A POLYCONVEX INTEGRAND; EULER–LAGRANGE EQUATIONS AND UNIQUENESS OF EQUILIBRIUM

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Abstract. In this manuscript we are interested in stored energy functionals W defined on the set of $d \times d$ matrices, which not only fail to be convex but satisfy $\lim_{\det \xi \to 0^+} W(\xi) = \infty$. We initiate a study which we hope would lead to a theory for the existence and uniqueness of minimizers of functionals of the form $E(\mathbf{u}) = \int_\Omega (W(\nabla \mathbf{u}) - \mathbf{F} \cdot \mathbf{u})dx$, as well as their Euler–Lagrange equations. The techniques developed here can be applied to a class of functionals larger than those considered in this manuscript, although we keep our focus on polyconvex stored energy functionals of the form $W(\xi) = f(\xi) + h(\det \xi)$ – such that $\lim_{t\to 0^+} h(t) = \infty$ – which appear in the study of Ogden material. We present a collection of perturbed and relaxed problems for which we prove uniqueness results. Then, we characterize these minimizers by their Euler–Lagrange equations.

1. Introduction

This manuscript describes a series of problems in the calculus of variations and sheds some light on them. Let $\Omega, \Lambda \subset \mathbb{R}^d$ be bounded convex sets and let $\mathbf{u}_0 : \Omega \to \Lambda$ be a diffeomorphism. We consider stored energy functionals of the form

$$W(\xi) = \begin{cases} f(\xi) + h(\det \xi) & \text{if} \quad \det \xi > 0 \\ \infty & \text{if} \quad \det \xi \le 0. \end{cases} \tag{1}$$

Here, $f \in C^1(\mathbb{R}^{d\times d})$ is strictly convex and is such that there exists $p \in (1, \infty)$ such that $f(\xi)$ behaves as $|\xi|^p$ for $|\xi|$ large enough. Motivated by the study of Ogden material [12] we impose that $h \in C^1(0, \infty)$ is convex and

$$\lim_{t\to 0^+} h(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{h(t)}{t} = \infty. \tag{2}$$

Given $\mathbf{F} : \Omega \to \mathbb{R}^d$ and $T > 0$, the variational problems we study are motivated by the challenging system of PDEs: find $\mathbf{u} : [0, T] \times \Omega \to \Lambda$ such that $\mathbf{u}(t, \cdot)(\Omega) = \Lambda$ and in the distributional sense,

$$\begin{cases} \partial_t \mathbf{u} + \operatorname{div}\big(DW(\nabla \mathbf{u})\big) & = \vec{0} \quad \text{in} \quad (0, T) \times \Omega \\ \mathbf{u}(0, \cdot) & = \mathbf{u}_0 \quad \text{in} \quad \Omega. \end{cases} \tag{3}$$

Because of the condition (2), it is neither known that (3) has a solution nor that there is a gradient flow solution for the energy $I_*(\mathbf{u}) := \int_\Omega \big(f(\nabla \mathbf{u}) + h(\det \nabla \mathbf{u}) - \mathbf{F} \cdot \mathbf{u}\big)dx$. If $p > d$, the variational problem

$$\min_{\mathbf{u}} \Big\{ \int_\Omega \big(f(\nabla \mathbf{u}) + h\big(\det \nabla \mathbf{u}\big) - \mathbf{F} \cdot \mathbf{u}\big)dx \mid \mathbf{u} \in W^{1,p}(\Omega, \Lambda), \;\; \mathbf{u}(\Omega) = \Lambda, \; \det \nabla \mathbf{u} > 0 \Big\}$$

has a minimizer u_*. One expects u_* to satisfies a system of equations which encodes more information (cf. (12)) than just being a stationary solution to (3). Making that statement rigorous remains open outstanding problems in the calculus of variations which we shed some light on.

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In this manuscript, we are concerned with a series of open problems closely related to the system of Equations in (3). The first one is to know if the minimizer of

$$\min_{(\beta,\mathbf{u})}\Big\{\int_\Omega \big(f(\nabla \mathbf{u}) + h(\beta) - \mathbf{F}\cdot\mathbf{u}\big)dx \mid \beta \in \det{}^* \nabla \mathbf{u}\Big\} \tag{4}$$

is unique. Here, the minimization in (4) is performed over the set of pairs $(\beta, \mathbf{u})$ such that $\mathbf{u} : \Omega \to \Lambda$ is a Borel map and $\beta : \Omega \to (0,\infty)$ is a Borel function. The notation $\beta \in \det^* \nabla \mathbf{u}$ means that $\mathbf{u}$ pushes $\beta\mathcal{L}^d$ forward to $\chi_\Lambda \mathcal{L}^d$, also denoted by $\mathbf{u}_\#(\beta\mathcal{L}^d) = \chi_\Lambda\mathcal{L}^d$. In other words,

$$\int_\Omega l(\mathbf{u})\beta dx = \int_\Lambda l(y)dy \qquad \forall l \in C_b(\mathbb{R}^d). \tag{5}$$

In fact, thanks to Remark 3.6 the problem in (4) can be interpreted as a relaxation of

$$\min_{\mathbf{u}}\Big\{\int_\Omega \Big(f(\nabla \mathbf{u}) + h\Big(\frac{|\det \nabla \mathbf{u}|}{N_{\mathbf{u}}(\mathbf{u})}\Big) - \mathbf{F}\cdot\mathbf{u}\Big)dx \mid \mathbf{u} \in W^{1,p}(\Omega,\Lambda), \;\; \mathbf{u}(\Omega) = \Lambda, \;\Big\}. \tag{6}$$

Here, $N_{\mathbf{u}}(y)$ is the cardinality of the pre–image $\mathbf{u}^{-1}\{y\}$. Endow $L^1(\Omega)$ and $W^{1,p}(\Omega,\mathbb{R}^d)$ with their respective weak topologies and endow $L^1(\Omega)\times W^{1,p}(\Omega,\mathbb{R}^d)$ with the product topology. The sublevel sets of the functional to be minimized in (4), when intersected with the set of $(\beta,\mathbf{u})$ such that $\beta \in \det^* \nabla\mathbf{u}$, are pre–compact. Since the functional to be minimized is lower semicontinuous for the topology, these facts ensure existence of a minimizer in (4).

The first condition imposed on h in (2) makes it a hard task to determine the Euler–Lagrange equations satisfied by the minimizers of (4). In contrast with the study done in [10], while the presence of f here makes it easy to readily obtain the existence of a minimizer in (4), it becomes the source of a tremendous complication when dealing with the uniqueness issue.

The second class of problems we study depends on a parameter τ and is a pertubation of (4) inspired by the so–called finite elements methods in numerical analysis. To describe these problems, we will start by introducing a family of spaces $\mathcal{S}^\tau$ and a family of operators $\nabla_{\mathcal{S}^\tau}$ (cf. Proposition 3.1), defined on an appropriate subset $\mathcal{U}_{\mathcal{S}^\tau}$ (cf. (26)) of the set of Borel maps $\mathbf{u} : \Omega \to \Lambda$. The operators $\nabla_{\mathcal{S}^\tau}$ depend on f and are defined in such a way that

$$\lim_{\tau\to 0} ||\nabla_{\mathcal{S}^\tau}\mathbf{u} - \nabla\mathbf{u}||_{L^p(\Omega)} = 0,$$

for all $\mathbf{u} \in W^{1,p}(\Omega,\mathbb{R}^d)$. Furthermore, when $p = 2$ and $f(\xi) \equiv |\xi|^2$, $\nabla_{\mathcal{S}^\tau}\mathbf{u}$ is the L^2–orthogonal projection of $\nabla\mathbf{u}$ onto $\mathcal{S}^\tau$ (cf. Remark 3.2). We consider the variational problem

$$\inf_{(\beta,\mathbf{u})}\int_\Omega \big(f(\nabla_{\mathcal{S}^\tau}\mathbf{u}) + h(\beta) - \mathbf{F}\cdot\mathbf{u}\big)dx, \tag{7}$$

where the infimum is performed over the set of pairs $(\beta,\mathbf{u})$ such that $\mathbf{u} \in \mathcal{U}_{\mathcal{S}^\tau}$ and $\beta \in \det^* \nabla\mathbf{u}$.

Observe first that unlike the analysis presented above, the sublevel sets of the functional $I_{\mathcal{S}^\tau}$ to be minimized in (7), are not known to be pre–compact for a topology for which $I_{\mathcal{S}^\tau}$ is lower semicontinuous. For instance, if $C > 0$ and $\tau > 0$ are fixed, the set $\{\mathbf{u} \in \mathcal{U}_{\mathcal{S}^\tau} \mid ||\nabla_{\mathcal{S}^\tau}\mathbf{u}||_{L^p(\Omega)} \le C\}$ is not a pre–compact set of $L^p(\Omega,\mathbb{R}^d)$ for the strong topology (cf. Remark 3.7). Regarding the uniqueness issue, there is no know metric for which the set of pairs $(\beta,\mathbf{u})$, over which we are minimizing, is strictly convex (or even convex) and so, there is no known type of convexity satisfies by the functional to be minimized in (7), which will ensure uniqueness of a minimizer. Under the assumption that $\mathbf{F}$ is non degenerate, i.e. $\mathbf{F}_\#(\chi_\Omega\mathcal{L}^d)$ is absolutely continuous with respect to Lebesgue measure, the main contribution of this manuscript is to prove the existence of a unique minimizer in (7). We completely characterize the minimizer of (7) by its Euler–Lagrange equations.

Our results are based on the discovery of a problem dual to (7), which could not have been found if one relies on the current theory in the calculus of variations.

When τ tends to 0, the infima in (7) tends to

$$\min_{\gamma\in\Gamma} \bar I(\gamma) \tag{8}$$

where

$$\bar I(\gamma) = \int_{\bar C} (f(\xi) + h(t) - \mathbf{F}(x)\cdot u)\gamma(dx, dt, du, d\xi);$$

here Γ is the set of measures on $\bar C$ where,

$$C = \Omega \times D \quad \text{and} \quad D = [0,\infty) \times \bar\Lambda \times \mathbb{R}^{d\times d},$$

satisfying

$$\int_{\bar C} f(\xi) d\gamma < \infty,$$

$$\int_{\bar C} b(x)d\gamma = \int_\Omega b dx, \qquad \int_{\bar C} tl(u)d\gamma = \int_\Lambda l dy, \qquad \int_{\bar C} \langle \xi, \psi(x)\rangle d\gamma = -\int_{\bar C} u\cdot \operatorname{div}\psi d\gamma \tag{9}$$

for all $b, l \in C_b(\mathbb{R}^d)$ and all $\psi \in C_c^\infty(\Omega, \mathbb{R}^{d\times d})$. The functions l and ψ will later play the role of Lagrange multipliers in a dual problem. When $\gamma = \delta_{(x,\beta(x),\mathbf{u}(x),\xi(x))}dx$, the constraint in (9) involving l ensures that $\beta \in \det^* \nabla\mathbf{u}$, while that involving ψ ensures that $\xi = \nabla \mathbf{u}$.

We can disintegrate any $\gamma \in \Gamma$ such that $\bar I(\gamma) < \infty$ in the following way (cf. [11] III–70) :

$$\int_{\bar C} L d\gamma = \int_\Omega dx \int_D L(u,t,x,\xi)\gamma^x(dt,du,d\xi), \quad \forall\, L \in C_c(\mathcal{R}).$$

We define the Borel maps $\beta_\gamma : \Omega \to [0,\infty)$, $\mathbf{u}_\gamma : \Omega \to \mathbb{R}^d$ and $U_\gamma : \Omega \to \mathbb{R}^{d\times d}$ by

$$\beta_\gamma(x) = \int_D t\gamma^x(dt,du,d\xi), \quad \mathbf{u}_\gamma(x) = \int_D u\gamma^x(dt,du,d\xi), \quad U_\gamma(x) = \int_D \xi\gamma^x(dt,du,d\xi). \tag{10}$$

Note that by Jensen's inequality

$$h(\beta_\gamma(x)) = h\Big(\int_D t\gamma^x(dt,du,d\xi)\Big) \le \int_D h(t)\gamma^x(dt,du,d\xi)$$

and so,

$$\int_\Omega h(\beta_\gamma(x))dx \le \int_{\bar C} h(t)\gamma(dx,dt,du,d\xi). \tag{11}$$

Similarly, by Jensen's inequality $U_\gamma \in L^p(\Omega, \mathbb{R}^{d\times d})$ and we have $\nabla\mathbf{u}_\gamma = U_\gamma$ (cf. Subsection 2.6) We prove that if γ_1 and γ_2 minimizes $\bar I$ over Γ then $\nabla\mathbf{u}_{\gamma_1} = \nabla\mathbf{u}_{\gamma_2}$.

Let $(l,k,\psi) \in \mathcal{C}\times\Sigma(\bar\Omega)$ (cf. Subsection 2.4), let k^∞ be the recession function of k. Given $\psi \in L^q(\Omega, \mathbb{R}^{d\times d})$ such that $\operatorname{div}\psi_{\mathbb{R}^d}$ is a Radon measure, let $\operatorname{div}^s\psi_{\mathbb{R}^d}$ be the singular part of $\operatorname{div}\psi_{\mathbb{R}^d}$ and set $g^s = |\operatorname{div}^s\psi_{\mathbb{R}^d}|$. We prove that if $\mathbf{u}_1 \in W^{1,p}(\Omega,\Lambda)$ satisfies

$$\mathbf{F} + \operatorname{div}^a\big(Df(\nabla\mathbf{u}_1)_{\mathbb{R}^d}\big) \in \partial k^*(\mathbf{u}_1), \quad h'(\beta_1) + l(\mathbf{u}_1) = 0 \quad \mathcal{L}^d - \text{a.e.} \tag{12}$$

and

$$\mathbf{u}_1 \in \partial k^\infty\big(\operatorname{div}^s Df(\nabla\mathbf{u}_1)_{\mathbb{R}^d}\big) \quad g^s - \text{a.e.} \tag{13}$$

then $\mathbf{u}_1$ is the unique minimizer of I over $W^{1,p}(\Omega,\Lambda)$ (cf. Theorem 6.8). Here, we have used $\psi = Df(\nabla\mathbf{u}_1)$. We conclude from the duality relation $\max -J = \inf I$ (cf. Theorem 6.4), that if every minimizer (l,k,ψ) of J such that $k = l^\sharp$ satisfies $k \in C^1(\mathbb{R}^d)$, then I has a unique minimizer over $W^{1,p}(\Omega, \bar\Lambda)$ (cf. Corollary 6.9). We don't know if there are other ways of drawing such a conclusion but using Lemma 4.4 (iii).

Set $f(\xi) \equiv \epsilon|\xi|^2/2$ and assume $h(t)$ equal to ∞ everywhere except at $t = 1$ and $h(1) = 0$. Formally, we have the following: $\mathbf{u}_1$ preserves Lebesgue measure and (12) reads off

$$\mathbf{F} = -\epsilon \triangle \mathbf{u}_1 + \nabla k^*(\mathbf{u}_1), \quad \mathcal{L}^d - \text{a.e.}.$$

When $\epsilon = 0$ we obtain the polar decomposition of $\mathbf{F}$ (cf. [2]) and when $\epsilon > 0$ we obtain a variant of that where $\mathbf{u}_1$ is differentiable.

The techniques developed in this manuscript can be applied to a class of functionals larger than those considered here. This includes functionals of the form $W(\xi) = |\xi|^2(\det \xi)^{-2/d}$, which appear in the study of Extremal mappings of finite distortion (cf. e.g. [3], [4] and the references therein).

2. Assumptions, notation and preliminaries

2.1. Main assumptions. Throughout this manuscript Ω and Λ are two convex bounded nonempty open subsets of $\mathbb{R}^d$. Without loss of generality, we assume that

$$\vec{0} \in \Lambda \quad \text{and} \quad \mathcal{L}^d(\Omega) = 1. \tag{14}$$

Let $r^* > 0$ be such that $\bar\Lambda$ is contained in the ball of radius r^*, centered at $\vec{0}$. We denote by $\nu_{\partial\Omega}$ the outward unit normal to $\partial\Omega$, which exists $\mathcal{H}^{d-1}$–a.e. Let ϱ_Λ be the Minkowski function of $\bar\Lambda$:

$$\varrho_\Lambda(u) = \inf_{t>0}\{t \mid u \in t\,\bar\Lambda\} \tag{15}$$

so that $\Lambda = \{\varrho_\Lambda < 1\}$ and $\partial\Lambda = \{\varrho_\Lambda = 1\}$. The support function of $\bar\Lambda$ is ϱ^o_Λ defined by

$$\varrho^o_\Lambda(v) = \sup_{u\in\bar\Lambda} u \cdot v.$$

We have ϱ^o_Λ is a convex function and for each $v \in \mathbb{R}^d$

$$\partial\varrho^o_\Lambda(v) = \{u \in \bar\Lambda \mid \varrho^o_\Lambda(v) = u \cdot v\} = \begin{cases} \{u \in \partial\Lambda \mid \varrho^o_\Lambda(v) = u \cdot v\} & \text{if } \; v \neq \vec{0} \\ \bar\Lambda & \text{if } \; v = \vec{0}. \end{cases} \tag{16}$$

We assume that $\mathbf{F} \in L^1(\Omega, \mathbb{R}^d)$ is a Borel vector field. Let $p, q \in (1, \infty)$ be such that $p^{-1} + q^{-1} = 1$. Let $f \in C^1(\mathbb{R}^d)$ be a strictly convex function such that there exists a constant $c > 0$ such that

$$c|\xi|^p \leq f(\xi) \leq c^{-1}(|\xi|^p + 1) \quad |Df(\xi)| \leq c^{-1}(|\xi|^{p-1} + 1) \quad \forall\, \xi \in \mathbb{R}^{d\times d}. \tag{17}$$

The Fenchel–Legendre transform of f is a strictly convex function of class C^1 which we denote by f^*. We assume the existence of a constant $\bar c > 0$ such that

$$\frac{1}{\bar c}|\xi|^q \leq f^*(\xi) \leq \frac{1}{\bar c}(|\xi|^q + 1), \quad |Df^*(\xi)| \leq \frac{1}{\bar c}(|\xi|^{q-1} + 1) \quad \forall\, \xi \in \mathbb{R}^{d\times d}. \tag{18}$$

Let $h \in C^2(0, +\infty)$ be a strictly convex function such that

$$\lim_{t\to 0^+} h(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{h(t)}{t} = \infty. \tag{19}$$

We extend h to $(-\infty, 0]$ by setting

$$h(t) = \infty \text{ if } \; t \leq 0. \tag{20}$$

Remark 2.1. The following are standard remarks.

(i) Since $h \in C^1(0, \infty)$ is strictly convex and (20) holds, $h^* \in C^1(\mathbb{R})$ and $(h^*)' > 0$.

(ii) By (19), $\lim_{s\to\infty} h^*(s)/s = \infty$ and $\lim_{s\to-\infty} h^*(s) = -\infty$.

(iii) If $h'' > 0$ then $h^* \in C^2(\mathbb{R})$ and its second derivative is positive.

