mathcal{P}_2(\mathbb{R}))$ is a separable space, it is easy to obtain a subsequence $\{\mathrm{dist}^{\epsilon_n}\}_n$ of these metrics which converges pointwise to a limit dist. The Γ-limit of $\mathcal{F}^\epsilon$ can be used to write a representation formula for dist. We use these results to study homogenization of Hamilton–Jacobi equations on the Wasserstein space.

The class of Lagrangians covered by our study includes those of systems of two interacting species in motion on the real line:

$$\mathcal{L}_4(\mu_1, \mu_2, \xi_1, \xi_2) = \frac{1}{4}\|\xi_1\|_{\mu_1}^2 + \frac{1}{4}\|\xi_2\|_{\mu_2}^2 - \int_{\mathbb{R}^2} W(x-y)\mu_1(\mathrm{d}x)\mu_2(\mathrm{d}y).$$

Here, W is a periodic, bounded function defined on the real line. We also consider Lagrangians of systems of interacting one-species (resp. non-interacting) particles:

$$\mathcal{L}_2(\mu,\xi)=\mathcal{L}_4(\mu,\mu,\xi,\xi), \quad \text{resp.} \quad \mathcal{L}_1(\mu,\xi)=\|v\|_\mu^2/2-\int_{\mathbb{R}} W\mathrm{d}\mu.$$

According to the rule $\mathcal{L}(\mu,\mathbf{v})=L(M,\mathbf{v}\circ M)$, the above Lagrangians give rise to

$$\begin{aligned} L_2(M,N)&=\frac{1}{2}\|N\|^2-\frac{1}{2}\iint_{X\times X} W(Mx-My)\mathrm{d}x\mathrm{d}y,\\ \text{resp.}\quad L_1(M,N)&=\frac{1}{2}\|N\|^2-\int_X W(Mx)\mathrm{d}x, \end{aligned} \tag{1.8}$$

where $\|\cdot\|$ denotes the $L^2(X;\mathbb{R}^d)$-norm. Our strategy is to prove under suitable hypotheses that the action of $L^\epsilon(M,N):=L(M/\epsilon,N)$, converges (in the Γ-convergence sense) on curves in $\mathcal{H}$ to the action of a homogenized Lagrangian $\bar{L}$. Most of the techniques of E [19] apply to our infinite dimensional system as well. We can prove that $\bar{L}_1(N)=\int_X \bar{l}_1\circ N\mathrm{d}x$, where l_1 is the Lagrangian defined over $\mathbb{R}^2$ by $l_1(x,v)=v^2/2-W(x)$ [19,15] whose effective Lagrangian $\bar{l}_1$ is computed "explicitly" in [15].

Two important consequences will emerge: we obtain the homogenization for the time-dependent Hamilton–Jacobi equations on $(0,\infty)\times\mathcal{P}_2(\mathbb{R})$ associated with $\mathcal{L}^\epsilon$. Indeed, it is a remarkable fact that one is able to use a Γ-convergence result for the "extended" Lagrangians L^ϵ to make inferences about $\mathcal{L}^\epsilon$. This comes as a consequence of the fact that the optimal paths/characteristics in the variational solutions for the Hamilton–Jacobi equations considered here lie pointwise in the set of monotone nondecreasing functions in $L^p(0,1)$ (this was observed in [10]) if the endpoints do so.

Secondly, we will obtain the convergence of action-minimizing solutions for systems of the type

$$\begin{cases} \partial_t\rho^\epsilon+\partial_x(\rho^\epsilon v^\epsilon)=0\\ \partial_t(\rho^\epsilon v^\epsilon)+\partial_x\big[\rho^\epsilon(v^\epsilon)^2\big]=-\epsilon^{-1}\rho^\epsilon A_i^\epsilon \end{cases} \tag{1.9}$$

to minimizers of the action $\int_0^T \bar{L}_i(\dot{M})\mathrm{d}t$, for

$$A_1^\epsilon(x)=V'\left(\frac{x}{\epsilon}\right) \quad \text{and} \quad A_2^\epsilon(t,x)=\int V'\left(\frac{x-y}{\epsilon}\right)\rho^\epsilon(t,\mathrm{d}y),$$

respectively.

Periodic potentials for Vlasov or Vlasov–Poisson systems occur, for example, in the case of ion lattices [1,14]. Such systems are part of mainstream studies of models of particle interaction with electric fields [3,13,17] etc. The length scales involved are accounted for by the epsilon-periodicity assumption. As usual, it is then natural to ask what the effective behavior of such a system is (as epsilon goes to zero). Extending our results to more general Lagrangians (almost periodic, or even stationary ergodic) is an interesting problem, not addressed in this work.

2. Homogenization on $\mathbf{L}^p(X)$

For any open set $\Omega\subset\mathbb{R}$ we denote by $\mathcal{H}_\Omega$ the set of $\phi\in W^{1,p}(\Omega;\mathbf{L}^p(X))$ which admit extensions $\bar{\phi}\in W^{1,p}_{\mathrm{loc}}(\mathbb{R};\mathbf{L}^p(X))$, and we write $\mathcal{H}$ in place of $\mathcal{H}_{(0,T)}$ when there is no confusion.

Let L be a Lagrangian on $\mathbf{L}^p(X) \times \mathbf{L}^p(X)$ satisfying the growth conditions

$$c\|N\|^p \le L(M,N) \le C\big(1+\|N\|^p\big), \quad \text{for all } M,\ N \in \mathbf{L}^p(X), \tag{2.1}$$

where c, C are given positive constants. We set $L^\epsilon(M,N) = L(M/\epsilon, N)$, and assume L is 1-periodic in M, i.e.

$$L(M+Z,N) = L(M,N) \quad \text{for all } Z \in \mathbf{L}^p_{\mathbb{Z}}(X),\ M,\ N \in \mathbf{L}^p(X), \tag{2.2}$$

where we adopt the notation

$$\mathbf{L}^p_{\mathbb{Z}}(Q) = \{Z \in \mathbf{L}^p(X) : Z(X) \subset \mathbb{Z}^d\}. \tag{2.3}$$

Also, we will use the sets

$$\mathbf{L}_{\mathbb{Z}}(X) = \{Z : X \to \mathbb{R}^d \text{ Lebesgue measurable } : Z(Q) \text{ is a finite subset of } \mathbb{Z}^d\} \tag{2.4}$$

and

$$\mathbf{L}_{\mathbb{Q}}(X) = \{Q : X \to \mathbb{R}^d \text{ Lebesgue measurable } : Q(X) \text{ is a finite subset of } \mathbb{Q}^d\}. \tag{2.5}$$

Given an open set $\Omega \subset \mathbb{R}$, we identify $\mathbf{L}^p(\Omega \times X)$ with $L^p(\Omega; \mathbf{L}^p(X))$. We refer to $t \in \Omega$ as the time variable and often write ϕ_t in place of $\phi(t,\cdot)$. This should not be confused with the partial derivative $\partial_t \phi$. If $\phi \in \mathcal{H}_\Omega$ we identify $\dot\phi$, the Fréchet derivative of $t \to \phi_t$, with the distributional derivative of ϕ with respect to the time variable.

2.1. The effective Lagrangian in $\mathbf{L}^p(X)$

Given $M_a, M_b \in \mathbf{L}^p(X)$, $\epsilon > 0$ and $a < b$ we set

$$C^\epsilon_{a,b}(M_a, M_b) = \inf\left\{\int_a^b L\left(\frac{\phi_t}{\epsilon}, \dot\phi_t\right) \mathrm{d}t : \phi \in \mathcal{H}_{(a,b)},\ \phi_a = M_a,\ \phi_b = M_b\right\}$$

and write $C_{a,b}$ in place of $C^1_{a,b}$. Observe that if $a < c < b$ and $M_c \in \mathbf{L}^p(X)$, then

$$C_{a,b}(M_a, M_b) \le C_{a,c}(M_a, M_c) + C_{c,b}(M_c, M_b) \tag{2.6}$$

and so, by (2.1), whenever $M_1, M_2, N \in \mathbf{L}^p(X)$ and $S > 2$ we have

$$\begin{aligned} C_{0,S}(M_a, M_1 + SN) \le\ & C_{0,1}(M_a, M_b) + C_{0,(S-2)}(M_b, M_2 + (S-2)N) \\ & + C(1 + \|2N + M_1 - M_2\|^p). \end{aligned}$$

Hence,

$$\liminf_{S\to\infty} \frac{C_{0,S}(M_a, M_1 + SN)}{S} \le \liminf_{S\to\infty} \frac{C_{0,S}(M_b, M_2 + SN)}{S}. \tag{2.7}$$

Since (M_a, M_1) and (M_2, M_b) are arbitrary, we conclude that the two limits in (2.7) coincide and so,

$$\bar L(N) := \liminf_{S\to\infty} \frac{C_{0,S}(M_a, M_1 + SN)}{S} \tag{2.8}$$

is independent of $M_a, M_1 \in \mathbf{L}^p(X)$. If $\epsilon, T > 0$ and $\phi \in \mathcal{H}$ then

$$\epsilon \int_0^{\frac{T}{\epsilon}} L(\phi_s^\epsilon, \dot{\phi}_s^\epsilon)\mathrm{d}s = \int_0^T L\left(\frac{\phi_t}{\epsilon}, \dot{\phi}_t\right)\mathrm{d}t, \tag{2.9}$$

where $\epsilon\phi_s^\epsilon := \phi_{\epsilon s}$. Hence, $\epsilon C_{0,T/\epsilon}(\phi_0/\epsilon, \phi_T/\epsilon) = C_{0,T}^\epsilon(\phi_0, \phi_T)$ and so, in light of (2.8),

$$\bar{L}(N) = \liminf_{\epsilon\to 0^+} \frac{C_{0,T}^\epsilon(\epsilon M, TN)}{T} = \liminf_{\epsilon\to 0^+} \frac{C_{0,T}^\epsilon(0, TN)}{T}. \tag{2.10}$$

Remark 2.1. Suppose $M \in \mathbf{L}^p(X)$, $Z \in \mathbf{L}_{\mathbb{Z}}(X)$ and $T > 0$. Then $T\bar{L}(Z/T) \leq C_{0,T}(M, M + Z)$. In particular, if $Q \in \mathbf{L}^p(X)$ and $T > 0$ are such that $TQ \in \mathbf{L}_{\mathbb{Z}}(X)$, then $C_{0,T}(M, M + TQ) \geq T\bar{L}(Q)$.

Proof. Let $\phi \in \mathcal{H}$ be such that $\phi_0 = \phi_T = 0$. Extend $t \to \phi_t$ to obtain a periodic function on $\mathbb{R}$, that we still denote by ϕ. Note that $t \to L(M + t/TZ + \phi_t, Z/T + \dot{\phi}_t)$ is T periodic, equal to M at $t = 0$ and equal to $M + jZ$ at $t = jT$. Thus,

$$\begin{aligned} &\fint_0^T L(M + t/TZ + \phi_t, Z/T + \dot{\phi}_t)\mathrm{d}t \\ &\quad = \fint_0^{jT} L(M + t/TZ + \phi_t, Z/T + \dot{\phi}_t)\mathrm{d}t \geq \frac{C_{0,jT}(M, M + jZ)}{jT}. \end{aligned}$$

We let j tend to ∞, use (2.8) and take into consideration that $\phi \in \mathcal{H}$ is arbitrary and satisfies $\phi_0 = \phi_T$ to conclude the proof of the remark.

2.2. *Time and space discretization of elements of $\mathcal{H}$*

In this subsection we recall a lemma and corollary which are, in different forms, well-known. Suppose $\phi \in \mathcal{H}$. Fix an integer $m > 1$ and set $t_j = j/mT = ih$ for $j = 0, \dots, m$. For $s \in [t_i, t_{i+1}]$ we set

$$\phi_s^m := \left(1 - \frac{s - t_i}{h}\right)\phi_{t_i} + \frac{s - t_i}{h}\phi_{t_{i+1}}.$$

Lemma 2.2. *We have*

$$\int_0^T |(\phi^m)'|^p(s)\mathrm{d}s \leq \int_0^T |\phi'|^p(s)\mathrm{d}s, \ \|\phi_t - \phi_t^m\|^p \leq 2h^{\frac{1}{p'}} C_\phi \quad \text{for all } t \in [0, T],$$

where $C_\phi^p := \int_0^T |\phi'|^p(s)\mathrm{d}s$. Hence, $\{\phi_t^m\}_N$ converges to ϕ_t in $\mathbf{L}^p(X)$ for $t \in [0, T]$. If $|\phi'| \leq C_\sigma$ almost everywhere on $(0, T)$ then $|(\phi^m)'| \leq C_\sigma$ almost everywhere on $(0, T)$.

Corollary 2.3. *Let $\phi \in \mathcal{H}$ and let $\delta > 0$. Then there exist a partition of X into finitely many parallel cubes $\{X_i\}_{i=1}^k$ of the same size and $\psi \in \mathcal{H}$ such that $\|\phi_t - \psi_t\| \leq \delta$ for all $t \in [0, T]$, and for t fixed ψ_t is constant on each cube, its range is contained in $\mathbb{Q}^d$ and*

$$\int_0^T \|\dot{\psi}_t\|^p\mathrm{d}t \leq \int_0^T \|\dot{\phi}_t\|^p\mathrm{d}t + \delta \tag{2.11}$$

and ψ can be chosen such that $|\psi'| \le \bar{C}_\sigma + 1$ and the number of cubes can be chosen to be of the form m^d.

2.3. Topologies on $\mathcal{H}_\Omega$

Let $\Omega \subset \mathbb{R}$ be an open set. We endow $\mathcal{H}_\Omega$ with the topology τ_Ω given by

$$\phi^n \overset{\tau_\Omega}{\to} \phi \iff \|\phi^n - \phi\|_{\mathbf{L}^p(\Omega\times Y)} \to 0 \text{ and}$$
$$\{\dot\phi^n\}_n, \{\phi^n\}_n \text{ are bounded in } \mathbf{L}^p(\Omega \times X) \tag{2.12}$$

for any $Y \Subset X$. Observe that $\{\phi^n\}_n \subset \mathcal{H}_\Omega$ converges to ϕ in $\mathcal{H}_\Omega$ if and only if the following three conditions are satisfied:

$$\{\phi^n\}_n \text{ converges to } \phi \text{ in } \mathbf{L}^p(\Omega \times Y), \quad \text{for all } Y \Subset X; \tag{2.13}$$
$$\{\phi^n\}_n \text{ converges weakly to } \phi \text{ in } \mathbf{L}^p(\Omega \times X); \tag{2.14}$$
$$\{\dot\phi^n\}_n \text{ converges weakly to } \dot\phi \text{ in } \mathbf{L}^p(\Omega \times X). \tag{2.15}$$

If (2.13)–(2.15) hold then

$$\int_\Omega \Big(\|\phi^n_t\|^p + \|\dot\phi^n_t\|^p\Big)\mathrm{d}t < \infty. \tag{2.16}$$

2.4. A Γ-convergence result on $\mathbf{L}^p(X)$; the Γ-limit for affine functions

Let us define

$$F^\epsilon(\phi, \Omega) = \int_\Omega L\left(\frac{\phi}{\epsilon}, \dot\phi\right)\mathrm{d}t \quad \text{and} \quad F(\phi, \Omega) = \int_\Omega \bar{L}(\dot\phi)\mathrm{d}t, \tag{2.17}$$

where $\Omega \subset \mathbb{R}$ is open and bounded, and $\phi \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}; \mathbf{L}^p(X))$. We shall prove that

$$F(\cdot, \Omega) = \Gamma(\tau_\Omega) \lim_{\epsilon\to 0} F^\epsilon(\cdot, \Omega) \quad \text{for any } \Omega \subset \mathbb{R} \text{ open and bounded.}$$

We start by recalling a standard compactness result on Γ-convergence (cf. e.g. [19]).

Proposition 2.4. *Let $\mathcal{S}$ be a set endowed with a topology τ and suppose that $\mathcal{S}$ is first countable. Let $\{F_k\}_{k\in\mathbb{N}}$ be a sequence of functionals from $\mathcal{S}$ into $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.*

The lower Γ-limit (resp. upper Γ-limit) of $\{F_k\}_{k\in\mathbb{N}}$ exists and is denoted by F^- (resp. F^+).

(i) *The Γ-limit of $\{F^k\}_{k\in\mathbb{N}}$ exists at ϕ if and only if $F^-(\phi) = F^+(\phi)$.*

(ii) *Suppose in addition that (X, τ) has a countable basis. Then there exists a subsequence $\{F^{k_n}\}_{n\in\mathbb{N}}$ and a functional $F^\infty : X \to \bar{\mathbb{R}}$ such that $\{F^{k_n}\}_{n\in\mathbb{N}}$ Γ-converges to F^∞. We write*

$$F^\infty = \Gamma(\tau) \lim_{n\to\infty} F^{k_n}.$$

In the next two subsections we will follow mostly the techniques in [19]. In many cases we skip details of proofs which either are no different from the arguments in the aforementioned reference, or can be adapted in a straightforward manner. The following result can be proved exactly as in [19]. Therefore, we only offer the main steps of the proof with no details.

Proposition 2.5. *There exists $\epsilon_n \to 0$ such that for any $\phi \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}; \mathbf{L}^p(X))$ there exists a regular Borel measure $F(\phi, \cdot)$ such that for any open, bounded $\Omega \subset \mathbb{R}$, we have*

$$F(\cdot, \Omega) = \Gamma(\tau) \lim_{n\to\infty} F^{\epsilon_n}(\cdot, \Omega) \tag{2.18}$$

and

$$c \int_{\Omega} \|\dot{\phi}\|^p \mathrm{d}t \le F(\phi, \Omega) \le C \int_{\Omega} (1 + \|\dot{\phi}\|^p) \mathrm{d}t. \tag{2.19}$$

Sketch of proof. Let $\mathcal{D}$ be the algebra generated by the open bounded intervals in $\mathbb{R}$ whose endpoints are rational numbers. Fix for a moment an integer $e > 0$ and $\Omega \in \mathcal{D}$. Let $\mathcal{H}^e$ be the set of $M \in \mathcal{H}_\Omega$ such that $\int_\Omega \|\dot{M}_t\|^p \mathrm{d}t \le e$ and denote by τ^e the restriction of the topology τ_Ω to $\mathcal{H}^e$. Note that $(\mathcal{H}^e, \tau^e)$ is a topological space which has a countable basis. Hence, there exists a sequence $\{\bar{\epsilon}_k\}_{k\in\mathbb{N}}$ converging to zero as k tends to ∞ such that

$$\Gamma(\tau^e) \lim_{k\to\infty} F^{\bar{\epsilon}_k}(\phi, \Omega) \quad \text{exists},$$

for $\phi \in \mathcal{H}^e$. The growth condition (2.1) ensures that

$$\Gamma(\tau) \lim_{k\to\infty} F^{\bar{\epsilon}_k}(\phi, \Omega) \quad \text{exists},$$

for $\phi \in \mathcal{H}^e$. Since $e \in \mathbb{N}$ is arbitrary and $\mathcal{D}$ is countable, using a diagonalization argument we obtain the following: there exists a sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ converging to zero as n tends to ∞ such that there exists

$$\Gamma(\tau) \lim_{n\to\infty} F^{\epsilon_n}(\cdot, \Omega)$$

for any $\Omega \in \mathcal{D}$. Let $\mathcal{O}_b$ be the set of bounded open sets $\Omega \subset \mathbb{R}$. Denote by $F^-(\cdot, \Omega)$ (resp. $F^+(\cdot, \Omega)$) the lower Γ-limit (resp. upper Γ-limit) of $\{F^{\epsilon_n}(\cdot, \Omega)\}_{n\in\mathbb{N}}$ for any open set Ω. According to the existence of the Γ-limit above, we have that $F^-(\phi, A) = F^+(\phi, A)$ for all $\phi \in \mathcal{H}$ and any $A \in \mathcal{D}$. Thus, for $\phi \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}; \mathbf{L}^p(X))$ we can define $F(\phi, \cdot)$ on $\mathcal{O}_b$ by setting

$$F(\phi, \Omega) = \sup_{A\in\mathcal{D}, A\Subset\Omega} F^-(\phi, A) = \sup_{A\in\mathcal{D}, A\Subset\Omega} F^+(\phi, A).$$

We would like to show that $F(\cdot, \Omega)$ is the Γ-limit of $F^{\epsilon_n}(\cdot, \Omega)$ for all $\Omega \in \mathcal{O}_b$. For fixed ϕ, one can easily show that $F^-(\phi, \cdot)$ is a superadditive set function. To show that $F^+(\phi, \cdot)$ is subadditive requires more work, but the same argument as in [19] applies. We claim that $F^-(\phi, \Omega) = F^+(\phi, \Omega) = F(\phi, \Omega)$ for any $\Omega \in \mathcal{O}_b$. For any $A \in \mathcal{D}$ with $A \Subset \Omega$ we have

$$F^+(\phi, \Omega) \le F^+(\phi, A) + F^+(\phi, \Omega \setminus A).$$

The growth properties (upper bound) on L ensure that for any $\delta > 0$ one can choose A appropriately such that $F^+(\phi, \Omega \setminus A) \le \delta$. Thus, since F^- is superadditive, we get that

$$F(\phi, A) = F^-(\phi, A) \le F^-(\phi, \Omega) \le F^+(\phi, \Omega) \le F^+(\phi, A) + \delta = F(\phi, A) + \delta.$$

Since δ is arbitrary, the claim is proved. For one thing, the equality of F^- and F^+ shows that F is the Γ-limit of $\{F^{\epsilon_n}\}_n$. Also, we get that $F(\phi, \cdot)$ is an increasing, inner regular, finitely additive set function. Thus, it can be extended to a Borel measure on $\mathbb{R}$. The growth estimate (2.19) follows immediately from (2.1).

We first ask the question of what F is when ϕ is affine in time. For this, we have the following proposition whose main conclusion is that $F(\phi, \cdot)$ is a dilation of the Lebesgue measure.

Proposition 2.6. *If $\phi(t, \cdot) = t\xi$ for some $\xi \in \mathbf{L}^p(X)$, then $F(\phi, \Omega) = \Phi(\xi)|\Omega|$ for some lower semicontinuous functional $\Phi : \mathbf{L}^p(X) \to \mathbb{R}$ satisfying*

$$c\|\xi\|^p \leq \Phi(\xi) \leq C(1 + \|\xi\|^p). \tag{2.20}$$

Proof. It is enough to show that if $\alpha \in \mathbb{R},\ f \in \mathbf{L}^p(X)$, then the first equality below

$$F(\phi, \Omega + \alpha) = F(\phi, \Omega) = F(\phi + f, \Omega) \tag{2.21}$$

holds for all $\Omega \in \mathcal{G}_b$ (open, bounded sets in $\mathbb{R}$ with negligible boundaries). The second equality will be needed towards the end of the section. Let $\phi_n \in W^{1,p}_{\text{loc}}(\mathbb{R}; \mathbf{L}^p(X))$ such that $\phi_n \overset{\tau_\Omega}{\to} \phi$ and $F(\phi, \Omega) = \lim F^{\epsilon_n}(\phi_n, \Omega)$. Now let $\varphi_n \in C_c(X; \mathbb{R}^d)$ such that $\|\varphi_n - (\alpha/\epsilon_n)\xi\| < 1$ for all n. Take $Z_n = [\varphi_n]$ (a vector consisting of the integer parts of all entries) and note that $Z_n \in \mathbf{L}_{\mathbb{Z}}(X)$. We have that $\|Z_n - \varphi_n\| < 1$ implies $\|Z_n - (\alpha/\epsilon_n)\xi\| < 2$, i.e. $\|\epsilon_n Z_n - \alpha\xi\| < 2\epsilon_n$. Put

$$N_n(s, x) = \phi_n(s - \alpha, x) + \epsilon_n Z_n(x), \quad n \geq 1.$$

Clearly, $N_n \overset{\tau_{\Omega+\alpha}}{\to} (s - \alpha)\xi + \alpha\xi$ in $\mathcal{H}_{\Omega+\alpha}$, and

$$F(\phi, \Omega) = \lim_{n\to\infty} \int_\Omega L\left(\frac{\phi_n}{\epsilon_n}, \dot{\phi}_n\right) \mathrm{d}t = \liminf_{n\to\infty} \int_{\Omega+\alpha} L\left(\frac{N_n}{\epsilon_n}, \dot{N}_n\right) \mathrm{d}t \geq F(\phi, \Omega + \alpha).$$

Of course, by symmetry, we obtain equality. As for the second equality in (2.21), let $P_n(t, x) = \phi_n(t, x) + \epsilon_n K_n$, where $K_n \in \mathbf{L}_{\mathbb{Z}}(X)$ is such that $\epsilon_n K_n \to f$ in $\mathbf{L}^p(X)$. Thus, $P_n \overset{\tau_\Omega}{\to} M + f$ and

$$F(\phi, \Omega) = \lim_{n\to\infty} \int_\Omega L\left(\frac{\phi_n}{\epsilon_n}, \dot{\phi}_n\right) \mathrm{d}t = \liminf_{n\to\infty} \int_{\Omega+\alpha} L\left(\frac{K_n}{\epsilon_n}, \dot{K}_n\right) \mathrm{d}t \geq F(\phi + f, \Omega).$$

Since here we have not used that ϕ is affine, we get the reverse inequality by symmetry again, so we have equality. The growth conditions on Φ are derived from (2.19) and its lower semicontinuity comes from the Γ-limit property of F.

We plan to identify Φ with $\bar{L}$. To be able to link $F(\cdot, \Omega)$ to $\bar{L}$ we need to make additional conditions on L. We assume that there exists $\Lambda > 0$ such that

$$L(M, N_1) - L(M, N_2) \leq \Lambda \int_{\{N_1 \neq N_2\}} |N_1|^p \mathrm{d}x \tag{2.22}$$

for all $M, N_1, N_2 \in \mathbf{L}^p(X)$. Also, L satisfies, for some continuous, monotone nondecreasing $\omega : \mathbb{R} \to \mathbb{R}$ such that $\omega(0) = 0$,

$$|L(M_1, N) - L(M_2, N)| \leq \omega\big(\mathcal{L}^d\big(\{M_1 \neq M_2\}\big)\big) \tag{2.23}$$

for all $M_1, M_2, N \in \mathbf{L}^p(X)$.

