ON SUMS OF FRACTIONAL PARTS

W. DUKE

Abstract. The difference between the sums of the fractional parts of integer multiples of two irrational numbers, when these irrationals differ by a rational number, is unbounded unless the difference is an integer.

1. Introduction

For $\alpha \in \mathbb{R}$ let $\{a\} = a - \lfloor a \rfloor$. For irrational α and a nonnegative integer n define

$$S(\alpha, n) = \sum_{k=1}^{n} (\{k\alpha\} - \tfrac{1}{2}).$$

This deceptively simple looking sum has been thoroughly studied for a long time,[1] but it still presents attractive unsolved problems. It is well known (see e.g. [10, p.104]) that $|S(\alpha, n)|$ is unbounded in n for a fixed irrational α. Since it is obvious that $S(\alpha, n) + S(\beta, n) = 0$ if $\alpha + \beta \in \mathbb{Z}$, a natural question arises: is it possible for

$$|S(\alpha, n) + S(\beta, n)|$$

to be bounded when α is irrational and $\alpha + \beta \in \mathbb{Q}$ is not an integer?

Theorem 1. *Suppose that α is irrational and that $\alpha+\beta$ is rational. Then the values of $|S(\alpha, n) + S(\beta, n)|$ are unbounded in n unless $\alpha + \beta \in \mathbb{Z}$, in which case it is zero.*

As a consequence, given any (irrational) real quadratic α, we have that $|S(\alpha, n) + S(\alpha', n)|$ is bounded if and only if $\alpha + \alpha' \in \mathbb{Z}$, where α' is the conjugate of α. Under the additional assumption that $\alpha\alpha' = 1$, this consequence was conjectured in [3, Conj. 6.17] in relation to some interesting problems in symplectic geometry about symplectic embeddings of ellipsoids (see also [12]).

The sum S arises in the problem of counting lattice points in a right triangle whose sides are on the positive axes. This connection is also behind its appearance in [3]. Suppose that $\alpha, \beta > 0$. Consider the counting function of lattice points inside the closed triangle Δ with vertices at $(0, 0), (0, \alpha)$ and $(0, \beta)$, when it is scaled by $t > 0$:

$$F(t) = \#(t\Delta \cap \mathbb{Z}^2).$$

A very special case of a well-known result of Ehrhart (see [2]) implies that for integers α, β and *integral* ℓ the function $F(\ell)$ is a quadratic polynomial in ℓ. Explicitly, when $\gcd(\alpha, \beta) = 1$, we have

$$F(\ell) = \tfrac{\alpha\beta}{2}\ell^2 + \tfrac{\alpha+\beta+1}{2}\ell + 1. \tag{1}$$

[1]See [10, IX, §2] for a summary of the classical literature on S up until about 1935. A more recent source is [11], especially Chapter 2. An elegant elementary approach to their theory was given in [15] (see the Math Review MR0006753 for some corrections). The book [1] contains a striking central limit theorem for $S(\alpha, n)$, when α is real quadratic.

For general rational α, β, Ehrhart's result is that (1) still holds provided we replace the coefficient of ℓ and 1 by certain periodic functions of ℓ having integral periods. Although they need not be constant, these periodic coefficients are clearly still bounded.

Suppose now that $\frac{\alpha}{\beta}$ is irrational and define, for any $t > 0$,

$$C(t) = F(t) - \left(\tfrac{\alpha\beta}{2}t^2 + \tfrac{1}{2}(\alpha+\beta)t\right). \tag{2}$$

By [7, Theorem A1] we have that $C(t) = o(t)$. When is $C(\ell)$ bounded for integers ℓ? For certain α, β, the answer follows easily from Theorem 1.

Corollary 1. *Suppose that $\alpha, \beta = \alpha'$ are the (real quadratic) solutions to*

$$ax^2 - bx + b = 0,$$

where $a, b \in \mathbb{Z}^+$ are such that $b^2 - 4ab > 0$ is not a square and $\gcd(a, b) = 1$*. Then $|C(\ell)|$ is bounded if and only if α, β are real quadratic integers, in which case $C(\ell) = 1$.*

Proof. For general $\alpha, \beta > 0$ and $m, n \in \mathbb{Z}^+$ we have the identity

$$C(\tfrac{m}{\alpha} + \tfrac{n}{\beta}) = 1 - S(\tfrac{\alpha}{\beta}, n) - S(\tfrac{\beta}{\alpha}, m).$$