2.2. **Divergence of extended functions.**

Matrices whose entries are signed measures. We denote by $\mathcal{M}(\bar\Omega, \mathbb{R}^d)$ the set of $m = (m_1, \cdots, m_d)$ such that m_i is a signed Radon measure on $\mathbb{R}^d$ supported by $\bar\Omega$. We define

$$||m||_{\mathcal{M}(\bar\Omega)} = \sup_{\mathbf{u}} \Big\{ \int_{\bar\Omega} \mathbf{u} \cdot m(dx) \mid \mathbf{u} \in C_c(\mathbb{R}^d, \mathbb{R}^d), |\mathbf{u}| \leq 1 \Big\}$$

Divergence of vector fields defined on Ω. If $\psi \in L^q(\Omega, \mathbb{R}^{d\times d})$, we denote by $\psi_{\mathbb{R}^d}$ the extension of ψ to $\mathbb{R}^d$ which is identically null on $\mathbb{R}^d \setminus \bar\Omega$. Assume there exists a constant $C > 0$ such that

$$\int_\Omega \langle \psi, \nabla \mathbf{u} \rangle dx \leq C ||\mathbf{u}||_{L^\infty(\mathbb{R}^d)} \tag{21}$$

for all $\mathbf{u} \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$. Equation (21) is equivalent to the fact that the distributional divergence of $\psi_{\mathbb{R}^d}$ is a Radon measure of finite total variation. The Riesz representation theorem (cf. e.g. [7]) provides us with a Radon measure μ on $\mathbb{R}^d$, supported by $\bar\Omega$, and a μ–measurable map $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ such that $|\sigma| = 1$ μ–a.e. and

$$\int_{\mathbb{R}^d} \sigma \cdot \mathbf{u} d\mu = \int_\Omega \langle \psi, \nabla \mathbf{u} \rangle dx \tag{22}$$

for all $\mathbf{u} \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$. Since μ is supported by $\bar\Omega$, standard approximation arguments yield (22) for $\mathbf{u} \in C^1(\bar\Omega, \mathbb{R}^d)$. We have $\sigma\mu = -\operatorname{div} \psi_{\mathbb{R}^d}$ is a vector whose components are signed Radon measures of finite masses and $||\operatorname{div} \psi_{\mathbb{R}^d}||_{\mathcal{M}(\bar\Omega)} = \mu(\mathbb{R}^d)$.

Let $\mathrm{Tr}_q : W^{1,q}(\Omega, \mathbb{R}^{d\times d}) \to L^q(\partial\Omega, \mathbb{R}^{d\times d}, \mathcal{H}^{d-1})$ be the trace operator. If $\psi \in W^{1,q}(\Omega, \mathbb{R}^{d\times d})$ then

$$\operatorname{div} \psi_{\mathbb{R}^d} = \operatorname{div} \psi \mathcal{L}^d|_\Omega - \mathrm{Tr}_q(\psi) \nu_{\partial\Omega} \mathcal{H}^{d-1}|_{\partial\Omega}. \tag{23}$$

Here, $\nu_{\partial\Omega}$ is the unit outward normal to $\partial\Omega$ and $\operatorname{div} \psi$ is the pointwise divergence of ψ in Ω.

Finite elements and Sobolev spaces. Throughout this manuscript either

$$\Sigma(\bar\Omega) = \{\psi \in L^q(\Omega, \mathbb{R}^{d\times d}) \mid \operatorname{div} \psi_{\mathbb{R}^d} \in \mathcal{M}(\bar\Omega, \mathbb{R}^d)\}, \quad \mathcal{S} = W_0^{1,q}(\Omega, \mathbb{R}^{d\times d}), \tag{24}$$

and

$$||\psi||_{\Sigma(\bar\Omega)} = ||\psi||_{L^q(\Omega)} + ||\operatorname{div} \psi_{\mathbb{R}^d}||_{\mathcal{M}(\bar\Omega)} \quad \text{for} \quad \psi \in \Sigma(\bar\Omega)$$

or $\mathcal{S} = \Sigma(\bar\Omega)$ are finite dimensional vector subspaces of $W_0^{1,\infty}(\Omega, \mathbb{R}^{d\times d})$:

$$\Sigma(\bar\Omega) \subset W_0^{1,\infty}(\Omega, \mathbb{R}^{d\times d}) \quad \text{and} \quad \dim \Sigma(\bar\Omega) < \infty. \tag{25}$$

In the latter case when (25) hold we further assume the following:

(i) there exists a finite collection of d–simplexes (closed and of nonempty interior) $\{C_r\}_{r=1}^n$, whose union contains $\bar\Omega$, and such that two distinct simplexes have disjoint interiors.

(ii) If $\psi = (\psi_{ij})_{ij} \in \Sigma(\bar\Omega)$, then the restriction of ψ_{ij} to any C_r is an affine function.

2.3. **Special displacements and weak determinants.**

A useful class of displacements. Suppose that $\Sigma(\bar\Omega) = \mathcal{S}$ are finite dimensional vector subspaces of $W_0^{1,\infty}(\Omega, \mathbb{R}^{d\times d})$, so that they are closed in $L^q(\Omega, \mathbb{R}^{d\times d})$ for the weak topology or $\mathcal{S}$ and $\Sigma(\bar\Omega)$ are given by (24).

We define

$$\mathcal{U}_\mathcal{S} = \Big\{ \mathbf{u} : \bar\Omega \to \bar\Lambda \text{ Borel map} \mid \exists C > 0 \text{ such that } \int_\Omega \mathbf{u} \cdot \operatorname{div} \psi \, dx \leq C ||\psi||_{L^q(\Omega)} \; \forall \psi \in \mathcal{S} \Big\}. \tag{26}$$

If we denote by $\mathcal{U}$ the set of Borel maps $\mathbf{u} : \bar{\Omega} \to \bar{\Lambda}$ that are in $W^{1,p}(\Omega, \mathbb{R}^d)$ then $\mathcal{U} \subset \mathcal{U}_{\mathcal{S}}$.

Using the terminology of [14], we denote the annihilator of $\mathcal{S}$ by

$$\mathcal{S}^{\perp} = \{\phi \in L^p(\Omega, \mathbb{R}^{d\times d}) \mid \int_\Omega \langle \phi, \psi \rangle = 0 \quad \forall \psi \in \mathcal{S}\}.$$

When $\dim \mathcal{S} < \infty$, $\mathcal{S}$ is closed for the strong topology and so, we have (cf. [14] Theorem 4.7),

$$\mathcal{S} = \{\psi \in L^q(\Omega, \mathbb{R}^{d\times d}) \mid \int_\Omega \langle \phi, \psi \rangle = 0 \quad \forall \phi \in \mathcal{S}^{\perp}\}. \tag{27}$$

Absolute value of determinants. Let $\mathbf{u} : \Omega \to \bar{\Lambda}$ be a measurable map and let $\beta : \bar{\Omega} \to [0, \infty]$ be a measurable function. We write

$$\beta \in \det{}^* \nabla \mathbf{u}$$

if

$$\int_\Omega \beta(x) l\big(\mathbf{u}(x)\big) dx = \int_\Lambda l dy \tag{28}$$

for all $l \in C_c(\mathbb{R}^d)$. In other words, $\mathbf{u}_{\#}(\beta \mathcal{L}^d) = \chi_\Lambda \mathcal{L}^d$. Observe that the set $\det^* \nabla \mathbf{u}$ depends on Λ and $\mathbf{u}$. Note that when the set $\det^* \nabla \mathbf{u}$ is non empty, then it is a convex subset of $L^1(\Lambda)$, contained in the sphere of radius $\mathcal{L}^d(\Lambda)$. Its intersection with any weakly compact subset of $L^1(\Omega)$ is also weakly compact. Since h is strictly convex and grows faster than linearly at ∞ we conclude that there exists a unique function, which we denote $\det^h \nabla \mathbf{u}$, which minimizes $\beta \to \int_\Omega h(\beta) dx$ over the set $\det^* \nabla \mathbf{u}$, provided that $\int_\Omega h(\beta) dx$ is not identically ∞ on $\det^* \nabla \mathbf{u}$. By the fact that h satisfies (19), the set where $\det^h \nabla \mathbf{u}$ vanishes, is of null measure.

Functionals to be minimized. Let $\mathbf{u} \in W^{1,p}(\Omega, \Lambda)$. When $\det^* \nabla \mathbf{u}$ is nonempty we set

$$I(\mathbf{u}) = \int_\Omega \Big(f(\nabla \mathbf{u}) + h(\det{}^h \nabla \mathbf{u}) - \mathbf{F} \cdot \mathbf{u} \Big) dx,$$

otherwise, we set $I(\mathbf{u}) = \infty$. Similarly, if $\mathbf{u} \in \mathcal{U}_{\mathcal{S}}$ and $\nabla_{\mathcal{S}} \mathbf{u}$ is the $\mathcal{S}$–pseudo projected gradient of $\mathbf{u}$ (cf. Proposition 3.1), we set

$$I_{\mathcal{S}}(\mathbf{u}) = \int_\Omega \Big(f(\nabla_{\mathcal{S}} \mathbf{u}) + h(\det{}^h \nabla \mathbf{u}) - \mathbf{F} \cdot \mathbf{u} \Big) dx$$

when the set of $\det^* \nabla \mathbf{u}$ is non empty, otherwise, we set $I_{\mathcal{S}}(\mathbf{u})$ to be ∞.

2.4. Dual variables and a dual functional.

Let $\mathcal{C}$ be the set of pairs (k, l) such that $k \in C(\mathbb{R}^d)$ is a Lipschitz function, $l : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is lower semicontinuous, $l \equiv \infty$ on $\mathbb{R}^d \setminus \bar{\Lambda}$ and

$$k(v) + l(u)t + h(t) \geq u \cdot v \tag{29}$$

for all $u, v \in \mathbb{R}^d$ and all $t > 0$.

The lowerscript and upperscript $\sharp$ operators. Let $k \in C(\mathbb{R}^d)$ be convex, $l : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ be lower semicontinuous such that the intersection of Λ and the effective domain of l contains an open ball of positive radius and $l \equiv \infty$ on the complement of $\bar{\Lambda}$. We define $l^\sharp$ to be the Legendre transform of $-h^*(-l)$ and we define $k_\sharp$ by the relation $h^*\big(-k_\sharp\big) = -k^*$. Note that if we set $h(1) = 0$ and

$h(t)=\infty$ for all $t \neq 1$ then $h^*(s) \equiv s$ and the lowerscript and upperscript $\sharp$ operators are nothing but the Legendre transform. We have

$$l^\sharp(v) = \sup_{t>0, u\in\bar\Lambda} \Big\{ u\cdot v - h(t) - tl(u) \mid t>0, u \in \bar\Lambda \Big\} = \sup_{t>0} t l^*\big(\frac{v}{t}\big) - h(t). \tag{30}$$

and

$$k_\sharp(u) = \sup_{t,v} \Big\{ \frac{u\cdot v - k(v) - h(t)}{t} \; t>0, v\in\mathbb{R}^d \Big\} = \sup_{t>0} \frac{k^*(u)-h(t)}{t} = -(h^*)^{-1}\big(-k^*(u)\big).$$

Therefore, $k_\sharp$ is the smallest function among the functions l such that $tl(u)+h(t) \geq k^*(u)$ for all $t>0$ and all $u \in \bar\Lambda$. In particular, if $u \in \Lambda$ then

$$k_\sharp(u) + h(1) \geq k^*(u) \geq -k(\vec{0}). \tag{31}$$

2.5. **A lower semicontinuity functional J.** Throughout this subsection we assume that $(\mathcal{S}, \Sigma(\bar\Omega))$ is given by either (24) or (25).

A dual functional. Suppose $(k,l) \in \mathcal{C}$ and $\psi \in \Sigma(\bar\Omega)$. Since l is lower semicontinuous and never assumes the value $-\infty$ on the compact set $\bar\Lambda$ while being identically ∞ outside $\bar\Lambda$, it is bounded below. The fact that k is Lipschitz ensures that the following expression is well–defined although it may be ∞ (cf. Subsection 7.1):

$$J(k,l,\psi) = \int_{\bar\Omega} \Big(f^*(\psi)dx + k\big(\operatorname{div}\psi_{\mathbb{R}^d} + \mathbf{F}\mathcal{L}^d\big)\Big) + \int_\Lambda l dy.$$

By (23) if $\psi \in W_0^{1,\infty}(\Omega, \mathbb{R}^{d\times d})$, $\operatorname{div}\psi_{\mathbb{R}^d} = \operatorname{div}\psi\mathcal{L}^d$, whereas by (107), if $\Lambda \subset \operatorname{dom} k^* \subset \bar\Lambda$ then $k^\infty = \varrho^o_{\bar\Lambda}$. By the definition of $\mathcal{C}$ if $(k,l) \in \mathcal{C}$ and $l = k_\#$ then $\operatorname{dom} l = \operatorname{dom} k^* \subset \bar\Lambda$ and by Lemma 4.6, if $\psi \in \Sigma(\bar\Omega)$ and $J(k,l,\psi)$ is finite then $\Lambda \subset \operatorname{dom} l$. In conclusion

$$\int_{\bar\Omega} k\big(\operatorname{div}\psi_{\mathbb{R}^d} + \mathbf{F}\mathcal{L}^d\big) = \begin{cases} \int_\Omega k\big(\operatorname{div}\psi + \mathbf{F}\big)dx & \text{if } \Sigma(\bar\Omega) \text{ is given } by \text{ (25)} \\ \int_{\bar\Omega} \varrho^o_{\bar\Lambda}\big(\operatorname{div}^s\psi\big) + \int_\Omega k\big(\operatorname{div}^a\psi + \mathbf{F}\big)dx & \text{if } \Sigma(\bar\Omega) \text{ is given } by \text{ (24).} \end{cases} \tag{32}$$

Lemma 2.2. *Suppose that $\psi_n \in \Sigma(\bar\Omega)$, $(k_n, l_n) \in \mathcal{C}$ are such that $k_n \in C(\mathbb{R}^d)$ and $l_n \in C(\Lambda)$ are convex, $l_n \equiv \infty$ on $\mathbb{R}^d \setminus \bar\Lambda$, and*

$$\sup_n ||\psi_n||_{\Sigma(\bar\Omega)} + ||l_n||_{L^1(\Omega)} < \infty. \tag{33}$$

Assume there exists $C>0$ such that for all integer $n \geq 1$ and all $v \in \mathbb{R}^d$

$$|k_n(v)| \leq C(|v|+1), \quad \operatorname{Lip}(k_n) \leq r^*, \quad \textit{and} \quad -C \leq \inf_\Lambda l_n. \tag{34}$$

Then

- (i) *there exist $(k,l) \in \mathcal{C}$, $\psi \in \Sigma(\bar\Omega)$ such that up to a subsequence $\{\psi_n\}_n$ converges weakly to ψ in $L^q(\Omega)$, $\{\,div\psi_{n\,\mathbb{R}^d}\}_n$ converges weak $*$ to $div\psi_{\mathbb{R}^d}$, $\{k_n\}_n$ converges locally uniformly to k on $\mathbb{R}^d$, $\{l_n\}_n$ converges locally uniformly to l on Λ.*
- (ii) *We have*
$$J(k,l,\psi) \leq \liminf_{n\to\infty} J(k_n, l_n, \psi_n).$$
- (iii) *If $k_n = l_n^\sharp$ for all n then $k = l^\sharp$.*

Proof. (ii) By (33) there exists a subsequence of $\{\psi_n\}_n$ that converges weakly to ψ in $L^q(\Omega)$ and such that $\{\operatorname{div}\psi_{n\,\mathbb{R}^d}\}_n$ converges weak $*$ to a vector value measure σ of finite total variations. One check that $\sigma = \operatorname{div}\psi_{\mathbb{R}^d}$ and $\psi \in \Sigma(\bar\Omega)$. Since $f^* \geq 0$ is convex, the theory of the calculus of variations (cf. e.g. [6]) ensures that

$$\liminf_{n\to\infty} \int_\Omega f^*(\psi_n)dx \geq \int_\Omega f^*(\psi)dx. \tag{35}$$

Since $\{l_n\}_n$ is bounded in $L^1(\Lambda)$ and l_n is convex, the theory of convex analysis [7] ensures existence of a subsequence of $\{l_n\}_n$ converges in $C_{loc}(\Lambda)$ to a convex function $\bar l \in C(\Lambda)$. Thanks to the uniform bound $-C \leq \inf_\Lambda l_n$ we may apply Fatou's Lemma to conclude that

$$\liminf_{n\to\infty} \int_\Lambda l_n dy \geq \int_\Lambda \bar l dy. \tag{36}$$

Similarly, there exists a subsequence of $\{k_n\}_n$ that converges in $C_{loc}(\mathbb{R}^d)$ to a convex function $k \in C(\mathbb{R}^d)$. We use (34) to obtain that $k_n^* \geq -k_n(\vec 0) \geq -C$. Because $(k_n, l_n) \in \mathcal{C}$, $l_n \geq k_n^* - h(1)$. Hence, $|k_n^*| \leq |h(1)| + |k_n(\vec 0)| + |l_n|$ and so, since the inequality in (34) controls $|k_n(\vec 0)|$, (33) yields $\sup_n ||k_n^*||_{L^1(\Omega)} < \infty$. By Lemma 7.6

$$\int_{\bar\Omega} k(\mathbf{F}\mathcal{L}^d + \operatorname{div}\psi_{\mathbb{R}^d}) \leq \liminf_{n\to\infty} \int_{\bar\Omega} k_n(\mathbf{F}\mathcal{L}^d + \operatorname{div}\psi_{n\,\mathbb{R}^d}). \tag{37}$$

We combine (35), (36) and (37) to conclude the proof of (ii).

(i) To conclude the proof of (i) it remains to show that $\bar l$ admits an extension l to $\mathbb{R}^d$ such that $(k, l) \in \mathcal{C}$. To achieve that goal, we define

$$l(u) = \inf_{\{w_n\}_n} \Big\{ \liminf_n l_n(w_n) \mid \{w_n\}_n \quad \text{converges to} \quad u \Big\} \quad \text{for} \quad u \in \bar\Lambda.$$

We have that l is lower semicontinuous and $l = \bar l$ on Λ. Since $(k_n, l_n) \in \mathcal{C}$ so does (k, l).

