On Ternary Quadratic Forms

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Dedicated to the memory of Arnold E. Ross

1 Introduction.

Let $q(x) = q(x_1, x_2, x_3)$ be a positive definite ternary quadratic form with integral coefficients. In 1946 Ross and Pall [RP] conjectured that every sufficiently large square-free integer that is represented by q modulo N for all N is in fact integrally represented. This conjecture was proven in 1988 [Du1] as an application of bounds for sums of Kloosterman sums given by Iwaniec [Iw] (see also [DS-P] and [Du2] for an exposition and references.)

In this paper I will quantify "sufficiently large" in this result in terms of the determinant D of q, which is defined by the 3×3 determinant

$$D = \det(\partial^2 q/\partial x_i \partial x_j)$$

and is known to be a positive even integer.

Theorem 1 *There is an absolute constant $c > 0$ so that $q(x) = n$ has an integral solution provided that*

$$n > c\, D^{337}$$

is square-free and that the congruence $q(x) \equiv n \pmod{8D^3}$ has a solution. The constant c is ineffective.

2 A uniform asymptotic formula.

Theorem 1 follows from a uniform asymptotic formula for the number of representations

$$r_q(n) = \#\{x \in \mathbb{Z}^3\, ;\, q(x) = n\}\,.$$

This quantity does not change if q is replaced by a $\mathbb{Z}$-equivalent form, that is one obtained from q by an invertible integral change of variables. The genus

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G of q consists of those forms that are equivalent to q under invertible variable changes over the p-adic integers, for all p. The genus G is known to split into finitely many $\mathbb{Z}$-equivalence classes. The average number of representations of n by forms in G is

$$r(n, G) = M(G)^{-1} \sum_{\{q\} \in G} \omega_q^{-1} r_q(n) \tag{1}$$

the sum being over $\mathbb{Z}$-inequivalent q in G. Here ω_q is the number of automorphs of q and

$$M(G) = \sum_{\{q\} \in G} \omega_q^{-1} \tag{2}$$

is the mass (or weight) of the genus G. We shall establish the following uniform asymptotic formula.

Theorem 2 *Fix $\epsilon > 0$. Then for n square-free we have*

$$r_q(n) = r(n, G) + O(D^{\frac{11}{2}} n^{\frac{1}{2} - \frac{1}{28}} (Dn)^{\epsilon})$$

where the implied constant depends only on ϵ.

It follows from the fundamental work of Siegel [Si] that for n square-free we have the formula

$$r(n, G) = \kappa\, L(1, \chi) \mathcal{N}(n; q) \sqrt{n/D}, \tag{3}$$

where $\chi(\cdot) = \left(\frac{-2Dn^*}{\cdot}\right)$ is the Kronecker symbol with n^* the discriminant of $\mathbb{Q}(\sqrt{n})$ and $L(s, \chi)$ is the Dirichlet L-function. Also

$$\mathcal{N}(n; q) = \#\{x \in (\mathbb{Z}/m\mathbb{Z})^3\,; q(x) \equiv n (\bmod\ m)\} m^{-2} \tag{4}$$

for $m = (2D)^3$ and $\kappa > 0$ is a constant. Theorem 1 is a consequence of the Theorem 2 together with the following lower bound, that is valid for all $\epsilon > 0$ under the condition that the congruence $q(x) \equiv n \,(\bmod\, 8D^3)$ has a solution:

$$r(n, G) \gg_{\epsilon} n^{\frac{1}{2}} D^{-\frac{13}{2}} (nD)^{-\epsilon}. \tag{5}$$

This follows from (3) and Siegel's lower bound for $L(1, \chi)$ and is where the ineffectivity of c in Theorem 1 arises.

As in [DS-P] one may refine Theorem 2 by replacing $r(n, G)$ by the associated quantity for the spinor genus of q. This allows one to weaken the condition on n to be that the square part of n be prime to the level of q (or has a limited ged with the level). Further refinements as in [DS-P] should be possible as well.

3 Weight 3/2 modular forms.

In this section we will reduce the proof of Theorem 2 to an upper bound for the L^2 norm of a cusp form by applying a known estimate from [DS-P] for the Fourier coefficients of a cusp form of weight 3/2.