For a proof see [16, Theorem I]. Our assumptions imply that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, so

$$C(\ell) = 1 - S(\tfrac{\alpha}{\beta}, \ell) - S(\tfrac{\beta}{\alpha}, \ell).$$

The result now follows from Theorem 1 after noting that α, β are real quadratic integers exactly when $a = 1$, while

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{(\alpha+\beta)^2}{\alpha\beta} - 2 = \frac{b}{a} - 2.$$

$\square$

Remark. Under the assumptions of Corollary 1, when α, β are real quadratic integers we have for $\ell \in \mathbb{Z}^+$ that

$$F(\ell) = \tfrac{b}{2}\ell^2 + \tfrac{b}{2}\ell + 1,$$

which is an example of the Ehrhart function of a pseudo-integral triangle (see [4]).

2. Proof of Theorem 1

Theorem 1 follows without difficulty from a result of Schoißengeier [17], which is a reformulation of one of Oren [13]. These papers give very useful developments of earlier work on local discrepancies of the sequence $\{k\alpha\}$, especially [6], [8], [9], [14]. Here "local" refers to the estimation of the discrepancy from uniform distribution of the sequence when measured with respect to a fixed interval or, more generally, with respect to integration of a fixed function.

Set $D(\alpha, \gamma, n) = S(\alpha + \gamma, n) - S(\alpha, n)$ where α is irrational and γ is rational. We want to show that $|D(\alpha, \gamma, n)|$ is unbounded unless $\gamma \in \mathbb{Z}$. Let $\gamma = \frac{p}{q} \in \mathbb{Q}$ be in reduced form with $q > 1$. We have

$$D(\alpha, \gamma, n) = \sum_{1 \le \ell \le n} \{\ell(\alpha + \tfrac{p}{q})\} - \{\ell\alpha\}.$$

Write $\ell = qk - r$ and $\delta = q\alpha$. We may assume that $1 \le r \le q-1$ and $1 \le k \le m$ if $n = mq - 1$. Note that terms with $r = 0$ are zero and can be omitted. By splitting into arithmetic progressions modulo q, it follows that

$$D(\alpha,\gamma,n) = \sum_{1\le r<q} \sum_{1\le k\le m} \{k\delta - r\alpha - \tfrac{pr}{q}\} - \{k\delta - r\alpha\}.$$

Next apply the elementary identity for $x \in \mathbb{R}$:

$$\{k\delta + x\} = \{k\delta\} + \chi_{[0,\{-x\})}(k\delta) - \{-x\},$$

where χ is the usual characteristic function made $\mathbb{Z}$-periodic. Thus

$$\begin{aligned} D(\alpha,\gamma,n) = \sum_{1\le k\le m} \Big(\sum_{1\le r<q} \chi_{[0,\{r\alpha+\frac{pr}{q}\})}(k\delta) - \chi_{[0,\{r\alpha\})}(k\delta) \Big) \\ - m \sum_{1\le r<q} \{r\alpha + \tfrac{pr}{q}\} - \{r\alpha\} \\ = \sum_{k\le m} f(k\delta) - m \int_0^1 f(x)dx, \end{aligned}$$

where f is a periodic step function. It follows from Cor. 3 of [17] that $D(\alpha,\gamma,n)$ is bounded if and only if f is in the space of periodic step functions generated by functions of the form $\chi_{I+\mathbb{Z}}(x)$, where $I \subset [0,1)$ is an interval whose length is in $\mathbb{Z} + q\alpha\mathbb{Z}$. Since α is irrational and $1 \le r < q$, we see that f is not in this space, proving Theorem 1. See Figure 1 for an illustration of a step function f that arises.

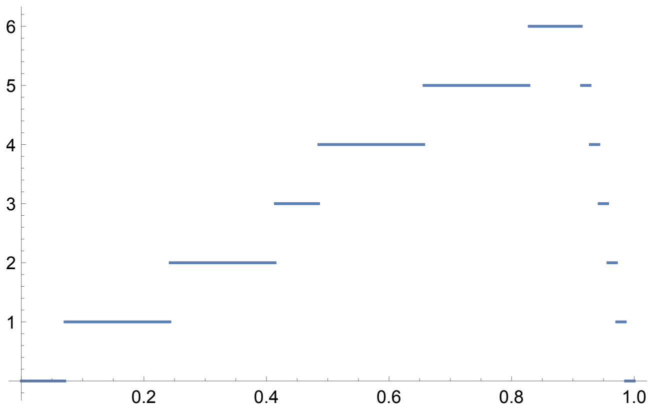

FIGURE 1. The step function f when $\alpha = \sqrt{2}$ and $\frac{p}{q} = \frac{4}{7}$