(iii) Suppose $k_n = l_n^\sharp$ for all n and fix $v \in \mathbb{R}^d$. We are to prove that $k(v) = l^\sharp(v)$. By (33) and the convexity property of l_n there exists a constant $\tilde C > C$ such that $|l_n(\vec 0)| \leq \tilde C$. Choose $t_n > 0$, $u_n \in \bar\Lambda$ such that $u_n \cdot v - t_n l_n(u_n) - h(t_n) = k_n(v)$. We have $-\tilde C \leq k_n(v) \leq u_n \cdot v - h(t_n) + \tilde C t_n$ and so, $-\tilde C + h(t_n) - \tilde C t_n \leq u_n \cdot v \leq r^*|v|$. Hence $\{t_n\}_n$ is contained in a compact subset on $(0, \infty)$ and so, we may assume without loss of generality that it converges to a point $t \in (0, \infty)$. As a consequence $\{l_n(u_n)\}_n$ is bounded and so, $\lim_n (t_n - t) l_n(u_n) = 0$. Similarly, we may assume that $\{u_n\}_n$ converges to some $u \in \bar\Lambda$. We have

$$k(v) = \lim_n k_n(v) = \limsup_n u_n \cdot v - t_n l_n(u_n) - h(t_n) = \limsup_n u_n \cdot v - t l_n(u_n) - h(t_n).$$

Thus, $k(v) \leq u \cdot v - t l(u) - h(t) \leq l^\sharp(v)$. Since $(k, l) \in \mathcal{C}$, $k \geq l^\sharp$ and so, $k(v) = l^\sharp(v)$. □

2.6. **Measures reminiscent to Young's measures.**

Enlarging the set of $\mathcal{U}_\mathcal{S}$; *a Young measure approach.* Let γ be a Radon measure on $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d}$ supported by

$$C = \Omega \times (0, \infty) \times \bar\Lambda \times \mathbb{R}^{d\times d}$$

and that satisfy the conditions

$$\int_C b(x)\gamma(dx, dt, du, d\xi) = \int_\Omega b\, dx \quad \forall\, b \in C_b(\mathbb{R}^d) \tag{38}$$

and

$$\int_C f(\xi)\gamma(dx,dt,du,d\xi) < \infty. \tag{39}$$

Since the Lebesgue measure of Ω has been normalized to 1 (cf. (14)), (38) ensures that not only γ is a probability measure, but we can apply the disintegration theorem to γ (cf. [11] III–70): we obtain a Borel map $x \to \gamma^x$ of Ω into the set of Borel probability measures on

$$D = (0,\infty) \times \bar{\Lambda} \times \mathbb{R}^{d\times d}$$

such that

$$\int_C L(x,t,u,\xi)\gamma(dx,dt,du,d\xi) = \int_\Omega dx \int_D L(x,t,u,\xi)\gamma^x(dt,du,d\xi)$$

for all $L \in C_c(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d})$. Since f is convex and (39) holds, by Jensen's inequality

$$\int_\Omega f(U_\gamma(x))dx \leq \int_\Omega \Big(\int_D f(\xi)\gamma^x(dt,du,d\xi)\Big)dx = \int_C f(\xi)\gamma(dx,dt,du,d\xi) < \infty,$$

where

$$U_\gamma(x) = \int_D \xi\gamma^x(dt,du,d\xi). \tag{40}$$

Thus the growth condition (17) on f implies that $U_\gamma \in L^p(\Omega,\mathbb{R}^{d\times d})$. Similarly, the fact that the support of γ in the u variables is contained in the convex set $\bar{\Lambda}$ yields that the function defined by

$$\mathbf{u}_\gamma(x) = \int_D u\gamma^x(dt,du,d\xi) \tag{41}$$

maps Ω into $\bar{\Lambda}$ up to a set of zero measure.

We define Γ to be the set of Radon measures on $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d}$ supported by C such that not only (38–39) hold, but

$$\int_C tl(u)\gamma(dx,dt,du,d\xi) = \int_\Lambda l dy \quad \forall\, l \in C_b(\mathbb{R}^d) \tag{42}$$

and

$$\int_C \langle \xi, \psi(x)\rangle \gamma(dx,dt,du,d\xi) = -\int_\Omega \mathbf{u}_\gamma \cdot \operatorname{div}\psi dx \quad \forall\, \psi \in C_c^\infty(\Omega,\mathbb{R}^{d\times d}). \tag{43}$$

By (43), $\mathbf{u}_\gamma \in W^{1,p}(\Omega,\mathbb{R}^d)$ and $\nabla\mathbf{u}_\gamma \equiv U_\gamma$. We have a natural "embedding"

$$\{(\beta,\mathbf{u}) \mid \mathbf{u} \in W^{1,p}(\Omega,\Lambda), \mathbf{u}_\#\beta = \chi_\Lambda\} \subset \Gamma \tag{44}$$

which to $(\beta,\mathbf{u})$ associates $\gamma \equiv \gamma^{(\beta,\mathbf{u})}$ defined for $L \in C_c(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d})$ by

$$\int_C L(x,t,u,\xi)\gamma(dx,dt,du,d\xi) = \int_\Omega L(x,\beta(x),\mathbf{u}(x),\nabla\mathbf{u}(x))dx.$$

We extend I to Γ to obtain the function

$$\bar{I}(\gamma) = \int_C \big(f(\xi) + h(t) - \mathbf{F}(x)\cdot u\big)\gamma(dx,dt,du,d\xi).$$

3. A pseudo–projection and a weak deternimant.

In this section, we consider a triangulation of Ω into d–simplexes. This provides us with a way of defining a pseudo–projection of gradient of vector fields which are not Sobolev maps. Throughout this section, $\mathcal{S} = \Sigma(\bar{\Omega})$ are the finite dimensional spaces as introduced in section 2.2 and so, they are closed in $L^q(\Omega,\mathbb{R}^{d\times d})$ for the weak topology.

3.1. **Pseudo–projected gradients.**

Proposition 3.1. *Given $\mathbf{u} \in \mathcal{U}_{\mathcal{S}}$, there exists a unique $G_0 \in L^p(\Omega, \mathbb{R}^{d\times d})$ such that*

$$\int_\Omega \mathbf{u}\, div\psi\, dx = -\int_\Omega \langle G_0, \psi\rangle dx \qquad \forall \psi \in \mathcal{S} \tag{45}$$

and $\psi_0 = Df(G_0) \in \mathcal{S}$. We write $G_0 = \nabla_{\mathcal{S}}\mathbf{u}$ and refer to it as the pseudo–projected gradient of $\mathbf{u}$ onto $\mathcal{S}$. It is the unique minimizer of $\int_\Omega f(G)dx$ over the set of $G \in L^p(\Omega, \mathbb{R}^{d\times d})$ satisfying (45).

Proof. 1. If $\mathbf{u} \in \mathcal{U}_{\mathcal{S}}$, then $\psi \to \int_\Omega \mathbf{u}\, \mathrm{div}\, \psi\, dx$ is a linear bounded operator on $\mathcal{S}$ for the $||\cdot||_{L^q}$–norm and so, it admits an extension L on $L^q(\Omega, \mathbb{R}^{d\times d})$ which has the same norm. By the Riesz representation theorem, the set $\mathcal{G}$ of all $G \in L^p(\Omega, \mathbb{R}^{d\times d})$ such that

$$L(\psi) = -\int_\Omega \langle G, \psi\rangle dx \tag{46}$$

for all $\psi \in \mathcal{S}$, is nonempty. Let $G_1 \in \mathcal{G}$. Observe that $\mathcal{S}^\perp$ is weakly closed. The growth condition (17) on f ensures that

$$\Big\{\phi \in \mathcal{S}^\perp \;\Big|\; \int_\Omega f(G_1 - \phi)dx \le C\Big\}$$

is weakly compact in $L^p(\Omega, \mathbb{R}^{d\times d})$ for every $C \in \mathbb{R}$. Since f is strictly convex, the theory of the direct methods of the calculus of variations (cf. e.g. [6]) ensures existence of a unique ϕ_0 which minimizes $\int_\Omega f(G_1 - \phi)dx$ over $\mathcal{S}^\perp$. Note that,

$$\mathcal{G} = G_1 + \mathcal{S}^\perp$$

and so, setting $G_0 = G_1 - \phi_0$, we observe that $G_0 \in \mathcal{G}$. Using the fact that L coincides with $\psi \to \int_\Omega \mathbf{u} \cdot \mathrm{div}\, \psi\, dx$ on $\mathcal{S}$, we infer that (45) holds.

If $\phi \in \mathcal{S}^\perp$ is arbitrary, $t \to \int_\Omega f(G_0 + t\phi)dx$ attains its minimum at $t = 0$ and so, its derivative is null at 0 :

$$0 = \int_\Omega \langle Df(G_0), \phi\rangle dx.$$

This, together with (27) yields $\psi_0 := Df(G_0) \in \mathcal{S}$, if we also use the second inequality in (18) and the fact that $G_0 \in L^p(\Omega, \mathbb{R}^{d\times d})$.

2. Suppose $G \in L^p(\Omega, \mathbb{R}^{d\times d})$ satisfies (45). Then $G - G_0 \in \mathcal{S}^\perp$. Since f is strictly convex, if $G \neq G_0$ then

$$\int_\Omega f(G)dx > \int_\Omega f(G_0)dx + \int_\Omega \langle G - G_0, \psi_0\rangle dx = \int_\Omega f(G_0)dx.$$

Thus, G_0 is uniquely determined and satisfies the required minimality property. □

Remark 3.2. It is apparent from the proof of Proposition 3.1 that if $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^d)$ then

(i) the following decomposition holds:

$$\nabla\mathbf{u} = \nabla_{\mathcal{S}}\mathbf{u} + \phi_0 \quad \text{where} \quad \phi_0 \in \mathcal{S}^\perp.$$

(ii) Unless $\nabla\mathbf{u} = \nabla_{\mathcal{S}}\mathbf{u}$,

$$\int_\Omega f(\nabla\mathbf{u})dx > \int_\Omega f(\nabla_{\mathcal{S}}\mathbf{u})dx.$$

(iii) When $f(\xi) \equiv |\xi|^2$ then $\nabla_{\mathcal{S}}\mathbf{u}$ is nothing but the orthogonal projection of $\nabla\mathbf{u}$ onto $\mathcal{S}$.

3.2. **Finite elements and approximation of gradients by pseudo–projected gradients.** Consider $\tau > 0$. By a triangulation of $\bar\Omega$ we mean that $\bar\Omega$ is covered by a finite number of subsets $\{C_r\}_{r=1}^{n(\tau)}$, called finite elements, whose union contains $\bar\Omega$ (possibly strictly). These finite elements are such that:

(i) each C_r is a d–simplex, i.e. the convexhull of $d+1$ points which are not contained in a hyperplane.

(ii) Any face of any d–simplex C_r either does not intersect $\bar\Omega$ or is a subset of the boundary $\partial\Omega$ or is a face of another d–simplex $C_r \neq C_{\bar r}$ with $r \neq \bar r$.

(iii) If $r \neq \bar r$ then the interior of C_r and $C_{\bar r}$ don't intersect.

(iv) Let h_r be the diameter of C_r and let ρ_r be the supremum of the diameters of the spheres inscribed in C_r. We assume that

$$\sup_{r=1,\cdots,n(\tau)} h_r^2\big(1+\rho_r^{-1}\big) = 0(\tau). \tag{47}$$

We consider X_τ the set of $\psi = (\psi)_{ij} \in W^{1,\infty}(\Omega, \mathbb{R}^{d\times d})$ such that the restriction of ψ_{ij} to any finite element C_r is an affine function. We define

$$\mathcal{S}^\tau = X_\tau \cap W_0^{1,\infty}(\Omega, \mathbb{R}^{d\times d})$$

$$P_{C_r} = \{\psi|_{C_r} \mid \psi \in X_\tau\}.$$

Let $\mathcal{T} \subset (0,\infty)$ be a set which has 0 as a point of accumulation and assume that $\tau < t$ implies

$$\mathcal{S}^t \subset \mathcal{S}^\tau. \tag{48}$$

For every $\phi \in W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$ there exists $\phi^t \in \mathcal{S}^t$ such that

$$\lim_{t\to 0} ||\phi - \phi^t||_{W^{1,q}(\Omega)} = 0. \tag{49}$$

Theorem 3.3. *If* $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^d)$ *then*

$$\lim_{\tau\to 0} ||\nabla\mathbf{u} - \nabla_{\mathcal{S}^\tau}\mathbf{u}||_{L^p(\Omega)} = 0.$$

Proof. It suffices to show that every subsequence of $\{\nabla_{\mathcal{S}^\tau}\mathbf{u}\}_\tau$ admits itself a subsequence which converges to $\nabla\mathbf{u}$ in $L^p(\Omega)$.

Since f satisfies the upper bound (17), Remark 3.2 (ii) implies that $\{\nabla^\tau\mathbf{u}\}_\tau$ is bounded in $L^p(\Omega)$ and so, up to a subsequence, it converges weakly to some $G \in L^p(\Omega)$. Fix $t \in \mathcal{T}$ and $\phi^t \in \mathcal{S}^t$. If $\tau \in \mathcal{T}$ is such that $\tau < t$, (48) implies

$$\int_\Omega \mathbf{u}\cdot\, \mathrm{div}\, \phi^t_{\mathbb{R}^d} dx = -\int_\Omega \langle \nabla_{\mathcal{S}^\tau}\mathbf{u}, \phi^t\rangle dx.$$

Thus, letting τ tend to 0 (up to a subsequence), we have

$$\int_\Omega \mathbf{u}\cdot\, \mathrm{div}\, \phi^t_{\mathbb{R}^d} dx = -\int_\Omega \langle G, \phi^t\rangle dx. \tag{50}$$

Fix an arbitrary $\phi \in C_0^\infty(\Omega, \mathbb{R}^{d\times d})$. Then select $\phi^t \in \mathcal{S}^t$ such that (49) holds and let t tend to 0 in (50) to conclude that

$$\int_\Omega \mathbf{u}\cdot\, \mathrm{div}\, \phi dx = -\int_\Omega \langle G, \phi\rangle dx.$$

Consequently, $G = \nabla\mathbf{u}$. The limit being independent of the subsequence of $\{\nabla_{\mathcal{S}^\tau}\mathbf{u}\}_\tau$ used, and the weak topology being metrizable, this proves that $\{\nabla_{\mathcal{S}^\tau}\mathbf{u}\}_\tau$ converges weakly to $\nabla\mathbf{u}$.

Since f is convex and bounded below, $G \to \int_\Omega f(G)dx$ is weakly lower semicontinuous on $L^p(\Omega, \mathbb{R}^{d\times d})$ (cf. e.g. [6]) and so,

$$\int_\Omega f(\nabla \mathbf{u})dx \le \liminf_{\tau\to 0} \int_\Omega f(\nabla_{\mathcal{S}^\tau} \mathbf{u})dx \le \limsup_{\tau\to 0} \int_\Omega f(\nabla_{\mathcal{S}^\tau} \mathbf{u})dx \le \int_\Omega f(\nabla \mathbf{u})dx.$$

Hence,

$$\int_\Omega f(\nabla \mathbf{u})dx = \lim_{\tau\to 0} \int_\Omega f(\nabla^\tau \mathbf{u})dx.$$

We use that f is strictly convex to infer that $\{\nabla_{\mathcal{S}^\tau} \mathbf{u}\}_\tau$ converges to $\nabla \mathbf{u}$ in $L^p(\Omega)$. □

3.3. **Properties of the absolute values of determinants in the weak sense.** Given $\mathbf{u} : \Omega \to \mathbb{R}^d$ and $y \in \mathbb{R}^d$, we denote by $N_{\mathbf{u}}(y)$ the cardinality of $\mathbf{u}^{-1}\{y\}$.

Remark 3.4. Suppose that $\mathbf{u} : \Omega \to \bar\Lambda$ and $\beta \in \det^* \nabla \mathbf{u}$. Then

- (i) If $C \subset \mathbb{R}^d$ is a compact set, then (28) holds for $l = \chi_C$. Furthermore, if $\beta > 0$, then $\mathbf{u}$ is nondegenerate in the sense that $\mathbf{u}_\#(\chi_\Omega \mathcal{L}^d)$ is absolutely continuous. As a consequence, (28) holds for $l = \chi_B$ when $B \subset \mathbb{R}^d$ is a bounded $\mathcal{L}^d$–measurable set.
- (ii) Suppose that $O \subset \Omega$ is such that $\mathcal{L}^d(\Omega \setminus O) = 0$. Then $\mathcal{L}^d(\mathbf{u}(O)) = \mathcal{L}^d(\bar\Lambda)$. As a consequence, if $\mathbf{u}(O)$ is $\mathcal{L}^d$–measurable, then $\mathcal{L}^d(\bar\Lambda \setminus \mathbf{u}(O)) = 0$.

Lemma 3.5. *Suppose that $r \in (d, \infty)$, $\mathbf{u} \in W^{1,r}(\Omega, \Lambda)$, Λ is the range of $\mathbf{u}$ and assume that $Z_{\mathbf{u}}$, the set of x such that $\det \nabla \mathbf{u}(x) = 0$ is of null Lebesgue measure. A Borel function $\beta : \Omega \to (0, \infty)$ belongs to $\det^* \nabla \mathbf{u}$ if and only if for almost every $y \in \Lambda$*

$$\sum_{x\in \mathbf{u}^{-1}(y)} \frac{\beta(x)}{|\det \nabla \mathbf{u}(x)|} = 1. \tag{51}$$

In particular, $|\det \nabla \mathbf{u}|/N_{\mathbf{u}}(\mathbf{u}) \in \det^* \nabla \mathbf{u}$.

Proof. By Sard's Theorem for Sobolev functions (cf. e.g. [8]) $\mathbf{u}(Z_{\mathbf{u}})$ is a set of null Lebesgue measure. If $l \in C(\mathbb{R}^d)$ and $\beta : \Omega \to (0, \infty)$ is a Borel function, by the area formula

$$\int_\Omega l(\mathbf{u}(x))\beta(x)dx = \int_\Omega l(\mathbf{u}(x)) \frac{\beta(x)}{|\det \nabla \mathbf{u}(x)|} |\det \nabla \mathbf{u}(x)|dx = \int_\Lambda l(y)\Big(\sum_{x\in \mathbf{u}^{-1}(y)} \frac{\beta(x)}{|\det \nabla \mathbf{u}(x)|}\Big)dy.$$

This proves the remark. □

Remark 3.6. Let $r > d$, suppose that $\mathbf{u} \in W^{1,r}(\Omega, \Lambda)$ is a map of $\bar\Omega$ into $\bar\Lambda$ such that $\mathbf{u}(\Omega)$ has full measure in Λ. Modifying $N_{\mathbf{u}}$ on a set of zero measure if necessary, we may assume that $N_{\mathbf{u}}$ is a Borel map. Without no longer requiring as in Lemma 3.5 that $Z_{\mathbf{u}}$ is of null measure, we still have $|\det \nabla \mathbf{u}|/N_{\mathbf{u}}(\mathbf{u}) \in \det^* \nabla \mathbf{u}$.