The proof of the following proposition is an easy consequence of Lemmas 2.8, 2.10 and 2.12.

Proposition 2.7. *Under (2.22) and (2.23) we have $\Phi = \bar{L}$ on $\mathbf{L}^p(X)$.*

We begin by proving the following result.

Lemma 2.8. *Then one has* $\bar{L}(\xi) \le \Phi(\xi)$ *for all* $\xi \in \mathbf{L}^p(X)$.

Proof. Take $\Omega = (0,1)$ and $\phi_n \in \mathcal{H}_\Omega$ such that

$$\phi_n \xrightarrow{\tau} 0 \quad \text{and} \quad F^{\epsilon_n}(t\xi + \phi_n, \Omega) \to \Phi(\xi).$$

Assume for the moment $\phi_n \in \mathcal{H}_\Omega^0 := W_0^{1,p}(\Omega; \mathbf{L}^p(X))$ and take $N_n(t,x) = \phi_n(\epsilon_n t, x)/\epsilon_n$, $\Omega_n = \Omega/\epsilon_n$. Now,

$$\begin{aligned}\Phi(\xi) &= \lim_{n\to\infty} \int_\Omega L\left(\frac{t\xi + \phi_n}{\epsilon_n}, \xi + \dot{\phi}_n\right)\mathrm{d}t = \lim_{n\to\infty} \epsilon_n \int_{\Omega_n} L(s\xi + N_n, \xi + \dot{N}_n)\mathrm{d}s \\ &\ge \liminf_{n\to\infty} \epsilon_n C_{1/\epsilon_n}(0, \xi/\epsilon_n) \ge \bar{L}(\xi),\end{aligned}$$

where we have used (2.10). In fact, we have also proved that $\liminf_{n\to\infty} \int_\Omega L\left(\frac{U_n}{\epsilon_n}, \dot{U}_n\right)\mathrm{d}t \ge \bar{L}(\xi)$ for any sequence $U_n \xrightarrow{\tau} t\xi$ such that $U_n - t\xi \in \mathcal{H}_0(\Omega)$. To eliminate this restriction (we need this inequality for sequences in $\mathcal{H}_\Omega$ only), we shall use a technique due to De Giorgi [16]. In our setting, due to the more delicate nature of the topology τ (strong L^p convergence is only local in space), a more subtle argument is needed. Let $\{U_n\} \subset \mathcal{H}_\Omega$ converge to $t\xi$ in the τ topology, and fix $\delta > 0$ sufficiently small. Let

$$S_0 = (\delta, 1-\delta) \times (\delta, 1-\delta)^d \subset \cdots \subset S_N \subset S = (0,1) \times (0,1)^d$$

be open, equidistant squares. Next we take

$$a_i : [0,1] \to [0,1], \qquad b_i : [0,1]^d \to [0,1],$$

smooth functions such that

$$a_i = b_i = 1 \quad \text{on } S_{i-1}, \qquad a_i = b_i = 0 \quad \text{on } S \setminus S_i \quad \text{and} \quad |\dot{a}_i(t)| \le (N+1)/\delta$$

for all $i = 1, \dots, N$. Then consider

$$U_n^i(t,x) = t\xi(x) + a_i(t)b_i(x)\big[U_n(t,x) - t\xi(x)\big],$$

so $U_n^i - t\xi \in \mathcal{H}_0(\Omega)$. Note that

$$\begin{aligned}F^{\epsilon_n}(U_n^i, \Omega) &= F^{\epsilon_n}(U_n, \Omega) + \int_0^1 \left[L\left(\frac{U_n}{\epsilon_n}, \dot{U}_n^i\right) - L\left(\frac{U_n}{\epsilon_n}, \dot{U}_n\right)\right]\mathrm{d}t \\ &\quad + \int_0^1 \left[L\left(\frac{U_n^i}{\epsilon_n}, \dot{U}_n^i\right) - L\left(\frac{U_n}{\epsilon_n}, \dot{U}_n^i\right)\right]\mathrm{d}t =: F^{\epsilon_n}(U_n, \Omega) + I_1^* + I_2^*.\end{aligned}$$

Write $S_i = I_i \times Q_i$ where $I_i \subset (0,1)^d$. Since $U_n^i(t,\cdot) = U_n(t,\cdot)$ on Q_{i-1} for all $t \in I_{i-1}$, we use (2.23) to infer that $|I_2^*| \le C\delta + \omega(C\delta)$. By (2.22),

$$I_1^* \le \Lambda \iint_{S \setminus S_{i-1}} |\dot{U}_n^i|^p \mathrm{d}x\mathrm{d}t.$$

On $S \setminus S_i$ we have $U_n^i = t\xi$, so

$$\begin{aligned}\iint_{S\setminus S_{i-1}} |\dot U_n^i|^p \mathrm{d}x\mathrm{d}t &\le \iint_{S\setminus S_i} |\xi(x)|^p \mathrm{d}x\mathrm{d}t + \iint_{S_i\setminus S_{i-1}} |\dot U_n^i|^p \mathrm{d}x\mathrm{d}t \\ &= C\delta + \iint_{S_i\setminus S_{i-1}} |\dot U_n^i|^p \mathrm{d}x\mathrm{d}t.\end{aligned}$$

Collecting all the above we get

$$F^{\epsilon_n}(U_n^i, \Omega) \le F^{\epsilon_n}(U_n, \Omega) + \Lambda \iint_{S_i\setminus S_{i-1}} |\dot U_n^i|^p \mathrm{d}x\mathrm{d}t + C\delta + \omega(C\delta). \tag{2.24}$$

Since

$$\begin{aligned}|\dot U_n^i|^p &= |\xi + \dot a_i b_i(U_n - t\xi) + a_i b_i(\dot U_n - \xi)|^p \\ &\le 3^{p-1}\left[|\xi|^p + \left(\frac{N+1}{\delta}\right)^p |U_n - t\xi|^p + |\dot U_n - \xi|^p\right],\end{aligned}$$

we infer

$$\begin{aligned}F^{\epsilon_n}(U_n^i, \Omega) \le\ & F^{\epsilon_n}(U_n, \Omega) + 0(\delta) + 3^{p-1}\Lambda\left(\frac{N+1}{\delta}\right)^p \iint_{S_i\setminus S_{i-1}} |U_n - t\xi|^p \mathrm{d}x\mathrm{d}t \\ &+ 3^{p-1}\Lambda \iint_{S_i\setminus S_{i-1}} |\dot U_n - \xi|^p \mathrm{d}x\mathrm{d}t + C\delta + \omega(C\delta).\end{aligned}$$

Since U_n converges to $t\xi$ locally in $\mathbf{L}^p(S)$, upon passing to liminf as $n \to \infty$, we obtain

$$\begin{aligned}&\liminf_{n\to\infty} F^{\epsilon_n}(U_n^i, \Omega) \\ &\quad \le \liminf_{n\to\infty} F^{\epsilon_n}(U_n, \Omega) + 3^{p-1}\Lambda \liminf_{n\to\infty} \iint_{S_i\setminus S_{i-1}} |\dot U_n - \xi|^p \mathrm{d}x\mathrm{d}t + C\delta + 0(\delta).\end{aligned}$$

Since $\{U_n^i - t\xi\} \subset \mathcal{H}_0(\Omega)$, we know that the left hand side is at least $\bar L(\xi)$. Thus, if we sum up from $i = 1$ to N and divide by N, we find

$$\bar L(\xi) \le \liminf_{n\to\infty} F^{\epsilon_n}(U_n) + \frac{3^{p-1}\Lambda}{N} \liminf_{n\to\infty} \iint_S |\dot U_n - \xi|^p \mathrm{d}x\mathrm{d}t + C\delta + 0(\delta).$$

Due to $U_n \xrightarrow{\tau} t\xi$, the integral is uniformly bounded in n, so, if we first let $N \to \infty$ and then $\delta \to 0$, we discover that

$$\bar L(\xi) \le \liminf_{n\to\infty} F^{\epsilon_n}(U_n, \Omega).$$

Thus, the proof is concluded.

If $R \in \mathbf{L}_{\mathbb{Q}}(X)$, then the map $t \to L(tR, R)$ is periodic, having as principal period the least common multiple of all the denominators of all (rational) entries of all vectors in the range of R, say, m. We next study $\bar L(R)$ for $R \in \mathbf{L}_{\mathbb{Q}}(X)$.

Lemma 2.9. *If $R \in \mathbf{L}_{\mathbb{Q}}(X)$, then*

$$\bar L(R) = \lim_{k\to\infty} \frac{C_{0,km}(0, kmR)}{km}.$$

We skip the proof of this lemma, as it needs no modification from the argument in [19]. On the basis of this result, we shall be able to obtain the opposite inequality, $\bar{L} \geq \Phi$, at first on $\mathbf{L}_{\mathbb{Q}}(X)$.

Lemma 2.10. *One has $\bar{L}(R) \geq \Phi(R)$ for all $R \in \mathbf{L}_{\mathbb{Q}}(X)$.*

Proof. Take $\delta > 0$. According to Lemma 2.9, we can find a positive integer k and $\phi \in \mathcal{H}_0(0, km)$ such that

$$\left|\bar{L}(R) - \fint_0^{km} L(sR + \phi, R + \dot{\phi})\mathrm{d}s\right| < \delta.$$

Set $\Omega = (0,1)$ and define $\tilde{\phi}$ as the periodic extension of ϕ on $\mathbb{R}$ and put $M_n(t,x) := tR(x) + \epsilon_n \tilde{\phi}(t/\epsilon_n, x)$. Then $M_n \overset{\tau_\Omega}{\to} tR$, so

$$\liminf_{n\to\infty} \int_0^1 L\left(\frac{M_n}{\epsilon_n}, \dot{M}_n\right)\mathrm{d}t \geq F(tR, \Omega) = \Phi(R).$$

We have

$$\int_0^1 L\left(\frac{M_n}{\epsilon_n}, \dot{M}_n\right)\mathrm{d}t = \fint_0^{T_n} L\big(tR + \tilde{\phi}, R + \dot{\tilde{\phi}}\big)\mathrm{d}t,$$

where $T_n := 1/\epsilon_n$. The map

$$t \to L\big(tR + \tilde{\phi}, R + \dot{\tilde{\phi}}\big)$$

is periodic, of period km. Thus, the last averaged integral from above becomes

$$\frac{[T_n/(km)]}{T_n} \int_0^{km} L\big(tR + \phi, R + \dot{\phi}\big)\mathrm{d}t + \frac{1}{T_n} \int_0^{T_n - N[T_n/(km)]} L\big(tR + \tilde{\phi}, R + \dot{\tilde{\phi}}\big)\mathrm{d}t,$$

which implies

$$\lim_{n\to\infty} \fint_0^{T_n} L\big(tR + \tilde{\phi}, R + \dot{\tilde{\phi}}\big)\mathrm{d}t = \fint_0^{km} L(sR + \phi, R + \dot{\phi})\mathrm{d}s.$$

We deduce $\delta + \bar{L}(R) > \Phi(R)$ and, since $\delta > 0$ is arbitrary, we conclude the proof.

Remark 2.11. Thus, if L satisfies the assumptions of Lemma 2.8 we have $\bar{L}(R) = \Phi(R)$ for all $\mathbf{L}_{\mathbb{Q}}(X)$. Before concluding that this is, in fact, true on the whole of $\mathbf{L}^p(X)$, we need one more lemma.

Lemma 2.12. *If $\xi_n \to \xi$ in $\mathbf{L}^p(X)$ and $\{\xi_n\}_{n=1}^\infty \subset \mathbf{L}_{\mathbb{Q}}(X)$, then*

$$\bar{L}(\xi) \leq \liminf_{n\to\infty} \bar{L}(\xi_n). \tag{2.25}$$

Furthermore, there exists $\xi_n \to \xi$ with $\{\xi_n\}_{n=1}^\infty \subset \mathbf{L}_{\mathbb{Q}}(X)$ such that

$$\bar{L}(\xi) = \lim_{n\to\infty} \bar{L}(\xi_n). \tag{2.26}$$

Proof. Note that (2.25) follows trivially from the previous lemmas and the lower semicontinuity of Φ. To prove (2.26), we start by selecting a sequence $T_n \to \infty$ and a sequence $\phi_n \in W_0^{1,p}(0, T_n; \mathbf{L}^p(X))$ such that

$$\bar{L}(\xi) = \lim_{n\to\infty} \fint_0^{T_n} L(t\xi + \phi_n, \xi + \dot{\phi}_n)\mathrm{d}t. \tag{2.27}$$

Next, as in [19], we let $A_n = [T_n] + 2$, $s_n = 1 - (T_n - [T_n])/2$. Then take $\zeta_n \in C_c(X; \mathbb{R}^d)$ such that $\|\zeta_n - \xi\| < 1/(2A_n)$. Define the vector field $[\cdot]_e$ by

$$[y]_{e,i} = 2k \quad \text{whenever } 2k-1 \le y_i < 2k+1 \text{ for } i = 1, \dots, d,$$

and let $\xi_n := [2A_n\zeta_n]_e/(2A_n)$. Note that

$$\|\xi_n - \xi\| < \frac{\sqrt{d}}{2A_n} \text{ and each component of } \xi_n \text{ is an integer divided by } A_n. \tag{2.28}$$

Now let $\psi_n \in W^{1,p}(0, A_n; \mathbf{L}^p(X))$ be given by

$$\psi_n(s,x) = \begin{cases} 0, & \text{if } 0 \le s \le s_n,\ x \in X \\ (s-s_n)\xi(x) + \phi_n(s-s_n, x), & \text{if } s_n \le s \le s_n + T_n,\ x \in X \\ \dfrac{s - s_n - T_n}{A_n - s_n - T_n}\{A_n\xi_n(x) - T_n\xi(x)\} + T_n\xi(x), & \text{if } s_n + T_n \le s \le A_n,\ x \in X. \end{cases}$$

Observe that $\psi_n - s\xi_n \in W_0^{1,p}(0, T_n; \mathbf{L}^p(X))$ and

$$\begin{aligned} \|\dot{\psi}_n(t,\cdot)\| &\le \frac{1}{s_n}\|A_n\xi_n - T_n\xi\| \le 2\|A_n\xi_n - T_n\xi\| \\ &\le 2A_n\|\xi_n - \xi\| + 2(A_n - T_n)\|\xi\| \le C \end{aligned}$$

for all $s_n + T_n \le t \le A_n$. Consequently,

$$\begin{aligned} \int_0^{A_n} L(\psi_n, \dot{\psi}_n)\mathrm{d}t &\le \int_0^{T_n+s_n} L(\psi_n, \dot{\psi}_n)\mathrm{d}t + \int_{s_n+T_n}^{A_n} L(\psi_n, \dot{\psi}_n)\mathrm{d}t \\ &\le \int_0^{T_n} L(t\xi + \phi_n, \xi + \dot{\phi}_n)\mathrm{d}t + C(A_n - s_n - T_n), \end{aligned}$$

which implies

$$\fint_0^{A_n} L(\psi_n, \dot{\psi}_n)\mathrm{d}t \le \frac{T_n}{A_n}\fint_0^{T_n} L(\phi_n, \dot{\phi}_n)\mathrm{d}t + C\frac{A_n - s_n - T_n}{A_n}$$

and so

$$\limsup_{n\to\infty} \fint_0^{A_n} L(\psi_n, \dot{\psi}_n)\mathrm{d}t \le \bar{L}(\xi). \tag{2.29}$$

But by Remark 2.1

$$\fint_0^{A_n} L(\psi_n, \dot{\psi}_n)\mathrm{d}t \ge \bar{L}(\xi_n), \quad \text{for all } n.$$

This, together with (2.25) and (2.29), yields (2.26).

Later we shall need the continuity of the map $\bar{L}$ with respect to the strong $\mathbf{L}^p(X)$ topology. This is obtained as a consequence of its convexity.

Proposition 2.13. *The map $\bar{L}$ is convex.*

Proof. Fix $Q_1,\ Q_2 \in \mathbf{L}_{\mathbb{Q}}(X)$, $\lambda \in [0, 1/2] \cap \mathbb{Q}$ and put $Q_3 := \lambda Q_1 + (1-\lambda)Q_2$. Take $\delta > 0$ arbitrary. Due to Lemma 2.9, there exists $T_0 > 0$ such that

$$\left| \inf_{\phi \in \mathcal{H}_0(0,T)} \fint_0^T L(tQ_i + \phi, Q_i + \dot{\phi})\mathrm{d}t - \bar{L}(Q_i) \right| \leq \frac{\delta}{2}$$

for $i = 1, 2, 3$ and $T \geq T_0$. Next, consider $T_1 \geq T_0$ such that $T_1 Q_1 \in \mathbf{L}_{\mathbb{Z}}(X)$ and set $T_2 := (-1 + 1/\lambda)T_1 \geq T_0$. We can find $\phi_i \in \mathcal{H}_{(0,T_i)}$ such that $\phi_i(0, \cdot) = 0$, $\phi_i(T_i, \cdot) = T_i Q_i$, and

$$\fint_0^{T_i} L(\phi_i, \dot{\phi}_i)\mathrm{d}t \leq \bar{L}(Q_i) + \delta$$

for $i = 1, 2$. Define $\tilde{\phi}$ to be the $\mathcal{H}_{(0,T_1+T_2)}$ map identically equal to ϕ_1 on $[0, T_1] \times X$ and to $T_1 Q1 + \phi_2(t - T_1, \cdot)$ on $[T_1, T_1 + T_2] \times X$. We obviously have $\tilde{\phi}(0, \cdot) = 0$ and $\tilde{\phi}(T_1 + T_2, \cdot) = (T_1 + T_2)Q_3$. Since

$$\int_0^{T_1+T_2} L(\tilde{\phi}, \dot{\tilde{\phi}})\mathrm{d}t = \int_0^{T_1} L(\phi_1, \dot{\phi}_1)\mathrm{d}t + \int_0^{T_2} L(\phi_2, \dot{\phi}_2)\mathrm{d}t$$

and $\delta > 0$ is arbitrary, one obtains

$$\bar{L}(Q_3) \leq \lambda \bar{L}(Q_1) + (1-\lambda)\bar{L}(Q_2).$$

We conclude by Lemma 2.12.

2.5. The Γ-limit

It is obvious from its definition that $\bar{L}$ is locally bounded. Therefore, its convexity implies the following:

Corollary 2.14. *The map $\bar{L}$ is locally Lipschitz on $\mathbf{L}^p(X)$.*

So far, we have proved that

$$F(\phi, \Omega) = \int_\Omega \bar{L}(\dot{\phi})\mathrm{d}t \tag{2.30}$$

for all $\phi(t, x) = t\xi(x) + u(x)$ (the addition of u is allowed due to the second equality in (2.21)), where $\xi,\ u \in \mathbf{L}^p(X)$. In general, we have:

Theorem 2.15. *The equality (2.30) holds for every $\phi \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}; \mathbf{L}^p(X))$.*

The proof of this theorem is exactly as in [19], provided that we clarify some issues related to the infinite dimensionality of our setting. This is what we do next.

Proof of Theorem 2.15. Due to the additivity property of F, we conclude (2.30) is also true for piecewise affine functions. To apply Egoroff's Theorem and obtain

$$F(\phi, \Omega) \leq \int_\Omega \bar{L}(\dot{\phi})\mathrm{d}t \tag{2.31}$$

just as in [19], we only need to show that there exists a sequence of affine functions ϕ_n that converges strongly in $\mathcal{H}_\Omega$ to ϕ. We refer the reader to Section 2.2.

For the reverse inequality, take any $\phi \in \mathcal{H}_{(a,b)}$ and recall that (see, e.g., [10]), if $\tilde{\phi}$ is the extension of ϕ to $\mathbb{R} \times X$ by $\tilde{\phi}(t, \cdot) = \phi(a, \cdot)$ if $t < a$ and $\tilde{\phi}(t, \cdot) = \phi(b, \cdot)$ if $t > b$, then the function

$$\alpha_t(s) = \begin{cases} \left\| \dfrac{\tilde{\phi}(t+s, \cdot) - \tilde{\phi}(t, \cdot)}{s} - \dot{\tilde{\phi}}(t, \cdot) \right\|, & \text{if } s \neq 0 \\ 0, & \text{if } s = 0 \end{cases}$$

is continuous on $\mathbb{R}$ for a.e. $t \in (a, b)$. In particular, for a.e. $t_0 \in \mathbb{R}$,

$$\begin{aligned} 0 &= \lim_{r \to 0^+} \frac{1}{r} \int_{-r}^{r} \left\| \frac{\tilde{\phi}(t_0+s, \cdot) - \tilde{\phi}(t_0, \cdot)}{s} - \dot{\tilde{\phi}}(t_0, \cdot) \right\|^p \mathrm{d}s \\ &= \lim_{r \to 0^+} \int_{t_0-1}^{t_0+1} \frac{1}{|t-t_0|^p} \left\| \frac{\tilde{\phi}(t_0 + r(t-t_0), \cdot) - \tilde{\phi}(t_0, \cdot)}{r} - (t-t_0)\dot{\tilde{\phi}}(t_0, \cdot) \right\|^p \mathrm{d}t. \end{aligned}$$

We deduce that

$$\lim_{r \to 0^+} \int_{t_0-1}^{t_0+1} \left\| \frac{\tilde{\phi}(t_0 + r(t-t_0), \cdot) - \tilde{\phi}(t_0, \cdot)}{r} - (t-t_0)\dot{\tilde{\phi}}(t_0, \cdot) \right\|^p \mathrm{d}t = 0,$$

i.e., as $r \to 0^+$,

$$\frac{\tilde{\phi}(t_0 + r(t-t_0), \cdot) - \tilde{\phi}(t_0, \cdot)}{r} - (t-t_0)\dot{\tilde{\phi}}(t_0, \cdot) \to 0 \quad \text{in } \mathbf{L}^p((t_0-1, t_0+1) \times X).$$

The rest of the argument follows exactly as in [19].

3. Hamilton–Jacobi equations on $\mathcal{P}_p(\mathbb{R})$; homogenization

We fix $T > 0$, $p > 1$ and set $X = (0, 1)$. We denote by $\|\cdot\|$ the standard $\mathbf{L}^p(0, 1)$ norm. Recall that $\mathcal{H} = AC^p(0, T; \mathbf{L}^p(0, 1))$. If $\mu \in \mathcal{P}_p(\mathbb{R})$, $L^p(\mu)$ stands for the set of $v : \mathbb{R} \to \mathbb{R}$ that are μ-measurable and such that $\|v\|_\mu^p := \int_{\mathbb{R}} |v|^p \mathrm{d}\mu$ is finite. We denote by $\mathcal{S}$ the set $AC^p(0, T; \mathcal{P}_p(\mathbb{R}))$ of p-absolutely continuous paths on the Wasserstein space and denote by W_p the p-Wasserstein distance. Some useful properties of the elements of $\mathcal{P}(\mathbb{R})$ are recalled in the Appendix (cf. Appendix A.1). We endow $\mathcal{S}$ with the following topology.

Definition 3.1. We say that $\{\sigma^n\}_n \subset \mathcal{S}$ τ_w-converges to σ in $\mathcal{S}$ if

$$\lim_{n \to \infty} \int_0^T W_1(\sigma_t^n, \sigma_t)\mathrm{d}t = 0 \tag{3.1}$$

$$\sup_{n \in \mathbb{N}} \int_0^T \left(W_p^p(\sigma_t^n, \delta_0) + |(\sigma^n)'|_t^p \right)\mathrm{d}t < \infty. \tag{3.2}$$

Remark 3.2. When $\mu \in \mathcal{P}_p(\mathbb{R})$, the $L^p(\mu)$-closure of $\{\varphi' : \varphi \in C_c^\infty(\mathbb{R})\}$ is $L^p(\mu)$. Thus, given $\sigma \in \mathcal{S}$ there exists up to a negligible set a unique velocity v for σ. In that case, if $\phi_t : (0, 1) \to \mathbb{R}$ is the unique monotone nondecreasing left continuous map such that $\phi_{t\#}\nu_0 = \sigma_t$ then $\phi \in \mathcal{H}$ and $\dot{\phi}_t = v_t \circ \phi_t$ for almost every $t \in (0, T)$ (cf. e.g. [10]). We say that ϕ is a flow associated with σ.

3.1. Properties of the weak topology τ

Remark 3.3. Let $A_n : (0,1) \to \mathbb{R}$ be a sequence of monotone nondecreasing functions for which there exists $C > 0$ such that $\|A_n\|_{L^p(0,1)} \le C$ for all $n \in \mathbb{N}$. Suppose $A : (0,1) \to \mathbb{R}$ is monotone nondecreasing. Then, for each $r \in (0, 1/2)$, $\{A_n\}_n$ is bounded in $L^\infty(r, 1-r)$ by a constant depending only on C and r. Furthermore, the following are equivalent:

(i) $\{A_n\}_n$ converges to A in $L^1(0,1)$.
(ii) $\{A_n\}_n$ converges to A locally in $L^p(0,1)$.

Proposition 3.4. *Let $\{\sigma^n\}_n \subset \mathcal{S}$ and let v^n be a velocity for σ^n. Let $\sigma \in \mathcal{S}$ and suppose that v is a velocity for it. Suppose $\phi_t^n : (0,1) \to \mathbb{R}$ is the unique monotone nondecreasing, left continuous map such that $\phi^n_{t\#}\nu_0 = \sigma_t^n$. Similarly, let ϕ_t be the unique monotone nondecreasing, left continuous map such that $\phi_{t\#}\nu_0 = \sigma_t$. Then the following assertions are equivalent:*

$$\{\sigma^n\}_n \ \tau_w\text{-converges to } \sigma \text{ in } \mathcal{S} \tag{3.3}$$

$$\{\phi^n\}_n \tau \text{ -converges to } \phi \text{ in } \mathcal{H}. \tag{3.4}$$

Proof. Recall that $\phi^n \in \mathcal{H}$ is equivalent to $\sigma^n \in \mathcal{S}$ and, as pointed in Remark 3.2, $\dot\phi_t^n = v_t^n \circ \phi_t^n$ for almost every $t \in (0,T)$. Hence,

$$\int_0^T \Big(W_p^p(\sigma_t^n, \delta_0) + |(\sigma^n)'|_t^p \Big)\mathrm{d}t = \int_0^T \Big(\|\phi_t^n\|^p + \|\dot\phi_t^n\|^p \Big)\mathrm{d}t \tag{3.5}$$

and so, $\{\dot\phi^n\}_n$ and $\{\phi^n\}_n$ are bounded in $L^p((0,T)\times(0,1))$ if and only if (3.2) holds. It remains to show that under (3.2), (3.1) holds if and only if $\{\phi_n\}_n$ converges locally to ϕ in $L^p((0,T)\times(0,1))$. Let $\bar C$ be such that $W_p^p(\sigma_t^n, \delta_0) = \|\phi_t^n\|^p \le \bar C$.