In case $q = 2$ the above calculation is quite transparent and the result is a consequence of [9] or [6]. For this, assume that $\frac{p}{q} = \frac{1}{2}$ and that $0 < \alpha < \frac{1}{2}$. Then

$$D(\alpha, \tfrac{1}{2}, 2m-1) = \sum_{1\le k\le m} \chi_{[\alpha,\alpha+\frac{1}{2})}(2k\alpha) - \tfrac{m}{2},$$

which is the local discrepancy of the sequence $\{2k\alpha\}$ for $1 \le k \le m$ in $[\alpha, \alpha + \frac{1}{2})$. By [9] this is unbounded since $\frac{1}{2} \notin \mathbb{Z} + 2\alpha\mathbb{Z}$. $\square$

Remark. An apparently quite difficult problem is to give criteria for the one-sided boundedness of $D(\alpha, \gamma, n)$. In particular, the possible one-sided boundedness of C from (2) is of interest for the problems of [3] mentioned above. This issue does not seem to have been extensively treated for general local discrepancies. Even simple local discrepancies like that of $\{2k\alpha\}$ in the interval $[\alpha, \alpha + \frac{1}{2})$ remain mysterious. Some results are given in [5].

Acknowledgements. I thank Peter Sarnak for informing me of the conjecture in [3] and Dan Cristofaro-Gardiner for some helpful comments. I also thank the referee for a useful correction and comment.

References

[1] Beck, J., Probabilistic Diophantine approximation. Randomness in lattice point counting. Springer Monographs in Mathematics. Springer, Cham, 2014. xvi+487 pp.

[2] Beck, M.; Robins, S. Computing the continuous discretely. Integer-point enumeration in polyhedra. Second edition. With illustrations by David Austin. Undergraduate Texts in Mathematics. Springer, New York, 2015. xx+285 pp.

[3] Cristofaro-Gardiner,D; Holm,T.S.; Mandini, A.; Pires A.R., On infinite staircases in toric symplectic four-manifolds, arXiv:2004.13062

[4] Cristofaro-Gardiner, D.; Li, T.X. ; Stanley, R. P. Irrational triangles with polynomial Ehrhart functions. Discrete Comput. Geom. 61 (2019), no. 2, 227–246.

[5] Dupain, Y.; Sós, V. T. On the one-sided boundedness of discrepancy-function of the sequence $\{na\}$. Acta Arith. 37 (1980), 363–374.

[6] Furstenberg, H.; Keynes, H.; Shapiro, L. Prime flows in topological dynamics. Israel J. Math. 14 (1973), 26–38.

[7] Hardy, G. H.; Littlewood, J. E. Some Problems of Diophantine Approximation: The Lattice-Points of a Right-Angled Triangle. Proc. London Math. Soc. (2) 20 (1921), no. 1, 15–36.

[8] Hecke, E. Über analytische Funktionen und die Verteilung von Zahlen mod. eins. Abh. Math. Sem. Univ. Hamburg 1 (1922), no. 1, 54–76., also in Werke.

[9] Kesten, H. On a conjecture of Erdös and Szüsz related to uniform distribution mod 1. Acta Arith. 12 (1966/67), 193–212.

[10] Koksma, J. F. Diophantische Approximationen. Reprint. Springer-Verlag, Berlin-New York, 1974. viii+157 pp.

[11] Kuipers, L.; Niederreiter, H. Uniform distribution of sequences. Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974. xiv+390 pp.

[12] McDuff, D. ; Schlenk, F. The embedding capacity of 4-dimensional symplectic ellipsoids. Ann. of Math. (2) 175 (2012), no. 3, 1191–1282.

[13] Oren, I, Admissible functions with multiple discontinuities. Israel J. Math. 42 (1982), no. 4, 353–360.

[14] Ostrowski, A. Notiz zur Theorie der Diophantischen Approximationen. Jahresbericht d.D.M.V., 36, 1927, pp. 178–180., 39, 1930, pp. 34-46, In Collected Math. Papers Vol. 3.

[15] Pillai, S. S. On a problem in Diophantine approximation. Proc. Indian Acad. Sci., Sect. A. 15, (1942). 177–189.

[16] Pillai, S. S. Lattice points in a right-angled triangle. II. Proc. Indian Acad. Sci., Sect. A. 17, (1943). 58–61.

[17] Schoißengeier, J., Regularity of distribution of (na)-sequences. Acta Arith. 133 (2008), no. 2, 127–157.

UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555
Email address: wdduke@ucla.edu