Proof. Observe that $\mathbf{u}(\Omega)$ is measurable, $\det \nabla \mathbf{u} \in L^1(\Omega)$, $N_{\mathbf{u}} \in L^1(\mathbb{R}^d)$ and so, since $\mathbf{u}(\Omega) \subset \Lambda$ we have

$$\int_\Omega |\det \nabla \mathbf{u}(x)| \bar l(\mathbf{u}(x))dx = \int_\Lambda \bar l(y) N_{\mathbf{u}}(y)dy$$

for all Borel function $\bar l \in L^\infty(\mathbb{R}^d)$ (cf. e.g. [8]). Hence for $l \in C_c(\mathbb{R}^d)$, since $N_{\mathbf{u}} \ge 1$ on Λ we have

$$\int_\Omega |\det \nabla \mathbf{u}| \frac{l}{N_{\mathbf{u}}}(\mathbf{u})dx = \int_\Lambda N_{\mathbf{u}} \frac{l}{N_{\mathbf{u}}} dy = \int_\Lambda l dy.$$

□

3.4. **An example: lack of compactness of $\{\mathbf{u}_\tau\}_{\tau>0}$ and a computation of $\det^h \nabla \mathbf{u}$.** define $w : (0,1) \to (-1,1)$ by

$$w(x) = \begin{cases} 4x & \text{if } 0 < x < \frac{1}{4} \\ -4x+2 & \text{if } \frac{1}{4} \le x \le \frac{3}{4} \\ 4x-4 & \text{if } \frac{3}{4} < x < 1. \end{cases} \tag{52}$$

Extend u periodically to $\mathbb{R}$. Set $u_\tau(x) = w(x/\tau - [x/\tau])$ where $[\cdot]$ denotes the greatest integer function and $\tau = 2^{-n}$ where n is an arbitrary nonnegative integer.

Remark 3.7. A control of the L^p norm of the pseudo–projected gradients of a collection of functions $\{u_\tau\}_\tau$, does not ensure any strong compactness property on $\{u_\tau\}_\tau$ in $L^p(0,1)$. Set $\tau = 2^{-n}$. Let $t = 2^{-m}$ where $m \le n$ and let $\mathcal{S}^t$ be the set of $\psi \in W^{1,\infty}_0(0,1)$ whose restriction to each interval $\big(it, (i+1)t\big)$ is affine, for $i = 0, \cdots, 2^m - 1$. We have

$$\int_0^1 u_\tau(x)\psi'(x) = \tau \int_0^1 w(x)dx \sum_{i=0}^{2^n-1} a_i = \int_0^1 w(x)dx \int_0^1 \psi'(x)dx = 0\big((\psi(1)-\psi(0)\big) = 0.$$

Thus, $\nabla_{\mathcal{S}^\tau} u_\tau = 0$ and so, $\nabla_{\mathcal{S}^t} u_\tau = 0$. We use again the fact that $\int_0^1 w(x)dx = 0$ to conclude that $\{u_\tau\}_\tau$ converges weakly $*$ in $L^\infty(0,1)$ to 0. But,

$$\int_0^1 |u_\tau|^p dx = \frac{1}{2}\int_{-1}^1 |x|^p dx = \frac{1}{p+1}.$$

This proves that $\{u_\tau\}_\tau$ in not pre–compact in $L^p(0,1)$ whenever $p \in [1,\infty)$.

Remark 3.8. If n is a nonnegative integer and $\tau = 2^{-n}$, then $\det^h \nabla \mathbf{u}_\tau = \beta_0$, where $\beta_0 \equiv 2$.

Proof. Observe that if $y \in (-1,1) \setminus \{0\}$ then $\mathbf{u}^{-1}\{y\}$ is a set of cardinality 2 and so, $\mathbf{u}_\tau^{-1}\{y\}$ is a set of cardinality $2 \cdot 2^n$. If $x \in \mathbf{u}^{-1}\{y\}$, $|\mathbf{u}'(x)| = 4$ and so, $|\mathbf{u}_\tau'(x)| = 4 \cdot 2^n$. We apply Remark 3.6 to conclude that $\beta_0 \in \det^* \nabla \mathbf{u}_\tau$.

It remains to show that β_0 is the element which minimizes $\int_0^1 h(\beta)dx$ over $\det^* \nabla \mathbf{u}_\tau$. Let $\beta \in \det^* \nabla \mathbf{u}_\tau$. We first use the fact that $|\mathbf{u}_\tau'(x)| = 4 \cdot 2^n$, we second use the co–area formula, we third use the fact that h is convex and finally use (51) to conclude that

$$\int_0^1 h(\beta)dx = \int_0^1 h(\beta)\frac{|\mathbf{u}_\tau'|}{2^{n+2}}dx = \frac{1}{2}\int_{-1}^1 \sum_{x\in \mathbf{u}^{-1}(y)} \frac{h(\beta(x))}{2^{n+1}}dy \ge \frac{1}{2}\int_{-1}^1 h\Big(\sum_{x\in \mathbf{u}^{-1}(y)} \frac{\beta(x)}{2^{n+1}}\Big)dy = \int_0^1 h(\beta_0)dx.$$

□

4. General estimates.

For $s \in \mathbb{R}$ we define

$$\lambda(s) = \mathcal{L}^d(\Lambda)s - \mathcal{L}^d(\Omega)h^*(s) + ||\mathbf{F}||_{L^1(\Omega)} r^*.$$

For $c \in \mathbb{R}$, we set

$$\lambda_c^+ = \sup_{s\in\mathbb{R}}\{s \mid \lambda(s) \ge -c\}, \qquad \lambda_c^- = \inf_{s\in\mathbb{R}}\{s \mid \lambda(s) \ge -c\}$$

By Remark 2.1 (ii) $\lim_{s\to\infty}\lambda(s) = -\infty$ and so, $\lambda_c^+ < \infty$. If t is a positive real number such that $t\mathcal{L}^d(\Omega) < \mathcal{L}^d(\Lambda)$, we use Young's inequality: $h^*(s) \ge -h(t) + ts$ to obtain

$$\limsup_{s\to-\infty}\lambda(s) \le \limsup_{s\to-\infty}\mathcal{L}^d(\Omega)h(t) + \Big(\mathcal{L}^d(\Lambda) - t\mathcal{L}^d(\Omega)\Big)s + ||\mathbf{F}||_{L^1(\Omega)} r^* = -\infty.$$

This proves that $\lambda_c^- > -\infty$.

4.1. **A special subset of $\mathcal{C}$.** We consider a special subset of $\mathcal{C}$, which we denote by $\partial\mathcal{C}_=$ and which consists of the pairs $(k,l) \in \mathcal{C}$ such that $k_\sharp = l$ and $l^\sharp = k$. We prove in Lemma 4.3 that if $k = l^\sharp$ then for every $v \in \mathbb{R}^d$ there exists $\bar u \in \Lambda \times \mathbb{R}^d$ and $\bar t \in (0,\infty)$ such that

$$k(v) + \bar t l(\bar u) + h(\bar t) = v \cdot \bar u.$$

The following Lemma and its proof can be found in [10].

Lemma 4.1. *For every Lipschitz function $k : \mathbb{R}^d \to \mathbb{R}$, $\big((k_\sharp)^\sharp\big)_\sharp = k_\sharp$ and so, if we set $\bar k = (k_\sharp)^\sharp$ and $\bar l = k_\sharp$ then $\bar k_\sharp = \bar l$ and $\bar l^\sharp = \bar k$.*

Remark 4.2. Assume $k \in C(\mathbb{R}^d)$ is a Lipschiz, $l : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is lower semicontinuous and $k(v) + tl(u) + h(t) \geq u \cdot v$ for all $(t,u,v) \in (0,\infty) \times \Lambda \times \mathbb{R}^d$. Assume $\psi \in \Sigma(\bar\Omega)$. The following hold:

(i) if $u \in \Lambda$ then
$$l(u) \geq |u| - k(\frac{u}{|u|}) - h(1).$$
In particular, l is bounded below on Λ. We use the fact that $\lim_{t\to\infty} h(t)/t = \infty$ to conclude that $l^\sharp$ assumes only finite values. Furthermore, $l^\sharp$ is convex as a supremum of aftine functions. The supremum in (30) being performed over the convex set $\bar\Lambda$, we obtain that the subdifferential of $l^\sharp$ is contained in $\bar\Lambda$ and so, $l^\sharp$ is r^*–Lipschitz. Similarly, $k_\sharp$ is convex and lower semicontinuous.

(ii) $(k, k_\sharp, \psi) \in \mathcal{C} \times \Sigma(\bar\Omega)$ and $(l^\sharp, l, \psi) \in \mathcal{C} \times \Sigma(\bar\Omega)$.

(iii) If $(l, \psi, k) \in \mathcal{C} \times \Sigma(\bar\Omega)$, then $l \geq k_\sharp$ and $k \geq l^\sharp$. Hence,
$$J(k,l,\psi) \geq J\big((k_\sharp)^\sharp, k_\sharp, \psi\big),\ J\big(k, k_\sharp, \psi\big).$$

(iv) If $J(k,l,\psi)$ is finite then so is $J\big((k_\sharp)^\sharp, k_\sharp, \psi\big)$ and so, the effective domain of $k_\sharp$ contains Λ.

Lemma 4.3. *Let $l : \mathbb{R}^d \to (-\infty,\infty]$ be a lower semicontinuous such that $l \equiv \infty$ on $\mathbb{R}^d \setminus \Lambda$, let $k : \mathbb{R}^d \to \mathbb{R}$ and let $v \in \mathbb{R}^d$. Then the following hold:*

(i) *the expression in (30) is a maximum.*

(ii) *$(\bar t, \bar u) \in (0,\infty) \times \bar\Lambda$ maximizes the expression in (30) if and only if*
$$h'(\bar t) + l(\bar u) = 0 \quad \textit{and} \quad \bar u \in \partial l^*(\frac{v}{\bar t}).$$

(iii) *If (ii) holds and $l^\sharp$ is differentiable at v, then $\nabla l^\sharp(v) = \bar u$ and so, $(\bar t, \bar u)$ is uniquely determined.*

(iv) *Suppose $\bar u \in \bar\Lambda$, $l(\bar u) = k_\sharp(\bar u)$, $\bar t > 0$ and $v \in \mathbb{R}^d$. Then $k(v) + \bar t l(\bar u) + h(\bar t) = \bar u \cdot v$ if and only if*
$$h'(\bar t) + l(\bar u) = 0 \quad \textit{and} \quad v \in \partial k^*(\bar u).$$

Proof. (i) The fact that the expression in (30) is a maximum is a consequence of the fact that Λ is bounded, l is lower semicontinuous and h satisfies (19).

(ii) Suppose $k(v) = l^\sharp(v)$ and $(\bar t, \bar u) \in (0,\infty) \times \bar\Lambda$ satisfies the system of equations in (ii). If $(t,u) \in (0,\infty) \times \bar\Lambda$ then

$$u \cdot v - tl(u) - h(t) = t(u \cdot \frac{v}{t} - l(u)) - h(t) \leq tl^*(\frac{v}{t}) - h(t) = \bar u \cdot v - tl(\bar u) - h(t).$$

But, by (19), $t \to tl(\bar u) + h(t)$ admits a minimizer at a point s such that $l(\bar u) + h'(s) = 0$. By Remark 2.1 (i) $s = \bar t$. We conclude that $(\bar t, \bar u)$ maximizes the expression in (30).

Conversely, suppose that $(\bar t, \bar u) \in (0,\infty) \times \bar\Lambda$ maximizes the expression in (30). Then, the derivative of $\bar u \cdot v - tl(\bar u) - h(t)$ with respect to t vanishes at $\bar t$ and so, the first identity in (ii) holds. Since $u \cdot v - \bar t l(u) - h(\bar t)$ attains its maximum on $\bar\Lambda$ at $\bar u$, so does $u \cdot (v/\bar t) - l(u)$. We use the fact that $l \equiv \infty$ on $\mathbb{R}^d \setminus \bar{\Lambda}$ to conclude that $u \cdot (v/\bar{t}) - l(u)$ attains its maximum on $\mathbb{R}^d$ at $\bar{u}$. Thus, the second statement in (ii) is satisfied.

(iii) In case $l^\sharp$ is differentiable at v, $w \to l^\sharp(w) - u \cdot w + \bar{t} l(\bar{u}) + h(\bar{t})$ attains its minimum at v and so, its gradient at v vanishes, i.e. $\nabla l^\sharp(v) = \bar{u}$.

(iv) Suppose $\bar{u} \in \bar{\Lambda}$ and $l(\bar{u}) = k_\sharp(\bar{u})$. Note that

$$k(w) + t k_\sharp(u) + h(t) \geq u \cdot w \ \ \forall\, (u, w, t) \quad \implies \quad t k_\sharp(u) + h(t) \geq k^*(u) \ \ \forall (u, t). \tag{53}$$

Suppose $k(v) + \bar{t} l(\bar{u}) + h(\bar{t}) = \bar{u} \cdot v$. We read off the first inequality in (30) that $k(w) + t k_\sharp(u) + h(t) - u \cdot w$ achieves its minimum at $(\bar{t}, \bar{u}, v)$ and so, its partial derivative with respect to t vanishes at $(\bar{t}, \bar{u}, v) : h'(\bar{t}) + l(\bar{u}) = 0$. Assume on the contrary that $k(v) + k^*(\bar{u}) > \bar{u} \cdot v$. Then,

$$k(v) + \bar{t} k_\sharp(\bar{u}) + h(\bar{t}) = k(v) + \bar{t} l(\bar{u}) + h(\bar{t}) = \bar{u} \cdot v < k(v) + k^*(\bar{u})$$

and so, simplifying by $k(v)$ we obtain

$$\bar{t} k_\sharp(\bar{u}) + h(\bar{t}) < k^*(\bar{u}),$$

which is at variance with the last assertion in (53). Hence, $k(v) + k^*(\bar{u}) = \bar{u} \cdot v$

Conversely, assume that

$$k(v) + k^*(\bar{u}) = \bar{u} \cdot v \quad \text{and} \quad h'(\bar{t}) + l(\bar{u}) = 0. \tag{54}$$

The fact that h satisfies (19) ensures that the last supremum in (30) is maximized at a point $t_0 > 0$ and the t–derivative of the functional maximized vanishes at t_0 :

$$l(\bar{u}) = k_\sharp(\bar{u}) = \frac{k^*(\bar{u}) - h(t_0)}{t_0}, \quad \frac{-t_0 h'(t_0) - (k^*(\bar{u}) - h(t_0))}{t_0^2} = 0. \tag{55}$$

The first equations in respectively (54) and (55) yield

$$k(v) + t_0 k_\sharp(\bar{u}) + h(t_0) = \bar{u} \cdot v, \tag{56}$$

while the two equations in (55) yield

$$h'(t_0) + l(\bar{u}) = 0. \tag{57}$$

Since h' is monotone increasing, the second assertion in (54) and (57) imply that $t_0 = \bar{t}$. We exploit that fact in (56) to conclude the proof of (iv). □

Lemma 4.4. *Let $(k_0, l_0) \in \mathcal{C}$ be such that $k_0 = l_0^\sharp$ and let $l \in C(\mathbb{R}^d)$. For $|\epsilon| < 1$, define $l_\epsilon = l_0 + \epsilon l$ and $k_\epsilon = l_\epsilon^\sharp$. Then*

(i)

$$\sup_{v,\epsilon} \Big\{ \frac{|k_\epsilon(v) - k_0(v)|}{|\epsilon|(|v| + 1)} \mid v \in \mathbb{R}^d, 0 < |\epsilon| \leq 1 \Big\} < \infty.$$

(ii) *Whenever k_0 is differentiable at v so that we can define $T_0(v)$ by $h'\big(T_0(v)\big) + l_0\big(\nabla k_0(v)\big)$,*

$$\lim_{\epsilon \to 0} \frac{k_\epsilon(v) - k_0(v)}{\epsilon} = -T_0(v) l\big(\nabla k_0(v)\big). \tag{58}$$

(iii) *If $\mathbf{b} \in L^1(\Omega, \mathbb{R}^d)$ is such that k_0 is differentiable at $\mathbf{b}(x)$ for almost every $x \in \Omega$, then*

$$\lim_{\epsilon \to 0} \int_\Omega \frac{k_\epsilon\big(\mathbf{b}(x)\big) - k_0\big(\mathbf{b}(x)\big)}{\epsilon} = -\int_\Omega T_0\big(\mathbf{b}(x)\big) l\Big(\nabla k_0\big(\mathbf{b}(x)\big)\Big).$$

Proof. (i) Fix $v \in \mathbb{R}^d$ and $|\epsilon| \leq 1$. By Lemma 4.3 (i) there exists $(t_\epsilon, u_\epsilon) \in (0,\infty) \times \bar{\Lambda}$ such that

$$-t_\epsilon l_\epsilon(u_\epsilon) - h(t_\epsilon) + v \cdot u_\epsilon = k_\epsilon(v) \geq -l_\epsilon(\vec{0}) - h(1). \tag{59}$$

Since $l_0(u_\epsilon) + h(1) + k_0(\vec{0}) \geq 0$, $|\epsilon| \leq 1$, $u_\epsilon \in \bar{\Lambda}$ and Λ is contained in the ball of radius $r^\star$ we conclude that

$$h(t_\epsilon) - t_\epsilon\big(h(1) + k_0(\vec{0}) + ||l||_{C(\bar{\Lambda})}\big) \leq r^*|v| + l_0(\vec{0}) + h(1) + ||l||_{C(\bar{\Lambda})}. \tag{60}$$

For each $\delta \geq 0$ define

$$\bar{t}(\delta) = \sup_{t>0}\{t \mid h(t) \leq r^*\delta + t\big(h(1) + k_0(\vec{0}) + ||l||_{C(\bar{\Lambda})}\big) + l_0(\vec{0}) + h(1) + ||l||_{C(\bar{\Lambda})}\}$$

Since $\lim_{t\to\infty} h(t)/t = \infty$ there exists a constant C^* such that for all $\delta \geq 0$ we have $\bar{t}(\delta) \leq C^*(\delta+1)$. Hence by (60), $t_\epsilon \leq \bar{t}(|v|) \leq C^*(|v|+1)$. By (59),

$$k_\epsilon(v) \leq k_0(v) - \epsilon t_\epsilon l(u_\epsilon) \quad \text{and} \quad k_0(v) \leq k_\epsilon(v) + \epsilon t_0 l(u_0).$$

Hence, if $0 < |\epsilon| \leq 1$

$$\frac{|k_\epsilon(v) - k_0(v)|}{|\epsilon|} \leq \max\{t_0, t_\epsilon\}||l||_{C(\bar{\Lambda})}.$$

This, together with the fact that $t_\epsilon \leq \bar{t}(|v|) \leq C^*(|v|+1)$ yields (i).

(ii) We refer the reader to Claim b in the proof of Theorem 2.3 [10].

(iii) By (i) there exists $C^* > 0$ such that if $0 < |\epsilon| \leq 1$ then

$$\Big|\frac{k_\epsilon\big(\mathbf{b}(x)\big) - k_0\big(\mathbf{b}(x)\big)}{\epsilon}\Big| \leq C^*(|\mathbf{b}| + 1).$$

Hence, since $\mathbf{b} \in L^1(\Omega, \mathbb{R}^d)$, we may use (ii) and apply the dominated convergence theorem to compute the limit in (iii). □

Lemma 4.5. *Let $k \in C(\mathbb{R}^d)$ be a convex Lipschitz function such that $k^* \in L^1(\Lambda)$ and $k^* \equiv \infty$ on $\mathbb{R}^d \setminus \bar{\Lambda}$. Then the map $\nabla k : dom \nabla k \to \bar{\Lambda}$ (resp. $\nabla \varrho^o_{\bar{\Lambda}} : \mathbb{R}^d \setminus \{\vec{0}\} \to \bar{\Lambda}$) has a Borel extension $K : \mathbb{R}^d \to \bar{\Lambda}$ (resp. $K^\infty : \mathbb{R}^d \to \bar{\Lambda}$) such that for all $v \in \mathbb{R}^d$, $K(v) \in \partial k(v)$ (resp. $K^\infty(v) \in \partial \varrho^o_{\bar{\Lambda}}(v)$).*

Proof. For each $v \in \mathbb{R}^d$ the sets $\partial k(v)$ and $\partial \varrho^o_{\bar{\Lambda}}(v)$ are compact, convex and contained in $\bar{\Lambda}$. Thus, the theory of multifunctions [5] ensures existence of Borel maps $K, K^\infty : \mathbb{R}^d \to \bar{\Lambda}$ which satisfy the conclusions of the lemma.