We assume in the sequel that (3.2) holds. By Remark A.1, every subsequence of $\{\sigma^n\}_n$ admits a subsequence $\{\sigma^{n_k}\}_k$ such that $\{\sigma_t^{n_k}\}_k$ converges to σ_t in $\mathcal{P}_q(\mathbb{R}^d)$ for every $t \in [0,T]$ and every $1 \le q < p$. The limit being independent of the subsequence, we conclude that $\{\sigma_t^n\}_n$ converges to σ_t in $\mathcal{P}_q(\mathbb{R}^d)$ for every $t \in [0,T]$ and every $1 \le q < p$. As $W_1(\sigma_t^n, \sigma_t) = \|\phi_t^n - \phi_t\|_{L^1(0,1)}$ we conclude that $\{\phi_t^n\}_n$ converges to ϕ_t in $L^1(0,1)$ for all $t \in [0,T]$. By Remark 3.3 and the fact that $W_p^p(\sigma_t^n, \delta_0) = \|\phi_t^n\|^p \le \bar C$ for all n, if $r \in (0,1/2)$ then $\{M_t^n\}_n$ converges to ϕ_t in $L^p(r, 1-r)$ for all $t \in [0,T]$ and $\{\phi^n\}_n$ is bounded in $L^\infty((0,T)\times(r,1-r))$. We use the Lebesgue dominated convergence theorem to conclude that $\{\phi^n\}_n$ converges to ϕ in $L^p((0,T)\times(r,1-r))$. Thus, $\{\phi^n\}_n$ converges locally to ϕ in $L^p((0,T)\times(0,1))$.

Conversely, suppose that $\{\phi^n\}_n$ converges locally to ϕ in $L^p((0,T)\times(0,1))$. Let C be the supremum over n of the expressions in (3.5). Let $r \in (0,1/2)$, set $S = (0,T) \setminus (r, 1-r)$ and denote by S^c the complement of S in $(0,T)\times(0,1)$. We have

$$\begin{aligned}\int_0^T W_1(\sigma_t^n, \sigma_t)\mathrm{d}t &\le \|\phi^n - \phi\|_{L^p(S)} (\mathcal{L}^2(S))^{\frac{1}{p'}} + \|\phi^n - \phi\|_{L^p(S^c)} (\mathcal{L}^2(S^c))^{\frac{1}{p'}} \\ &\le \|\phi^n - \phi\|_{L^p(S)} (\mathcal{L}^2(S))^{\frac{1}{p'}} + 2C^{\frac{1}{p}} (2rT)^{\frac{1}{p'}}.\end{aligned}$$

This proves that $\limsup_{n\to\infty} \int_0^T W_1(\sigma_t^n, \sigma_t)\mathrm{d}t \le 2C^{\frac{1}{p}}(2rT)^{\frac{1}{p'}}$. Since $r \in (0,1/2)$ is arbitrary, we conclude that (3.1) holds.

Lemma 3.5. *Let $\{\sigma\} \cup \{\sigma^n\}_n \subset \mathcal{S}$ and let v^n (resp. v) be the velocity for σ^n (resp. σ). Suppose $\phi_t^n : (0,1) \to \mathbb{R}$ (resp. $\phi_t : (0,1) \to \mathbb{R}$) is the unique monotone nondecreasing left continuous map such that $\phi_{t\#}^n \nu_0 = \sigma_t^n$ (resp. $\phi_{t\#}\nu_0 = \sigma_t$). Let $\tilde{\phi}^n$ (resp. $\tilde{\phi}$) be a flow associated with σ^n (resp. σ). Suppose $\{\tilde{\phi}^n\}_n$ τ-converges to $\tilde{\phi}$ in $\mathcal{H}$. Then $\{\phi^n\}_n$ τ-converges to ϕ.*

Proof. As ϕ^n and $\tilde{\phi}^n$ are flows associated with σ^n we may substitute ϕ^n by $\tilde{\phi}^n$ in (3.5) to obtain that $\{\dot{\tilde{\phi}}_n\}_n$ and $\{\tilde{\phi}_n\}_n$ are bounded in $L^p((0,T)\times(0,1))$. We use that $\|\phi_t^n - \phi_t\|_{L^1(0,1)} \le \|\tilde{\phi}_t^n - \tilde{\phi}_t\|_{L^1(0,1)}$ (cf. e.g. [10]) to conclude that

$$\lim_{n\to\infty}\int_0^T W_1(\sigma_t^n, \sigma_t)\mathrm{d}t$$
$$= \lim_{n\to\infty}\int_{(0,T)\times(0,1)} |\phi^n - \phi|\mathrm{d}z\mathrm{d}t \le \lim_{n\to\infty}\int_{(0,T)\times(0,1)} |\tilde{\phi}^n - \tilde{\phi}|\mathrm{d}z\mathrm{d}t = 0.$$

We use Proposition 3.4 to conclude that $\{\phi^n\}_n$ τ-converges to ϕ in $\mathcal{H}$.

3.2. The effective Lagrangian on $\mathcal{P}_p(\mathbb{R})$

As in Section 2 we assume that L is a Lagrangian on $L^p(X)\times L^p(X)$ satisfying the growth conditions (2.1) and the periodicity condition (2.2). We further assume the following invariance property: if $M, \bar{M} \in L^p(X)$ satisfy $\bar{M}_\#\nu_0 = M_\#\nu_0 =: \mu$ and $v \in L^p(\mu)$ then

$$L(M, v\circ M) = L(\bar{M}, v\circ\bar{M}). \tag{3.6}$$

We define the Lagrangian $\mathcal{L}$ by

$$\mathcal{L}(\mu, v) = L(M, v\circ M). \tag{3.7}$$

We denote by $\mathcal{TP}_p(\mathbb{R})$ the set of pairs (μ, v) such that $\mu \in \mathcal{P}_p(\mathbb{R})$ and $v \in L^p(\mu)$. In [2] $\mathcal{TP}_p(\mathbb{R})$ is referred to as the tangent bundle of $\mathcal{P}_p(\mathbb{R})$. For $\sigma \in \mathcal{S}$, v the velocity for σ and $\epsilon > 0$ we define

$$\mathcal{F}^\epsilon(\sigma) = \int_0^T \mathcal{L}\Big(D_\#^{1/\epsilon}\sigma_t, v_t\circ D^\epsilon\Big)\mathrm{d}t.$$

We do not display the dependence of $\mathcal{F}^\epsilon$ on v, as the velocity of minimal norm for σ is uniquely determined. We set

$$\mathcal{F}(\sigma) = \int_0^T \bar{L}\big(v_t\circ\phi_t\big)\mathrm{d}t,$$

where ϕ_t is the unique monotone nondecreasing left continuous map that pushes ν_0 forward to σ_t.

Theorem 3.6. *The family $\{\mathcal{F}^\epsilon\}_\epsilon$ $\Gamma(\tau)$-converges to $\mathcal{F}$ as ϵ tends to* 0.

Proof. Fix $\sigma \in \mathcal{S}$ and let v be the velocity of minimal norm for σ. Let $\{\sigma^n\}_n$ be a sequence that τ-converges to σ in $\mathcal{S}$ and denote by v^n the velocity of minimal norm for σ^n. Let $\phi^n \in AC^p(0,T;\mathcal{P}_p(\mathbb{R}))$ be such that ϕ_t^n is left continuous, monotone nondecreasing and pushes ν_0 forward to σ_t^n. By Proposition 3.4, $\{\phi^n\}_n$ τ-converges to ϕ, where ϕ_t is the left continuous, monotone nondecreasing map that pushes ν_0 forward to σ_t. Recall that ϕ is a flow associated with σ and ϕ^n is a flow associated with σ^n (cf. [10]). Let $\{\mathcal{F}^{\epsilon_n}\}_n$ be a subsequence of $\{\mathcal{F}^{\epsilon}\}_\epsilon$. Using the identity $\dot{\phi}_t^n = v_t^n \circ \phi_t^n$ we obtain

$$\begin{aligned}\mathcal{F}^{\epsilon_n}(\sigma^n) &= \int_0^T \mathcal{L}\big((D^{1/\epsilon}\circ\phi_t^n)_\# \nu_0, v_t\circ D^\epsilon\big)\mathrm{d}t \\ &= \int_0^T L\big(D^{1/\epsilon}\circ\phi_t^n, v_t^n\circ\phi_t^n\big)\mathrm{d}t = F^{\epsilon_n}(\phi^n). \qquad (3.8)\end{aligned}$$

By Theorem 2.15, $\{F^{\epsilon_n}\}_\epsilon$ $\Gamma(\tau)$-converges to F as n tend to ∞. Hence, by (3.8)

$$\liminf_{n\to\infty}\mathcal{F}^{\epsilon_n}(\sigma^n) = \liminf_{n\to\infty}F^{\epsilon_n}(\phi^n) \ge \int_0^T \bar{L}(\dot{\phi}_t)\mathrm{d}t = \int_0^T \bar{L}(v_t\circ\phi_t)\mathrm{d}t = \mathcal{F}(\sigma). \qquad (3.9)$$

As we may choose $\{\sigma^n\}_n$ such that the inequality in (3.9) turns into an equality, we conclude that $\mathcal{F}$ is the lower $\Gamma(\tau)$-limit of $\{F^\epsilon\}_\epsilon$. Similarly, $\mathcal{F}$ is the upper $\Gamma(\tau)$-limit of $\{F^\epsilon\}_\epsilon$, which concludes the proof.

3.3. Homogenization for Hamilton–Jacobi equations in the Wasserstein space

Let us fix $\Lambda > 0$ and consider the extension to $\mathcal{TP}_2(\mathbb{R})$ of the classical mechanical Lagrangian, i.e.

$$\mathcal{L}(\mu,\xi) = \Lambda\|\xi\|_\mu^2 - \mathcal{W}(\mu) \quad \text{for all } (\mu,\xi)\in\mathcal{TP}_2(\mathbb{R}),$$

which corresponds to

$$L(M,N) = \Lambda\|N\|^2 - W(M) \quad \text{for all } (M,N)\in[L^2(X)]^2 \qquad (3.10)$$

via $\mathcal{L}(M_\#\nu_0,\xi) = L(M,\xi\circ M)$ for $M\in L^2(X)$. We hope that the reader will be convinced that the results proved in this section hold even if L is not necessarily of this form but simply satisfies the assumptions from the previous section and, additionally, it is continuous in both variables and nondecreasing in $\|N\|$.

As an application of Theorem 3.6, below we provide a proof of the fact that the viscosity solutions given by the backward Lax–Oleinik formula (see [9]) for the time-dependent Hamilton–Jacobi equations corresponding to the Lagrangians $\mathcal{L}^\epsilon$ (defined below) converge to the backward Lax–Oleinik solution for

$$\partial_t U(t,\mu) + \bar{\mathcal{H}}(\mu,\nabla_W U(t,\mu)) = 0, \qquad U(0,\mu) = U_0(\mu), \qquad (3.11)$$

where ∇_W denotes the Wasserstein gradient (cf. [9]) and

$$\bar{\mathcal{H}}(\mu,\zeta) = \sup_{\xi\in L^2(X)}\big\{\langle\zeta,\xi\rangle_{L^2(X)} - \bar{\mathcal{L}}(\mu,\xi)\big\} \quad \text{for all } \zeta\in L^2(X) \qquad (3.12)$$

is the Legendre transform of $\bar{\mathcal{L}}$. The latter can be defined by (3.7) due to the fact that the Lagrangian in (3.10) satisfies (3.6). Furthermore, $\bar{L}$ satisfies the necessary conditions for the existence theory in [9] to apply to (3.11). We assume that

$$u_0 : L^2(X)\to\mathbb{R} \text{ is bounded below}, \quad W \text{ is bounded, nonpositive and 1-periodic}, \qquad (3.13)$$

and satisfies, for some continuous nondecreasing $\omega:\mathbb{R}\to\mathbb{R}$ such that $\omega(0)=0$,

$$|W(M_1) - W(M_2)| \le \omega\big(\mathcal{L}^1\big(\{M_1\ne M_2\}\big)\big) \qquad (3.14)$$

for all $M_1, M_2 \in L^2(X)$. Thus, the conditions (2.1), (2.22), (2.23) on L are all satisfied. We also assume that

$$W \text{ is } L^2_{\mathrm{loc}}(X)\text{-upper semicontinuous}, \qquad u_0 \text{ is } L^2_{\mathrm{loc}}(X)\text{-lower semicontinuous}. \tag{3.15}$$

Denote by $\mathcal{M}$ the set of all monotone nondecreasing functions in $L^2(X)$. Let $L^\epsilon(M, N) := L(M/\epsilon, N)$ for $\epsilon > 0$, so

$$\mathcal{L}^\epsilon(\mu, \xi) := \mathcal{L}(D_\#^{1/\epsilon}\mu, \xi \circ D^\epsilon).$$

The following minima exist and are equal (cf. [10]):

$$\begin{aligned} U^\epsilon(t, \mu) &= \min_{\sigma(t)=\mu} \left\{ U_0(\sigma(0)) + \int_0^t \mathcal{L}^\epsilon(\sigma_s, \dot\sigma_s)\mathrm{d}s \right\} \\ &= \min_{\phi \in \mathcal{C}_t(\cdot, M_\mu; \mathcal{M})} \left\{ U_0(\phi_0) + \int_0^t L^\epsilon(\phi_s, \dot\phi_s)\mathrm{d}s \right\} \\ &= U_0(M^\epsilon(0)) + \int_0^t L^\epsilon(M^\epsilon(s), \dot M^\epsilon(s))\mathrm{d}s, \end{aligned}$$

where M_μ is the monotone rearrangement of ν_0 into μ and $\mathcal{C}_t(M, N; \mathcal{M})$ is the set of all curves $\phi \in H^1(0, T; \mathcal{M})$ such that $\phi_0 = M, \phi_t = N$. According to [9], U^ϵ is a viscosity solution for

$$\partial_t U(t, \mu) + \mathcal{H}^\epsilon(\mu, \nabla_W U(t, \mu)) = 0, \qquad U(0, \mu) = U_0(\mu), \tag{3.16}$$

where the Legendre transform $\mathcal{H}^\epsilon$ of $\mathcal{L}^\epsilon$ is defined as in (3.12) with $\mathcal{L}^\epsilon$ replacing $\bar{\mathcal{L}}$. Below we prove the convergence of these viscosity solutions to the corresponding viscosity solution for (3.11).

Proposition 3.7. *Assume u_0 is continuous with respect to the $L^2_{\mathrm{loc}}(X)$ strong topology and let $M_T \in \mathcal{M}$. Let $\epsilon_n \downarrow 0$ and $M^n \in \mathcal{C}(\cdot, M_T)$ be a minimizer for*

$$\left\{ \mathcal{A}^n(S) := u_0(S(0)) + \int_0^T L^{\epsilon_n}(S(s), \dot S(s))\mathrm{d}s : S \in \mathcal{C}(\cdot, M_T) \right\}.$$

Then, possibly up to a subsequence, $M^n \xrightarrow{\tau} M$ for some $M \in H^1(0, T; \mathcal{M})$ which is a minimizer for

$$\left\{ \mathcal{A}(S) := u_0(S(0)) + \int_0^T \bar L(\dot S(s))\mathrm{d}s : S \in \mathcal{C}(\cdot, M_T) \right\}.$$

Furthermore, as $n \to \infty$,

$$\begin{aligned} &\min_{S \in \mathcal{C}_T(\cdot, M_T)} \left\{ u_0(S(0)) + \int_0^T L^{\epsilon_n}(S(s), \dot S(s))\mathrm{d}s \right\} \\ &\quad \to \min_{S \in \mathcal{C}_T(\cdot, M_T)} \left\{ u_0(S(0)) + \int_0^T \bar L(\dot S(s))\mathrm{d}s \right\}. \end{aligned} \tag{3.17}$$

Proof. Recall that, in fact, $M^n \in \mathcal{C}(M_0, M_T; \mathcal{M})$. Since $\dot M^n$ is clearly bounded in $L^2((0, T) \times X)$, we deduce that a subsequence $\{M^n\}_n$ converges in the τ topology to some $M \in \mathcal{C}_T(M_0, M_T; \mathcal{M})$. But we know that the actions of L^{ϵ_n} Γ-converge with respect to the

τ-topology to the action of $\bar{L}$. All that is left to show is the $\Gamma(\tau)$-convergence of $\mathcal{A}^n$ to $\mathcal{A}$. According to Proposition 2.4 in [19], it is enough to prove that the map $S \to u_0(S(0))$ is continuous on $H^1(0,T;L^2(X))$ with respect to the τ-topology. But $S_n \xrightarrow{\tau} S$ implies (see [10]) that $S_n(s) \to S(s)$ in $L^2_{\mathrm{loc}}(X)$-strong for all $s \in [0,t]$. In particular, $S_n(0) \to S(0)$ in $L^2_{\mathrm{loc}}(X)$, so $u_0(S_n(0)) \to u_0(S(0))$.

4. Applications to 1D pressureless gas dynamics

4.1. Some special Lagrangians

Note that L_1 and L_2 defined in (1.8) satisfy (2.1), (2.22) and (2.23). We shall prove:

Proposition 4.1. *One has*

$$\bar{L}_1(\xi) = \int_X \bar{l}_1 \circ \xi \,\mathrm{d}x, \tag{4.1}$$

where $\bar{l}_1$ is the effective Lagrangian associated with $l_1(x,v) = v^2/2 - V(x)$.

Proof. If $M \in H^1_0(0,T;L^2(X))$, then $M(\cdot,x) \in H^1_0(0,T)$ for a.e. $x \in X$. Since

$$⨍_0^T L_1(M + t\xi, \dot{M} + \xi)\mathrm{d}t = \int_X ⨍_0^T \left\{\frac{1}{2}|\dot{M}(t,x) + \xi(x)|^2 - V(M(t,x) + t\xi(x))\right\}\mathrm{d}t\mathrm{d}x,$$

we infer

$$\begin{aligned}
&\inf_M ⨍_0^T L_1(M + t\xi, \dot{M} + \xi)\mathrm{d}t \\
&\quad \geq \int_X \inf_{M(\cdot,x)} ⨍_0^T \left\{\frac{1}{2}|\dot{M}(t,x) + \xi(x)|^2 - V(M(t,x) + t\xi(x))\right\}\mathrm{d}t\mathrm{d}x.
\end{aligned}$$

Thus, Fatou's Lemma yields

$$\bar{L}_1(\xi) \geq \int_X \bar{l}_1 \circ \xi \,\mathrm{d}x.$$

To prove the opposite inequality, we take a sequence of discrete $\xi_n \to \xi$ in $L^2(X)$ and a.e. such that ξ_n is constantly $\xi_n^i \in \mathbb{R}$ on (c_{i-1}, c_i) (a regular partition of X), $i = 1, \ldots, n$. Clearly,

$$\begin{aligned}
\inf_M ⨍_0^T L_1(M + t\xi_n, \dot{M} + \xi_n)\mathrm{d}t &\leq \frac{1}{n} \inf_{\mathbf{r} \in H^1_0(0,T;\mathbb{R}^n)} \sum_{i=1}^n ⨍_0^T l_1(r_i + t\xi_n^i, \dot{r}_i + \xi_n^i)\mathrm{d}t \\
&= \frac{1}{n} \sum_{i=1}^n \inf_{r \in H^1_0(0,T;\mathbb{R})} ⨍_0^T l_1(r + t\xi_n^i, \dot{r} + \xi_n^i)\mathrm{d}t.
\end{aligned}$$

Passing to lim inf as $T \to \infty$, we obtain

$$\bar{L}_1(\xi_n) \leq \int_X \bar{l}_1 \circ \xi_n \mathrm{d}x.$$

Due to the continuity of $\bar{L}_1$ over $L^2(X)$ and of $\bar{l}_1$ over $\mathbb{R}$, we can pass to the limit as $n \to \infty$ to conclude the proof.

We introduce the Lagrangian $L_4 : [L^2(X)]^2 \times [L^2(X)]^2 \to \mathbb{R}$ given by

$$L_4(M_1, M_2, N_1, N_2) = \frac{1}{4}\|N_1\|^2 + \frac{1}{4}\|N_2\|^2 - \iint_{X^2} V(M_1(x) - M_2(y))\mathrm{d}y\mathrm{d}x. \tag{4.2}$$

As we shall see later, this corresponds to a system of two interacting species of particles. Arguing as in the proof of Proposition 4.1, one can show that

$$\bar{L}_4(\xi_1, \xi_2) = \iint_{X^2} \bar{l}_2(\xi_1(x), \xi_2(y))\mathrm{d}y\mathrm{d}x \quad \text{for any } \xi_1,\ \xi_2 \in L^2(X). \tag{4.3}$$

In the case of the two-particle (same species) interaction Lagrangian, we can rewrite it as

$$L_2(M, N) = \iint_{X^2} \left\{ \frac{1}{4}|N(x)|^2 + \frac{1}{4}|N(y)|^2 - V\big(M(x) - M(y)\big) \right\} \mathrm{d}y\mathrm{d}x. \tag{4.4}$$

Thus, the natural question is that of whether $\bar{L}_2(\xi)$ coincides with $\bar{L}_4(\xi, \xi) = \iint_{X^2} \bar{l}_2(\xi(x), \xi(y))\mathrm{d}y\mathrm{d}x$, where l_2 is the Lagrangian on $\mathbb{R}^2 \times \mathbb{R}^2$ given by $l_2(\mathbf{x}, \mathbf{v}) = |\mathbf{v}|^2/4 - V(x_1 - x_2)$. The answer is, in general, negative. The following lemma will be used in the sequel.

Lemma 4.2. *Let $T > 0$ and $\alpha \in \mathbb{R}$ such that $\alpha T \in \mathbb{Z}$. Then there is a unique solution for $\ddot{r} = -4V'(r)$ with boundary conditions $r(0) = 0,\ r(T) = \alpha T$. This solution is precisely the unique minimizer*

$$\operatorname{argmin}\left\{ \int_0^T l(r, \dot{r})\mathrm{d}t : r \in H^1(0, T),\ r(0) = 0,\ r(T) = \alpha T \right\},$$

where

$$l : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad l(x, v) = v^2/8 - V(x). \tag{4.5}$$

Proof. It is easy to prove that the minimizer exists, it is of class C^3 and its Euler–Lagrange equation is exactly the ODE above. Let us assume that $\alpha > 0$ (the other cases are handled similarly). Any solution satisfies

$$\frac{1}{8}|\dot{r}(t)|^2 + V(r(t)) = \frac{1}{8}|\dot{r}(0)|^2$$

because $r(0) = 0$ and $V(0) = 0$. We infer, due to $V \le 0$, $\alpha > 0$ and the continuity of $\dot{r}$,

$$\dot{r}(t) = \sqrt{m_0^2 - 8V(r(t))}, \quad \text{where } m_0 = \dot{r}(0) > 0.$$

Indeed, note that $\dot{r}(0)$ cannot be nonpositive. If $F(t) := \int_0^t (m_0^2 - 8V(s))^{-1/2}\mathrm{d}s$, then $F(r(t)) = t$ for all $t \in [0, T]$. In particular, since $r(T) = \alpha T$ is a positive integer and F' is periodic, then $1/\alpha = \int_0^1 (m_0^2 - 8V(s))^{-1/2}\mathrm{d}s$, so $\dot{r}(0)$ is uniquely determined. So, by uniqueness for the initial-value problem, we conclude the proof.

We are now ready to justify the following somewhat surprising result.

Proposition 4.3. *One has*

$$\bar{l}_2(a, b) = \frac{1}{8}(a + b)^2 + \bar{l}(a - b) \quad \textit{for all } a,\ b \in \mathbb{R}. \tag{4.6}$$

Proof. Let us consider minimizing

$$\int_0^T l_2(\mathbf{r},\dot{\mathbf{r}})\mathrm{d}t = \int_0^T \left\{\frac{1}{4}|\dot r_1|^2 + \frac{1}{4}|\dot r_2|^2 - V(r_1 - r_2)\right\}\mathrm{d}t$$

over $H^1(0,T;\mathbb{R}^2)$ with $\mathbf{r}(0) = \mathbf{0}$, $\mathbf{r}(T) = T\mathbf{c} \in \mathbb{Z}^2$. Clearly, the minimizer $\mathbf{r}$ exists, is in C^3, and satisfies the Euler–Lagrange system

$$\ddot r_1 = -2V'(r_1 - r_2), \qquad \ddot r_2 = -2V'(r_2 - r_1) \quad \text{with } \mathbf{r}(0) = \mathbf{0},\ \mathbf{r}(T) = T\mathbf{c}.$$

Since V is even, by adding the two equations we infer $\ddot r_1 + \ddot r_2 = 0$, so

$$\dot r_1 + \dot r_2 = c_1 + c_2 \quad \text{on } [0,T]. \tag{4.7}$$

By subtracting the equations we get that $r := r_1 - r_2$ satisfies $\ddot r = -4V'(r)$. But $r(0) = 0$ and $r(T) = T(c_1 - c_2) =: Tc \in \mathbb{Z}$, so, according to Lemma 4.2, r is the unique minimizer of $\int_0^T l(r,\dot r)\mathrm{d}t$ with the prescribed endpoints. Therefore, due to (4.7),

$$\begin{aligned}\min_{u(0)=0,u(T)=Tc} \fint_0^T l(u,\dot u)\mathrm{d}t &= \fint_0^T \left\{\frac{1}{8}|\dot r_1 - \dot r_2|^2 - V(r_1 - r_2)\right\}\mathrm{d}t \\ &= \min_{\mathbf{r}(0)=\mathbf{0},\mathbf{r}(T)=T\mathbf{c}} \fint_0^T l_2(\mathbf{r},\dot{\mathbf{r}})\mathrm{d}t - \frac{1}{8}(c_1+c_2)^2.\end{aligned}$$

We conclude that (4.6) is true for any $a,\ b \in \mathbb{Q}$ by taking $\liminf$ as $T \uparrow \infty$. We finish the argument by using the continuity of $\bar l$ and $\bar l_2$.