The level of q is the smallest $N \in \mathbb{Z}^+$ so that NA^{-1} is even integral, where $A = (\partial^2 q/\partial x_i \partial x_j)$. It is known that (see [Le] p. 404)

$$N \,|\, 2D \,|\, 4N^2 . \tag{6}$$

Let $a(n) = r(n, G) - r_q(n)$. Then by [Si] we have that

$$f(z) = \sum_{n=1}^{\infty} a(n) e(nz) \in S_{\frac{3}{2}}(\Gamma_0(N), \psi) ,$$

the space of cusp forms of weight $\frac{3}{2}$ for $\Gamma_0(N)$ with character ψ, a real character mod N. By the proof of Lemma 2 in [DS-P] we have that for n square-free

$$|a(n)| \ll_\epsilon \|f\| N^{\frac{1}{2} - \frac{1}{28} + \epsilon} \tag{7}$$

where

$$\|f\|^2 = \int_{\Gamma_0(N)\backslash H} |f|^2 y^{\frac{3}{2}} \, d\mu$$

with $d\mu = dx\, dx/y^2$. To proceed, we must estimate $\|f\|$. Note that we are using $\Gamma_0(N)$ instead of $\Gamma_1(N)$ as is done in Lemma 2 of [DS-P]. A variant of the argument used in the next result occurred in [DFI].

Lemma 1 *Let $f(z) = \sum_{n=1}^{\infty} a(n) e(nz)$ be a cusp form of weight $\frac{3}{2}$ in $S_{\frac{3}{2}}(\Gamma_0(N), \psi)$. Then*

$$\|f\|^2 \ll \Gamma(\alpha)\, d(N)\, N^{2\alpha} \sum_{n=1}^{\infty} |a(n)|^2 n^{-\alpha},$$

where $\alpha > \frac{1}{2}$ is any number so that the series converges. Here $d(\cdot)$ is the divisor function and the implied constant is absolute.

Lemma 1 will be proven in the next section. Granting this, to finish the proof of Theorem 2 we need an upper bound for $a(n) = r(n, G) - r_q(n)$ that is uniform in D, hence in N. From (1) we have

$$|a(n)| \ll M(G) r(n, G) \tag{8}$$

since $\omega_q \le 48$ ([Ne] p. 180). Now from (2)

$$M(G) \le h_3(D)$$

where $h_3(D)$ is the number of all classes of q with determinant D. By Minkowski reduction theory (see [Ca]) we have the crude bound

$$h_3(D) \ll \sum_{\substack{0<a\le b\le c \\ abc\le 2D}} a^2 b \ll D^2 \,.$$

Thus from (3) and (8)

$$|a(n)| \ll_\epsilon N^{\frac{1}{2}} D^{\frac{3}{2}} (nD)^\epsilon$$

since from (4)

$$\mathcal{N}(n;q) \ll D^\epsilon \,.$$

Taking $\epsilon = 2 + 2\epsilon$ in the lemma gives

$$\|f\|^2 \ll N^4 D^3 (ND)^{4\epsilon}$$

and so Theorem 2 follows by (6) and (7).

4 A bound for the L^2 norm of a cusp form.

In this section we will prove Lemma 1 and thus complete the proof of Theorem 2. Let $\mathcal{D}$ be the standard fundamental domain for the modular group $\Gamma = \Gamma_0(1)$

$$\mathcal{D} = \{ z = x + iy;\ |x| \le 1/2 \text{ and } |z| \ge 1 \} \,.$$

A fundamental domain for $\Gamma_0(N)$ is $\cup_\sigma \sigma\mathcal{D}$, where σ runs over coset representatives for $\Gamma_0(N)$ in Γ. Thus

$$\|f\|^2 = \sum_\sigma \int_{\mathcal{D}} |f(\sigma z)|^2 \mathrm{Im}(\sigma z)^{\frac{3}{2}} d\mu. \tag{9}$$