4.2. Coercitivity of the restriction of J to a subset of the boundary of $\mathcal{C}$.

Lemma 4.6. *Let $c \in \mathbb{R}$ and $(k, l, \psi) \in \mathcal{C} \times \Sigma(\bar{\Omega})$ be such that $l = k_\sharp$ and $J(k, l, \psi) \leq c$. Then*

(i)
$$-\lambda_c^+ \leq \inf_\Lambda l \leq m := \frac{1}{\mathcal{L}^d(\Lambda)} \int_\Lambda l \leq -\lambda_c^-.$$

(ii)
$$||l||_{L^1(\Lambda)} \leq c + r^*||\mathbf{F}||_{L^1(\Omega)} + \mathcal{L}^d(\Lambda)|\lambda_c^+| - \mathcal{L}^d(\Omega)h^*(\lambda_c^-).$$

(iii) *There exists a constant $C \equiv C_r(c)$ depending only on c and r such that*

$$\sup_{\Lambda^r}|l|, \quad \sup_{\Lambda^r}|\nabla l| \leq C_r(c), \quad \textit{where} \quad \Lambda^r = \{y \in \Lambda \mid \; dist(x, \partial\Lambda) \geq r\}.$$

(iv) *If the closed ball $B_\epsilon(\vec{0})$, centered at the origin and of radius ϵ is contained in Λ^r, then $k(v) \geq \epsilon|v| + h^*(C_r(c))$ for all $v \in \mathbb{R}^d$.*

Proof. (i) By Remark 4.2, l assumes only finite values on Λ. Since l is convex, it is locally Lipschitz on Λ (cf. e.g [7]). By (29),

$$k(v) \geq u \cdot v + h^*(-l(u)) \tag{61}$$

for all $u \in \Lambda$ and $v \in \mathbb{R}^d$. For $\mathbf{u}(x) \equiv u$ we have (cf. subsection 7.1)

$$\int_{\bar\Omega} k(\mathbf{F} + \operatorname{div} \psi_{\mathbb{R}^d}) \geq \int_{\bar\Omega} \Big(\mathbf{u} \cdot (\mathbf{F} + \operatorname{div} \psi_{\mathbb{R}^d}) + h^*(-l(\mathbf{u})) \Big) = u \cdot \int_{\bar\Omega} \mathbf{F} dx + \mathcal{L}^d(\Omega) h^*\big(-l(u)\big)$$

and so, since Λ is contained in the ball of radius r^*

$$\int_{\bar\Omega} k(\mathbf{F} + \operatorname{div} \psi_{\mathbb{R}^d}) dx \geq -r^* ||\mathbf{F}||_{L^1(\Omega)} + \mathcal{L}^d(\Omega) h^*(-l(u)) \tag{62}$$

We use the fact that $f^* \geq 0$ in (62) to obtain that

$$J(k, l, \psi) \geq -r^* ||\mathbf{F}||_{L^1(\Omega)} + \mathcal{L}^d(\Omega) h^*(-l(u)) + \int_\Lambda l dy. \tag{63}$$

We can use $\inf l$ in place of $l(u)$ in (63) to conclude that

$$J(k, l, \psi) \geq -r^* ||\mathbf{F}||_{L^1(\Omega)} + \mathcal{L}^d(\Omega) h^*(-\inf l) + \mathcal{L}^d(\Lambda) \inf l = -\lambda(-\inf l).$$

If $J(k, l, \psi) \leq c$ then $\lambda(-\inf l) \geq -c$ and so, $\lambda_c^- \leq -\inf l \leq \lambda_c^+$. Since l is continuous on Λ and the latter set is convex, it is connected and so, there exists $u \in \Lambda$ such that $l(u) = m$. Using that specific u in (63) we conclude as before that m satisfies the desired inequalities.

(ii) We have just established that $l + \lambda_c^+ \geq 0$. Thus by (63),

$$\int_\Lambda |l + \lambda_c^+| dy \leq c + r^* ||\mathbf{F}||_{L^1(\Omega)} - \mathcal{L}^d(\Omega) h^*(-\inf l) \leq c + r^* ||\mathbf{F}||_{L^1(\Omega)} - \mathcal{L}^d(\Omega) h^*(\lambda_c^-), \tag{64}$$

which yields (ii).

(iii) Since l is convex, (ii) implies (iii) (cf. e.g. [7]).

(iv) Assume the closed ball $B_\epsilon(\vec{0})$, centered at the origin and of radius ϵ is contained in Λ^r. We set $u = \epsilon v/|v|$ in (61), use (i) and the fact that h^* is monotone increasing to conclude the proof of (iv). ☐

Corollary 4.7. *Let $c \in \mathbb{R}$ and $(k, l, \psi) \in \mathcal{C} \times \Sigma(\bar\Omega)$ be as in Lemma 4.6. Then*

(i)
$$\int_\Omega f^*(\psi) dx + \mathcal{L}^d(\Lambda) \lambda_c^+ + \mathcal{L}^d(\Omega) h^*(\lambda_c^-) \leq c + r^* ||\mathbf{F}||_{L^1(\Omega)}$$

(ii) *There exists a constant $C \equiv C(r, \epsilon, c)$ which depends only on c, r and ϵ but is independent of k, l and ψ, such that*
$$\int_{\bar\Omega} |k(\mathbf{F} + \operatorname{div} \psi_{\mathbb{R}^d})| dx \leq C + r^* ||\mathbf{F}||_{L^1(\Omega)}.$$

(iii) *Further assume that $k = l^\sharp$, $\alpha > 0$ and*
$$\frac{C + r^* ||\mathbf{F}||_{L^1(\Omega)}}{\alpha}, \; \frac{C + r^* ||\mathbf{F}||_{L^1(\Omega)} - h^*\big(C_r(c)\big) \mathcal{L}^d(\Omega)}{\alpha \epsilon} < \frac{1}{2} \mathcal{L}^d(\Omega), \tag{65}$$

Then
$$|k(v)| \leq \alpha(1 + r^*) + r^* |v|. \tag{66}$$

Proof. (i) We use (62), Lemma 4.6 (i) and the fact that h^* is monotone increasing (cf. Remark 2.1) to obtain that

$$\int_{\bar\Omega} k(\mathbf{F} + \operatorname{div} \psi_{\mathbb{R}^d})dx \geq -r^*||\mathbf{F}||_{L^1(\Omega)} + \mathcal{L}^d(\Omega)h^*(\lambda_c^-). \tag{67}$$

This, together with Lemma 4.6 (i) completes the proof of (i).

(ii) By Lemma 4.6 (iv)

$$\int_{\bar\Omega}\Big|k(\mathbf{F}+\operatorname{div}\psi_{\mathbb{R}^d})-\epsilon|\mathbf{F}+\operatorname{div}\psi_{\mathbb{R}^d}|-h^*\big(C_r(c)\big)\Big|dx = \int_{\bar\Omega}\Big(k(\mathbf{F}+\operatorname{div}\psi_{\mathbb{R}^d})-\epsilon|\mathbf{F}+\operatorname{div}\psi_{\mathbb{R}^d}|-h^*\big(C_r(c)\big)\Big)$$

We use the triangle inequality and the fact that $f^* \geq 0$, then add and substract $\int_{\bar\Omega} l dy$ to conclude that

$$\begin{aligned} \int_{\bar\Omega} |k(\mathbf{F} + \operatorname{div}\psi_{\mathbb{R}^d})|dx &\leq \int_{\bar\Omega} k(\mathbf{F} + \operatorname{div}\psi_{\mathbb{R}^d})dx + f^*(\psi) + |h^*\big(C_r(c)\big)| - h^*\big(C_r(c)\big) \\ &\leq c - \int_{\bar\Omega} l dy + |h^*\big(C_r(c)\big)| - h^*\big(C_r(c)\big). \end{aligned}$$

This, together with Lemma 4.6 (ii) yields the desired result.

(iii) Let $\operatorname{div}^a\psi_{\mathbb{R}^d}$ be the absolutely continuous part of $\operatorname{div}\psi_{\mathbb{R}^d}$. By (ii)

$$\mathcal{L}^d\Big\{|k(\mathbf{F} + \operatorname{div}^a\psi_{\mathbb{R}^d})| \geq \alpha\Big\} \leq \frac{C + r^*||\mathbf{F}||_{L^1(\Omega)}}{\alpha}$$

and so,

$$\mathcal{L}^d\Big\{|k(\mathbf{F} + \operatorname{div}^a\psi_{\mathbb{R}^d})| < \alpha\Big\} \geq \mathcal{L}^d(\Omega) - \frac{C + r^*||\mathbf{F}||_{L^1(\Omega)}}{\alpha} \tag{68}$$

Using Lemma 4.6 (iv) to obtain a lower bound on $|k(\mathbf{F} + \operatorname{div}^a\psi_{\mathbb{R}^d})|$, thanks to (ii) we conclude that

$$\int_{\bar\Omega} |\mathbf{F} + \operatorname{div}^a\psi_{\mathbb{R}^d}|dx \leq \frac{C + r^*||\mathbf{F}||_{L^1(\Omega)} - h^*\big(C_r(c)\big)\mathcal{L}^d(\Omega)}{\epsilon}.$$

As above, we conclude that

$$\mathcal{L}^d\Big\{|\mathbf{F} + \operatorname{div}^a\psi_{\mathbb{R}^d}| < \alpha\Big\} \geq \mathcal{L}^d(\Omega) - \frac{C + r^*||\mathbf{F}||_{L^1(\Omega)} - h^*\big(C_r(c)\big)\mathcal{L}^d(\Omega)}{\alpha\epsilon}. \tag{69}$$

Take α so that (65) holds to conclude that the set of $x \in \Omega$ such that

$$|k\big(\mathbf{F}(x) + \operatorname{div}^a\psi_{\mathbb{R}^d}(x)\big)| < \alpha \quad \text{and} \quad |\mathbf{F}(x) + \operatorname{div}^a\psi_{\mathbb{R}^d}(x)| < \alpha \tag{70}$$

is of positive measure. By Remark 4.2 (i), k is r^*–Lipschitz. Let x be such that (70) holds and set $v_0 = \mathbf{F}(x) + \operatorname{div}^a\psi_{\mathbb{R}^d}(x)$. We use that

$$|k(v) - k(v_0)| \leq r^*|v - v_0| \quad \text{and} \quad |k(v_0)|,\ |v_0| < \alpha$$

to obtain (66). □

4.3. **Comparing the graphs of $-J$ and $I_{\mathcal{S}}$.** Throughout this subsection, we fix $\tau \in \mathcal{T} \subset (0,\infty)$ (cf. Subsection 3.2) and set $\mathcal{S} = \mathcal{S}^\tau$. We assume that $\Sigma(\bar\Omega) = \mathcal{S}$ is a finite dimensional vector space as given in (25).

Lemma 4.8. *Let $(k,l,\psi) \in \mathcal{C} \times \Sigma(\bar\Omega)$ be such that $k \in C(\mathbb{R}^d)$ is a convex Lipschitz function such that $\Lambda \subset \operatorname{dom} k^* \subset \bar\Lambda$. If $\mathbf{u} \in \mathcal{U}_{\mathcal{S}}$ and $\beta : \Omega \to (0,\infty)$ are such that $\beta \in \det^* \nabla\mathbf{u}$ then*

$$-J(k,l,\psi) \leq \int_\Omega \big(f(\nabla_{\mathcal{S}}\mathbf{u}) + h(\beta) - \mathbf{F}\cdot\mathbf{u}\big)dx. \tag{71}$$

Equality holds in (71) if and only if $\nabla_{\mathcal{S}}\mathbf{u} = Df^*(\psi)$,

$$\mathbf{u}(x) \in \partial k(\mathbf{F} + \operatorname{div}\psi) \quad \textit{and} \quad l(\mathbf{u}) + h'(\beta) = 0 \quad \mathcal{L}^d \textit{ a.e.} \tag{72}$$

Proof. Recall that by (23) $\operatorname{div}\psi_{\mathbb{R}^d} = \operatorname{div}\psi\mathcal{L}^d$. By Remark 4.2 we may assume without loss of generality that $l = k_\#$. We use (80) to obtain

$$\int_{\bar\Omega} k(\mathbf{F}+\operatorname{div}\psi)dx \geq \int_{\bar\Omega}\big(\mathbf{u}\cdot(\mathbf{F}+\operatorname{div}\psi) - h(\beta) - \beta l(\mathbf{u})\big)dx = \int_{\bar\Omega}\big(\mathbf{u}\cdot\mathbf{F} - \langle\nabla_{\mathcal{S}}\mathbf{u},\psi\rangle - h(\beta) - \beta l(\mathbf{u})\big). \tag{73}$$

Rearranging the terms in (73) and using $\beta \in \det^* \nabla\mathbf{u}$ we conclude that

$$-J(k,l,\psi) \leq \int_{\bar\Omega}\big(-f^*(\psi) + h(\beta) - \mathbf{F}\cdot\mathbf{u} + \langle\nabla_{\mathcal{S}}\mathbf{u},\psi\rangle\big). \tag{74}$$

Unless $\nabla_{\mathcal{S}}\mathbf{u} = Df^*(\psi)$ a.e.,

$$\int_{\bar\Omega}\Big(-f^*(\psi) + h(\beta) - \mathbf{F}\cdot\mathbf{u} + \langle\nabla\mathbf{u},\psi\rangle\Big) < \int_{\bar\Omega}\Big(f(\nabla\mathbf{u}) + h(\beta) - \mathbf{F}\cdot\mathbf{u}\Big).$$

The above calculations show that equality holds in (71) if and only if $\nabla_{\mathcal{S}}\mathbf{u} = Df^*(\psi)$ holds and

$$k(\mathbf{F} + \operatorname{div}\psi) + \beta l(\mathbf{u}) + h(\beta) = \mathbf{u}\cdot(\mathbf{F} + \operatorname{div}\psi) \quad \mathcal{L}^d \text{ a.e.} \tag{75}$$

We exploit Lemma 4.3 (iii) to conclude that (75) is equivalent to (72). □

4.4. **Existence of optimizers in the dual problem of projected pseudo–gradient.**

Theorem 4.9. *There exists $(k_0, l_0, \psi_0) \in \mathcal{C} \times \Sigma(\bar\Omega)$ that minimizes J over $\mathcal{C} \times \Sigma(\bar\Omega)$ and such that $k_{0\sharp} = l_0$ and $l_0^\sharp = k_0$.*

Proof. Observe first that there exists $(k,l,\psi) \in \mathcal{C} \times \Sigma(\bar\Omega)$ such that $J(k,l,\psi) < \infty$. Lemma 6.2 ensures that when $\Sigma(\bar\Omega)$ is the infinite dimensional space given by (24), then the infimum of J over $\mathcal{C} \times \Sigma(\bar\Omega)$ is finite. Similarly, Lemma 4.8 ensures that when $\Sigma(\bar\Omega)$ is the finite dimensional space given by (25), then infimum of J over $\mathcal{C} \times \Sigma(\bar\Omega)$ is not $-\infty$. Let $\{(k^n, l^n, \psi^n)\}_n$ be a minimizing sequence of J over $\mathcal{C} \times \Sigma(\bar\Omega)$. Using the fact that the infimum of J over $\mathcal{C} \times \Sigma(\bar\Omega)$ is finite and combining Remark 4.2 (iii) and Lemma 4.1, we may assume without loss of generality that

$$k^n_\sharp = l^n, \quad l^{n\,\sharp} = k^n \quad \text{and} \quad \sup_n J(k^n, l^n, \psi^n) =: c < \infty.$$

We use Lemma 4.6 to conclude that

$$\sup_n \int_\Lambda |l^n| dy < \infty, \quad \inf_n \inf_\Lambda l_n > -\infty. \tag{76}$$

We use Corollary 4.7 to obtain that

$$\sup_n \int_\Lambda f^*(\psi^n)dx, \quad \sup_n \int_{\bar\Omega} |k^n(\mathbf{F} + \operatorname{div}\psi^n_{\mathbb{R}^d})| < \infty. \tag{77}$$

Furthermore, Lemma 4.6 (iv) and Corollary 4.7 provide us with constants $\epsilon_0, C_0 > 0$ and $\epsilon_1 \in \mathbb{R}$ independent on n such that

$$\sup_n |k^n(v)| \leq C_0(|v|+1), \quad \inf_n k^n(v) \geq \epsilon_0 |v| - \epsilon_1 \tag{78}$$

for $v \in \mathbb{R}^d$. We use the first inequality in (77) and the lower bound on f^* as provided by (18), we record the inequality in (76), exploit (77) and (78) to conclude that

$$\sup_n ||\psi_n||_{\Sigma(\bar\Omega)} + ||l_n||_{L^1(\Lambda)} < \infty \quad \text{and} \quad \inf_n \inf_\Lambda l_n > -\infty \quad \text{and} \quad \mathrm{Lip}\,(k^n) \leq r^*. \tag{79}$$

This, together with Lemma 2.2 implies that J admits a minimizer (l_0, ψ_0, k_0) over $\mathcal{C} \times \Sigma(\bar\Omega)$. Combining Remark 4.2 (iii) and Lemma 4.1 and substituting (k_0, l_0, ψ_0) by $((k_{0\sharp})^\sharp, k_{0\sharp}, \psi_0)$ if necessary, we preserve the minimizer property while ensuring that $k_{0\sharp} = l_0$ and $l_0^\sharp = k_0$. □

5. A minimization problem involving a pseudo–projected gradient.

Throughout this subsection, we fix $\tau \in \mathcal{T} \subset (0,\infty)$ (cf. Subsection 3.2) and set $\mathcal{S} = \mathcal{S}^\tau$. We assume that $\Sigma(\bar\Omega) = \mathcal{S}$ is a finite dimensional vector space as given in (25) and

$$\mathbf{F} \;\text{ is nondegenerate}$$

in the sense that whenever $N \subset \mathbb{R}^d$ is a set of null Lebesgue measure, so is $\mathbf{F}^{-1}(N)$. Note that when $\Sigma(\bar\Omega)$ is given by (25) and $\mathbf{u} \in \mathcal{U}_\mathcal{S}$ (cf. Proposition 3.1), $\nabla_\mathcal{S}\mathbf{u}$ satisfies the property:

$$\int_\Omega \langle \psi, \nabla_\mathcal{S} \mathbf{u}\rangle dx = -\int_{\bar\Omega} \mathbf{u} \cdot \mathrm{div}\, \psi_{\mathbb{R}^d} \tag{80}$$

and all $\psi \in \mathcal{S}$.