We next introduce a new one-dimensional Lagrangian $\tilde l(x,v) = v^2/4 - V(x)$. Note that, if $a \in \mathbb{R}$,

$$\begin{aligned}\bar{\tilde l}(a) &= \liminf_{T\to\infty} \min_{r(0)=0,r(T)=aT} \fint_0^T \left\{\frac{1}{8}|\dot r|^2 + \frac{1}{8}|\dot r|^2 - V(r)\right\}\mathrm{d}t \\ &\geq \frac{1}{8}a^2 + \bar l(a)\end{aligned} \tag{4.8}$$

because $\fint_0^T |\dot r|^2\mathrm{d}t \geq a^2$. We are now ready to prove:

Proposition 4.4. (1) *For every $\xi \in L^2(X)$,*

$$\bar L_2(\xi) \geq \iint_{X^2} \bar l_2(\xi(x),\xi(y))\mathrm{d}y\mathrm{d}x. \tag{4.9}$$

(2) *Unless $V \equiv 0$, there exists $\xi \in L^2(X)$ such that*

$$\bar L_2(\xi) > \iint_{X^2} \bar l_2(\xi(x),\xi(y))\mathrm{d}y\mathrm{d}x. \tag{4.10}$$

Proof. It is easy to see that $\bar L_4(\xi,\xi) \leq \bar L_2(\xi)$ for all $\xi \in L^2(X)$ (the infima are taken over a smaller class for $\bar L_2$).

We shall prove (4.10) by contradiction. Assume it is false. Then, due to (4.9), we have that

$$\bar L_2(\xi) = \iint_{X^2} \bar l_2(\xi(x),\xi(y))\mathrm{d}y\mathrm{d}x \quad \text{for all } \xi \in L^2(X). \tag{4.11}$$

Let us take an arbitrary $M \in H^1(0, T; L^2(X))$ with $M(0, \cdot) = 0$, $M(T, x) = T\xi(x)$ and write

$$\begin{aligned}\fint_0^T L_2(M, \dot{M})\mathrm{d}t &= \fint_0^T \iint_{X^2} \left\{\frac{1}{4}|\dot{M}(t,x) - \dot{M}(t,y)|^2 - V(M(t,x) - M(t,y))\right\}\mathrm{d}y\mathrm{d}x\mathrm{d}t \\ &\quad + \frac{1}{2}\fint_0^T \left(\int_X \dot{M}(t,x)\mathrm{d}x\right)^2 \mathrm{d}t \\ &\geq \iint_{X^2} \min_{r(0)=0, r(T)=(\xi(x)-\xi(y))T} \left\{\fint_0^T \tilde{l}(r, \dot{r})\mathrm{d}t\right\}\mathrm{d}y\mathrm{d}x + \frac{1}{2}\left(\int_X \xi \mathrm{d}x\right)^2.\end{aligned}$$

Passing to lim inf as $T \to \infty$ and using Fatou's Lemma, we deduce

$$\bar{L}_2(\xi) \geq \iint_{X^2} \bar{\bar{l}}(\xi(x) - \xi(y))\mathrm{d}y\mathrm{d}x + \frac{1}{2}\left(\int_X \xi \mathrm{d}x\right)^2, \tag{4.12}$$

which, together with (4.11) and Proposition 4.3, gives

$$\frac{1}{8}\iint_{X^2} |\xi(x) - \xi(y)|^2 \mathrm{d}y\mathrm{d}x \geq \iint_{X^2} \left[\bar{\bar{l}}(\xi(x) - \xi(y)) - \bar{l}(\xi(x) - \xi(y))\right]\mathrm{d}y\mathrm{d}x$$

for all $\xi \in L^2(X)$. Thus, $a^2/8 \geq \bar{\bar{l}}(a) - \bar{l}(a)$ for all $a \in \mathbb{R}$. But this and (4.8) imply

$$a^2/8 = \bar{\bar{l}}(a) - \bar{l}(a) \quad \text{for all } a \in \mathbb{R}. \tag{4.13}$$

By definition, we also have $\bar{\bar{l}}(a) = \bar{l}(a\sqrt{2})$ for all $a \in \mathbb{R}$. So, (4.13) implies

$$a^2/4 = \bar{\bar{l}}(a\sqrt{2}) - \bar{\bar{l}}(a) \quad \text{for all } a \in \mathbb{R}. \tag{4.14}$$

The Legendre transform of $\tilde{l}$ is $\tilde{h}(x, p) = p^2 + V(x)$. Its effective Hamiltonian is computed in [15] as

$$\bar{\tilde{h}}(p) = \begin{cases} 0 & \text{if } |p| \leq \langle\sqrt{-V}\rangle \\ \lambda & \text{if } |p| = \langle\sqrt{-V + \lambda}\rangle \end{cases}$$

so, since $\bar{\tilde{l}}$ is its Legendre transform, its subgradient at 0, $\partial^-\bar{\tilde{l}}(0)$, has a maximal element $p_0 \geq \langle\sqrt{-V}\rangle > 0$ unless $V \equiv 0$ (trivial case). If we divide (4.14) by $a\sqrt{2}$ for $a > 0$ and let $a \to 0^+$ we obtain $p_0 = p_0/\sqrt{2}$ which contradicts $p_0 > 0$.

Remark 4.5. It is not clear whether equality in (4.12) holds for all $\xi \in L^2(X)$.

4.2. Homogenization for action-minimizing solutions

Fix $T > 0$ and $\rho_0, \rho_T \in \mathcal{P}_2(\mathbb{R})$. One consequence of the analysis in the previous section is that if we consider action-minimizing solutions (shown to exist) of

$$\begin{cases} \partial_t \rho^\epsilon + \partial_x(\rho^\epsilon v^\epsilon) = 0 \\ \partial_t(\rho^\epsilon v^\epsilon) + \partial_x\left[\rho^\epsilon (v^\epsilon)^2\right] = -\frac{1}{\epsilon}\rho^\epsilon A_i^\epsilon \end{cases} \tag{4.15}$$

with prescribed endpoints $\rho^\epsilon(0) = \rho_0$ and $\rho^\epsilon(T) = \rho_T$, then the corresponding optimal maps $M_{\rho^\epsilon(t)}$ converge in $L^2_{\mathrm{loc}}(X)$ to $M(t)$, where $M \in H^1(0,T;L^2(X))$ minimizes the action $\int_0^T \bar{L}_i(\dot{M})\mathrm{d}t$. Recall that here

$$A_1^\epsilon(x) = V'\left(\frac{x}{\epsilon}\right) \quad \text{and} \quad A_2^\epsilon(t,x) = \int V'\left(\frac{x-y}{\epsilon}\right)\rho^\epsilon(t,\mathrm{d}y),$$

respectively. Let $\rho(t) =: M(t)_\# \nu_0$. It follows that $\rho^\epsilon(t,\cdot)$ converges to $\rho(t,\cdot)$ in the p-Wasserstein distance for any $1 \le p < 2$. So far, we have not been able to identify the limiting system of equations satisfied by (ρ, v) in the case of the two-particle interaction Lagrangian L_2. We can only prove:

Theorem 4.6. *Let $(\rho^\epsilon, v^\epsilon)$ be action-minimizing solutions for*

$$\begin{cases} \partial_t\rho^\epsilon + \partial_x(\rho^\epsilon v^\epsilon) = 0 \\ \partial_t(\rho^\epsilon v^\epsilon) + \partial_x\left[\rho^\epsilon (v^\epsilon)^2\right] = -1/\epsilon \rho^\epsilon V'\left(\frac{x}{\epsilon}\right) \end{cases} \tag{4.16}$$

with prescribed $\rho^\epsilon(0) = \rho_0$ and $\rho^\epsilon(T) = \rho_T$. Then, the limiting ρ is the geodesic in the Wasserstein space connecting ρ_0 and ρ_T. Thus, (ρ, v) satisfies the pressureless Euler system

$$\begin{cases} \partial_t\rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho v) + \partial_x(\rho v^2) = 0. \end{cases} \tag{4.17}$$

Proof. Since we know that $t \to M(t)$ minimizes

$$\int_0^T \bar{L}_1(\dot{M})\mathrm{d}t = \int_0^T \int_X \bar{l}_1(\dot{M}(t,x))\mathrm{d}x\mathrm{d}t,$$

(equality due to Proposition 4.1) it is enough to show that $\bar{l}_1$ is strictly convex to conclude that $M(t,x) = (1-t/T)M_0(x) + (t/T)M_T(x)$. The Legendre transform of l_1 is $h_1(x,p) = p^2/2 + V(x)$. Just as before we recall the computation in [15] to write down its effective Hamiltonian

$$\bar{h}_1(p) = \begin{cases} 0 & \text{if } |p| \le \langle\sqrt{-2V}\rangle \\ \lambda & \text{if } |p| = \langle\sqrt{-2V+2\lambda}\rangle. \end{cases}$$

So,

$$p = \sqrt{2}\int_0^1 \sqrt{\bar{h}_1(p) - V(x)}\mathrm{d}x \quad \text{for } p \ge \langle\sqrt{-2V}\rangle =: p_0.$$

We infer that $\bar{h}_1$ is differentiable at all p with $p > p_0$ and

$$\bar{h}_1'(p) = \frac{1}{\int_0^1 \frac{\mathrm{d}x}{\sqrt{\bar{h}_1(p)-V(x)}}} \quad \text{for } p > p_0.$$

Since $\bar{h}_1$ is even, it is enough show that

$$\lim_{p\to p_0^-}\int_0^1 \frac{\mathrm{d}x}{\sqrt{\bar{h}_1(p) - V(x)}} = \infty$$

in order to conclude that $\bar{h}_1$ is differentiable at $\pm p_0$. But V is C^2 and $V(0) = V'(0) = 0$, so there exists $C > 0$ such that $\sqrt{-V(x)} \le Cx$ on [0, 1]. This, combined with the fact that $\bar{h}_1(p) \downarrow 0$ as $p \to {p_0}^-$, yields that the limit in the last display is, indeed, ∞. Since $\bar{h}_1 = 0$ on $[-p_0, p_0]$, we obtain that $\bar{h}_1$ is differentiable everywhere on $\mathbb{R}$. Thus, $\bar{l}_1$ is strictly convex.

Likewise, $\bar{l}$ (for l defined in (4.5)) is strictly convex. In view of Proposition 4.3, we infer the strict convexity of $\bar{l}_2$ and then, due to (4.3), the strict convexity of $\bar{L}_4$. Thus, we may state the following true result.

Theorem 4.7. *Let $(\rho_i^\epsilon, v_i^\epsilon)$, $i = 1, 2$, be action-minimizing solutions for*

$$\begin{cases} \partial_t \rho_i^\epsilon + \partial_x(\rho_i^\epsilon v_i^\epsilon) = 0, \\ \partial_t(\rho_i^\epsilon v_i^\epsilon) + \partial_x\big[\rho_i^\epsilon (v_i^\epsilon)^2\big] = -1/\epsilon \rho_i^\epsilon A_j^\epsilon, \quad i \ne j \\ A_i^\epsilon(t, x) = 2\displaystyle\int_{\mathbb{R}} V'\left(\frac{x - y}{\epsilon}\right)\rho_i^\epsilon(t, \mathrm{d}y) \end{cases} \tag{4.18}$$

with prescribed $\rho_i^\epsilon(0) = \rho_{i,0}$ and $\rho_i^\epsilon(T) = \rho_{i,T}$, $i = 1, 2$. Then, the limiting ρ_i are geodesics in the Wasserstein space connecting $\rho_{i,0}$ and $\rho_{i,T}$. Thus, (ρ_i, v_i) decouple to each satisfy the pressureless Euler system (4.17) for $i = 1, 2$.

To prove existence for the action-minimizing solutions $(\rho_i^\epsilon, v_i^\epsilon)$ for fixed endpoints $\rho_i^\epsilon(0) = \rho_{i,0}$ and $\rho_i^\epsilon(T) = \rho_{i,T}$, we consider minimizing the action

$$\int_0^T L_4\left(\frac{\mathbf{M}}{\epsilon}, \dot{\mathbf{M}}\right)\mathrm{d}t$$

over the set $\mathbf{M} = (M_1, M_2) \in H^1(0, T; \mathbf{L}^2(X))$ such that M_i are prescribed and monotone nondecreasing at $0, T$. Then we adapt the proof (existence of minimizing paths) in [9] to get the existence of minimizers. The corresponding Euler–Lagrange system is

$$\begin{cases} \ddot{M}_1(t, x) = -\dfrac{2}{\epsilon}\displaystyle\int_X V'\left(\frac{M_1(t, x) - M_2(t, y)}{\epsilon}\right)\mathrm{d}y \\ \ddot{M}_2(t, x) = -\dfrac{2}{\epsilon}\displaystyle\int_X V'\left(\frac{M_2(t, x) - M_1(t, y)}{\epsilon}\right)\mathrm{d}y \end{cases} \tag{4.19}$$

and a minimizing pair $(M_1(t, \cdot), M_2(t, \cdot))$ consists of monotone nondecreasing maps in $L^2(X)$ for all $t \in [0, T]$. Let $\varphi_i \in C_c^1((0, T) \times X)$ and multiply the equations in the above system by $\varphi_i(t, M_i(t, x))$ for $i = 1, 2$ respectively. Then integrate on $[0, T] \times X$ by performing integration by parts in time for the left hand side. After that, we use Proposition 4.2 in [10] to obtain, upon setting $\rho_i^\epsilon(t, \cdot) = M_i(t, \cdot)_{\#}\nu_0$, the formulation of (4.18) in the sense of distributions. To show some detail:

$$\begin{aligned} \int_0^T \int_X \ddot{M}_i \varphi_i(t, M_i)\mathrm{d}z\mathrm{d}t &= -\int_0^T \int_X \big[\dot{M}_i \partial_t \varphi_i(t, M_i) + |\dot{M}_i|^2 \partial_x \varphi_i(t, M_i)\big]\mathrm{d}z\mathrm{d}t \\ &= -\int_0^T \int_X \big[v_i(t, M_i)\partial_t \varphi_i(t, M_i) + v_i(t, M_i)^2 \partial_x \varphi_i(t, M_i)\big]\mathrm{d}z\mathrm{d}t \\ &= -\int_0^T \int_{\mathbb{R}} \big[v_i(t, x)\partial_t \varphi_i(t, x) + v_i^2(t, x)\partial_x \varphi_i(t, x)\big]\rho_i(t, \mathrm{d}x)\mathrm{d}t. \end{aligned}$$

But the ith equation in (4.19) also gives

$$\int_0^T \int_X \ddot{M}_i \varphi_i(t, M_i)\mathrm{d}z\mathrm{d}t = -\frac{2}{\epsilon}\int_0^T \iint_{X^2} V'\left(\frac{M_i(t,z) - M_j(t,z')}{\epsilon}\right)\mathrm{d}z'\mathrm{d}z\mathrm{d}t$$
$$= -\frac{2}{\epsilon}\int_0^T \iint_{\mathbb{R}^2} V'\left(\frac{x-y}{\epsilon}\right)\rho_j(t,\mathrm{d}y)\rho_i(t,\mathrm{d}x)\mathrm{d}t,$$

where $i \neq j$. By equating the two we get the desired result (note that we dropped the ϵ superscripts). Note that the continuity equations hold automatically due to $\dot{M}_i(t,\cdot) = v_i(t, M_i(t,\cdot))$ (again, see Proposition 4.2 in [10]).

5. Γ-limits on $\mathcal{P}_p(\mathbb{R}^d)$

Throughout this subsection $T > 0$ is prescribed and ν_0 is the Lebesgue measure on $X = (0,1)^d$. We assume that W is a nonnegative, bounded, real-valued function defined on $\mathbf{L}^p(X)$ which satisfies $W(M) = W(M^*)$ whenever $M, M^* \in \mathbf{L}^p(X)$ are such that $M_\#\nu_0 = M^*_\#\nu_0$. We also assume

$$W \text{ is continuous on } \mathbf{L}^p(X) \text{ endowed with the uniform convergence topology.} \tag{5.1}$$

For $\mu \in \mathcal{P}_p(\mathbb{R}^d)$, $\mathbf{L}^p(\mu)$ is the set of $\xi : \mathbb{R}^d \to \mathbb{R}^d$ that are μ-measurable and such that $\|\xi\|_\mu^p =: \int_{\mathbb{R}^d} |\xi|^p \mathrm{d}\mu < \infty$. We denote by $\mathcal{T}_\mu\mathcal{P}_p(\mathbb{R}^d)$ the closure of $\{\nabla\varphi : \varphi \in C_c^\infty(\mathbb{R}^d)\}$ in $\mathbf{L}^p(\mu)$. The union of the sets $\{\mu\} \times \mathcal{T}_\mu\mathcal{P}_p(\mathbb{R}^d)$ is denoted by $\mathcal{T}\mathcal{P}_p(\mathbb{R}^d)$ (cf. e.g. [2]).

We set

$$L(M,N) := \frac{1}{p}\|N\|^p + W(M), \qquad \mathcal{L}(\mu,\xi) := L(M, \xi\circ N), \qquad \mathcal{W}(\mu) := W(M)$$

for $(\mu,\xi) \in \mathcal{T}\mathcal{P}_p(\mathbb{R}^d)$ and $M, N \in \mathbf{L}^p(X)$ such that $M_\#\nu_0 = \mu$. We assume that $c = 1/p$ and C are such that L satisfies (2.1) and (2.2) and we suppose that (2.23) holds. Note that

$$|\mathcal{W}(\mu_1) - \mathcal{W}(\mu_2)| \le \omega\Big(\mathcal{L}^d\{M_1 \neq M_2\}\Big) \tag{5.2}$$

for any $\mu_1, \mu_2 \in \mathcal{P}_p(\mathbb{R}^d)$ and any $M_1, M_2 \in \mathbf{L}^p(X)$ such that $\mu_i = M_{i\#}\nu_0,\ i = 1,2$.

If $\Omega \subset (0,T)$ is an open set with negligible boundary, we denote as $AC^p(\Omega;\mathcal{P}_p(\mathbb{R}^d))$ the set of p-absolutely continuous paths of Ω into $\mathcal{P}_p(\mathbb{R}^d)$. If $\sigma \in AC^p(\Omega;\mathcal{P}_p(\mathbb{R}^d))$ and $\mathbf{v}$ is the velocity of minimal norm for σ, we define

$$\mathcal{F}^\epsilon(\sigma;\Omega) = \int_\Omega \mathcal{L}(D_\#^{1/\epsilon}\sigma_t, \mathbf{v}_t \circ D^\epsilon)\mathrm{d}t.$$

We simply write $\mathcal{F}^\epsilon(\sigma,\mathbf{v})$ in place of $\mathcal{F}^\epsilon(\sigma;\Omega)$ when $\Omega = (0,T)$.

Remark 5.1. Let $\{\mu^k\}_k \subset \mathcal{P}_p(\mathbb{R}^d)$ be a bounded sequence and O^k be the unique optimal map (optimality being measured against the cost $|\cdot|^p$, unless otherwise specified) that pushes ν_0 forward to μ^k (cf. e.g. [4,8]).

(i) Suppose $\{\mu^k\}_k$ converges narrowly to μ. The set $\Gamma_0(\nu_0,\mu)$, which consists of the optimal measures which have ν_0 and μ^k as marginals, reduces to $\{\mathbf{id} \times O\}$ (cf. e.g. [8]). Here, O is the unique optimal map that pushes ν_0 forward to μ. Proposition 7.1.3 [2] ensures that

$\{(\mathbf{id} \times O^k)_{\#}\nu_0\}_k$ narrowly converges to $\mathbf{id} \times O$. As a consequence, whenever $1 \le q < p$, $\{O^k\}_k$ converges to O in $\mathbf{L}^q(X)$.

(ii) Further assume that $\{\mu^k\}_k$ converges to μ in $\mathcal{P}_p(\mathbb{R}^d)$. Then

$$\limsup_{k\to\infty} |\|O^k\| - \|O\|| = \limsup_{k\to\infty} |W_p(\delta_0, \mu^k) - W_p(\delta_0, \mu)| \\ \le \limsup_{k\to\infty} W_p(\mu, \mu^k) = 0. \tag{5.3}$$

As $\{O^k\}_k$ is a bounded subset of $\mathbf{L}^p(X)$ by (i), it converges weakly to O in $\mathbf{L}^p(X)$. Hence, by using (5.3) we conclude that $\{O^k\}_k$ converges to O in $\mathbf{L}^p(X)$.

5.1. The effective Lagrangian on $\mathcal{P}_p(\mathbb{R}^d)$

As in (1.4), we define

$$\bar{\mathcal{L}}(\mu, \xi) = \liminf_{T\to\infty} \frac{\mathcal{C}_{0,T}\big(\delta_0, (T\xi)_{\#}\mu\big)}{T} = \liminf_{T\to\infty} \frac{\mathcal{C}_{0,T}\big(\delta_0, D^T \circ \xi_{\#}\mu\big)}{T}.$$

Let $\sigma \in AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ and let $\mathbf{v}$ be a velocity for σ. Setting

$$\sigma_s^\epsilon := D_{\#}^{1/\epsilon}\sigma_{\epsilon s}, \qquad \mathbf{v}_s^\epsilon := \mathbf{v}_{\epsilon s} \circ D^\epsilon,$$

we have that $\mathbf{v}^\epsilon$ is a velocity for σ^ϵ and $\int_0^T \mathcal{L}(\sigma_t, \mathbf{v}_t)\mathrm{d}t = \epsilon \int_0^T \mathcal{L}^\epsilon(\sigma_s^\epsilon, \mathbf{v}_s^\epsilon)\mathrm{d}s$. As σ is arbitrary, we conclude that

$$\mathcal{C}_{0,T}(\sigma_0, \sigma_T) = \epsilon\mathcal{C}^\epsilon_{0,T/\epsilon}(D_{\#}^{1/\epsilon}\sigma_0, D_{\#}^{1/\epsilon}\sigma_T).$$

In particular,

$$\frac{\mathcal{C}_{0,T}(\delta_0, \sigma_T)}{T} = \frac{\mathcal{C}^\epsilon_{0,T}\big(D_{\#}^{1/\epsilon}\sigma_0, D^{T/\epsilon} \circ D_{\#}^{T^{-1}}\sigma_T\big)}{\epsilon/T}. \tag{5.4}$$

Note that

$$\frac{1}{p}\|\mathbf{v}_t\|_{\sigma_t}^p \le \mathcal{L}^\epsilon(D_{\#}^{1/\epsilon}\sigma, \mathbf{v} \circ D^\epsilon) = \frac{1}{p}\|\mathbf{v}_t\|_{\sigma_t}^p + \mathcal{W}(D_{\#}^{1/\epsilon}\sigma) \le C(\|\mathbf{v}_t\|_{\sigma_t}^p + 1)$$

and so,

$$cW_p^p(\sigma_a, \sigma_b) \le \mathcal{F}^\epsilon\big(\sigma; (a, b)\big) \tag{5.5}$$

for any $\sigma \in AC^p(a, b; \mathcal{P}_p(\mathbb{R}^d))$. If, in addition, σ is a geodesic of constant speed then

$$\mathcal{F}^\epsilon(\sigma; (a, b)) \le C(b - a)\left(1 + \frac{W_p^p(\sigma_a, \sigma_b)}{(b - a)^p}\right). \tag{5.6}$$

For real numbers $a < b$ and $\sigma_a, \sigma_b \in \mathcal{P}_p(\mathbb{R}^d)$ we define

$$\mathcal{C}^\epsilon_{a,b}(\sigma_a, \sigma_b) = \inf_\sigma \int_a^b \mathcal{L}(D_{\#}^{1/\epsilon}\sigma_t, \mathbf{v}_t \circ D^\epsilon)\mathrm{d}t, \tag{5.7}$$

where the infimum is performed over the set of $(\sigma, \mathbf{v})$ such that $\mathbf{v}$ is a velocity for $\sigma \in \mathcal{P}(\sigma_a, \sigma_b)$.

Lemma 5.2. *$\mathcal{W}$ is continuous for the narrow convergence on bounded subsets of $\mathcal{P}_p(\mathbb{R}^d)$.*

Proof. Suppose $\{\mu^k\}_k$ is a bounded sequence of $\mathcal{P}_p(\mathbb{R}^d)$ which converges narrowly to μ in $\mathcal{P}_p(\mathbb{R}^d)$. Let $O, O^k : X \to \mathbb{R}^d$ be such that $O^k_\# \nu_0 = \mu^k$ and $O_\# \nu_0 = \mu$ optimally against $|\cdot|^p$. Remark 5.1 ensures that $\{O^k\}_k$ converges to O in $\mathbf{L}^1(X)$ and so for $\delta > 0$ there exist a set $E \subset X$ such that $\mathcal{L}^d(X \setminus E) \le \delta$ and up to a subsequence which we do not relabel, $\{O^k\}_k$ converges uniformly to O on E. Let $\bar{O}^k$ be equal to O^k on E and 0 on $X \setminus E$. We define $\bar{O}$ in a similar manner. We exploit (5.2) to obtain

$$|\mathcal{W}(\mu^k) - \mathcal{W}(\mu)| \le |W(\bar{O}^k) - W(\bar{O})| + 2\omega(\delta).$$

We now use (5.1) by letting $k \to \infty$, then letting δ tend to zero we conclude the proof.