For each σ we have a Fourier expansion in the cusp $\sigma(\infty)$:

$$f(\sigma z) = \varepsilon_\sigma \sum_{n>0} a_\sigma(n) e\left(\frac{nz}{w_\sigma}\right)$$

where $|\varepsilon_\sigma| = 1$ and w_σ is the width of the cusp. Thus

$$\begin{aligned}\int_{\mathcal{D}} |f(\sigma z)|^2 (\mathrm{Im}\ \sigma z)^{\frac{3}{2}} d\mu &\le \int_{\frac{\sqrt{3}}{2}}^{\infty} \int_0^{w_\sigma} |f(\sigma z)|^2 (\mathrm{Im}\ \sigma z)^{\frac{3}{2}} d\mu \\ &= w_\sigma \int_{\frac{\sqrt{3}}{2}}^{C_\sigma} \sum_n |a_\sigma(n)|^2 e^{-4\pi n y/w_\sigma} y^{-1/2} dy \\ &= 2 \int_{\frac{\sqrt{3}}{2}}^{C_\sigma} \int_{\frac{-w_\sigma}{2}}^{\frac{w_\sigma}{2}} |f(\sigma z)|^2 (\mathrm{Im}\ \sigma z)^{\frac{3}{2}} d\mu\end{aligned} \tag{10}$$

where $C_\sigma = \frac{\sqrt{3}}{2} + w_\sigma C$ and $C > 0$ is an absolute constant so that

$$\int_t^\infty e^{-y} y^{-\frac{1}{2}} dy \leq 2 \int_t^{t+C} e^{-y} y^{-\frac{1}{2}} dy$$

for all $t > 0$. We may choose the coset representations

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \text{ with } \gamma \mid N \text{ and } |\delta| \leq \frac{\gamma}{2}.$$

For such σ the cusp $\sigma(\infty) = \frac{\alpha}{\gamma}$ has width $w_\sigma = N/(\gamma^2, N) \leq N/\gamma$, so that

$$C_\sigma = \frac{\sqrt{3}}{2} + w_\sigma C \leq \frac{\sqrt{3}}{2} + \frac{NC}{\gamma}.$$

Now for z in range of (10) we have

$$\operatorname{Im}(\sigma z) = y((\gamma x + \delta)^2 + \gamma^2 y^2)^{-1} \geq 4y((N+\gamma)^2 + \gamma^2 y^2)^{-1} \gg C^{-1} N^{-2}$$

and

$$\left| \operatorname{Re}(\sigma z) - \frac{\alpha}{\gamma} \right| = \gamma^{-2} \left| x + \frac{\delta}{\gamma} \right| \left(\left(x + \frac{\delta}{\gamma} \right)^2 + y^2 \right)^{-1} \qquad \ll \gamma^{-2} y^{-1} \leq 1.$$

Thus changing variables z to $\sigma^{-1} z$ in (10) we get

$$\int_D |f(\sigma z)|^2 (\operatorname{Im} \sigma z)^{\frac{3}{2}} \, d\mu \ll \int_{C^{-1}N^{-2}} \int_0^1 |f(z)|^2 \, y^{\frac{3}{2}} d\mu$$

and hence from (9)

$$\|f\|^2 \ll N d(N) \int_{C^{-1}N^{-2}}^\infty \int_0^1 |f(z)|^2 \, y^{\frac{3}{2}} \, d\mu \tag{11}$$

since the index of $\Gamma_0(N)$ in Γ is

$$N \prod_{p|N} \left(1 + \frac{1}{p} \right) \ll N d(N).$$

Putting the Fourier expansion of $f(z)$ into (11) we get

$$\begin{aligned} \|f\|^2 \ll N d(N) \sum_{n=1}^\infty |a(n)|^2 \int_{C^{-1}N^{-2}}^\infty e^{-4\pi n y} \, y^{\frac{1}{2}} \frac{dy}{y} \\ = C^{-\frac{1}{2}} d(N) \sum_{n=1}^\infty |a(n)|^2 \int_1^\infty e^{-\frac{4\pi n t}{CN^2}} t^{\frac{1}{2}} \frac{dt}{t} \\ = C^{-\frac{1}{2}} d(N) \sum_{n=1}^\infty |a(n)|^2 \int_1^\infty e^{-\frac{4\pi n t}{CN^2}} t^{\alpha} \, \frac{dt}{t} \end{aligned}$$

for any $\alpha > \frac{1}{2}$, and this is

$$\leq C^{-\frac{1}{2}} d(N) \sum_{n=1}^{\infty} |a(n)|^2 \int_0^{\infty} e^{-\frac{4\pi nt}{CN^2}} t^{\alpha} \; \frac{dt}{t}$$

$$\ll \Gamma(\alpha) N^{2\alpha} d(N) \sum_{n=1}^{\infty} |a(n)|^2 n^{-\alpha}$$

for any $\alpha > \frac{1}{2}$ so that the series converges.

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