5.1. Dual of the problem with projected pseudo–gradient.

The goal of this subsection is to established the duality relation:

$$\sup_{(k,l,\psi) \in \mathcal{C} \times \Sigma(\bar\Omega)} -J(k,l,\psi) = \min_{\mathbf{u} \in \mathcal{U}_\mathcal{S}} I_\mathcal{S}(\mathbf{u}).$$

Let (k_0, l_0, ψ_0) be a minimizer of J such that $l_0^\sharp = k_0$ (cf. Theorem 4.9).

Remark 5.1. Let $C \subset \bar\Omega$ be a closed set of null Lebesgue measure such that $\psi_0 \in C^1(\bar\Omega \setminus C)$. Since $\mathbf{F}$ is nondegenerate and $\mathrm{div}\,\psi_0$ is piecewise constant, then $\mathbf{F} + \mathrm{div}\,\psi_0$ is nondegenerate.

Let $N \subset \bar\Omega$ be a Borel set of null Lebesgue measure that contains C and the preimage of $\partial\Lambda \cup (\mathbb{R}^d \setminus \mathrm{dom}\,\nabla k_0)$ by $\mathbf{F} + \mathrm{div}\,\psi_0$. Existence of N is ensured by Remark 5.1.

We set

$$\Omega_0 = \bar\Omega \setminus N.$$

Definition 5.2. As in Lemma 4.5 let $K_0 : \mathbb{R}^d \to \bar\Lambda$ be a Borel map which extend ∇k_0. We define a Borel map $T_0 : \mathbb{R}^d \to (0,\infty)$ to be the unique solution to the equation

$$h'\big(T_0(v)\big) + l_0\big(K_0(v)\big) = 0.$$

By Lemma 4.3, whenever k_0 is differentiable at v and $\nabla k_0(v) \in \Omega$, we have

$$k_0(v) + T_0(v) l_0\big(U_0(v)\big) + h\big(T_0(v)\big) = v \cdot K_0(v). \tag{81}$$

Proposition 5.3. *Define on the set* Ω_0 *the functions*

$$\beta_0 = T_0\big(\mathbf{F} + \mathit{div}\,\psi_0\big) \quad \text{and} \quad \mathbf{u}_0 = K_0\big(\mathbf{F} + \mathit{div}\,\psi_0\big). \tag{82}$$

(i) *We have that the range of* $\mathbf{u}_0$ *is contained in* $\bar\Lambda$, $\beta_0 > 0$ $\mathcal{L}^d$*–a.e. and* $\beta_0 \in \det^* \nabla \mathbf{u}_0$.

(ii) *We have that* $\mathbf{u}_0 \in \mathcal{U}_\mathcal{S}$ *and* $\nabla_\mathcal{S}\mathbf{u}_0 = Df^*(\psi_0)$.

(iii) *Finally, we have* $-J(k_0, l_0, \psi_0) = I_\mathcal{S}(\mathbf{u}_0)$.

Proof. (i) Since $k_0 = l_0^\sharp$, by Remark 4.2 the range of K_0 is contained in $\bar\Lambda$ and so, the range of $\mathbf{u}_0$ is contained in $\bar\Lambda$. Fix $l \in C_c(\mathbb{R}^d)$ and for $\epsilon \in (-1,1)$ define $l_\epsilon = l_0 + \epsilon l$ and $k_\epsilon = l_\epsilon^\sharp$. By Lemma 4.4, $\epsilon \to J(k_\epsilon, l_\epsilon, \psi_0)$ is differentiable at 0 and so, since its attains its minimum at 0, its derivative there must vanish:

$$0 = \int_\Lambda l dy - \int_\Omega T_0(\mathbf{F} + \operatorname{div}\psi_0) l\big(U_0(F + \operatorname{div}\psi_0)\big). \tag{83}$$

Using the definition of $\mathbf{u}$ and β_0 as provided by (82) and taking into account that $l \in C_c(\mathbb{R}^d)$ is arbitrary in (83), we conclude that $\beta_0 \in \det^* \nabla\mathbf{u}_0$.

(ii) For $\psi \in \mathcal{S}$ and $\epsilon \in (-1,1)$ define $\psi_\epsilon = \psi_0 + \epsilon\psi$. Since $\Sigma(\bar\Omega) = \mathcal{S}$ is a vector space, $\psi_\epsilon \in \Sigma(\bar\Omega)$. We have

$$J(k_0, l_0, \psi_\epsilon) = J(k_0, l_0, \psi_0) + \epsilon \int_{\bar\Omega} A_\epsilon$$

where

$$A_\epsilon = \frac{k_0\big(\mathbf{F} + \operatorname{div}\psi_0 + \epsilon \operatorname{div}\psi\big) - k_0\big(\mathbf{F} + \operatorname{div}\psi_0\big)}{\epsilon} + \frac{f^*\big(\psi_0 + \epsilon\psi\big) - f^*\big(\psi_0\big)}{\epsilon}.$$

We use that k_0 is r^*–Lipschitz and the condition on Df^* imposed in (18) to obtain that

$$|A_\epsilon| \le r^* || \operatorname{div}\psi||_{L^\infty(\Omega)} + \frac{1}{\bar c}\Big\| |\psi|(|\psi_0| + |\psi|)^{q-1} + 1\Big\|_{L^\infty(\Omega)}.$$

Hence, we may apply the Lebesgue Dominated Convergence Theorem to the integral of A_ϵ to obtain that $\epsilon \to J(k_0, l_0, \psi_\epsilon)$ is differentiable at 0 and so, since it attains its minimum at 0, its derivative vanishes there:

$$\int_\Omega \operatorname{div}\psi \cdot \mathbf{u}_0 dx + \int_\Omega \langle Df^*(\psi_0), \psi\rangle dx = 0. \tag{84}$$

The fact that (84) holds for any arbitrary $\psi \in \mathcal{S}$ yields that $\mathbf{u} \in \mathcal{U}_\mathcal{S}$ and

$$\nabla_\mathcal{S}\mathbf{u}_0 = Df^*(\psi_0) \quad \mathcal{L}^d - \text{ a.e.} \tag{85}$$

(iii) By the definition of K_0 and the definition of $\mathbf{u}_0$ in (82)

$$k_0(\mathbf{F} + \operatorname{div}\psi_0) + k_0^*(\mathbf{u}_0) = \mathbf{u}_0 \cdot (\mathbf{F} + \operatorname{div}\psi_0), \quad \text{on} \quad \Omega_0. \tag{86}$$

Similarly, by Definition 5.2 and the definition of β in (82)

$$h'(\beta_0) + l_0(\mathbf{u}_0) = 0, \quad \mathcal{L}^d - \text{a.e.} \tag{87}$$

We combine (85–87) and apply Lemma 4.8 to conclude that $-J(k_0, l_0, \psi_0) = I_\mathcal{S}(\mathbf{u}_0)$. □

Theorem 5.4 (Existence and uniqueness of a minimizer)**.** *The following hold:*

(i) *The minimizer* $\mathbf{u}$ *of* $I_\mathcal{S}$ *over* $\mathcal{U}_\mathcal{S}$ *is unique. In particular, it satisfies* $|\det^h \nabla\mathbf{u}| > 0$ *and*

$$h'\big(|\det^h \nabla\mathbf{u}|\big) + l_0(\mathbf{u}) = 0, \quad \mathbf{u} = \nabla k_0(\mathbf{F} + \mathit{div}\,\psi_0), \quad \nabla_\mathcal{S}\mathbf{u} = Df^*(\psi_0) \quad a.e.$$

(ii) *If we further assume that the image of any Borel subset of* Ω *by* F *is* $\mathcal{L}^d$*–measurable, then*

$$\mathcal{L}^d\big(\Lambda \setminus \mathbf{u}(\Omega)\big) = 0.$$

(iii) *We have* $\mathcal{L}^d(\Omega \setminus \Omega_1) = 0$*, where* Ω_1 *is the largest subset of* Ω_0 *such that*

$$\mathbf{u}(x) = \mathbf{u}(y), \quad x, y \in \Omega_1 \qquad \Longrightarrow \qquad |\det^h \nabla\mathbf{u}|(x) = |\det^h \nabla\mathbf{u}|(y).$$

Proof. (i) Let (k_0, l_0, ψ_0) be the minimizer of J found in Theorem 4.9 so that $l_0^\sharp = k_0$, $k_{0\sharp} = l_0$. Lemma 4.8 (i) and Proposition 5.3 ensure that the pair $(\beta_0, \mathbf{u}_0)$ given in Proposition 5.3 is the unique minimizer of

$$\int_\Omega \big(f(\nabla_{\mathcal{S}}\mathbf{u}) + h(\beta) - \mathbf{F}\cdot\mathbf{u}\big)dx$$

over the set of $(\beta, \mathbf{u})$ such that $\mathbf{u} \in \mathcal{U}_{\mathcal{S}}$ and $\beta \in \det^* \nabla\mathbf{u}$. Since $\int_\Omega h(\det^h \nabla\mathbf{u}_0)dx \le \int_\Omega h(\beta_0)dx$ we conclude that $\beta_0 = \det^h \nabla\mathbf{u}_0$ since otherwise, we would have $\int_\Omega h(\det^h \nabla\mathbf{u}_0)dx < \int_\Omega h(\beta_0)dx$.

(ii) Assume that the image of any Borel subset of Ω by $\mathbf{F}$ is $\mathcal{L}^d$–measurable. Then $(\mathbf{F}+\operatorname{div}\psi_0)(\Omega_0)$ is $\mathcal{L}^d$–measurable. Since

$$\mathcal{L}^d\Big((\mathbf{F}+\operatorname{div}\psi_0)(\Omega_0)\Big) = \sup_K\{\mathcal{L}^d(K) \mid K \subset (\mathbf{F}+\operatorname{div}\psi_0)(\Omega_0), K \text{ is compact}\},$$

there exists a countable collection $\{K_n\}_n$ of compact sets such that

$$K_n \subset K_{n+1} \subset (\mathbf{F}+\operatorname{div}\psi_0)(\Omega_0) \subset \operatorname{dom}\nabla k_0$$

for $n \ge 1$, and

$$\mathcal{L}^d(N) = 0, \quad \text{where} \quad N = (\mathbf{F}+\operatorname{div}\psi_0)(\Omega_0) \setminus \cup_{n=1}^\infty K_n.$$

The set

$$\Omega_1 = (\mathbf{F}+\operatorname{div}\psi_0)^{-1}\big(\cup_{n=1}^\infty K_n\big)$$

is a Borel subset of Ω of full measure. Since the K_n are contained in the range of $\mathbf{F}+\operatorname{div}\psi_0$ we conclude that

$$(\mathbf{F}+\operatorname{div}\psi_0)(\Omega_1) = \cup_{n=1}^\infty K_n.$$

Thus,

$$\mathbf{u}_0(\Omega_1) = \nabla k_0(\mathbf{F}+\operatorname{div}\psi_0)(\Omega_1) = \nabla k_0\big(\cup_{n=1}^\infty K_n\big) = \cup_{n=1}^\infty \nabla k_0(K_n)$$

is a Borel set since the restriction of ∇k_0 to $\operatorname{dom}\nabla k_0$ is continuous and $K_n \subset \operatorname{dom}\nabla k_0$ is compact. We eventually invoke Remark 3.4 (ii) to conclude that $\nabla k_0(\mathbf{F}+\operatorname{div}\psi_0)(\Omega_1)$ is a Borel set of full measure in $\bar\Lambda$.

(iii) is a direct consequence of (i). □

We have identified necessary conditions satisfied by the minimizer of $I_{\mathcal{S}}$ over $\mathcal{U}_{\mathcal{S}}$. The goal of the next theorem is to show that these condition completely characterize the minimizer of $I_{\mathcal{S}}$ over $\mathcal{U}_{\mathcal{S}}$.

Corollary 5.5 (Minimizers are characterized by Euler–Lagrange equations)**.** *Suppose the image of any Borel subset of Ω by $\mathbf{F}$ is $\mathcal{L}^d$–measurable. Let $\bar{\mathbf{u}} : \Omega \to \bar\Lambda$ be a nondegenerate $\mathcal{L}^d$–measurable map belonging to $\mathcal{U}_{\mathcal{S}}$ and suppose the function $\psi := Df(\nabla_{\mathcal{S}}\bar{\mathbf{u}})$ belongs to $\mathcal{S}$. Suppose $\beta : \Omega \to (0,\infty)$ an $\mathcal{L}^d$–measurable function such that $\beta \in \det^* \nabla\bar{\mathbf{u}}$. Let Ω_e be the largest subset of Ω such that*

$$\bar{\mathbf{u}}(x) = \bar{\mathbf{u}}(y), \quad x, y \in \Omega_e \qquad \Longrightarrow \qquad \beta(x) = \beta(y).$$

Let $l : \bar{\mathbf{u}}(\Omega_e) \to \mathbb{R}$ be univoquely defined by $l(\bar{\mathbf{u}}(x)) + h'(\beta(x)) = 0$ for $x \in \Omega_e$. Suppose that

(i) *$\mathcal{L}^d(\Omega \setminus \Omega_e) = 0$ and $\mathcal{L}^d\big(\Lambda \setminus \bar{\mathbf{u}}(\Omega_e)\big) = 0$.*

(ii) *The function l has a lower semicontinuous extension we still denote $l : \bar\Lambda \to (-\infty, \infty]$, which is continuous in Λ.*

(iii) *Set $k = l^\sharp$ and suppose that $\bar{\mathbf{u}} = \nabla k(\mathbf{F} + \operatorname{div}\psi)$ $\mathcal{L}^d$–a.e. on Ω.*

Then $\beta = |\det^h \nabla\mathbf{u}|$ $\mathcal{L}^d$–a.e., and $\bar{\mathbf{u}}$ minimizes $I_{\mathcal{S}}$ over $\mathcal{U}_{\mathcal{S}}$.

Proof. Since $(l, k) \in \mathcal{C}$ and $\psi \in \Sigma(\bar\Omega)$ and the second property in (iii) holds, thanks to Lemma 4.8 and Theorem 5.4 – using the facts that $\psi := Df(\nabla_{\mathcal{S}}\bar{\mathbf{u}})$ and $l(\bar{\mathbf{u}}(x)) + h'(\beta(x)) = 0$ a.e. – we obtain that $\beta = \det^h \nabla\bar{\mathbf{u}}$ $\mathcal{L}^d$–a.e., and $\bar{\mathbf{u}}$ minimizes $I_{\mathcal{S}}$ over $\mathcal{U}_{\mathcal{S}}$. □

6. The variational problems when $\tau = 0$.

Throughout this subsection we assume that $\mathbf{F} \in L^1(\mathbb{R}^d, \mathbb{R}^d)$ and that $\Sigma(\bar\Omega)$ is the infinite dimensional vector space given by (24).

6.1. **Divergence and Sobolev maps.** Let $\mathbf{u} \in W^{1,p}(\Omega, \bar\Lambda)$ and for $n \geq 1$ integer, let Ω_n be the set of x in $\mathbb{R}^d$ such that $\varrho_\Omega(x) < n$. We consider the extension of $\mathbf{u}$ set to be $\mathbf{u}\big(x/\varrho_\Omega(x)\big)$ in $\Omega_3 \setminus \bar\Omega$ to obtain a $W^{1,p}(\Omega, \bar\Lambda)$–function which we still denote by $\mathbf{u}$. Let $\rho_\epsilon(x) = \epsilon^{-d}\rho(\epsilon^{-1}x)$ be a standard mollifier and set

$$\mathbf{u}_\epsilon(x) = \int_\Omega \rho_\epsilon(x-y)\mathbf{u}(y)dy.$$

We have $\{\mathbf{u}_\epsilon\}_\epsilon \subset C^\infty(\bar\Omega_2, \mathbb{R}^d)$. If $\psi \in \Sigma(\bar\Omega)$, since the support of $\operatorname{div}\psi_{\mathbb{R}^d}$ is contained in $\bar\Omega$, extending $\mathbf{u}_\epsilon$ to obtain a map in $C^\infty_c(\mathbb{R}^d, \mathbb{R}^d)$, we have

$$\int_\Omega \langle \nabla \mathbf{u}_\epsilon, \psi\rangle dx = -\int_{\bar\Omega} \mathbf{u}_\epsilon \cdot \operatorname{div}\psi_{\mathbb{R}^d}(dx). \tag{88}$$

Let $g = |\operatorname{div}\psi_{\mathbb{R}^d}| + \mathcal{L}^d$, let $g^a \geq \mathcal{L}^d|_{\bar\Omega}$ be the absolutely continuous part of g and let g^s be the singular part. Choose a Borel set $B_1 \subset \bar\Omega$ such that

$$g^a(B_1) = g^s(\bar\Omega \setminus B_1) = 0.$$

Recall that $\{\mathbf{u}_\epsilon\}_\epsilon$ converges to $\mathbf{u}$ in $W^{1,p}(\Omega, \bar\Lambda)$ and there exists a Borel set B_2 of null Lebesgue measure such that $\{\mathbf{u}_\epsilon(x)\}_\epsilon$ converges to $\mathbf{u}(x)$ whenever $x \in \bar\Omega \setminus B_2$. Since $\{\mathbf{u}_\epsilon\}_\epsilon \subset L^\infty(\bar\Omega, \bar\Lambda; g)$ and the range of $\mathbf{u}_\epsilon$ is contained in the convex closed set $\bar\Lambda$, there exists a subsequence $\{\mathbf{u}_{\epsilon_n}\}_n$ which converges in the weak $\star$ topology to a Borel map $\mathbf{u}_\psi : \bar\Omega \to \bar\Lambda$. We substitute ϵ by ϵ_n in (88) and let n tend to ∞ to obtain

$$\int_\Omega \langle \psi, \nabla \mathbf{u}\rangle dx = -\int_{\bar\Omega} \mathbf{u}_\psi \cdot \operatorname{div}_{\mathbb{R}^d}\psi(dx). \tag{89}$$

Replacing B_2 by another Borel set of null Lebesgue measure which we still denote by B_2 and setting $B = B_1 \cup B_2$, we have

$$g^a(B) = g^s(\bar\Omega \setminus B) = 0 \qquad \text{and} \qquad \mathbf{u} = \mathbf{u}_\psi \quad \text{on} \quad \Omega \setminus B.$$

Remark 6.1. If $\mathbf{u} \in W^{1,p}(\Omega, \bar\Lambda) \cap C(\bar\Omega, \mathbb{R}^d)$ then $\mathbf{u}_\psi = \mathbf{u}$ is independent of ψ.

6.2. **The dual of the full variational problem.** Let Γ be the set of Borel measures introduced in subsection 2.6. For $\gamma \in \Gamma$, recall that $\mathbf{u}_\gamma$ and U_γ are defined respectively in (40) and (41), and satisfy the properties: $\mathbf{u}_\gamma \in W^{1,p}(\Omega, \bar\Lambda)$ and $U_\gamma = \nabla \mathbf{u}_\gamma$.