Lemma 5.3. *Given $\sigma_a, \sigma_b \in \mathcal{P}_p(\mathbb{R}^d)$, there exists $\sigma^* \in AC^p(a, b; \mathcal{P}_p(\mathbb{R}^d))$, a minimizer in (5.7). Let $\mathbf{v}^*$ be the velocity of minimal norm for σ^*. Then there exists a set $\mathcal{N} \subset (a, b)$ of zero measure and a map $\mathbf{v} : (a, b) \times \mathbb{R}^d \to \mathbb{R}^d$ such that $\mathbf{v}_t = \mathbf{v}^*_t$ for all $t \in (a, b) \setminus \mathcal{N}$ and*

$$t \to \frac{1}{p'} \|\mathbf{v}_t\|^p_{\mathbf{L}^p(\sigma^*_t)} + \mathcal{W}(D^{1/\epsilon}_\# \sigma^*_t)$$

is time independent. In particular,

$$\sup_{t \in (a,b)} \|\mathbf{v}_t\|^p_{\mathbf{L}^p(\sigma^*_t)} \le pC(b-a)\left(1 + \frac{W^p_p(\sigma_a, \sigma_b)}{(b-a)^p}\right) + p^2 \sup |\mathcal{W}|.$$

Proof. Let $\{\sigma^k\}_k$ be a minimizing sequence in (5.7). Let σ be a geodesic of constant speed connecting σ_a to σ_b and let $\mathbf{v}$ be the velocity of minimal norm for it. We assume without loss of generality that

$$\int_a^b \mathcal{L}(D^{1/\epsilon}_\# \sigma_t, \mathbf{v}_t \circ D^\epsilon) \mathrm{d}t \ge \int_a^b \mathcal{L}(D^{1/\epsilon}_\# \sigma^k_t, \mathbf{v}^k_t \circ D^\epsilon) \mathrm{d}t.$$

We exploit (5.5), (5.6), and the fact that $\sigma^k_a = \sigma_a$, and use Remark A.1 to conclude that $\{\sigma^k\}_k$ is bounded in $AC^p(a, b; \mathcal{P}_p(\mathbb{R}^d))$. In light of the same remark, we may assume without loss of generality the existence of a $\sigma^* \in AC^p(a, b; \mathcal{P}_p(\mathbb{R}^d))$ such that $\{\sigma^k_t\}_k$ converges to σ^*_t in $\mathcal{P}_q(\mathbb{R}^d)$ for every $q \in [1, p)$ and every $t \in [0, T]$. Let $\mathbf{v}^*$ be the velocity of minimal norm for σ^*. Note that $\sigma^*_a = \sigma_a$ and $\sigma^*_b = \sigma_b$. By Lemma 5.2, $\{\mathcal{W}(\sigma^k_t)\}_k$ converges to $\mathcal{W}(\sigma^*_t)$ for every $t \in [0, T]$. As $\{\mathcal{W}(\sigma^k_t)\}_k$ is bounded uniformly in t and n, we use the Lebesgue dominated convergence theorem to conclude that

$$\lim_{k \to \infty} \int_a^b \mathcal{W}(D^{1/\epsilon}_\# \sigma^k_t) \mathrm{d}t = \int_a^b \mathcal{W}(D^{1/\epsilon}_\# \sigma^*_t) \mathrm{d}t.$$

Then one can reproduce verbatim the proof of Proposition 3 [10] given in the case $p = 2$ to infer

$$\liminf_{k \to +\infty} \int_a^b \|\mathbf{v}^k_t\|^p_{\mathbf{L}^p(\sigma^k_t)} \mathrm{d}t \ge \int_a^b \|\mathbf{v}^*_t\|^p_{\mathbf{L}^p(\sigma^*_t)} \mathrm{d}t,$$

which is the last ingredient needed for concluding that σ^* is a minimizer in (5.7).

We have that the distributional derivative of $t \to 1/p' \|\mathbf{v}_t\|^p_{\mathbf{L}^p(\sigma^*_t)} + \mathcal{W}(D^{1/\epsilon}_\# \sigma^*_t)$ is null (cf. Proposition 3.11 [9] for the case $p = 2$). This proves that there exists a set $\mathcal{N} \subset (0, T)$ of zero measure and a map $\mathbf{v} : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d$ such that $\mathbf{v}_t = \mathbf{v}_t^*$ for all $t \in (0, T) \setminus \mathcal{N}$ and $t \to 1/p' \|\mathbf{v}_t\|^p_{\mathbf{L}^p(\sigma_t^*)} + \mathcal{W}(D_\#^{1/\epsilon} \sigma_t^*)$ is time independent. By (5.6), the set of t such that

$$\frac{1}{p'} \|\mathbf{v}_t\|^p_{\mathbf{L}^p(\sigma_t^*)} - \mathcal{W}(D_\#^{1/\epsilon} \sigma_t^*) \le C(b-a)\left(1 + \frac{W_p^p(\sigma_a, \sigma_b)}{(b-a)^p}\right)$$

is of positive measure. We use that $1/p' \|\mathbf{v}_t\|^p_{\mathbf{L}^p(\sigma_t^*)} + \mathcal{W}(D_\#^{1/\epsilon} \sigma_t^*)$ is time independent to conclude the proof of the lemma.

5.2. $\mathcal{P}_p(\mathbb{R}^d)$ as a quotient space of $\mathbf{L}^p(X)$

Proposition 5.4. *Let $M \in \mathbf{L}^p(X)$ and fix $\delta > 0$. Then:*

(i) *There exist $\bar{M} \in \mathbf{L}^p(X)$ and a partition of X into parallel cubes $\{X_i\}_{i=1}^k$ of the same size such that $\|\bar{M} - M\| \le \delta$, $|\bar{M}| \le |M| + 1$, $\bar{M}$ is a constant value m_i on each X_i, $m_i \in \mathbb{Q}$ and $m_i \ne m_i$ whenever $i \ne j$.*

(ii) *There exists $\tilde{M} \in \mathbf{L}^p(X)$ such that $|\tilde{M}| \le |M| + 2$, $\|\tilde{M} - M\| \le \delta$, $\tilde{M}$ is invertible, and both M and $\tilde{M}^{-1}$ are Borel maps. Furthermore, $\mathcal{L}^d(\tilde{M}^{-1}(A)) = 0$ whenever $A \subset \mathbb{R}^d$ and $\mathcal{L}^d(A) = 0$.*

Proof. (i) Clearly, there exists $M^* \in \mathbf{L}^p(X)$ and a partition of X into parallel cubes $\{X_i\}_{i=1}^k$ of the same size such that $|\bar{M}| \le |M| + 1/2$, $\|M^* - M\| \le \delta/2$, M^* is a constant value m_i^* on each X_i and $\{m_i^*\}_{i=1}^k \subset \mathbb{Q}$. Next, choose a set $\{m_i\}_{i=1}^k \subset \mathbb{Q}$ of cardinality k such that $2|m_i - m_i^*| \le \min\{1, \delta\}$. Then $\bar{M} = \sum_{i=1}^k m_i \chi_{X_i}$ satisfies the required property.

(ii) In light of (i), we may assume without loss of generality that there exists a partition of X into parallel cubes $\{X_i\}_{i=1}^k$ of the same size such that M is a constant value m_i on each X_i, $m_i \in \mathbb{Q}$ and $m_i \ne m_j$ whenever $i \ne j$. Set $\tilde{M}z = Mz + z/a$ where $a > 0$ is such that

$$a \min_{i \ne j} |m_i - m_j| > \sqrt{d}, \qquad 2\sqrt{d} < a\delta, \quad a > \sqrt{d}.$$

The first inequality above ensures that $\tilde{M}$ is one-to-one, the second one gives that $\|\tilde{M} - M\| \le \delta$, while the third one ensures that $|\tilde{M}| \le |M| + 1$. Clearly, both M and $\tilde{M}^{-1}$ are Borel maps and $\mathcal{L}^d(\tilde{M}^{-1}(A)) = 0$ whenever $A \subset \mathbb{R}^d$ and $\mathcal{L}^d(A) = 0$.

The push forward operator $M \to M_\# \nu_0$ defines an equivalence relation on $\mathbf{L}^p(X)$. The goal of this subsection is to show that $M_\# \nu_0 = \tilde{M}_\# \nu_0$ if and only if $\inf_{\mathcal{G}} \|M - \tilde{M} \circ G\| = 0$ where $\mathcal{G}$ is the non-commutative group which consists of maps $G : X \to X$ such that G is one-to-one $\mathcal{L}^d$-almost everywhere onto X, and G, G^{-1} are Borel maps that push ν_0 forward to itself.

Theorem 5.5. *If $M, \tilde{M} \in \mathbf{L}^p(X)$, then the following are equivalent:*

(i) $M_\# \nu_0 = \tilde{M}_\# \nu_0$;

(ii) $(\exists)\{G^n\}_n \subset \mathcal{G}$ *such that* $\lim_{n\to\infty} \|M - \tilde{M} \circ G^n\| = 0$.

Proof. We will prove that (i) implies (ii). The reverse implication is easy. Set $\mu = M_\# \nu_0$ and suppose that (i) holds. Let O be the unique p-optimal Borel map that pushes ν_0 forward to μ (i.e. optimality is measured against the cost function $|\cdot|^p$). We may assume without loss of generality that $M = O$. Let $\{M^n\}_n \subset L^\infty(X)$ be a sequence of Borel maps converging to $\tilde{M}$ in $\mathbf{L}^p(X)$

such that M^n is invertible, both M^n and its inverse are Borel maps and $\mathcal{L}^d\Big((M^n)^{-1}(A)\Big) = 0$ whenever $A \subset \mathbb{R}^d$ is a Borel measurable set such that $\mathcal{L}^d(A) = 0$. Such a sequence of maps exists by Proposition 5.4.

Let $\{O^n\}_n \subset C(\mathbb{R}^d)$ be a sequence of maps such that $O^n_{\#}\nu_0 = M^n_{\#}\nu_0 := \mu^n$ and O^n is p-optimal. Such a sequence exists by the mass transportation theory (cf. e.g. [8]) and O^n is essentially invertible. Set

$$F^n := (O^n)^{-1} \circ M^n, \qquad G^n := (M^n)^{-1} \circ O^n.$$

Note that F^n and G^n are Borel maps as compositions of Borel maps. They preserve the Lebesgue measure and are both one-to-one almost everywhere from X onto X. In fact, G^n is essentially the inverse of F^n. The convergence of $\{M^n\}_n$ to $\tilde{M}$ in $\mathbf{L}^p(X)$ ensures that of $\{\mu^n\}_n$ to μ in $\mathcal{P}_p(\mathbb{R}^d)$ and so, by Remark 5.1, $\{O^n\}_n$ converges to O in $\mathbf{L}^p(X)$. Note that

$$\tilde{M} \circ G^n - M = \tilde{M} \circ G^n - M^n \circ G^n + O^n - O.$$

Using the triangular inequality and the fact that G^n preserves Lebesgue measure we obtain

$$\limsup_{n\to\infty} \|M - \tilde{M} \circ G^n\| \le \limsup_{n\to\infty} \|M^n - \tilde{M}\| + \|O^n - O\| = 0.$$

Remark 5.6. Let $W : \mathbf{L}^p(X) \to \mathbb{R}$ be a continuous map. By Theorem 5.5, if $W(M) = W(M \circ G)$ for all $M \in \mathbf{L}^p(X)$ and all $G \in \mathcal{G}$, then $W(M) = W(\bar{M})$ for all $M, \bar{M} \in \mathbf{L}^p(X)$ satisfying $M_{\#}\nu_0 = \tilde{M}_{\#}\nu_0$. The converse of this statement clearly holds even if W is not continuous.

5.3. Homogenization on $\mathcal{P}_p(\mathbb{R}^d)$

Lemma 3.5 suggests that in higher dimensions we consider a special subset of $AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ on which we introduce a special topology.

Definition 5.7. Let $\sigma \in AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ and suppose $\mathbf{v}$ is the velocity of minimal norm for σ. Suppose $\phi \in \mathcal{H}$ is such that

$$\text{(i) } \phi_{t\#}\nu_0 = \sigma_t \quad \text{for every } t \in [0, T]; \tag{5.8}$$

$$\text{(ii) } \dot{\phi}_t = \mathbf{v}_t \circ \phi_t \quad \text{for a.e. } t \in (0, T). \tag{5.9}$$

We say that ϕ is a flow associated with $(\sigma, \mathbf{v})$. We denote by $\mathcal{S}_p(\mathbb{R}^d)$ the set of σ such that $\sigma \in AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$, $\mathbf{v}$ is the velocity of minimal norm for σ, and there is a flow ϕ associated with $(\sigma, \mathbf{v})$.

Definition 5.8. Let $\sigma \in \mathcal{S}_p(\mathbb{R}^d)$ and suppose $\mathbf{v}$ is the velocity of minimal norm for σ. Let $\{\sigma^n\}_n \subset \mathcal{S}_p(\mathbb{R}^d)$ and suppose $\mathbf{v}^n$ is the velocity of minimal norm for σ^n. We say that $\{\sigma^n\}_n$ τ-converges to σ if there exists a flow ϕ^n associated with $(\sigma^n, \mathbf{v}^n)$ and a flow ϕ associated with $(\sigma, \mathbf{v})$ such that $\{\phi^n\}_n$ τ-converges to ϕ in $AC^p(0, T; \mathbf{L}^p(X))$.

Definition 5.9. (i) We say that a subset $\mathcal{A}$ of $AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ is bounded if

$$\sup_{\mathcal{A}} \int_0^T \Big(W_p^p(\sigma_t, \delta_0) + |\sigma'|^p(t)\Big)\mathrm{d}t < \infty.$$

(ii) We say that a bounded sequence $\{\sigma^n\}_n$ τ_w-converges to σ in $AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$ if

$$\lim_{n\to\infty}\int_0^T W_1(\sigma_t^n,\sigma_t)\mathrm{d}t = 0.$$

Corollary 5.10. *Suppose $\{\sigma^n\}_n \subset AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$ τ_w-converges to (σ) in $AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$. Let O_t^n be the p-optimal map that pushes ν_0 forward to σ_t^k. Similarly, let O_t be the p-optimal map that pushes ν_0 forward to σ_t. If $q\in[1,p)$, then $\{O_t^n\}_n$ converges to O in $\mathbf{L}^q(X)$. As a consequence, for each $t\in[0,T]$, $\{\sigma_t^n\}_n$ converges to σ_t in $\mathcal{P}_q(\mathbb{R}^d)$.*

Proof. By Remark A.1, σ^n is $1/p'$-Hölder continuous uniformly in n and so we exploit the fact that $\lim_{n\to\infty}\int_0^T W_1(\sigma_t^n,\sigma_t)\mathrm{d}t = 0$ to conclude that every subsequence of $\{\sigma^n\}_n$ admits a subsequence $\{\sigma^{n_k}\}_k$ such that not only does $\sigma_t^{n_k}$ converge to σ_t in $\mathcal{P}_1(\mathbb{R}^d)$, but also it converges in $\mathcal{P}_q(\mathbb{R}^d)$ for each $t\in[0,T]$. By Remark 3.3, $\{O^{n_k}\}_k$ converges to O in $\mathbf{L}^q(X)$. The limit being independent of the subsequence that we started with, we conclude that, in fact, for the whole sequence we have that $\{O_t^n\}_n$ converges to O in $\mathbf{L}^q(X)$. We use that $W_q(\sigma_t^n,\sigma_t)\le \|O_t^n - O_t\|_{\mathbf{L}^q(X)}$ to conclude that for each $t\in[0,T]$, $\{\sigma_t^n\}_n$ converges to σ_t in $\mathcal{P}_q(\mathbb{R}^d)$.

Remark 5.11. Every τ-convergent sequence in $\mathcal{S}_p(\mathbb{R}^d)$ is τ_w-convergent in $AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$.

Proof. Assume $\{\sigma^n\}_n$ τ-converges to σ in $\mathcal{S}_p(\mathbb{R}^d)$. Let $\mathbf{v}^n$ (resp. $\mathbf{v}$) be the velocity of minimal norm for σ^n (resp. σ). Let ϕ^n (resp. ϕ) be a flow associated with σ^n (resp. σ) such that (2.13)–(2.15) hold. Let $Y\Subset X$ be an open set. Then

$$\int_0^T W_1(\sigma_t^n,\sigma_t)\mathrm{d}t \le \mathcal{L}^d(Y)^{\frac{1}{p'}}\|\phi^n-\phi\|_{\mathbf{L}^p((0,T)\times Y)} + \mathcal{L}^d(X\setminus Y)^{\frac{1}{p'}}\sup_n\|\phi^n-\phi\|_{\mathbf{L}^p((0,T)\times X)}.$$

We let n tend to ∞, use (2.13) and then let Y tend to X to obtain that the left hand side tends to zero. As $W_p(\sigma_t^n,\sigma_t)\le\|\phi_t^n-\phi_t\|$, we conclude that $|(\sigma^n)'(t)|\le|\dot\phi_t^n|$ for almost every $t\in(0,T)$. Thus,

$$\int_0^T W_p^p(\sigma_t^n,\delta_0)\mathrm{d}t = \|\phi^n\|^p_{\mathbf{L}^p((0,T)\times X)},\qquad \int_0^T |(\sigma^n)'(t)|^p\mathrm{d}t \le \|\dot\phi^n\|^p_{\mathbf{L}^p((0,T)\times X)}.$$

Hence, (2.14)–(2.15) hold.

5.4. A topology stronger than τ_w yielding the same Γ-limit

Recall that ω is the continuous, monotone nonincreasing function introduced in (5.2) such that $\omega(0)=0$.

Let $K\equiv K^m\in C^1(\mathbb{R}^d;\mathbb{R}^d)$ be a 1-Lipschitz function such that $|K(x)|\le m$ for all $x\in\mathbb{R}^d$, $K(x)=0$ for $|x|\ge m+2$ and $K(x)=x$ for $|x|\le m$. As K is 1-Lipschitz, so is the operator $\mu\to K_\#\mu$ of $\mathcal{P}_q(\mathbb{R}^d)$ into itself, for any $q\in[1,\infty)$. If $t,s\in[0,T]$ then

$$W_q(K_\#\sigma_t,K_\#\sigma_s)\le W_q(\sigma_t,\sigma_s),\qquad W_q(K_\#\mu,\mu)\le 2\left(\int_{|x|\ge m}|x|^q\mu(\mathrm{d}x)\right)^{1/q}. \tag{5.10}$$

Lemma 5.12. *If $\sigma \in AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$, then for any map $\phi \in \mathbf{L}^p((0,T)\times X)$ such that $\phi_{t\#}\nu_0 = \sigma_t$ we have*

$$\mathcal{F}^\epsilon(\sigma) \ge \mathcal{F}^\epsilon(K_\#\sigma) - \int_0^T \omega\Big(\mathcal{L}^d\{|\phi_t| \ge m\}\Big)\mathrm{d}t \ge \mathcal{F}^\epsilon(K_\#\sigma) - \int_0^T \omega\left(\frac{W_1(\sigma_t,\delta_0)}{m}\right)\mathrm{d}t.$$

Proof. Note that the first inequality in the lemma implies the second one. Our task reduces then to proving the first inequality. By (5.10), $\sigma \to K_\#\sigma$ is 1-Lipschitz and maps $AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$ into itself. Thus, $|(K_\#\sigma)'|(t)$ is at most $|\sigma'|(t)$ almost everywhere on $(0,T)$. Fix $a>0$ and $M \in \mathbf{L}^p(X)$ such that $M_\#\nu_0 = \mu \in \mathcal{P}_p(\mathbb{R}^d)$. We have

$$\begin{aligned}&|\mathcal{W}(D^a_\#\mu) - \mathcal{W}((D^a\circ K)_\#\mu)| = |W(D^a\circ M) - W(D^a\circ K\circ M)|\\ &\quad\le \omega\Big(\mathcal{L}^d\{D^a\circ M \ne D^a\circ K\circ M\}\Big) \le \omega\Big(\mathcal{L}^d\{|M|\ge m\}\Big).\end{aligned}$$

This is the last ingredient for concluding the proof of the lemma.

Corollary 5.13. *Suppose $\sigma \in AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$ and σ_t is supported in the ball of radius $R<m$ centered at the origin for every $t\in[0,T]$. Then there exists $\{\sigma^n\}_n$ τ_w-converging to σ such that for every $t\in[0,T]$ and every $n\in\mathbb{N}$, σ^n_t is supported in the ball of radius $m+2$ centered at the origin, and $\mathcal{F}_w(\sigma) = \lim_{n\to\infty}\mathcal{F}^{\epsilon_n}(\sigma^n)$. As a consequence, $\{\sigma^n_t\}_n$ converges to σ_t in $\mathcal{P}_p(\mathbb{R}^d)$ for every $t\in[0,T]$.*

Proof. Let $\{\bar\sigma^n\}_n$ be a sequence τ_w-converging to σ and such that $\mathcal{F}(\sigma) = \lim_{n\to\infty}\mathcal{F}^{\epsilon_n}(\bar\sigma^n)$. Set $\sigma^n_t := K_\#\bar\sigma^n_t$. By (5.10) and the fact that $|(\sigma^n)'|(t) \le |\bar\sigma'|(t)$ almost everywhere on $(0,T)$, $\{\sigma^n\}_n$ τ_w-converges to $K_\#\sigma = \sigma$. Let O^n_t be the p-optimal map that pushes ν_0 forward to $\bar\sigma^n_t$ and, similarly, let O_t be the p-optimal map that pushes ν_0 forward to σ_t. Corollary 5.10 gives that $\{O^n_t\}_n$ converges to O_t in $\mathbf{L}^1(X)$ and so, since $|O_t| \le R < m$, we conclude that $\lim_{n\to\infty}\mathcal{L}^d\{|O^n_t|\ge m\} = 0$. We use Lemma 5.12 to conclude that

$$\begin{aligned}\mathcal{F}(\sigma) &= \lim_{n\to\infty}\mathcal{F}^{\epsilon_n}(\bar\sigma^n) \ge \liminf_{n\to\infty}\left\{\mathcal{F}^{\epsilon_n}(\sigma^n) - \int_0^T \omega\Big(\mathcal{L}^d\{|O^n_t|\ge m\}\Big)\mathrm{d}t\right\}\\ &\ge \mathcal{F}(K_\#\sigma) = \mathcal{F}(\sigma).\end{aligned}$$

By Corollary 5.10, $\{\bar\sigma^n_t\}_n$ converges to σ_t in the W_1-metric and so, $\{\sigma^n_t\}_n$ converges to σ_t in the W_1-metric. Note that σ^n_t is supported by the ball of radius $m+2$ centered at the origin. Hence, $\{\sigma^n_t\}_n$ converges to σ_t in the W_p-metric.

Corollary 5.14. *Let $\sigma \in AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$. The following hold:*

(i) *There exists a sequence $\{\sigma^m\}_m$ τ_w-converging to σ such that $\mathcal{F}_w(\sigma) = \lim_{m\to\infty}\mathcal{F}_w(\sigma^m)$ and for each $m\in\mathbb{N}$ there exists $r^m>0$ such that for all $t\in(0,T)$, σ^m_t is contained in the ball of radius r^m, centered at the origin.*

(ii) *There exists a sequence $\{\sigma^n\}_n$ τ_w-converging to σ such that $\mathcal{F}_w(\sigma) = \lim_{n\to\infty}\mathcal{F}^{\epsilon_n}(\sigma^n)$ and $\{\sigma^n_t\}_n$ converges to σ_t in $\mathcal{P}_p(\mathbb{R}^d)$ for every $t\in[0,T]$.*

Proof. (i) We now denote K by K^m to display its dependence on m. Let $\sigma \in AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$, set $\sigma^m_t = K^m_\#\sigma_t$ and let $\delta>0$ be arbitrary. By the fact that $\sigma_0\in\mathcal{P}_p(\mathbb{R}^d)$, Remark A.1 yields $\sup_t W_p(\sigma_t,\delta_0) < \infty$. Thus, for m large enough, $\omega\Big(\frac{W_1(\sigma_t,\delta_0)}{m}\Big) \le \delta$. We exploit Lemma 5.12 to obtain that $\mathcal{F}_w(\sigma) \geq \mathcal{F}_w(\sigma^m) - \delta$. As $\mathcal{F}_w$ is τ_w-lower semicontinuous and $\{\sigma^m\}_m$ τ_w-converges to σ, we conclude that

$$-\delta + \limsup_{m\to+\infty} \mathcal{F}_w(\sigma^m) \leq \mathcal{F}_w(\sigma) \leq \liminf_{m\to+\infty} \mathcal{F}_w(\sigma^m).$$

As $\delta > 0$ is arbitrary, we conclude the proof of (i).

(ii) For each $k \in \mathbb{N}$, choose $k \leq m_k \in \mathbb{N}$ large enough that $\mathcal{F}_w(\sigma) \geq \mathcal{F}_w(K_\#^{m_k}\sigma) - 1/k$ and set $\sigma_t^k := K_\#^{m_k}\sigma_t$. By Corollary 5.13, we may choose a sequence $\{\sigma^{k,n}\}_n$ that τ_w-converges to σ^k and such that

$$\mathcal{F}_w(\sigma^k) = \lim_{n\to\infty} \mathcal{F}^{\epsilon_n}(\sigma^{k,n}) \quad \text{and} \quad \lim_{n\to\infty} W_p(\sigma_t^{k,n}, \sigma_t^k) = 0$$

for each $t \in [0, T]$. Thanks to Egoroff's Theorem, there exists a decreasing sequence of Lebesgue measurable sets $N_1 \supset N_2 \supset \cdots \supset N_k \supset \cdots$ such that $\mathcal{L}^1(N_k) < 1/k$ and there exists n_k such that

$$\mathcal{F}_w(\sigma^k) \geq \mathcal{F}^{\epsilon_n}(\sigma^{k,n}) - \frac{1}{k} \quad \text{and} \quad W_p(\sigma_t^{k,n}, \sigma_t^k) < \frac{1}{k}$$

for all $n \geq n_k$ and all $t \notin N_k$. We exploit (5.10) and the triangle inequality to obtain that

$$W_p(\sigma_t^{k,n_k}, \sigma_t) \leq W_p(\sigma_t^{k,n_k}, \sigma_t^k) + W_p(\sigma_t^k, \sigma_t) \leq \frac{1}{k} + 2\left(\int_{|x|\geq k} |x|^p \sigma_t(\mathrm{d}x)\right)^{\frac{1}{p}},$$

for all $t \notin N := \cap_{k=1}^\infty N_k$. Hence, $\lim_{k\to\infty} W_p(\sigma_t^{k,n_k}, \sigma_t) = 0$ for all $t \notin N$. As $\mathcal{W} \geq 0$, we have that

$$\int_0^T |(\sigma^{k,n_k})'|^p(t)\mathrm{d}t \leq p\mathcal{F}^{\epsilon_{n_k}}(\sigma^{k,n_k}) \leq p\mathcal{F}_w(\sigma) + p.$$

We use this uniform bound in n and k and Remark A.1 to conclude that the sequence $\{\sigma^{k,n_k}\}_k$ is bounded in $AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ and $\{\sigma^{k,n_k}\}_k$ is uniformly Hölder continuous. Hence,

$$\lim_{k\to\infty} W_p(\sigma_t^{k,n_k}, \sigma_t) = 0 \quad \text{for all } t \in [0, T].$$

Using that $W_1 \leq W_p$, we conclude that $\{\sigma^{k,n_k}\}_k$ τ_w-converges to σ. As a consequence,

$$\mathcal{F}_w(\sigma) \leq \liminf_{k\to\infty} \mathcal{F}^{\epsilon_{n_k}}(\sigma^{k,n_k}).$$

But we also have

$$\mathcal{F}_w(\sigma) \geq \limsup_{k\to\infty} \mathcal{F}^{\epsilon_{n_k}}(\sigma^{k,n_k}),$$

which concludes the proof of the corollary.