Let $(l, \psi, k) \in \mathcal{C} \times \Sigma(\bar\Omega)$. We use (89) to obtain

$$\int_{\bar\Omega} (\mathbf{u}_\gamma)_\psi \cdot \operatorname{div}\psi_{\mathbb{R}^d} dx = -\int_\Omega \langle \nabla \mathbf{u}_\gamma, \psi\rangle dx = -\int_C \langle \xi, \psi(x)\rangle \gamma(dx, dt, du, d\xi). \tag{90}$$

Similarly,

$$\int_{\bar\Omega} \mathbf{u}_\gamma \cdot \mathbf{F}(x) dx = \int_C u \cdot \mathbf{F}(x)\gamma(dx, dt, du, d\xi). \tag{91}$$

We apply Jensen's inequality and then use the fact that $(l, k) \in \mathcal{C}$ to obtain

$$\int_{\bar\Omega} k^*(\mathbf{u}_\gamma) dx \leq \int_C k^*(u)\gamma(dx, dt, du, d\xi) \leq \int_C \big(tl(u) + h(t)\big)\gamma(dx, dt, du, d\xi). \tag{92}$$

By Young's inequality

$$\int_C \langle \xi, \psi\rangle \gamma(dx,dt,du,d\xi) < \int_C \big(f(\xi)+f^*(\psi)\big)\gamma(dx,dt,du,d\xi) \tag{93}$$

unless $Df^*(\psi(x)) = \xi$ γ–a.e., in which case, equality holds.

Lemma 6.2. *We have $-J(k,l,\psi) \le \bar I(\gamma)$ and the inequality is strict unless $Df^*(\psi(x)) = \xi$–γ a.e.*

Proof. We may assume without loss of generality that $J(k,l,\psi) < \infty$ and by Remark 4.2 we may assume that $l = k_\#$. These facts allow us to exploit the representation formula (32) and use (89) to obtain that

$$\begin{aligned}\int_{\bar\Omega} k(\mathbf{F}\mathcal{L}^d + \operatorname{div}\psi_{\mathbb{R}^d}) &= \int_\Omega k(\mathbf{F} + \operatorname{div}^a\psi)dx + \int_{\bar\Omega} \varrho^o_{\bar\Lambda}(\operatorname{div}^s\psi) \\ &\ge \int_\Omega \Big((\mathbf{u}_\gamma)_\psi\cdot(\mathbf{F}+\operatorname{div}^a\psi) - k^*(\mathbf{u}_\gamma)\Big)dx + \int_{\bar\Omega}(\mathbf{u}_\gamma)_\psi\cdot(\operatorname{div}^s\psi) \\ &= \int_\Omega \Big((\mathbf{u}_\gamma)_\psi\cdot\mathbf{F} - k^*(\mathbf{u}_\gamma)\Big)dx + \int_{\bar\Omega}(\mathbf{u}_\gamma)_\psi\cdot(\operatorname{div}\psi_{\mathbb{R}^d}).\end{aligned}$$

This, together with (11), (90) and (91) implies

$$\begin{aligned}\int_{\bar\Omega} k(\mathbf{F}\mathcal{L}^d + \operatorname{div}\psi_{\mathbb{R}^d}) \ \ge\ & \int_C u\cdot\mathbf{F}(x)\gamma(dx,dt,du,d\xi) - \int_C\big(h(t)+tl(u)\big)\gamma(dx,dt,du,d\xi) \\ & - \int_C \langle\xi,\psi(x)\rangle\gamma(dx,dt,du,d\xi).\end{aligned}$$

We use the fact that $f(\xi)+f^*(\psi)\ge\langle\xi,\psi(x)\rangle$ and that the inequality is strict unless $Df^*(\psi(x)) = \xi$ to conclude the proof. □

Let $\tau\in\mathcal{T}\subset(0,\infty)$ and $\mathcal{S}^\tau$ be as in Section 5. Let $\mathbf{u}^\tau\in\mathcal{U}_{\mathcal{S}^\tau}$ be the unique minimizer of $I_{\mathcal{S}^\tau}$ over $\mathcal{U}_{\mathcal{S}^\tau}$ (cf. Theorem 5.4). Let $(k^\tau,l^\tau)\in\mathcal{C}$ and $\psi^\tau\in\mathcal{S}^\tau$ be such that (cf. Proposition 5.3) $-J(k^\tau,l^\tau,\psi^\tau) = I_{\mathcal{S}^\tau}(\mathbf{u}^\tau)$, $l^{\tau\,\sharp} = k^\tau$ and $k^\tau_\sharp = l^\tau$. Set

$$\bar\psi\equiv 0,\quad \bar l\equiv 0,\quad \bar k(v) = \bar l^\sharp(v) = \varrho^o_{\bar\Lambda}(v) - \inf_{t>0} h(t),\quad c = J(\bar k,\bar l,\bar\psi).$$

Since $\bar\psi\in\mathcal{S}^\tau$ and $(\bar k,\bar l)\in\mathcal{C}$ we conclude that $J(k^\tau,l^\tau,\psi^\tau)\le c$ and so, by Lemma 4.6 and Corollary 4.7

$$\sup_{\tau\in\mathcal{T}} ||\psi^\tau||_{L^q(\Omega)} + ||\operatorname{div}\psi^\tau||_{\mathcal{M}(\bar\Omega)} + ||l^\tau||_{L^1(\Omega)} < \infty \tag{94}$$

and there exist constants $C,\epsilon>0$ independent of τ such that

$$\epsilon|v| - C \le k^\tau(v) \le r^*|v| + C,\quad -C\le\inf_\Lambda l^\tau. \tag{95}$$

Lemma 6.3. *There exists a sequence $\{\tau_n\}_n\subset\mathcal{T}$ decreasing to 0 as n tends to ∞ such that $\{\psi^{\tau_n}\}_n$ converges weakly in $L^q(\Omega)$ to a map ψ^0 in $L^q(\Omega,\mathbb{R}^{d\times d})$. In addition, $\psi^0\in\Sigma(\bar\Omega)$ and $\{\,div\psi^{\tau_n}\mathcal{L}^d\}_n$ converges weak $*$ to $div\psi^0_{\mathbb{R}^d}$ in $\mathcal{M}(\bar\Omega)$. Furthermore, $\{k^{\tau_n}\}_n$ converges locally uniformly to a convex function $k^0\in C(\mathbb{R}^d)$, $\{l^{\tau_n}\}_n\subset C(\Lambda)$ converges locally uniformly to a convex function $l^0\in C(\Lambda)$ which admits an extension over $\bar\Lambda$ we still denote l^0 such that $k^0 = (l^0)^\sharp$ – in particular, $(k^0,l^0)\in\mathcal{C}$ – and*

$$\liminf_{n\to\infty} J(k^{\tau_n},l^{\tau_n},\psi^{\tau_n}) \ge J(k^0,l^0,\psi^0).$$

Proof. Note that $\mathcal{S}^\tau \subset \Sigma(\bar\Omega)$. Thanks to (94) and (95) we may apply Lemma 2.2 to conclude this proof. □

Theorem 6.4. *There exists $(k, l, \psi) \in \mathcal{C} \times \Sigma(\bar\Omega)$ and $\gamma \in \Gamma$ such that $k = l^\sharp$ and*

$$-J(k,l,\psi) = \max_{\mathcal{C}\times\Sigma(\bar\Omega)} -J = \min_{\Gamma} \bar I = \bar I(\gamma).$$

Proof. We first observe that, since every $\mathbf{F} \in L^1(\Omega, \mathbb{R}^d)$ can be approximated by a sequence of non–degenerate maps $\{\mathbf{F}_n\}_n \subset L^\infty(\Omega, \mathbb{R}^d)$ we may assume in the sequel, without loss of generality, that $\mathbf{F}$ is non–degenerate and bounded. In light of Lemma 6.2 it suffices to find $\gamma \in \Gamma$ such that $\bar I(\gamma) \le -J(k,l,\psi)$ where $(k,l) \in \mathcal{C}$ and $\psi \in \Sigma(\bar\Omega)$ are given by Lemma 6.3.

We define on $\bar C$ the measure γ^τ given by

$$\int_{\bar C} L d\gamma^\tau = \int_\Omega L\big(x, \det{}^* \nabla \mathbf{u}^\tau(x), \mathbf{u}^\tau(x), \nabla_{\mathcal{S}^\tau}\mathbf{u}^\tau(x)\big)dx \quad \text{for} \quad L \in C_c(\bar C).$$

Claim 1. $\{\gamma^\tau\}_\tau$ is pre–compact for the narrow convergence.

Proof of Claim 1. Let $\bar{\mathbf{u}} \in W^{1,\infty}(\Omega, \Lambda)$ be a homeomorphism of $\bar\Omega$ onto $\bar\Lambda$ such that $\det \nabla\bar{\mathbf{u}} > 0$, $\det \nabla\bar{\mathbf{u}} + (\det \nabla\bar{\mathbf{u}})^{-1} \in L^\infty(\Omega)$. By Lemma 3.5 $\{\det \nabla\bar{\mathbf{u}}\} = \det{}^* \nabla\bar{\mathbf{u}}$. We use the minimality property of $\mathbf{u}^\tau$ and by the bound in Remark 3.2 (ii) to obtain

$$\int_{\bar C} \big(f(\xi) + h(t) - \mathbf{F}(x)\cdot u\big)\gamma^\tau(dx,dt,du,d\xi) = I_{\mathcal{S}^\tau}(\mathbf{u}^\tau) \le I_{\mathcal{S}^\tau}(\bar{\mathbf{u}}) \le I(\bar{\mathbf{u}}) < \infty.$$

Thus, since the range of $\mathbf{u}^\tau$ is contained in $\bar\Lambda$ and the latter set in contained in the ball of radius r^*

$$\int_{\bar C} \big(f(\xi) + h(t)\big)\gamma^\tau(dx,dt,du,d\xi) \le I(\bar{\mathbf{u}}) + r^*||\mathbf{F}||_{L^1(\Omega)}. \tag{96}$$

By (17) and (19) the sub–level sets of $(x,t,u,\xi) \to f(\xi) + h(t)$ are compact subsets of $\bar C$ and so, we learn from (96) that $\{\gamma^\tau\}_{\tau\in\mathcal{T}}$ is a pre–compact subset of the set of Borel probability measures, for the narrow convergence. Let γ be a point of accumulation of $\{\gamma^\tau\}_{\tau\in\mathcal{T}}$ for the narrow convergence.

Claim 2. We claim that $\gamma \in \Gamma$.

Proof of Claim 2. Since $(x,t,u,\xi) \to f(\xi) + h(t)$ is lower semicontinuous and bounded below (cf. e.g. [1]), (96) implies that

$$\int_{\bar C} \big(f(\xi) + h(t)\big)\gamma(dx,dt,du,d\xi) \le \lim\inf_{\tau\to 0^+} \int_{\bar C} \big(f(\xi) + h(t)\big)\gamma^\tau(dx,dt,du,d\xi) \le I(\mathbf{u}) + r^*||\mathbf{F}||_{L^1(\Omega)}.$$

This, together with the first inequality in (19) implies that γ is supported by a compact subset of C and $\int_{\bar C} f(\xi)\gamma(dx,dt,du,d\xi) < \infty$. Because γ^τ satisfies the first two identities in (9), so does its point of accumulation γ.

It remains to check that the third identity in (9) holds to conclude that $\gamma \in \Gamma$. The set $\{(\mathbf{u}^\tau, \nabla_{\mathcal{S}^\tau}\mathbf{u}^\tau)\}_{\tau\in\mathcal{T}}$ is bounded in $L^p(\Omega,\mathbb{R}^d) \times L^p(\Omega, \mathbb{R}^{d\times d})$ and so, we may find a sequence $\{\tau_n\}_n$ extracted from the subsequence of Lemma 6.3, decreasing to 0 such that $\{(\mathbf{u}^{\tau_n}, \nabla_{\mathcal{S}^{\tau_n}}\mathbf{u}^{\tau_n})\}_n$ converges weakly to some $(\mathbf{u}, U)$ in $L^p(\Omega,\mathbb{R}^d) \times L^p(\Omega, \mathbb{R}^{d\times d})$ and $\mathcal{S}^{\tau_n} \subset \mathcal{S}^{\tau_{n+1}}$ for all $n \in \mathbb{N}$. Let $\phi \in C_c^\infty(\Omega, \mathbb{R}^{d\times d})$ and let $\{\phi^\tau\}_{\tau\in\mathcal{T}}$ be as in (49). Fix $n_0 \in \mathbb{N}$. Since $\phi^{\tau_n} \in \mathcal{S}^{\tau_{n_0}}$ for all $n \ge n_0$ we conclude that for all $n \ge n_0$

$$\int_\Omega \mathbf{u}^{\tau_n}\cdot \operatorname{div}\phi^{\tau_{n_0}} = -\int_\Omega \langle \phi^{\tau_{n_0}}, \nabla_{\mathcal{S}^{\tau_n}}\mathbf{u}^{\tau_n}\rangle \tag{97}$$

Letting first n tend to ∞ and then n_0 tend to ∞, we use (49) to conclude that

$$\int_\Omega \mathbf{u}\cdot\, \mathrm{div}\,\phi = -\int_\Omega \langle \phi, U\rangle.$$

Since ϕ is an arbitrary element of $C_c^\infty(\Omega,\mathbb{R}^{d\times d})$ we conclude that $\mathbf{u}\in W^{1,p}(\Omega,\mathbb{R}^d)$ and $U=\nabla\mathbf{u}$. Let $\mathbf{v}\in C_c(\mathbb{R}^d,\mathbb{R}^d)$ and $V\in C_c(\mathbb{R}^d,\mathbb{R}^{d\times d})$. We have

$$\int_\Omega \mathbf{u}\cdot\mathbf{v}dx = \lim_n \int_\Omega \mathbf{u}_n^\tau\cdot\mathbf{v}dx = \lim_n \int_{\bar{C}} u\cdot\mathbf{v}(x)\gamma_n^\tau(dx,dt,du,d\xi) = \int_{\bar{C}} u\cdot\mathbf{v}(x)\gamma(dx,dt,du,d\xi)$$

and so, $\mathbf{u}=\mathbf{u}_\gamma$. Similarly, one checks that

$$\int_\Omega \langle U,V\rangle dx = \int_{\bar{C}} \langle \xi, V\rangle\gamma(dx,dt,du,d\xi)$$

and so, $U=U_\gamma$. The equalities $\mathbf{u}=\mathbf{u}_\gamma$, $U=U_\gamma$ and $U=\nabla\mathbf{u}$ imply that the third identity in (9) holds, and this concludes the proof of Claim 2.

Claim 3. We claim that $\bar{I}(\gamma)=-J(k,l,\psi)$.

Proof of Claim 3. Since $(t,\xi)\to f(\xi)+h(t)$ is lower semicontinuous on $[0,\infty)\times\mathbb{R}^{d\times d}$ and bounded below and since $\mathbf{F}(x)\cdot u$ is bounded, we have that $\bar{I}$ is lower semicontinuous for the narrow convergence (cf. e.g. [1]). This, together with the lower semicontinuity property of J as provided by Lemma 6.3 implies

$$\bar{I}(\gamma)\le \liminf_{n\to\infty}\bar{I}(\gamma^{\tau_n}) = \liminf_{n\to\infty} I_{\mathcal{S}^{\tau_n}}(\mathbf{u}^{\tau_n}) = \liminf_{n\to\infty} -J(k^{\tau_n},l^{\tau_n},\psi^{\tau_n})\le -J(k,l,\psi).$$

□

Lemma 6.5. *Let (k,l,ψ) and γ be the optima found in Theorem 6.4 and let $\mathbf{u}$ be the u–moment of γ – denoted by $\mathbf{u}_\gamma$ in (10) – as introduced in the above proof. Then*

$$k(\mathbf{F}+\,div^a\psi_{\mathbb{R}^d}) + k^*(\mathbf{u}) = (\mathbf{F}+\,div^a\psi_{\mathbb{R}^d})\cdot\mathbf{u} \quad and \quad Df(\nabla\mathbf{u})=\psi \quad \mathcal{L}^d - a.e. \tag{98}$$

and

$$k(l^s)=\mathbf{u}_\psi \quad g^s - a.e. \quad where \quad l^s := \frac{d(\,div^s\psi_{\mathbb{R}^d})}{dg^s},\quad g^s := |\,div^s\psi_{\mathbb{R}^d}|. \tag{99}$$

Proof. Recall that by Theorem 5.4

$$\int_{\bar\Omega} k^{\tau_n}(\mathbf{F}+\mathrm{div}\,\psi^{\tau_n}) + \int_\Omega (k^*)^{\tau_n}(\mathbf{u}^{\tau_n})dx = \int_\Omega \mathbf{u}^{\tau_n}\cdot(\mathbf{F}+\mathrm{div}\,\psi^{\tau_n})dx = \int_\Omega \mathbf{u}^{\tau_n}\cdot\mathbf{F}dx - \int_{\bar\Omega}\langle\nabla_{\mathcal{S}^{\tau_n}}\mathbf{u}^{\tau_n},\psi^{\tau_n}\rangle dx$$

and so, since $Df(\nabla_{\mathcal{S}^{\tau_n}}\mathbf{u}^{\tau_n})=\psi^{\tau_n}$ we have

$$\int_{\bar\Omega}\big(k^{\tau_n}(\mathbf{F}+\,\mathrm{div}\,\psi^{\tau_n}) + (k^*)^{\tau_n}(\mathbf{u}^{\tau_n}) + f(\nabla_{\mathcal{S}^{\tau_n}}\mathbf{u}^{\tau_n})) + f^*(\psi^{\tau_n})\big)dx = \int_\Omega \mathbf{u}^{\tau_n}\cdot\mathbf{F}dx \tag{100}$$

Recall that $\{\nabla_{\mathcal{S}^{\tau_n}}\mathbf{u}^{\tau_n}\}_n$ converges weakly to $\nabla\mathbf{u}$ in $L^p(\Omega,\mathbb{R}^{d\times d})$ and f is convex bounded below; $\{\psi^{\tau_n}\}_n$ converges weakly to ψ in $L^q(\Omega,\mathbb{R}^{d\times d})$ and f^* is convex bounded below. Thus, the standard theory of the calculus of variations (cf. e.g. [6]) ensures that

$$\liminf_n \int_\Omega f(\nabla_{\mathcal{S}^{\tau_n}}\mathbf{u}^{\tau_n}))dx \ge \int_\Omega f(\nabla\mathbf{u})dx \quad \text{and} \quad \liminf_n\int_\Omega f^*(\psi^{\tau_n})dx\ge\int_\Omega f^*(\psi)dx \tag{101}$$

By Lemma 7.6

(102)

$$\liminf_n \int_{\bar\Omega} k^{\tau_n}(\mathbf{F}+\,\mathrm{div}\,\psi^{\tau_n}) \ge \int_{\bar\Omega} k(\mathbf{F}\mathcal{L}^d+\,\mathrm{div}\,\psi) \quad \text{and} \quad \liminf_n\int_\Omega (k^{\tau_n})^*(\mathbf{u}^{\tau_n})dx \ge \int_\Omega k^*(\mathbf{u})dx.$$

We combine (100), (101) and (102) to conclude that

$$
\begin{aligned}
& \int_{\bar\Omega} k(\mathbf{F} + \operatorname{div}\psi) + \int_\Omega \big(k^*(\mathbf{u}_\gamma) + f(\nabla \mathbf{u}_\gamma) + f^*(\psi)\big) dx \\
\leq \;& \liminf_n \int_{\bar\Omega} \big(k^{\tau_n}(\mathbf{F} + \operatorname{div}\psi^{\tau_n}) + (k^{\tau_n})^*(\mathbf{u}^{\tau_n}) + f(\nabla_{\mathcal{S}^{\tau_n}} \mathbf{u}^{\tau_n}) + f^*(\psi^{\tau_n})\big) dx \\
\leq \;& \limsup_n \int_{\bar\Omega} \big(k^{\tau_n}(\mathbf{F} + \operatorname{div}\psi^{\tau_n}) + (k^{\tau_n})^*(\mathbf{u}^{\tau_n}) + f(\nabla_{\mathcal{S}^{\tau_n}} \mathbf{u}^{\tau_n}) + f^*(\psi^{\tau_n})\big) dx \\
= \;& \int_\Omega \mathbf{u} \cdot \mathbf{F} dx \\
= \;& \int_{\bar\Omega} (\mathbf{u}_\gamma)_\psi \cdot (\mathbf{F}\mathcal{L}^d + \operatorname{div}\psi_{\mathbb{R}^d}) dx + \int_\Omega \langle \nabla \mathbf{u}, \psi \rangle dx.
\end{aligned}
$$

This proves (98) and (99).

Remark 6.6. Observe that by the inequalities in (103), we have established that

$$
\lim_n \int_\Omega f(\nabla_{\mathcal{S}^{\tau_n}} \mathbf{u}^{\tau_n}) dx = \int_\Omega f(\nabla \mathbf{u}) dx, \text{ and } \lim_n \int_\Omega f^*(\psi^{\tau_n})\big) dx = \int_\Omega f^*(\psi) dx.
$$

6.3. Sufficient condition for uniqueness.

Remark 6.7. By Theorem 6.4, there exists (k, l, ψ) that minimizes J over $\mathcal{C} \times \Sigma(\bar\Omega)$. Since $\bar I$ extends I we have $\inf_\Gamma \bar I \leq \inf_{W^{1,p}(\Omega,\Lambda)} I$. Let $\mathbf{v}$ a minimizer of I over $W^{1,p}(\Omega, \Lambda)$ and assume $\inf_\Gamma \bar I = \inf_{W^{1,p}(\Omega,\Lambda)} I$. Observe that if $\beta = \det^h \nabla \mathbf{v}$ then

$$
\gamma^{\beta,\mathbf{v}} := (\mathrm{id} \times \beta \times \mathbf{v} \times \nabla \mathbf{v})_\#(\mathcal{L}^d|_\Omega) \in \Gamma
$$

and

$$
\bar I(\gamma^{\beta,\mathbf{v}}) = \int_\Omega \big(f(\nabla \mathbf{v}) + h(\beta) - \mathbf{v} \cdot \mathbf{F}\big) dx = I(\mathbf{v}) = \inf_{W^{1,p}(\Omega,\Lambda)} I.
$$

Thus, $\gamma^{\beta,\mathbf{v}}$ minimizes $\bar I$ over Γ and so, by Lemma 6.2, $\nabla \mathbf{v} = Df^*(\psi)$. Since Λ coincides with $\mathbf{v}(\bar\Omega)$ up to a set of null measure and Ω is connected, we conclude that $\mathbf{v}$ is uniquely determined. Hence, it is the unique minimizer of I over $W^{1,p}(\Omega, \Lambda)$.

Throughout the remaining of this subsection we assume that $(l, k, \psi) \in \mathcal{C} \times \Sigma(\bar\Omega)$ is such that $l = k_\sharp$. We denote by $\operatorname{div}^s \psi_{\mathbb{R}^d}$ (resp. $\operatorname{div}^a \psi_{\mathbb{R}^d}$) the singular (resp. absolutely continuous) part of $\operatorname{div}\psi$ and set

$$
g^s = |\operatorname{div}^s \psi_{\mathbb{R}^d}| \quad \text{and} \quad l^s = \frac{d(\operatorname{div}^s \psi_{\mathbb{R}^d})}{dg^s}.
$$

Theorem 6.8. *Suppose* $\mathbf{u} \in W^{1,p}(\Omega, \Lambda)$, $\beta \in \det^* \nabla \mathbf{u}$, *satisfy the conditions*

$$
\nabla \mathbf{u} = Df^*(\psi), \quad \mathbf{u} \in \partial k(\mathbf{F} + \mathit{div}^a \psi), \quad h'(\beta) + l(\mathbf{u}) = 0 \quad \mathcal{L}^d - a.e. \tag{103}
$$

and

$$
\mathbf{u}_\psi \in \partial k^\infty(l^s) \quad g^s - a.e. \tag{104}
$$

Then $\mathbf{u}$ *minimizes* I *over* $W^{1,p}(\Omega, \Lambda)$ *and any other minimizer of* I *over* $W^{1,p}(\Omega, \Lambda)$ *coincides with* $\mathbf{u}$ $\mathcal{L}^d$*–a.e.*

Proof. By Lemma 4.3, (103) implies

$$k(\mathbf{F}+\operatorname{div}^a\psi)+\beta l(\mathbf{u})+h(\beta)=\mathbf{u}\cdot(\mathbf{F}+\operatorname{div}^a\psi)$$

and so,

$$\int_\Omega\big(k(\mathbf{F}+\operatorname{div}^a\psi)+\beta l(\mathbf{u})+h(\beta)\big)dx=\int_\Omega\mathbf{u}\cdot(\mathbf{F}+\operatorname{div}^a\psi)dx. \tag{105}$$

Similarly, we use (104) and (107) to conclude that

$$\int_{\bar\Omega}k^\infty(\operatorname{div}^s\psi)=\int_{\bar\Omega}\mathbf{u}\cdot\operatorname{div}^s\psi(dx). \tag{106}$$

We combine (32), (105), (106) and use the fact that $\beta\in\det^*\nabla\mathbf{u}$ to conclude that

$$\int_{\bar\Omega}k(\mathbf{F}+\operatorname{div}\psi)+\int_\Omega h(\beta)dx+\int_\Lambda ldy=\int_\Omega\mathbf{u}_\psi\cdot(\mathbf{F}\mathcal{L}^d+\operatorname{div}\psi)=\int_\Omega\mathbf{u}\cdot\mathbf{F}dx-\int_\Omega\langle\nabla\mathbf{u},\psi\rangle dx$$

Since $\nabla\mathbf{u}=Df^*(\psi)$ we have

$$\int_{\bar\Omega}k(\mathbf{F}+\operatorname{div}\psi)+\int_\Omega h(\beta)dx+\int_\Lambda ldy=\int_\Omega\mathbf{u}\cdot\mathbf{F}dx-\int_\Omega\big(f^*(\psi)+f(\nabla\mathbf{u})\big)dx$$

which is equivalent to

$$-J(k,l,\psi)=\int_\Omega\big(f(\nabla\mathbf{u})+h(\beta)-\mathbf{u}\cdot\mathbf{F}\big)dx\geq I(\mathbf{u}).$$

This, together with Remark 6.7 implies that $\mathbf{u}$ is the unique minimizer of I over $W^{1,p}(\Omega,\Lambda)$. □

Corollary 6.9. *Suppose all the minimizers (k,l,ψ) of J over $\mathcal{C}\times\Sigma(\bar\Omega)$ such that $k=l^\sharp$ are such that k is differentiable at $\mathbf{F}(x)+\,div\,\psi(x)$ for almost every $x\in\Omega$ (for example $k\in C^1(\mathbb{R}^d)$). Then $\bar{\mathbf{u}}:=\nabla k(\mathbf{F}+\,div^a\psi)$ is the unique minimizer of I over $W^{1,p}(\Omega,\bar\Lambda)$.*

Proof. Let (k,l,ψ) and γ be the optima found in Lemma 6.5 and let $\mathbf{u}_\gamma$ be the u–moment of γ. Since $\bar{\mathbf{u}}=\mathbf{u}_\gamma$ $\mathcal{L}^d$–almost everywhere on Ω, we have $\bar{\mathbf{u}}_\psi=(\mathbf{u}_\gamma)_\psi$. Let $\beta>0$ be the unique function defined by $h'(\beta)+l(\mathbf{u})=0$. By Lemma 6.5 and Theorem 6.8, it suffices to prove that $\beta\in\det^*\nabla\bar{\mathbf{u}}$ to conclude that $\bar{\mathbf{u}}$ is the unique minimizer of I over $W^{1,p}(\Omega,\bar\Lambda)$. But Proposition 5.3 (i) gives that $\beta\in\det^*\nabla\bar{\mathbf{u}}$. □

7. Appendix

7.1. **Integrals functionals on the set of Radon measures.** Let $\mathcal{M}(\bar\Omega)$ be the set of $m=(m_1,\cdots,m_d)$ such that $m_1,\cdots,m_d$ are signed Radon measures on $\mathbb{R}^d$, supported by $\bar\Omega$.

Definition 7.1. The following definitions can be found respectively in [13] and [15].

(i) The recession function of a convex function $k\in C(\mathbb{R}^d)$ is $k^\infty:\mathbb{R}^d\to(-\infty,\infty]$ given by

$$k^\infty(v)=\lim_{t\to\infty}\frac{k(v_0+tv)}{t}\qquad(v\in\mathbb{R}^d)\quad\text{where}\quad v_0\in\mathbb{R}^d\text{ is arbitrary.}$$

(ii) If $m\in\mathcal{M}(\bar\Omega)$, m^a is the absolutely continuous part of m and $m^s=m-m^a$,

$$\int_{\bar\Omega}k(m)=\int_{\bar\Omega}k(m^a)dx+\int_{\bar\Omega}k^\infty(m^s)=\int_{\bar\Omega}k(m^a)dx+\int_{\bar\Omega}k^\infty\Big(\frac{dm^s}{d\mu}\Big)d\mu,$$

where μ is any Radon measure such that $|m^s|<<\mu$. In particular, we can take $\mu=|m^s|$.

Remark 7.2. Observe that if $k \in C(\mathbb{R}^d)$ is a convex function, then k^∞ is homegeneous of degree 1. If in addition k is a Lipschitz function, then k^∞ assumes only finite values and so, it is also a Lipschitz function. Denote by D the domain of k^* and let ϱ_D^o be the support function of D:

$$\varrho_D^o(v) = \sup_u\{u \cdot v \mid u \in D\} = \sup_u\{u \cdot v \mid u \in \overline{D}\}.$$

If $u \in D$ then $k(tv)/t \geq v \cdot u - k^*(u)/t$ and so, $k^\infty(v) \geq v \cdot u$. Since $u \in D$ is arbitrary, we conclude that $k^\infty(v) \geq \varrho_D^o(v)$. Conversely, if $t > 0$ and $u_t \in \partial k(tv)$, we have $k(tv)/t = v \cdot u_t - k^*(u_t)/t \leq \varrho_D^o(v) + k(\vec{0})/t$. Letting t tend to ∞ we conclude that if $\Lambda \subset \operatorname{dom} k^* \subset \bar\Lambda$ then

$$k^\infty = \varrho_{\bar D}^o \quad \text{and} \quad \partial k^\infty(v) = \Big\{u \in \bar\Lambda \mid u \cdot v = k^\infty(v)\Big\}. \tag{107}$$

It is convenient to have other representation formulas for $\int_{\bar\Omega} k(m)$, useful for instance for proving the uniform lower semicontinuity property appearing in Lemma 7.6.

Define

$$\mathcal{K}_1(m) = \sup_{\mathbf{u} \in C_c(\mathbb{R}^d, \mathbb{R}^d)} \int_{\bar\Omega} \Big(\mathbf{u} \cdot m(dx) - k^*(\mathbf{u})\Big), \quad \mathcal{K}_2(m) = \sup_{\mathbf{u} \in C(\bar\Omega, \bar\Lambda)} \int_{\bar\Omega} \Big(\mathbf{u} \cdot m(dx) - k^*(\mathbf{u})\Big),$$

and

$$\mathcal{K}_3(m) = \max_{\mathbf{u} \in \mathbf{B}} \int_{\bar\Omega} \Big(\mathbf{u} \cdot m(dx) - k^*(\mathbf{u})\Big),$$

where, $\mathbf{B}$ is the set of bounded Borel maps $\mathbf{u} : \bar\Omega \to \bar\Lambda$.

Remark 7.3. If $k \in C(\mathbb{R}^d)$ is a convex Lipschitz function, $k^* \in L^1(\Lambda)$, $k^* \equiv \infty$ on $\mathbb{R}^d \setminus \bar\Lambda$ and $m \in \mathcal{M}(\bar\Omega)$, then

$$\int_{\bar\Omega} k(m) = \mathcal{K}_1(m) = \mathcal{K}_2(m) = \mathcal{K}_3(m).$$

Remark 7.4. Let k, m be as in Remark 7.3. For every Borel map $\mathbf{v} : \Omega \to \bar\Lambda$ and $\epsilon > 0$, we may find $\lambda' \in (0, 1)$ and $\mathbf{u} \in C(\bar\Omega, \mathbb{R}^d)$ such that $\mathbf{u}(\bar\Omega) \subset \{\varrho_\Lambda \leq \lambda'\} \subset\subset \Lambda$ and

$$\int_{\bar\Omega} k(m) \leq \epsilon + \int_{\bar\Omega} \Big(\mathbf{u} \cdot m(dx) - k^*(\mathbf{u})\Big).$$

Lemma 7.5. *Let $k \in C(\mathbb{R}^d)$ be a convex function that is e_0–Lipschitz and such that the interior of $\operatorname{dom} k^*$ contains the closed ball $\bar B_r$, centered at the origin and of radius $r > 0$. Then $\int_{\bar\Omega} k(m) < \infty$ if and only if $\int_{\bar\Omega} k_0(m) < \infty$ where $k_0(v) \equiv |v|$. In that case there exists a constant e_r which depends only on the measure of Λ, r and any number bigger than $||k^*||_{L^1(B_{2r})}$ such that*

$$r \int_{\bar\Omega} k_0(m) - e_r \leq \int_{\bar\Omega} K(m) \leq e_0 \int_{\bar\Omega} k_0(m) + e_r.$$

Proof. To obtain the above upper bound, we use the fact that k is e_0–Lipschitz. The lower bound is a direct consequence of the fact that

$$k(v) \geq v \cdot \frac{rv}{|v|} - k^*(\frac{rv}{|v|})$$

and the fact that the theory of convex analysis ensures that (cf. e.g. [7]) $||k^*||_{L^\infty(B_r)}$ is bounded by a constant which depends only on r and $||k^*||_{L^1(B_{2r})}$. $\square$

Lemma 7.6. *Let $\{k^n\}_n \subset C(\mathbb{R}^d)$ be a sequence of convex e_0–Lipschitz functions converging pointwise to k and suppose $\{(k^n)^*\}_n$ is a bounded sequence in $L^1(\Lambda)$. If $m \in \mathcal{M}(\bar\Omega)$ and $\{m^n\}_n \subset \mathcal{M}(\bar\Omega)$ converges distributionally to m, then*

$$\int_{\bar\Omega} k(m) \leq \liminf_{n\to\infty} \int_{\bar\Omega} k^n(m^n).$$

Proof. To prove the theorem, we may assume without loss of generality that

$$l := \sup_n \int_{\bar\Omega} k^n(m^n) < \infty. \tag{108}$$

Observe first that $k : \mathbb{R}^d \to \mathbb{R}$ is convex and e_0–Lipschitz as a pointwise limit of convex and e_0–Lipschitz functions. The sequence $\{(k^n)^*\}_n$ being bounded in $L^1(\Lambda)$, the theory of convex analysis (cf. e.g. [7]) ensures that $\{(k^n)^*\}_n$ is bounded in $W^{1,\infty}(O)$ is O is a open set such that $\bar O \subset \Lambda$. In particular, if $\lambda \in (0,1)$ since $\{u \in \Lambda \mid \varrho(u) \leq \lambda\}$ is a compact set contained in Λ, we use the Ascoli–Arzela Theorem to obtain that up to a subsequence, $\{k_n^*\}_n$ converges uniformly on $\{\varrho \leq \lambda\}$ to some function. From the fact that $\{k_n\}_n$ converges pointwise to k we infer that k^* must be that function and in fact, the whole sequence $\{k_n^*\}_n$ converges to k^*. Since, $\lambda \in (0,1)$ is arbitrary, we conclude that $\{k_n^*\}_n$ converges pointwise to k^* on Λ. Similarly, $\{k^n\}_n$ converges uniformly to k on compact sets. In particular, $\{k^n\}_n$ is bounded in $C(\bar B_1)$ where $\bar B_1$ is the closed unit ball centered at the origin. Thus, if $u \in \Lambda$,

$$k_n^*(u) \geq u \cdot \frac{u}{|u|} - k_n(\frac{u}{|u|}) \geq |u| - \sup_n ||k_n||_{C(\bar B_1)}.$$

With that lower bound at hand, since Λ is a bounded set, we may apply Fatou's Lemma and obtain

$$\int_\Lambda k^* dy \leq \liminf_{n\to\infty} \int_\Lambda (k^n)^* dy < \infty$$

and k^* is bounded below.

Let $r > 0$ be such that the closed ball centered at the origin and of radius $2r$, is contained in Λ. By Lemma 7.5

$$\int_{\bar\Omega} |m| \leq \lim \inf_{n\to\infty} \int_{\bar\Omega} |m^n| \leq \sup_{n\in\mathbb{N}} \int_{\bar\Omega} |m^n| < \infty.$$

This, together with Lemma 7.5 again, shows that $\int_{\bar\Omega} k(m)$ is finite. Thus, given $\epsilon > 0$ arbitrary, we may use Remark 7.4 to infer existence of $\mathbf{u}_\epsilon \in C_c(\bar\Omega, \bar\Lambda)$ such that the range of $\mathbf{u}_\epsilon$ is contained in a compact set of the form $\{\varrho \leq \lambda\} \subset \Lambda$ and

$$\int_{\bar\Omega} k(m) \leq \epsilon + \int_{\bar\Omega} \Big(\mathbf{u}_\epsilon \cdot m - k^*(\mathbf{u}_\epsilon)\Big). \tag{109}$$

Because $\{k_n^*\}_n$ converges uniformly to k^* on $\{\varrho \leq \lambda\}$, it follows that

$$\int_{\bar\Omega} \Big(\mathbf{u}_\epsilon \cdot m - k^*(\mathbf{u}_\epsilon)\Big) = \lim_{n\to\infty} \int_{\bar\Omega} \Big(\mathbf{u}_\epsilon \cdot m^n - (k^n)^*(\mathbf{u}_\epsilon)\Big) \leq \lim_{n\to\infty} \int_{\bar\Omega} k^n(m^n).$$

We combine this, together with (109) and take into account that $\epsilon > 0$ is arbitrary to conclude the proof. □

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