Remark 5.15. Let τ^w be the topology obtained by replacing $\lim_{n\to\infty} \int_0^T W_1(\sigma_t^n, \sigma_t)\mathrm{d}t = 0$ by the stronger condition $\lim_{n\to\infty} \int_0^T W_p^p(\sigma_t^n, \sigma_t)\mathrm{d}t = 0$ in Definition 5.9. We have proved above that the $\Gamma(\tau^w)$-limit of $\mathcal{F}^{\epsilon_n}$ is still $\mathcal{F}_w$.

5.5. Representation of the effective Lagrangians on $\mathcal{P}_p(\mathbb{R}^d)$ in terms of that on $\mathbf{L}^p(X)$

Remark 5.16. Let $\sigma \in AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ and let $\mathbf{v}$ be a velocity for σ. We extend σ outside $[0, T]$ by setting $\sigma_t = \sigma_0$ if $t < 0$ and $\sigma_t = \sigma_T$ if $t > T$. Similarly, we extend $\mathbf{v}$ by setting $\mathbf{v}_t = 0$ if $t < 0$ or $t > T$. Note that for any $a < b$, the extension (still denoted by σ) belongs to $AC^p(a, b; \mathcal{P}_p(\mathbb{R}^d))$ and the extension of $\mathbf{v}$ remains a velocity for σ. If $\mathbf{v}$ is the velocity of minimal norm, then so is its extension.

(i) One can use a rescaled family of spatial mollifiers $\rho^\epsilon > 0$ and define

$$\sigma_t^\epsilon = \rho^\epsilon * \sigma_t \quad \text{and} \quad \sigma_t^\epsilon \mathbf{v}_t^\epsilon = \rho^\epsilon * (\sigma_t \mathbf{v}_t).$$

We have $\|\mathbf{v}_t^\epsilon\|_{\sigma_t^\epsilon} \leq \|\mathbf{v}_t\|_{\sigma_t}$ (cf. e.g. Lemma 8.1.9 [2]). Then one uses a rescaled family of time mollifiers ν^δ supported in $[-\delta, \delta]$ and defines

$$\sigma_t^{\epsilon,\delta} = \nu^\delta * \sigma_t^\epsilon, \qquad \sigma_t^{\epsilon,\delta} \mathbf{v}_t^{\epsilon,\delta} = \nu\delta * (\sigma_t^\epsilon \mathbf{v}_t^\epsilon).$$

We have $\|\mathbf{v}_t^{\epsilon,\delta}\|_{\sigma_t^{\epsilon,\delta}} \leq \|\mathbf{v}_t^\epsilon\|_{\sigma_t^\epsilon}$ (cf. e.g. [7, Section 5.3]). We have that

$$\sigma_t^\epsilon \to \sigma_t \quad \text{in } \mathcal{P}_p(\mathbb{R}^d), \qquad \sigma^\epsilon \mathbf{v}^\epsilon \to \sigma \mathbf{v} \text{ narrowly and } \|\mathbf{v}_t^\epsilon\|_{\sigma_t^\epsilon} \to \|\mathbf{v}_t\|_{\sigma_t}$$

as $\epsilon \to 0$. Similarly,

$$\sigma_t^{\epsilon,\delta} \to \sigma_t^\epsilon \quad \text{in } \mathcal{P}_p(\mathbb{R}^d), \qquad \sigma^{\epsilon,\delta} \mathbf{v}^{\epsilon,\delta} \to \sigma^\epsilon \mathbf{v}^\epsilon \text{ narrowly and } \|\mathbf{v}_t^{\epsilon,\delta}\|_{\sigma_t^{\epsilon,\delta}} \to \|\mathbf{v}_t^\epsilon\|_{\sigma_t^\epsilon}$$

as $\delta \to 0$. Furthermore,

$$\sigma^{\epsilon,\delta} \in C^\infty\big([0, T] \times \mathbb{R}^d\big) \quad \text{and} \quad \mathbf{v}^{\epsilon,\delta} \in C^\infty\big([0, T] \times \mathbb{R}^d; \mathbb{R}^d\big).$$

(ii) In conclusion, for each integer $k \geq 1$ there exist $\{\sigma^k\}_k \subset AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ and velocities $\mathbf{v}^k$ for σ^k such that $\sigma^k \in C^\infty\big([0, T] \times \mathbb{R}^d\big)$ and $\mathbf{v}^k \in C^\infty\big([0, T] \times \mathbb{R}^d; \mathbb{R}^d\big)$. Furthermore $\sigma_t^k \to \sigma_t$ in $\mathcal{P}_p(\mathbb{R}^d)$ for each $t \in [0, T]$, $\sigma^k \mathbf{v}^k \to \sigma \mathbf{v}$ narrowly and $\|\mathbf{v}_t^k\|_{\sigma_t^k} \to \|\mathbf{v}_t\|_{\sigma_t}$ as $k \to \infty$. As $\mathcal{W}$ is bounded, the Lebesgue dominated convergence theorem yields

$$\lim_{k \to \infty} \int_0^T \mathcal{L}(\sigma_t^k, \mathbf{v}_t^k) \mathrm{d}t = \int_0^T \mathcal{L}(\sigma_t, \mathbf{v}_t) \mathrm{d}t.$$

(iii) By Proposition 8.1.8 [2], there exists $\{S^k\}_k \subset W^{1,p}(0, T; \mathbf{L}^p(\sigma_0^k))$ such that

$$\dot{S}_t^k(x) = v_t^k \circ S_t^k(x) \quad \text{and} \quad S_0^k(x) = x \quad \text{for a.e. } x \in \mathbb{R}^d \text{ and } t \in [0, T]. \tag{5.11}$$

Since $\sigma_0^k > 0$ and

$$\int_0^T \Big(\sup_B |\mathbf{v}_t^k| + \mathrm{Lip}(\mathbf{v}_t^k, B) \Big) \mathrm{d}t < \infty$$

for every ball $B \subset \mathbb{R}^d$, we conclude that, in fact, (5.11) holds for every x. Furthermore, S_t^k is invertible and $S_{t\#}^k \nu_0 = \sigma_t^k$.

Theorem 5.17. *If $\mu \in \mathcal{P}_p(\mathbb{R}^d)$, $V \in \mathbf{L}^p(\mu)$, $M \in \mathbf{L}^p(X)$ is any map that pushes the Lebesgue measure restricted to X forward to $V_\# \mu := \nu$ then $\bar{L}(M) = \bar{\mathcal{L}}(\mu, V)$.*

Proof. 1. Let $(\sigma, \mathbf{v})$ be such that $\mathbf{v}$ is a velocity for $\sigma \in \mathcal{P}(\delta_0, D^T_\# \nu)$, where we recall the definition of D^T: $D^T x = Tx$. For $k \geq 1$ integer we choose σ^k and $\mathbf{v}^k$ as in Remark 5.16. We set $\nu^k_t := D^{T^{-1}}_\# \sigma^k_t$ and $\nu_t := D^{T^{-1}}_\# \sigma_t$ so that

$$\lim_{k\to\infty} W^p_p(\nu^k_t, \nu_t) = 0. \tag{5.12}$$

Fix $\delta > 0$ and k such that

$$\int_0^T \mathcal{L}(\sigma_t, \mathbf{v}_t)\mathrm{d}t \geq \int_0^T \mathcal{L}(\sigma^k_t, \mathbf{v}^k_t)\mathrm{d}t - \delta. \tag{5.13}$$

Let S^k be as in Remark 5.16. Let $O^k_T : X \to \mathbb{R}^d$ be the unique $|\cdot|^p$-optimal map that pushes the Lebesgue measure restricted to X forward to σ^k_T (cf. e.g. [8] Theorem 3.7) and, similarly, let $O : X \to \mathbb{R}^d$ be the unique $|\cdot|^p$-optimal map that pushes the Lebesgue measure restricted to X forward to ν. Set

$$M^k_t := S^k_t \circ (S^k_T)^{-1} \circ D^T \circ O^k_T$$

so that M^k_t pushes the Lebesgue measure restricted to X forward to σ^k_t. By (5.11),

$$\dot{M}^k_t = \mathbf{v}^k_t \circ M^k_t$$

for $t \in [0, T]$ and $\{M^k\}_k \subset \mathcal{H}$. We have

$$\int_0^T \mathcal{L}(\sigma^k_t, \mathbf{v}^k_t)\mathrm{d}t = \int_0^T L(M^k_t, \dot{M}^k_t)\mathrm{d}t \geq C_{0,T}(M^k_0, O^k_T). \tag{5.14}$$

By (2.6),

$$\begin{aligned} C_{0,T}(M_0, TO) &\leq C_{0,1}(0, M^k_0) + C_{1,T-2}(M^k_0, (T-3)O^k_{T-3}) \\ &\quad + C_{T-2,T-1}\big((T-3)O^k_{T-3}, (T-3)O\big) + C_{T-1,T}\big((T-3)O, TO\big) \\ &\leq C_{0,T-3}\big(M^k_0, (T-3)O^k_{T-3}\big) \\ &\quad + C\Big(3 + \|M^k_0\|^p + (T-3)^p\|O^k_T - O\|^p + 3^p\|O\|^p\Big), \end{aligned}$$

where we have used (2.1). By Remark 5.1 and (5.12), $\{O^k\}_k$ converges strongly to O in $\mathbf{L}^p(X)$.

We use Remark 5.16 and the fact that M^k_0 pushes the Lebesgue measure restricted to X forward to σ^k_0 to conclude that

$$\|M^k_0\| = W_p(\sigma^k_0, \delta_0) = 0(1/k)$$

and so, $\{M^k_0\}_k$ converges to 0 in $\mathbf{L}^p(X)$. Similarly, letting first k tend to ∞ in the previous inequality we obtain

$$C_{0,T}(M_0, T\ O) \leq \liminf_{k\to\infty} C_{0,T-3}\big(M^k_0, (T-3)\ O^k_{T-3}\big) + 3^p C\|O\|^p.$$

We combine this together with (5.13) and (5.14) and use that σ is an arbitrary path starting at 0 and ending at $D^T_\# \nu$, and that $\delta > 0$ is arbitrary, to conclude that

$$\mathcal{C}_{0,T}(\delta_0, D^T_\# \nu) \geq C_{0,T+3}\big(M_0, (T+3)\ O\big) - 3^p C\|O\|^p. \tag{5.15}$$

Dividing both sides of (5.15) by T and letting T tend to ∞ we conclude that $\bar{\mathcal{L}}(\mu, \mathbf{v}) \geq \bar{L}(O)$. As $\bar{L}$ is locally Lipschitz and so is continuous, we use Theorem 5.5 and the fact that $M_\#\nu_0 = O_\#\nu_0$ to conclude that $\bar{\mathcal{L}}(\mu, V) \geq \bar{L}(M)$.

Conversely, let $\phi \in \mathcal{H}$ be such that $\phi_0 \equiv 0$ and $\phi_T = TM$. We exploit Corollary 2.3 and the fact that $\mathcal{W}$ is continuous to obtain $\phi^k \in \mathcal{H}$ and a partition of X into parallel cubes $\{X_i\}_{i=1}^k$ of the same size such that $\|\phi_t^k - \phi_t\| \le 1/k$, ϕ_t is constant on each cube for $t \in [0,T]$ and

$$\int_0^T L(\phi_t, \dot{\phi}_t)\mathrm{d}t \ge \int_0^T L(\phi_t^k, \dot{\phi}_t^k)\mathrm{d}t - \frac{1}{k}. \tag{5.16}$$

Set $\sigma_t^{k,T}$ as the push forward by ϕ_t^k of the Lebesgue measure restricted to X and let σ_t^T be the push forward by ϕ_t, so

$$W_p^p(\sigma_t^{k,T}, \sigma_t) \le \|\phi_t^k - \phi_t\|^p \le 1/k^p.$$

We have added the subscript T to emphasize the T-dependence of $\sigma^{k,T} : [0,T] \to \mathcal{P}_p(\mathbb{R}^d)$. By Lemma A.4, there exists a velocity $\mathbf{w}^k$ for $\sigma^{k,T}$ such that $\dot{\phi}_t^k = \mathbf{w}_t^k \circ \phi_t^k$ for almost every $t \in (0,T)$. We use (5.16) and the fact that

$$\int_0^T \mathrm{d}t \int_X \|\dot{\phi}_t^k\|^p \mathrm{d}z = \int_0^T \mathrm{d}t \int_X \|\mathbf{w}_t^k\|^p \mathrm{d}\sigma_t^{k,T} \quad \text{and} \quad W(\phi_t^k) = \mathcal{W}(\sigma_t^{k,T})$$

for all $t \in [0,T]$ to conclude that

$$\int_0^T L(\phi_t, \dot{\phi}_t)\mathrm{d}t \ge \mathcal{C}_{0,T}(\sigma_0^{k,T}, \sigma_T^{k,T}) - \frac{1}{k}. \tag{5.17}$$

But, as above,

$$\begin{aligned}\mathcal{C}_{0,T}(\delta_0, \sigma_T^T) \le\ & \mathcal{C}_{0,1}(\delta_0, \sigma_0^k) + \mathcal{C}_{0,T-3}(\sigma_0^{k,T-3}, \sigma_{T-3}^{k,T-3}) \\ & + \mathcal{C}_{0,1}(\sigma_{T-3}^{k,T-3}, \sigma_{T-3}^{T-3}) + \mathcal{C}_{0,1}(\sigma_{T-3}^{T-3}, \sigma_T^T).\end{aligned} \tag{5.18}$$

By (2.1),

$$\mathcal{C}_{0,1}(\delta_0, \sigma_0^k) \le C\big(1 + W_p^p(\delta_0, \sigma_0^k)\big) = C\big(1 + \|M_0^k\|^p\big).$$

Similarly,

$$\mathcal{C}_{0,1}\big(\sigma_{T-3}^{k,T-3}, \sigma_{T-3}^{T-3}\big) \le C\left(1 + \frac{1}{k^p}\right), \qquad \mathcal{C}_{0,1}(\sigma_{T-3}^{T-3}, \sigma_T^T) \le C\big(1 + 3^p\|M\|^p\big).$$

Letting k tend to ∞ in (5.18), we conclude that

$$\mathcal{C}_{0,T}(\delta_0, \sigma_T^T) \le \liminf_{k\to\infty} \mathcal{C}_{0,T-3}(\sigma_0^{k,T-3}, \sigma_{T-3}^{k,T-3}) + \mathcal{C}_{0,1}(\sigma_{T-3}^{T-3}, \sigma_T^T). \tag{5.19}$$

As ϕ is an arbitrary path such that $\phi_0 \equiv 0$ and $\phi_T = TM$, (5.17) and (5.19) imply

$$C_{0,T}(0, TM) \ge \mathcal{C}_{0,T}\Big(\delta_0, (D^T \circ V)_\# \nu_0\Big).$$

Dividing both sides of the inequality by T and letting T tend to ∞ we have $\bar{L}(M) \ge \bar{\mathcal{L}}(\mu, V)$.

Corollary 5.18. *Let $\sigma \in \mathcal{S}_p(\mathbb{R}^d)$ and let $\mathbf{v}$ be the velocity of minimal norm for σ. Whenever $\phi \in AC^p(0,T;\mathbf{L}^p(X))$ and $\phi_{t\#}\nu_0 = \sigma_t$ for all $t \in [0,T]$, we have*

$$\bar{L}(\mathbf{v}_t \circ \phi_t) = \bar{\mathcal{L}}(\sigma_t, \mathbf{v}_t) \quad \forall t \in [0,T].$$

Proof. Let $\bar{\phi} \in AC^p(0, T; \mathbf{L}^p(X))$ be such that $\bar{\phi}_{t\#}\nu_0 = \sigma_t$ for all $t \in [0, T]$. We have

$$(\mathbf{v}_t \circ \phi_t)_{\#}\nu_0 = \mathbf{v}_{t\#}\sigma_t = (\mathbf{v}_t \circ \bar{\phi}_t)_{\#}\nu_0.$$

This, together with Theorem 5.17 yields the conclusion.

As a consequence, one can define the functional $\mathcal{F}$ on $AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ as in (1.5). We are now ready to identify the $\Gamma(\tau)$-limit of $\{\mathcal{F}^\epsilon\}_\epsilon$ as $\epsilon \to 0$.

Theorem 5.19. *Consider $\mathcal{F}$ given by (1.5), where $\mathbf{v}$ is the velocity of minimal $\mathbf{L}^p(\sigma)$-norm associated with σ. We have*

$$\mathcal{F} = \Gamma(\tau) \lim_{\epsilon \to 0} \mathcal{F}^\epsilon \quad on\ \mathcal{S}_p(\mathbb{R}^d). \tag{5.20}$$

Proof. Consider an arbitrary sequence of positive numbers $\{\epsilon_n\}_n$ that converges to 0. Let $(\sigma, \mathbf{v}) \in \mathcal{S}_p(\mathbb{R}^d)$ and take $\{(\sigma^n, \mathbf{v}^n)\}_n \subset \mathcal{S}_p(\mathbb{R}^d)$ to be any sequence that τ-converges to $(\sigma, \mathbf{v})$. Choose $\{\phi\} \cup \{\phi^n\}_n \subset AC^p(0, T, \mathbf{L}^p(X))$ such that ϕ (resp. ϕ^n) is a flow associated with $(\sigma, \mathbf{v})$ (resp. $(\sigma^n, \mathbf{v}^n)$) such that $\{\phi^n\}_n$ τ-converges to ϕ in $AC^p(0, T, \mathbf{L}^p(X))$. We have

$$\begin{aligned}\liminf_{n\to\infty} \mathcal{F}^{\epsilon_n}(\sigma^n) &= \liminf_{n\to\infty} F^{\epsilon_n}(\phi^n) \ge F(\phi) = \int_0^T \bar{L}(\dot{\phi}_t)\mathrm{d}t = \int_0^T \bar{L}(\mathbf{v}_t \circ \phi_t)\mathrm{d}t \\ &= \int_0^T \bar{\mathcal{L}}(\sigma_t, \mathbf{v}_t)\mathrm{d}t,\end{aligned} \tag{5.21}$$

where we have used Theorem 2.15 and Corollary 5.18.

Next, choose a sequence $\{\psi^n\}_n \subset AC^p(0, T, \mathbf{L}^p(X))$ that τ-converges to ϕ and such that

$$\lim_{n\to\infty} F^{\epsilon_n}(\psi^n) = F(\phi) = \int_0^T \bar{L}(\dot{\phi}_t)\mathrm{d}t = \int_0^T \bar{L}(\mathbf{v}_t \circ \phi_t)\mathrm{d}t. \tag{5.22}$$

Since $\mathcal{W}$ is continuous (cf. Lemma 5.2) and bounded on $\mathcal{P}_p(\mathbb{R}^d)$, thanks to Corollary 2.3 we may assume without loss of generality that ψ^n is of the form $\psi_t^n = \sum_{j=1}^{m_n} x_t^{j,n}\chi_{C^{j,n}}$, where $\{C^{j,n}\}_{j=1}^{m_n}$ is a partition of X consisting of squares of the same size. Lemma A.4 provides a family $\mathbf{v}^n : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d$ of Borel maps such that $\dot{\psi}_t^n = \mathbf{v}_t^n \circ \psi_t^n$ for almost every $t \in (0, T)$ and $\mathbf{v}^n$ is the velocity of minimal norm for $\sigma^n \in AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ defined by $\sigma_t^n = \psi_{t\#}^n\nu_0$. We have that $\{\sigma^n\}_n$ τ-converges to σ in $\mathcal{S}_p(\mathbb{R}^d)$ and so, combining this fact with (5.22), we obtain

$$\lim_{n\to\infty} \mathcal{F}^{\epsilon_n}(\sigma^n) = \lim_{n\to\infty} F^{\epsilon_n}(\psi^n) = \int_0^T \bar{L}(\mathbf{v}_t \circ \phi_t)\mathrm{d}t = \mathcal{F}(\sigma). \tag{5.23}$$

This proves the theorem.

Let $\sigma \in AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ and let $\mathbf{v}$ be a velocity for σ. Fix $0 < s < T/2$, $\bar{\sigma}_0, \bar{\sigma}_T \in \mathcal{P}_p(\mathbb{R}^d)$ and modify σ on $[0, s] \cup [T - s, T]$ to obtain the path $\bar{\sigma} \in AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ defined the following way: on $[0, s]$ we set $\bar{\sigma}$ to be the geodesic of constant speed in $\mathcal{P}_p(\mathbb{R}^d)$ connecting $\bar{\sigma}_0$ to σ_s. Similarly, on $[T-s, T]$ we set $\bar{\sigma}$ to be the geodesic of constant speed in $\mathcal{P}_p(\mathbb{R}^d)$ connecting σ_{T-s} to $\bar{\sigma}_T$. We denote by $\bar{\mathbf{v}}$ the geodesic velocity on $[0, s] \cup [T - s, T]$ and set $\bar{\mathbf{v}}$ to coincide with v on $[s, T - s]$. We use (5.6) and the fact that $\mathcal{L} \ge 0$ to obtain

$$\int_0^T \mathcal{L}^\epsilon(\sigma_t, \mathbf{v}_t)\mathrm{d}t \ge \int_0^T \mathcal{L}^\epsilon(\bar{\sigma}_t, \bar{\mathbf{v}}_t)\mathrm{d}t - Cs\left(2 + \frac{W_p^p(\sigma_s, \bar{\sigma}_0) + W_p^p(\bar{\sigma}_T, \sigma_{T-s})}{s^p}\right). \tag{5.24}$$

Lemma 5.20. *If $\{\sigma_t^n\}_n$ converges to σ_t in $\mathcal{P}_p(\mathbb{R}^d)$ for $t = 0, T$, then*

$$\liminf_{n\to\infty} \mathcal{C}^{\epsilon_n}(\sigma_0^n, \sigma_T^n) \geq \liminf_{n\to\infty} \mathcal{C}^{\epsilon_n}(\sigma_0, \sigma_T).$$

Proof. By Lemma 5.3, there exists $\{\sigma^n\}_n \subset AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ such that $\mathcal{C}^{\epsilon_n}(\sigma_0^n, \sigma_T^n) = \mathcal{F}^{\epsilon_n}(\sigma^n)$ and $E < \infty$ such that $|(\sigma^n)'|(t) \leq E$ for every $n \in \mathbb{N}$ and almost every $t \in (0, T)$. Let $\bar{\sigma}^n$ be the sequence obtained by modifying σ as above on $[0, s] \cup [T - s, T]$, such that $\bar{\sigma}_0^n = \sigma_t$ for $t = 0, T$. By (5.24),

$$\begin{aligned}\liminf_{n\to\infty} \mathcal{C}^{\epsilon_n}(\sigma_0^n, \sigma_T^n) &\geq \liminf_{n\to\infty} \mathcal{F}^{\epsilon_n}(\bar{\sigma}^n) - Cs\left(2 + \frac{W_p^p(\sigma_s, \sigma_0) + W_p^p(\sigma_T, \sigma_{T-s})}{s^p}\right) \\ &\geq \liminf_{n\to\infty} \mathcal{C}^{\epsilon_n}(\sigma_0, \sigma_T) - Cs(2 + 2E^p).\end{aligned}$$

We let s tend to 0 to conclude the proof.

Remark 5.21. Let $\xi \in \mathbf{L}^p(X)$ and define $\sigma_t = (t\xi)_\# \nu_0$. Then $\sigma \in AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ and $(t, x) \to \mathbf{v}_t(x) = x/t$ is the velocity of minimal norm for σ. Set $\phi_t(z) = t\xi(z)$ for $t \in [0, T]$ and $z \in X$. We have that $\phi \in AC^p(0, T; \mathbf{L}^p(X))$ and ϕ is a flow associated with $(\sigma, \mathbf{v})$.

Proof. Fix $0 \leq s \leq t \leq T$. We have

$$W_p^p(\sigma_t, \delta_0) = t^p \|\xi\|^p \quad \text{and} \quad W_p(\sigma_s, \sigma_t) \leq (t - s)\|\xi\|^p,$$

which proves that $\sigma \in AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$.

Setting $\mathbf{v}_t(x) = x/t$, we obtain that $\mathbf{v}$ is a Borel map such that $t \to \|\mathbf{v}_t\|_{\sigma_t}$ belongs to $L^p(0, T)$. Furthermore, if $F \in C_c^\infty((0, T) \times \mathbb{R}^d)$, using that $\sigma_t = (t\xi)_\# \nu_0$ and that $\mathbf{v}_t(t\xi(z)) = \xi(z)$, we obtain

$$\begin{aligned}&\int_0^T \mathrm{d}t \int_{\mathbb{R}^d} \Big(\partial_t F + \langle \nabla F; \mathbf{v}_t \rangle\Big) \mathrm{d}\sigma_t \\ &\quad = \int_0^T \mathrm{d}t \int_X \Big(\partial_t F\big(t, t\xi(z)\big) + \langle \nabla F(t, t\xi(z)); \mathbf{v}_t(t\xi(z)) \rangle\Big) \mathrm{d}z \\ &\quad = \int_0^T \mathrm{d}t \int_X \frac{\mathrm{d}}{\mathrm{d}t}\Big(F(t, t\xi(z))\Big) \mathrm{d}z = 0,\end{aligned}$$

which proves that $\mathbf{v}$ is a velocity for σ. Set $\varphi(x) = t^{\frac{1}{1-p}}/p'|x|^{p'}$ so that $\nabla\varphi|\nabla\varphi|^{p-2} = \mathbf{v}_t \in \mathbf{L}^p(\sigma_t)$. This proves that $\mathbf{v}_t \in \mathcal{T}_{\sigma_t}\mathcal{P}_p(\mathbb{R}^d)$ and so, $\mathbf{v}$ is the velocity of minimal norm for σ. One readily checks that $\mathbf{v}_t \circ \phi_t = \dot{\phi}_t$ to conclude the proof of the remark.

Theorem 5.22. *Suppose $\mathcal{F}^{\epsilon_n}$ $\Gamma(\tau_w)$-converges to $\mathcal{F}_w$ and $\Gamma(\tau)$-converges to $\mathcal{F}$. Let $\xi \in \mathbf{L}^p(X)$ and set $\sigma_t = (t\xi)_\# \nu_0$. Then $\mathcal{F}_w(\sigma) = \mathcal{F}(\sigma) = T\bar{L}(\xi)$.*

Proof. By Remark 5.11, τ_w is weaker than τ and so, $\mathcal{F}_w(\sigma) \leq \mathcal{F}(\sigma)$. Set $\phi_t = D^t \circ \xi$. By Remark 5.21, $D^{t^{-1}}$ is the velocity of minimal norm for σ_t and, as $D^t \circ \xi = t\xi$ pushes ν_0 forward to σ_t, we use Theorem 5.19 to conclude that $\mathcal{F}(\sigma) = \int_0^T \bar{L}\big(D^{t^{-1}} \circ D^t \circ \xi\big)\mathrm{d}t = T\bar{L}(\xi)$. It remains to show the reverse inequality. Let $\{\sigma^n\}_n \subset AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ be a sequence that τ_w-converges to σ and such that $\mathcal{F}_w(\sigma) = \lim_{n\to\infty} \mathcal{F}^{\epsilon_n}(\sigma^n)$. By Corollary 5.14, we may assume that $\lim_{n\to\infty} W_p(\sigma_t^n, \sigma_t) = 0$ for all $t \in [0, T]$. Fix $s \in (0, T)$ and let $\bar{\sigma}^n \in AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$

be obtained by modifying σ^n in the following way: on $[0, s]$, $\bar{\sigma}^n$ is the geodesic of constant speed starting at $\sigma_0 = \delta_0$ and ending at σ_s^n. On $[s, T-s]$, $\bar{\sigma}^n$ coincides with σ^n. On $[T-s, T]$, $\bar{\sigma}^n$ is the geodesic of constant speed starting at σ_{T-s}^n and ending at σ_T. By (5.24),

$$\mathcal{F}_w(\sigma) \geq \limsup_{n\to\infty} \mathcal{F}^{\epsilon_n}(\bar{\sigma}^n) - C(2+e) \geq \limsup_{n\to\infty} \mathcal{C}_{0,T}^{\epsilon_n}(\sigma_0, \sigma_T) - C(2+e)s, \tag{5.25}$$

where we have used that $W_p^p(\sigma_0, \sigma_s) = W_p^p(\sigma_{T-s}, \sigma_T) = s^p e$ where $e = \int_X |z|^p \mathrm{d}z$. As $s \in (0, T)$ is arbitrary, we conclude that $\mathcal{F}(\sigma) \geq \limsup_{n\to\infty} \mathcal{C}_{0,T}^{\epsilon_n}(\sigma_0, \sigma_T)$. We now use (5.4) to conclude that

$$\begin{aligned}\mathcal{F}_w(\sigma) &\geq T \liminf_{n\to\infty} \frac{\mathcal{C}_{0,T/\epsilon_n}(\delta_0, D^{T/\epsilon_n} \circ D_\#^{T^{-1}} \sigma_T)}{T/\epsilon_n} = T \liminf_{n\to\infty} \frac{\mathcal{C}_{0,T/\epsilon_n}(\delta_0, D^{T/\epsilon_n} \circ \xi_\# \nu_0)}{T/\epsilon_n} \\ &\geq T\mathcal{L}(\nu_0, \xi).\end{aligned}$$

This, together with Theorem 5.17, yields the proof.

6. A particular case of homogenization on $\mathcal{P}_p(\mathbb{R}^d)$

Throughout this section we assume that $T > 0$ and $1 < p < \infty$, $W \in C(\mathbb{T}^d)$ is even and satisfies $W(x) \leq W(0) = 0$ for $x \in \mathbb{R}^d$. We set $X = (0,1)^d$ and define

$$L(M, N) = \frac{1}{p}\|N\|^p - \int_{X\times X} W(Mz - Mw)\mathrm{d}w\mathrm{d}z,$$
$$L^\epsilon(M, N) = \frac{1}{p}\|N\|^p - \int_{X\times X} W\left(\frac{Mz - Mw}{\epsilon}\right)\mathrm{d}w\mathrm{d}z,$$

for $M, N \in \mathbf{L}^p(X)$ and $\epsilon > 0$. Similarly, we define

$$\mathcal{L}(\mu, \xi) = \frac{1}{p}\|\xi\|_\mu^p - \int_{\mathbb{R}^d\times\mathbb{R}^d} W(x-y)\mu(\mathrm{d}x)\mu(\mathrm{d}y),$$
$$\mathcal{L}^\epsilon(\mu, \xi) = \frac{1}{p}\|\xi\|_\mu^p - \int_{\mathbb{R}^d\times\mathbb{R}^d} W\left(\frac{x-y}{\epsilon}\right)\mu(\mathrm{d}x)\mu(\mathrm{d}y),$$

for $\mu \in \mathcal{P}_p(\mathbb{R}^d)$ and $\xi \in \mathbf{L}^p(\mu)$ so that $\mathcal{L}^\epsilon(\mu, \xi) = \mathcal{L}(D_\#^{1/\epsilon}\mu, \xi \circ D^\epsilon)$.

Remark 6.1. Note that if $c \in \mathbb{R}$ and $\phi \in AC^p(0, T; \mathbf{L}^p(X))$ is such that $\phi_0 = \phi_T \equiv 0$, then using that $W \leq 0$ we have

$$\int_0^T L(ct + \phi_t, c + \dot{\phi}_t)\mathrm{d}t \geq \int_0^T L(ct, c)\mathrm{d}t = \frac{1}{p}T|c|^p.$$

Thus, $\bar{L}(c) = |c|^p/p$. Let $c^* = c|c|^{p-2}$ and let $\bar{H}$ be the Legendre transform on $\bar{L}$. Then

$$\bar{H}(c^*) \geq -\bar{L}(c) + c^* \cdot c = \frac{1}{p'}|c^*|^{p'},$$

which proves that $\bar{H}(c^*) = |c^*|^{p'}/p'$.

6.1. Actions restricted to paths consisting of Dirac masses

If m is a positive integer, we denote by $\mathcal{P}^m(\mathbb{R}^d)$ the set of measures μ which are averages of m Dirac masses: $1/m \sum_{i=1}^m \delta_{x^i}$, where $\{x^i\}_{i=1}^m \subset \mathbb{R}^d$. For $\mu_a, \mu_b \in \mathcal{P}^m(\mathbb{R}^d)$ we set

$$\mathcal{C}^\epsilon_{a,b;m}(\mu_a, \mu_b) = \inf_\sigma \left\{ \int_a^b \mathcal{L}^\epsilon(\sigma_t, \mathbf{v}_t)\mathrm{d}t : \sigma \in AC^p(a, b; \mathcal{P}^m(\mathbb{R}^d)),\ \sigma_a = \mu_a, \sigma_b = \mu_b \right\}.$$

Following verbatim the proof of Lemma 3.4 [11], one obtains that on $\mathcal{P}^m_p(\mathbb{R}^d) \times \mathcal{P}^m_p(\mathbb{R}^d)$,

$$\mathcal{C}^\epsilon_{a,b;m} = \mathcal{C}^\epsilon_{a,b;\kappa m} \quad \text{for any integer } \kappa \geq 1. \tag{6.1}$$

Lemma 6.2. *If $\mu_a, \mu_b \in \mathcal{P}^m_p(\mathbb{R}^d)$, then $\mathcal{C}^\epsilon_{a,b;m}(\mu_a, \mu_b) = \mathcal{C}^\epsilon_{a,b}(\mu_a, \mu_b)$.*

Proof. We may assume without loss of generality that $\epsilon = 1$, $a = 0$ and $b = T$. Let σ be the minimizer of $\mathcal{F}(\cdot, (0, T))$ over the set of paths in $AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ which have σ_0 and σ_T as endpoints, as provided by Lemma 5.3. It suffices to show that for every $\delta > 0$,

$$\int_0^T \mathcal{L}(\sigma_t, \mathbf{v}_t)\mathrm{d}t \geq \mathcal{C}_{0,T;m}(\mu_0, \mu_T) - \delta.$$

Lemma 5.3 ensures the existence of a constant $C_\sigma > 1$ such that $\|\mathbf{v}_t\|_{\sigma_t} \leq C_\sigma - 1$ for almost every $t \in (0, T)$. Here, $\mathbf{v}$ is a velocity for σ. We use Remark 5.16 to find a sequence $\{\sigma^k\}_k \subset AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ and $\mathbf{v}^k$ a velocity for σ^k such that $\sigma^k \in C^\infty([0, T] \times \mathbb{R}^d)$, $\mathbf{v}^k \in C^\infty([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$, $\sigma^k_t \to \sigma_t$ in $\mathcal{P}_p(\mathbb{R}^d)$ for each $t \in [0, T]$ and

$$\int_0^T \mathcal{L}(\sigma^k_t, \mathbf{v}^k_t)\mathrm{d}t \leq \int_0^T \mathcal{L}(\sigma_t, \mathbf{v}_t)\mathrm{d}t + \frac{\delta}{4} \quad \text{for } k \geq k_0.$$

The following property is also ensured: $\|\mathbf{v}^k_t\|_{\sigma^k_t} \leq \|\mathbf{v}_t\|_{\sigma_t} \leq C_\sigma - 1$.

Let S^k and M^k be as in the proof of Theorem 5.17, so that

$$M^k \in AC^p(0, T; \mathbf{L}^p(X)), \qquad \int_0^T \mathcal{L}(\sigma^k_t, \mathbf{v}^k_t)\mathrm{d}t = \int_0^T L(M^k_t, \dot{M}^k_t)\mathrm{d}t, \qquad M^k_{t\#}\nu_0 = \sigma^k_t$$

for all $t \in [0, T]$ and $\|\dot{M}^k\| = \|\mathbf{v}^k_t\|_{\sigma^k_t} \leq \|\mathbf{v}_t\|_{\sigma_t}$ for a.e. $t \in (0, T)$. We use Corollary 2.3 and the fact that $\mathcal{W}$ is continuous and bounded to obtain a sequence $\{\phi^k\}_k \subset AC^p(0, T; \mathbf{L}^p(X))$ such that for each k fixed there is a partition of X into finitely many parallel subcubes of the same size such that ϕ^k_t is constant on each subcube and

$$\int_0^T L(\phi^k_t, \dot{\phi}^k_t)\mathrm{d}t \leq \int_0^T L(M^k_t, \dot{M}^k_t)\mathrm{d}t + \frac{\delta}{4}, \qquad \|M^k_t - \phi^k_t\| \leq \frac{1}{k} \quad \text{for all } t \in [0, T].$$

For k fixed, we may assume that the total number of subcubes is of the form $(l_k m)^d$, where l_k is a natural number. Set $\bar{\sigma}^k_t = \phi^k_{t\#}\nu_0$ so that $\tilde{\sigma}^k \in AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$. By Lemma A.4, there exists $\tilde{\mathbf{v}}^k$, a velocity for $\tilde{\sigma}^k$, such that $\dot{\phi}^k_t = \tilde{\mathbf{v}}^k_t \circ \phi^k_t$ for almost every $t \in (0, T)$. Fix $s \in (0, T)$ and let $\bar{\sigma}^k$ be defined in the following way: for $t \in [0, s]$ it is the geodesic of constant speed connecting μ_0 to $\tilde{\sigma}^k_s$. For $t \in [s, T - s]$, $\bar{\sigma}^k_t$ coincides with $\tilde{\sigma}^k_t$. For $t \in [T - s, T]$, $\bar{\sigma}^k_t$ is the geodesic of constant speed connecting σ^k_{T-s} to μ_T. By (5.24),

$$\int_0^T \mathcal{L}(\tilde{\sigma}^k_t, \tilde{\mathbf{v}}^k_t)\mathrm{d}t \geq \int_0^T \mathcal{L}(\bar{\sigma}^k_t, \bar{\mathbf{v}}^k_t)\mathrm{d}t - Cs\left(2 + \frac{W^p_p(\mu_0, \tilde{\sigma}^k_s) + W^p_p(\mu_T, \tilde{\sigma}^k_{T-s})}{s^p}\right). \tag{6.2}$$

We use the triangle inequality and the fact that $\|\dot{\phi}^k_t\| \leq C_\sigma$ to obtain

$$\begin{aligned} W_p(\mu_0, \tilde{\sigma}^k_s) &\leq W_p(\mu_0, \sigma^k_0) + W_p(\sigma^k_0, \tilde{\sigma}^k_0) + W_p(\tilde{\sigma}^k_0, \tilde{\sigma}^k_s) \\ &\leq W_p(\mu_0, \sigma^k_0) + \|M^k_0 - \phi^k_0\| + sC_\sigma \end{aligned}$$

and so, for k large enough, $W^p_p(\mu_0, \tilde{\sigma}^k_s) \leq 2sC_\sigma$. Similarly, for k large enough, $W^p_p(\mu_T, \tilde{\sigma}^k_{T-s}) \leq 2sC_\sigma$. We use (6.2) to obtain, for s small enough,

$$\begin{aligned} \int_0^T \mathcal{L}(\tilde{\sigma}^k_t, \tilde{\mathbf{v}}^k_t)\mathrm{d}t &\geq \int_0^T \mathcal{L}(\bar{\sigma}^k_t, \bar{\mathbf{v}}^k_t)\mathrm{d}t - \frac{\delta}{4} \geq \mathcal{C}^\epsilon_{0,T;(l_k m)^d}(\mu_0, \mu_T) - \frac{\delta}{4} \\ &= \mathcal{C}^\epsilon_{0,T;m}(\mu_0, \mu_T) - \frac{\delta}{4}, \end{aligned} \tag{6.3}$$

where we have used (6.1). This finishes the proof.

6.2. The lower bound of the Γ-limit on $\mathcal{P}_p(\mathbb{R}^d)$

Let $l, m > 1$ be integers, and let $\{x^j_i\} \subset \mathbb{Q}^d$ where $i = 0, \ldots, 2^l$ and $j = 1, \ldots, m^d$. Let $\sigma \in AC_p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ be such that $\sigma_t = \frac{1}{m^d}\sum_{j=1}^{m^d} \delta_{x^j_t}$, where

$$x^j_t := x^j_i + \left(\frac{2^l t}{T} - i\right)\left(x^j_{i+1} - x^j_i\right) \quad \text{whenever } t \in [s_i, s_{i+1}].$$

Here, $s_i := Ti/2^l$ for $i = 0, \ldots, 2^l - 1$.

Theorem 6.3. *Let $\mathbf{v}$ be the unique velocity for σ and let $\bar{\phi}$ be a flow associated with $(\sigma, \mathbf{v})$. Then*

$$\mathcal{F}_w(\sigma) = \int_0^T \bar{L}(\mathbf{v}_t \circ \bar{\phi}_t)\mathrm{d}t.$$

Proof. Note that $\sigma \in \mathcal{S}_p(\mathbb{R}^d)$. By Remark 5.11, τ_w is weaker than τ and so $\mathcal{F}_w(\sigma) \leq \mathcal{F}(\sigma)$. It remains to show the reverse inequality.

By Corollary 5.14, there exists a sequence $\{\sigma^n\}_n$ τ_w-converging to σ such that $\{\sigma^n_t\}_n$ converges to σ_t in $\mathcal{P}_p(\mathbb{R}^d)$ for every $t \in [0, T]$ and $\mathcal{F}_w(\sigma) = \lim_{n\to\infty} \mathcal{F}^{\epsilon_n}(\sigma^n)$. We exploit Appendix A.2 and use the fact that $\mathcal{W}$ is continuous and bounded (cf. Lemma 5.2) to assume without loss of generality that for each n there exists an integer m_n such that σ^n_t is of the form $1/m_n \sum_{i=1}^{m_n} \delta_{x^{i,n}_t}$. By Appendix A.4, we may assume that $t \to x^{i,n}_t$ belongs to $W^{1,p}(0, T; \mathbb{R}^d)$. By Appendix A.3, there exist a Borel velocity $\mathbf{v}^n$ for σ^n and a flow $\phi^n \in AC^p(0, T, \mathbf{L}^p(X))$ associated with $(\sigma^n, \mathbf{v}^n)$. Fix a positive integer $k > l$. For $i = 0, \ldots, 2^k$ we

$$t_i = ih \quad i = 0, \ldots, 2^k, \qquad h = \frac{T}{2^k}.$$

Note that $\mathbf{v}^n$ is the velocity of minimal norm for σ^n. We have

$$\mathcal{F}(\sigma) = \lim_{n\to\infty} \mathcal{F}^{\epsilon_n}(\sigma^n) = \lim_{n\to\infty} F^{\epsilon_n}(\phi^n) = \lim_{n\to\infty} \sum_{i=0}^{2^k-1} \int_{t_i}^{t_{i+1}} \mathcal{L}^{\epsilon_n}(\sigma_t^n, \mathbf{v}_t^n)\mathrm{d}t$$

$$\geq \sum_{i=0}^{2^k-1} \liminf_{n\to\infty} \mathcal{C}^{\epsilon_n}_{t_i,t_{i+1}}\big(\sigma^n_{t_i}, \sigma^n_{t_{i+1}}\big) \geq \sum_{i=0}^{2^k-1} \liminf_{n\to\infty} \mathcal{C}^{\epsilon_n}_{t_i,t_{i+1}}\big(\sigma_{t_i}, \sigma_{t_{i+1}}\big) \tag{6.4}$$

$$= \sum_{i=0}^{2^k-1} \liminf_{n\to\infty} \epsilon_n \mathcal{C}_{0,\frac{h}{\epsilon_n}}\Big(D_\#^{1/\epsilon_n}\sigma_{t_i}, D_\#^{1/\epsilon_n}\sigma_{t_{i+1}}\Big). \tag{6.5}$$

We have used Lemma 5.20 and the fact that $\{\sigma_t^n\}$ converges to σ_t for $t = t_i, t_{i+1}$ to obtain the last expression in (6.4). To obtain (6.5) we have used (2.9). We exploit Lemma 5.3 and then Lemma 6.2 to find a minimizer $\bar\sigma^n \in AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ such that for $t \in [0, T]$, $\bar\sigma_t^n$ is of the form

$$\bar\sigma_t^n = \frac{1}{m^d}\sum_{j=1}^{m^d} \delta_{\bar x_t^{j,n}} \quad \text{and} \quad \bar\sigma_{t_i} = \sigma_{t_i}$$

for $i = 0, 1, \ldots, 2^k - 1$ and

$$\epsilon_n \mathcal{C}_{0,\frac{h}{\epsilon_n}}\Big(D_\#^{1/\epsilon_n}\bar\sigma_{t_i}, D_\#^{1/\epsilon_n}\bar\sigma_{t_{i+1}}\Big) = \int_{t_i}^{t_{i+1}} \mathcal{L}(D_\#^{1/\epsilon_n}\bar\sigma, \bar{\mathbf{v}}_t^n)\mathrm{d}t,$$

where $\bar{\mathbf{v}}^n$ is the unique velocity for $\bar\sigma^n$. Thanks to Appendix A.4, we may assume that $t \to \bar x_t^{j,n}$ is of class $W^{1,p}(0, T; \mathbb{R}^d)$. By Lemma A.2, there exists a flow $\bar\phi^n$ associated with $\bar\sigma^n$. Furthermore, there are m^d parallel subcubes of X of the same size such that for each subcube, $\bar\phi_t^n$ is the point $\bar x_t^{i,n}$ for some i. We have

$$\epsilon_n \mathcal{C}_{0,\frac{h}{\epsilon_n}}\Big(D_\#^{1/\epsilon_n}\bar\sigma_{t_i}, D_\#^{1/\epsilon_n}\bar\sigma_{t_{i+1}}\Big) = \int_{t_i}^{t_{i+1}} L\left(\frac{\bar\phi_t^n}{\epsilon_n}, \dot{\bar\phi}_t^n\right)\mathrm{d}t = \epsilon_n C_{0,\frac{h}{\epsilon_n}}\left(\frac{\bar\phi_{t_i}^n}{\epsilon_n}, \frac{\bar\phi_{t_{i+1}}^n}{\epsilon_n}\right).$$

By (2.6), for any $P > 0$ we have

$$C_{0,\frac{h}{\epsilon_n}}\left(\frac{\bar\phi_{t_i}^n}{\epsilon_n}, \frac{\bar\phi_{t_{i+1}}^n}{\epsilon_n}\right) \geq -C_{0,1}\left(\left[\frac{1}{P\epsilon_n}\right]P\bar\phi_{t_i}^n, \frac{\bar\phi_{t_i}^n}{\epsilon_n}\right) - C_{\frac{h}{\epsilon_n},\frac{h}{\epsilon_n}+1}\left(\frac{\bar\phi_{t_{i+1}}^n}{\epsilon_n}, \left[\frac{1}{P\epsilon_n}\right]P\bar\phi_{t_{i+1}}^n\right)$$

$$+ C_{0,\frac{h}{\epsilon_n}+2}\left(\left[\frac{1}{P\epsilon_n}\right]P\bar\phi_{t_i}^n, \left[\frac{1}{P\epsilon_n}\right]P\bar\phi_{t_{i+1}}^n\right) \tag{6.6}$$

$$\geq -6C + C_{0,\frac{h}{\epsilon_n}+2}\left(\left[\frac{1}{P\epsilon_n}\right]P\bar\phi_{t_i}^n, \left[\frac{1}{P\epsilon_n}\right]P\bar\phi_{t_{i+1}}^n\right). \tag{6.7}$$

We have used (2.1) to obtain (6.7). By Remark 2.1,

$$C_{0,\frac{h}{\epsilon_n}+2}\left(\left[\frac{1}{P\epsilon_n}\right]P\bar\phi_{t_i}^n, \left[\frac{1}{P\epsilon_n}\right]P\bar\phi_{t_{i+1}}^n\right) \geq \left(2 + \frac{h}{\epsilon_n}\right)\bar L(q_n),$$

$$\text{where } q^n = \frac{\left[\frac{1}{P\epsilon_n}\right] P\left(\bar{\phi}^n_{t_{i+1}} - \bar{\phi}^n_{t_i}\right)}{\frac{h}{\epsilon_n} + 2}. \tag{6.8}$$

Although $\bar{\phi}^n_{t_i}$ depends on n, note that, if $t_i \in [s_{i^*}, s_{i^*+1}]$, the range of $\bar{\phi}^n_{t_i}$ is the subset of finite cardinality $\{x^j_{i^*} + \left(2^{l-k}i - i^*\right)(x^j_{i^*+1} - x^j_{i^*})\}_{j=1}^{m^d}$ contained in $\mathbb{Q}^d$. The union of these sets depends on k but is independent of n. Hence, $\{\bar{\phi}^n_{t_i}\}_n$ is bounded in $L^\infty(0, T; \mathbf{L}^p(X))$ and there exists an integer P_k independent of n such that the range of $P_k\bar{\phi}^n_{t_i}$ is contained in $\mathbb{Z}^d$ for $i = 0, 1, \dots, 2^k - 1$. We also have

$$q^n = \frac{P_k\left(\left[\frac{1}{P_k\epsilon_n}\right] - \frac{1}{P_k\epsilon_n}\right)(\bar{\phi}^n_{t_{i+1}} - \bar{\phi}^n_{t_i}) + \frac{1}{\epsilon_n}(\bar{\phi}^n_{t_{i+1}} - \bar{\phi}^n_{t_i})}{\frac{h}{\epsilon_n} + 2}. \tag{6.9}$$

Hence,

$$\left\| q^n - \frac{\bar{\phi}^n_{t_{i+1}} - \bar{\phi}^n_{t_i}}{h + 2\epsilon_n} \right\| \le \epsilon_n \frac{\|\bar{\phi}^n_{t_{i+1}} - \bar{\phi}^n_{t_i}\|}{h + 2\epsilon_n} P_k, \tag{6.10}$$

which implies that $\{q^n\}_n$ is bounded in $\mathbf{L}^p(X)$. As $\bar{L}$ is convex and assumes only finite values, it is locally Lipschitz. Up to a subsequence which we do not relabel, $\{\bar{\phi}^n_{t_i}\}_n$ converges in $\mathbf{L}^p(X)$ to some $\phi^k_{t_i}$ which depends on k and that pushes ν_0 forward to σ_{t_i}. Furthermore, $\phi^k_{t_i}$ is a constant on each subcube. Observe that $\lim_{k\to\infty} W_p(\phi^k_{t_i\#}\nu_0, \sigma_{t_i}) = 0$. We combine (6.5)–(6.8) and (6.10) to conclude that

$$\mathcal{F}(\sigma) \ge \sum_{i=0}^{2^k-1} h\bar{L}\left(\frac{\phi^k_{t_{i+1}} - \phi^k_{t_i}}{h}\right) = \int_0^T \bar{L}(\dot{\phi}^k_t)\mathrm{d}t. \tag{6.11}$$

For $t \in [t_i, t_{i+1}]$ we have set

$$\phi^k_t = \left(1 - \frac{t - t_i}{h}\right)\phi^k_{t_i} + \frac{t - t_i}{h}\phi^k_{t_{i+1}}.$$

When $t_i,\ t_{i+1} \in [s_{i^*}, s_{i^*+1}]$ (and for each $i \in \{0, 1, \dots, 2^k - 1\}$ there exists such an $i^* \in \{0, 1, \dots, 2^l - 1\}$), we have

$$|\bar{\phi}^n_{t_{i+1}} - \bar{\phi}^n_{t_i}| \le 2^l h|x^j_{i^*+1} - x^j_{i^*}|.$$

Hence, $\{\phi^k_{t_i}\}_k$ is bounded in $W^{1,\infty}(0, T; \mathbf{L}^p(X))$. Let $E > 0$ be a constant such that $|(\phi^k)'|(t) \le E$ for almost every $t \in (0, T)$. The sequence $\{\phi^k_{t_i}\}_k$ converges strongly in $L^\infty(0, T; \mathbf{L}^p(X))$ to some ϕ such that ϕ_t is constant on each subcube and $|(\phi)'|(t) \le E$ for almost every $t \in (0, T)$. Set $\nu_t = \phi_{t\#}\nu_0$ and observe that $\nu_{t_i} = \sigma_{t_i}$. Let $t \in [0, T)$. For k fixed, we have $t \in [t_i, t_{i+1})$ for some $i \in \{0, 1, \dots, 2^k - 1\}$ and

$$W_p(\sigma_t, \nu_t) \le W_p(\sigma_t, \sigma_{t_i}) + W_p(\nu_{t_i}, \nu_t) \le 2E(t - t_i) \le 2^{1-k}E.$$

This proves that $\nu = \sigma$. Let $\mathbf{v}$ be the unique velocity for σ so that ϕ is a flow associated with $(\sigma, \mathbf{v})$. By (6.11),

$$\mathcal{F}(\sigma) \geq \int_0^T \bar{L}(\dot{\phi}_t)\mathrm{d}t = \int_0^T \bar{L}(\mathbf{v}_t \circ \phi_t)\mathrm{d}t = \int_0^T \bar{L}(\mathbf{v}_t \circ \bar{\phi}_t)\mathrm{d}t.$$

To obtain the last inequality we have used that $(\mathbf{v}_t \circ \phi_t)_\# \nu_0 = \mathbf{v}_{t\#}\sigma_t = (\mathbf{v}_t \circ \bar{\phi}_t)_\# \nu_0$, and that $\bar{L}$ is continuous on $\mathbf{L}^p(X)$, and then applied Remark 5.6.

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Appendix

A.1. *A review of p-absolutely continuous curves in $\mathcal{P}_p(\mathbb{R}^d)$ (cf. e.g. [2])*

For $\mu \in \mathcal{P}_p(\mathbb{R}^d)$ the $\mathbf{L}^p(\mu)$-closure of $\{\nabla\varphi : |\nabla\varphi|^{q-2}\ \varphi \in C_c^\infty(\mathbb{R}^d)\}$ is denoted by $\mathcal{T}_\mu\mathcal{P}_p(\mathbb{R}^d)$ where $q = p/(p-1)$. Equivalently, $\mathcal{T}_\mu\mathcal{P}_p(\mathbb{R}^d)$ is the set of $\mathbf{v}$ such that $\mathbf{v}|\mathbf{v}|^{p-2}$ belongs to the $\mathbf{L}^q(\mu)$-closure of $\{\nabla\varphi : \varphi \in C_c^\infty(\mathbb{R}^d)\}$.

Let $\sigma \in AC^p(a, b; \mathcal{P}_p(\mathbb{R}^d))$. According to [2, Section 8.3] there exists a Borel vector field $v : (a, b) \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\text{(i) } \partial_t\sigma + \nabla \cdot (\sigma\mathbf{v}) = 0 \quad \text{in } \mathcal{D}((a, b) \times \mathbb{R}^d) \tag{A.1}$$

$$\text{(ii) } \mathbf{v}_t \in \mathbf{L}^p(\sigma_t) \quad \text{for a.e. } t \in (a, b) \tag{A.2}$$

$$\text{(iii) } \int_a^b \|\mathbf{v}_t\|_{\sigma_t}^p \mathrm{d}t < \infty. \tag{A.3}$$

We say that $\mathbf{v}$ is a velocity for σ. We can choose $\mathbf{v}$ such that $\mathbf{v}_t \in \mathcal{T}_{\sigma_t}\mathcal{P}_p(\mathbb{R}^d)$ for almost every $t \in (0, T)$. In the latter case, $\mathbf{v}$ is uniquely determined and is called the velocity of minimal norm for σ, i.e. if $\mathbf{w}$ is another velocity for σ, then $\|\mathbf{v}_t\|_{\sigma_t} \leq \|\mathbf{w}_t\|_{\sigma_t}$ for almost every $t \in (0, T)$. For almost every t, the metric derivative

$$|\sigma'|(t) := \lim_{h \to 0} \frac{W_p(\sigma_{t+h}, \sigma_t)}{|h|}$$

exists and is equal to the $\|\cdot\|_{\sigma_t}$-norm of the velocity of minimal norm at t. Here, W_p stands for the p-Wasserstein metric. For $0 \leq s < t \leq T$ and $C_\sigma^p := \int_0^T |\sigma'|^p(l)\mathrm{d}l$ we have

$$W_p(\sigma_s, \sigma_t) \leq \int_s^t |\sigma'|(l)\mathrm{d}l \leq |t-s|^{\frac{1}{p'}} C_\sigma. \tag{A.4}$$

Given $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$ we denote by $\mathcal{P}(\mu, \nu)$ the set of $\sigma \in AC^p(a, b; \mathcal{P}_p(\mathbb{R}^d))$ such that $\sigma_a = \mu$ and $\sigma_b = \nu$.

Remark A.1. Let $\{\sigma^n\}_n \subset AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ and let $\mathbf{v}^n$ be its velocity of minimal norm. Suppose there exists $C > 0$ such that $\int_0^T \|\mathbf{v}_t^n\|_{\sigma_t^n}^p \mathrm{d}t \leq C^p$ for all $n \in \mathbb{N}$. Then:

(i) For $0 \leq s < t \leq T$ we have $W_p(\sigma_t^n, \sigma_s^n) \leq C|t-s|^{1/p'}$ for all $n \in \mathbb{N}$. The following four assertions are equivalent: (1) $\sup_{n,t} W_p(\sigma_t^n, \delta_0) < \infty$, (2) $\sup_n W_p(\sigma_0^n, \delta_0) < \infty$, (3) there exists $\{t_n\}_n \subset [0,T]$ such that $\sup_n W_p(\sigma_{t_n}^n, \delta_0) < \infty$ and (4) $\{\sigma^n\}_n$ is bounded in $AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$.

(ii) If $q \in [1,p)$, then $AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$ compactly embeds in $C^{0,1/p}\big([0,T];\mathbb{R}^d\big)$. More precisely, if $\{\sigma^n\}_n \subset AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$ is bounded, then there exist an increasing sequence $\{n_k\}_k$ and a curve $\sigma \in AC^p(0,T;\mathcal{P}_p(\mathbb{R}))$ such that for every $t \in [0,T]$, $\{\sigma_t^{n_k}\}_k$ converges to σ_t in $\mathcal{P}_q(\mathbb{R}^d)$.

Proof. The uniform Hölder continuity is given by (A.4). Clearly (4) implies (3). Suppose next that (3) holds. Then

$$W_p(\sigma_t^n, \delta_0) \leq W_p(\sigma_t^n, \sigma_{t_n}^n) + W_p(\sigma_{t_n}^n, \delta_0) \leq T^{\frac{1}{p'}} C + \sup_n W_p(\sigma_{t_n}^n, \delta_0)$$

which proves (4). One concludes easily that the four assertions are equivalent. This concludes the proof of (i).

Consider the topology given by the narrow convergence, which is a Hausdorff topology, compatible with W_p in the sense of [2] Section 2.1. We apply Proposition 3.3.1 [2] to obtain an increasing sequence $\{n_k\}_k$ and $\sigma \in AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$ such that $\{\sigma_t^{n_k}\}_k$ converges narrowly to σ_t for almost every $t \in [0,T]$. As $\sup_{n,t} W_p(\sigma_t^n, \delta_0) < \infty$, $\{\sigma_t^{n_k}\}_k$ converges to σ_t in $\mathcal{P}_q(\mathbb{R}^d)$ for almost every $t \in [0,T]$. The uniform Hölder continuity of σ^n yields that in fact $\{\sigma_t^{n_k}\}_k$ converges to σ_t in $\mathcal{P}_q(\mathbb{R}^d)$ for every $t \in [0,T]$.

A.2. *Time discretization of p-absolutely continuous curves*

Let $\sigma \in AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$ and $\mathbf{v}$ be a velocity for σ. Set $C_\sigma^p := \int_0^T |\sigma'|^p(s)\mathrm{d}s$. Fix an integer $n > 1$ and set $t_j = j/nT = ih$ for $j = 0,\dots,n$. Choose $\gamma_i \in \Gamma(\sigma_{t_i}, \sigma_{t_{i+1}})$ such that

$$W_p^p(\sigma_{t_i}, \sigma_{t_{i+1}}) = \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p \gamma_i(\mathrm{d}x,\mathrm{d}y).$$

Let $\pi_1, \pi_2 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be the projections defined by $\pi_1(x,y) = x$, $\pi_2(x,y) = y$. For $s \in [t_i, t_{i+1}]$ we set

$$\sigma_t^n := \left(\left(1 - \frac{t-t_i}{h}\right)\pi_1 + \frac{t-t_i}{h}\pi_2\right)_{\#}\gamma_i.$$

It is well-known that $\sigma^n \in AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$ and it is a geodesic of constant speed on $[t_i, t_{i+1}]$. Its velocity of minimal norm is $\mathbf{v}^n$, satisfying

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} F\left\langle\left(1 - \frac{t-t_i}{h}\right)\pi_1 + \frac{t-t_i}{h}\pi_2; \frac{y-x}{h}\right\rangle \gamma_i(\mathrm{d}x,\mathrm{d}y) = \int_{\mathbb{R}^d} \langle F; \mathbf{v}_t^n\rangle \mathrm{d}\sigma_t^n.$$

We compile all the well-known facts in the following lemma (cf. e.g. [2]):

Lemma A.2. *We have $W_p(\sigma_t, \sigma_t^n) \le 2h^{1/p'} C_\sigma$, for $t \in [0, T]$. Furthermore,*

$$\begin{aligned} &\int_0^T \|\mathbf{v}_t^n\|_{\sigma_t^n}^p \mathrm{d}t \le C_\sigma^p \quad \textit{and} \\ &\|\mathbf{v}_t^n\|_{\sigma_t^n} = |(\sigma^n)'|(t) = \frac{W_p(\sigma_{t_i}, \sigma_{t_{i+1}})}{h} \le \frac{\int_{t_i}^{t_{i+1}} |\sigma'|(s)\mathrm{d}s}{h} \end{aligned} \tag{A.5}$$

whenever $t_i \le s < t \le t_{i+1}$.

Remark A.3. Lemma A.2 asserts that $\{\sigma_t^n\}_n$ converges to σ_t in $\mathcal{P}_p(\mathbb{R}^d)$. Hence, if $\mathcal{W} : \mathcal{P}_p(\mathbb{R}^d) \to \mathbb{R}$ is continuous and bounded, the Lebesgue dominated convergence theorem yields

$$\lim_{n\to\infty} \int_0^T \mathcal{W}(\sigma_t^n)\mathrm{d}t = \int_0^T \mathcal{W}(\sigma_t)\mathrm{d}t.$$

In light of the first inequality in (A.5) we obtain that for every $\delta, a > 0$ we have

$$\int_0^T \big(1/p\|\mathbf{v}_t^n\|_{\sigma_t^N}^p - \mathcal{W}(D_\#^a \sigma_t^n)\big)\mathrm{d}t \le \delta + \int_0^T \big(1/p\|\mathbf{v}_t\|_{\sigma_t}^p - \mathcal{W}(D_\#^a \sigma_t)\big)\mathrm{d}t$$

for n large enough, where $D^a x = a\, x$.

Fix an $m > 1$ and a real number $\delta > 0$. For $i = 1, \dots, n$ let $\{x_i^j\}_{j=1}^m \subset \mathbb{R}^d$ and set

$$\bar\sigma_{t_i} := \frac{1}{m}\sum_{j=1}^m \delta_{x_i^j}, \qquad \bar\gamma_i = \frac{1}{m}\sum_{j=1}^m \delta_{(x_i^j, x_i^{j+1})}.$$

Assume

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p \bar\gamma_i(\mathrm{d}x, \mathrm{d}y) = W_p^p\big(\bar\sigma_{t_i}, \bar\sigma_{t_{i+1}}\big), \quad W_p^p\big(\sigma_{t_i}, \bar\sigma_{t_i}\big) \le \bar\delta^p$$

where $\bar\delta$ is to be chosen soon. As above, we define

$$\bar\sigma_t^n := \left(\left(1 - \frac{t-t_i}{h}\right)\pi_1 + \frac{t-t_i}{h}\pi_2\right)_\# \bar\gamma_i.$$

Without repeating once the above listed facts, we have $\bar\sigma^n \in AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ and it is a geodesic of constant speed on $[t_i, t_{i+1}]$. Let $\bar{\mathbf{v}}^n$ be the velocity of minimal norm for $\bar\sigma^n$. Then

$$\|\bar{\mathbf{v}}_t^n\|_{\sigma_t^n} = |(\bar\sigma^n)'|(t) = W_p(\bar\sigma_{t_i}, \bar\sigma_{t_{i+1}})/h$$

and so,

$$\int_0^T \|\bar{\mathbf{v}}_t^n\|_{\bar\sigma_t^n}^p \mathrm{d}t = h\sum_{i=1}^n \frac{W_p^p(\bar\sigma_{t_i}, \bar\sigma_{t_{i+1}})}{h^p}.$$

We choose $\bar\delta < \min\{\delta, hC_\sigma\}$ small enough and m large enough that

$$W_p^p(\bar\sigma_{t_i}, \bar\sigma_{t_{i+1}}) \le W_p^p(\sigma_{t_i}, \sigma_{t_{i+1}}) + h^{p-1}\delta/n$$

to conclude that

$$\int_0^T \|\bar{\mathbf{v}}_t^n\|_{\bar\sigma_t^n}^p \mathrm{d}t \le h\sum_{i=1}^n \frac{W_p^p(\sigma_{t_i}, \sigma_{t_{i+1}})}{h^p} + \delta = \int_0^T \|\mathbf{v}_t^n\|_{\sigma_t^n}^p \mathrm{d}t + \delta. \tag{A.6}$$

For $t \in [t_i, t_{i+1}]$ we have

$$W_p(\bar{\sigma}_t^n, \sigma_t^n) \le W_p(\bar{\sigma}_t^n, \bar{\sigma}_{t_i}^n) + W_p(\bar{\sigma}_{t_i}^n, \sigma_{t_i}^n) + W_p(\sigma_{t_i}^n, \sigma_t^n)$$

and so,

$$W_p(\bar{\sigma}_t^n, \sigma_t^n) \le 3\bar{\delta} + 2W_p(\sigma_{t_i}^n, \sigma_{t_{i+1}}^n) \le 5hC_\sigma. \tag{A.7}$$

A.3. Eulerian and Lagrangian coordinates for spatially discrete flows

Suppose $C^1, \dots, C^n$ is a partition of X up to a negligible set and $x^1, \dots, x^n \in W^{1,p}(0, T; \mathbb{R}^d)$. Set

$$\phi_t = \sum_{i=1}^n x_t^i \chi_{C^i}, \qquad \sigma_t = \frac{1}{n}\sum_{i=1}^n \delta_{x_t^i}.$$

Note that $\sigma_t = \phi_{t\#}\nu_0$.

Lemma A.4. (i) *We have $\phi \in AC^p(0, T; \mathbf{L}^p(X))$ and there exists a Borel map $\mathbf{v} : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d$ such that $\dot{\phi}_t = \mathbf{v}_t \circ \phi_t$ for almost every $t \in (0, T)$.*
(ii) *We have $\sigma \in AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ and $\mathbf{v}$ is the unique velocity for σ up to a negligible set. In other words, ϕ is a flow associated with $(\sigma, \mathbf{v})$.*

Proof. (i) The fact that $\phi \in AC^p(0, T; \mathbf{L}^p(X))$ is straightforward to establish. The set of $t \in (0, T)$ such that each one of the x^i is differentiable at t and $x_t^i = x_t^j$ implies $\dot{x}_t^i = \dot{x}_t^j$ for all i, j is a set of full measure in $(0, T)$ (cf. e.g. [5]). Thus, there exists a Borel subset $\mathcal{A} := \mathcal{T} \times X \subset (0, T) \times X$ of full $\mathcal{L}^{1+d}|_{(0,T)\times X}$ measure such that

$$\phi_t(x) = \phi_t(y) \quad \text{implies} \quad \dot{\phi}_t(x) = \dot{\phi}_t(y) \quad \text{whenever } (t, x),\ (t, y) \in \mathcal{A}.$$

By Corollary 2.3 [18], there exists a Borel map $\mathbf{v} : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d$ such that $\dot{\phi}_t = \mathbf{v}_t \circ \phi_t$ for almost every $t \in (0, T)$.

(ii) As $\sigma_t = \phi_{t\#}\nu_0$, we have for $s, t \in [0, T]$ that $W_p(\sigma_t, \sigma_s) \le \|\phi_t - \phi_s\|$ and so, since $\phi \in AC^p(0, T; \mathbf{L}^p(X))$, we conclude that $\sigma \in AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$. Furthermore, using that $\dot{\phi}_t = \mathbf{v}_t \circ \phi_t$, we obtain for $F \in C_c^1((0, T) \times \mathbb{R}^d)$

$$\begin{aligned}\int_0^T \mathrm{d}t \int_{\mathbb{R}^d} \Big(\partial_t F_t + \langle \nabla F_t; \mathbf{v}_t \rangle\Big) \mathrm{d}\sigma_t &= \int_0^T \mathrm{d}t \int_{\mathbb{R}^d} \Big(\partial_t F_t \circ \phi_t + \langle \nabla F_t \circ \phi_t; \dot{\phi}_t \rangle\Big) \mathrm{d}\nu_0 \\ &= \int_{\mathbb{R}^d} (F_T \circ \phi_T - F_0 \circ \phi_0) \mathrm{d}\nu_0 = 0.\end{aligned}$$

This proves that $\mathbf{v}$ is a velocity for σ. If $\mathbf{w}$ is another velocity for σ we have $\nabla \cdot ((\mathbf{v}_t - \mathbf{w}_t)\sigma_t) = 0$ for almost every $t \in (0, T)$. But for each $t \in [0, T]$, σ_t is supported by finitely many points and so the closure of $\{\nabla\varphi : \varphi \in C_c^\infty(\mathbb{R}^d)\}$ equals $\mathbf{L}^p(\sigma_t)$ (cf., e.g., Example 2.8 [7]). This proves that $\mathbf{w}_t = \mathbf{v}_t \sigma_t$-almost everywhere. Hence, $\mathbf{v} = \mathbf{w}$ up to a negligible set.

A.4. A time regularity result for spatially discrete p-absolutely continuous curves in $\mathcal{P}_p(\mathbb{R}^d)$

As in Appendix A.2, let $\sigma \in AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ and let $\mathbf{v}$ be a velocity for σ. Suppose in addition that there exists $x^1, \dots, x^n : [0, T] \to \mathbb{R}^d$ such that $\sigma_t = 1/n \sum_{j=1}^n \delta_{x^j(t)}$.

Note that, if for each $t \in (0, T)$ we choose a permutation of n letters $\tau(t, \cdot)$, then $\sigma_t = \sum_{j=1}^{n} \delta_{x^{\tau(t,j)}(t)}$. Unless $\tau(t, \cdot)$ is chosen appropriately, $t \to x^{\tau(t,j)}(t)$ may not be in $W^{1,p}(0, T; \mathbb{R}^d)$. For instance, let $x_=t$ for all rational $t \in [0, T]$ and $x_t = 1 - t$ for all irrational $t \in [0, T]$. Similarly, let $y_t = t$ for all irrational $t \in [0, T]$ and $y_t = 1 - t$ for all rational $t \in [0, T]$. We have $t \to (1/2)(\delta_{x_t} + \delta_{y_t})$ belongs to $AC^p(0, T; \mathcal{P}_p(\mathbb{R}))$ whereas neither x nor y is continuous. The goal of this section is to show that for an appropriate choice of $\tau, t \to x^{\tau(t,j)}(t)$ is in $W^{1,p}(0, T; \mathbb{R}^d)$. Combining this with Appendix A.3 we obtain the existence of a flow ϕ associated with $(\sigma, \mathbf{v})$. In other words, $\phi \in AC^p(0, T; \mathbf{L}^p(X))$ and $\dot{\phi}_t = \mathbf{v}_t \circ \phi_t$ for almost every $t \in (0, T)$.

Let σ^N and $\gamma_{i,s}^N$ be as in Appendix A.2. We start ordering the set $\{x_0^j\}_{j=1}^n$ any way we want. The standard mass transportation theory ensures that we can choose an ordering of $\{x_{t_i}^j\}_{j=1}^n$, $i = 0, \dots, N$, such that γ_i^N is of the form $\gamma_i^N = \frac{1}{n} \sum_{j=1}^{n} \delta_{(x_{t_i}, x_{t_{i+1}})}$. For $t \in [t_i, t_{i+1}]$ we set

$$x_t^j = \left(1 - \frac{t - t_i}{h}\right) x_{t_i} + \frac{t - t_i}{h} x_{t_{i+1}}.$$

We denote by $\phi^N(t)$ the vector in $\mathbb{R}^{nd}$ whose j component is x_t^j and by $\|\cdot\|_{nd}$ the euclidean norm on $\mathbb{R}^{nd}$. We have $\phi^N \in W^{1,p}(0, T; \mathbb{R}^{nd})$ and by (A.5),

$$\|\dot{\phi}^N\|_{nd}^p \le n \int_0^T |(\sigma^N)'|^p(s)\,\mathrm{d}s \le \int_0^T |\sigma'|^p(s)\,\mathrm{d}s.$$

Hence there exists a subsequence $\{\Phi^{N_k}\}_k$ that converges weakly to some ϕ in $W^{1,p}(0, T; \mathbb{R}^{nd})$, converges strongly in $L^p(0, T; \mathbb{R}^{nd})$ and converges pointwise in $(0, T)$. Thus $\{\sigma_t^{N_k}\}_k$ converges in $\mathcal{P}_p(\mathbb{R}^d)$ to $\tilde{\sigma}_t := \frac{1}{n} \sum_{j=1}^{n} \delta_{\phi_t^j}$. Lemma A.2 gives that $\sigma_t = \tilde{\sigma}_t$.

References

[1] A. Alastuey, Mean-field kinetic theory of a classical electron gas in a periodic potential. I. Formal solution of the Vlasov equation, J. Stat. Phys. 48 (1987) 839–871.

[2] L. Ambrosio, N. Gigli, G. Savaré, Gradient flows in metric spaces and the Wasserstein spaces of probability measures, in: Lectures in Mathematics, Birkhäuser, ETH, Zurich, 2005.

[3] B. Ayuso, A.J. Carrillo, C.W. Shu, Discontinuous Galerkin methods for the one-dimensional Vlasov–Poisson system, Kinet. Relat. Models 4 (2011) 955–989.

[4] Y. Brenier, Polar factorization and monotone rearrangement of vector-valued functions, Comm. Pure Appl. Math. 44 (1991) 375–417.

[5] L.C. Evans, R.F. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, 1992.

[6] A. Fathi, Weak KAM theory in Lagrangian dynamics, Preliminary Version, Lecture Notes, 2003.

[7] W. Gangbo, H.K. Kim, T. Pacini, Differential forms on Wasserstein space and infinite-dimensional Hamiltonian systems, Mem. Amer. Math. Soc. 211 (993) (2011).

[8] W. Gangbo, R. McCann, The geometry of optimal transportation, Acta Math. 177 (1996) 113–161.

[9] W. Gangbo, T. Nguyen, A. Tudorascu, Hamilton–Jacobi equations in the Wasserstein space, Methods Appl. Anal. 15 (2) (2008) 155–184.

[10] W. Gangbo, T. Nguyen, A. Tudorascu, Euler–Poisson systems as action minimizing paths in the Wasserstein space, Arch. Ration. Mech. Anal. 192 (3) (2009) 419–452.

[11] W. Gangbo, A. Tudorascu, A weak KAM theorem; from finite to infinite dimension, Edizioni Della Normale, Scuola Normale Superiore Pisa, 2010.

[12] W. Gangbo, A. Tudorascu, Lagrangian dynamics on an infinite-dimensional torus; a weak KAM theorem, Adv. Math. 224 (1) (2010) 260–292.

[13] Y. Guo, The Vlasov–Poisson–Landau system in a periodic box, J. Amer. Math. Soc. (2011) http://dx.doi.org/10.1090/S0894-0347-2011-00722-4.
[14] M.A. Herrero, E. Zuazua, Current Trends in Applied Mathematics, Editorial Complutense, 1996.
[15] P.L. Lions, G. Papanicolaou, S.R.S. Varadhan, Homogenization of Hamilton–Jacobi equations, CCA, 1988 (unpublished).
[16] S. Muller, Homogenization of nonconvex integral functionals and elastic materials, Arch. Ration. Mech. Anal. 99 (1987) 189–212.
[17] T. Nguyen, A. Tudorascu, Pressureless Euler/Euler–Poisson systems via adhesion dynamics and scalar conservation laws, SIAM J. Math. Anal. 40 (2) (2008) 754–775.
[18] A. Tudorascu, On the velocities of flows consisting of cyclically monotone maps, Indiana Univ. Math. J. 59 (3) (2010) 929–956.
[19] W. E, A class of homogenization problems in the calculus of variations, Comm. Pure Appl. Math. XLIV (1991) 733–759.