Number fields with large class groups

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Abstract

After a review of the quadratic case, a general problem about the existence of number fields of a fixed degree with extremely large class numbers is formulated. This problem is solved for abelian cubic fields. Then some conditional results proven elsewhere are discussed about totally real number fields of a fixed degree, each of whose normal closure has the symmetric group as Galois group.

1 Introduction.

It was Littlewood who first addressed the question of how large the class number h of an imaginary quadratic field $\mathbb{Q}(\sqrt{d})$ can be as a function of $|d|$ as $d \to -\infty$ through fundamental discriminants. In 1927 [14] he showed, assuming the generalized Riemann hypothesis (GRH), that for all fundamental $d < 0$

$$h \leq 2(c + \mathrm{o}(1))|d|^{\frac{1}{2}} \log\log|d|, \tag{1}$$

where $c = e^{\gamma}/\pi$, where γ is Euler's constant. Furthermore he showed, still under GRH, that there are infinitely many such d with

$$h \geq (c + \mathrm{o}(1))|d|^{\frac{1}{2}} \log\log|d|. \tag{2}$$

He deduced these results from the class number formula of Dirichlet, which for $d < -4$ states that

$$h = \frac{|d|^{\frac{1}{2}}}{\pi} L(1, \chi)$$

for $\chi(\cdot) = \left(\frac{d}{\cdot}\right)$ the Kronecker symbol, and corresponding results about the Dirichlet L-value $L(1, \chi)$. His analysis is based on the idea that under GRH one can approximate $\log L(s, \chi)$ by a very short sum over primes of the type[1]

$$\log L(1, \chi) = \sum_{p \leq (\log|d|)^{\frac{1}{2}}} \chi(p)p^{-1} + \mathrm{O}(1). \tag{3}$$

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[1] In fact, Littlewood used a more elaborate approximation in order to obtain good constants.

This easily gives (1) up to the constant, since

$$\sum_{p\leq x} p^{-1} \sim \log\log x$$

as $x \to \infty$. To prove (2) using (3), again up to the constant, one must produce infinitely many $d < 0$ with $\chi(p) = 1$ for all p with $p \leq (\log|d|)^{\frac{1}{2}}$, which is easily accomplished. Later Chowla [3, 4] proved (2) unconditionally.

In questions about the size of the class number of real quadratic fields the distribution of the regulator $R = \log\varepsilon$, for ε a fundamental unit for the field, causes new problems in the class number formula for $d > 0$:

$$h = \frac{|d|^{\frac{1}{2}}}{R} L(1,\chi).$$

For example, the conjecture of Gauss that $h = 1$ for infinitely many $d > 0$ is at present insurmountable, even under GRH. However, the obvious bound $R \gg \log d$ and Littlewood's [14] GRH result $L(1,\chi) \ll \log\log d$ give, for $d > 0$, the conditional upper bound

$$h \ll d^{\frac{1}{2}}(\log\log d/\log d).$$

In 1977 Montgomery-Weinberger [16] proved unconditionally that

$$h \gg d^{\frac{1}{2}}(\log\log d/\log d)$$

for infinitely many $d > 0$, thereby showing that this conditional upper bound is best possible, up to the constant. They make use of the special sequence of d, studied first by Chowla, namely square-free $d = 4t^2+1$, for which $\varepsilon = 2t+\sqrt{d}$ and hence $R \leq \log d$. They use zero-density estimates for Dirichlet L-functions to allow short approximations to enough $\log L(1,\chi)$, even in a quadratic sequence of discriminants, to produce large values of $L(1,\chi)$.

2 A general problem

Surprisingly little is known about such extremal class number problems for number fields of degree $n > 2$. First we must define a suitable family of number fields with which to work.

Given a number field K of degree n let G be the Galois group of its normal closure $\hat{K}$ over $\mathbb{Q}$. If $H \subseteq G$ is the subgroup fixing K pointwise, then G acts on G/H as a permutation group. Given an ordering of the cosets of G/H we thus associate to K an embedding $G \hookrightarrow S_n$. As is well known, the resulting cycle structure of the Frobenius of an unramified prime determines its splitting type in K and that of complex conjugation (the Frobenius at ∞) the signature of K.

Given any subgroup $G \subseteq S_n$ we define $\mathcal{K}(G)$ to be the set of all number fields K of degree n that have an ordering of their cosets so that their associated embedding is G. In order for there to be a uniform upper bound for the class number we need to further restrict the family $\mathcal{K}(G)$. Suppose, for example, that G is *primitive* as a permutation group (see [24]) and that $\mathcal{K}^+(G)$ consists of the totally positive fields in $\mathcal{K}(G)$. By the class number formula for $K \in \mathcal{K}^+(G)$ we have

$$h = \frac{d^{1/2}}{2^{n-1}R} L(1,\chi) \tag{4}$$

where $d = \mathrm{disc}(K)$ is the discriminant, R is the regulator and

$$L(s,\chi) = \zeta_K(s)/\zeta(s) \tag{5}$$

is an Artin L-function, $\zeta_K(s)$ being the Dedekind zeta function of K. In general, it is not even known that $L(s,\chi)$ is entire, although it is conjectured that it is (Artin), and further that GRH holds for these $L(s,\chi)$. If we assume this, then the method of Littlewood shows that

$$L(1,\chi) \ll (\log\log d)^{n-1}.$$

Remak [17] proved that if K contains no non-trivial subfields (which is true if G is primitive) then

$$R \gg (\log d)^{n-1}, \tag{6}$$

with the implied constants depending only on n. Thus, under GRH, we have the upper bound for the class number of $K \in \mathcal{K}^+(G)$

$$h \ll d^{\frac{1}{2}} (\log\log d/\log d)^{n-1}. \tag{7}$$

More generally, by prescribing the location in G of complex conjugation and applying the more general estimates for regulators given by Silverman [20], we could formulate such an estimate for an imprimitive G and an arbitrary signature. In any case, it seems to be extremely difficult to prove any result as strong as (7) unconditionally.

The problem arises to show that (7) is sharp, that is, that there exist $K \in \mathcal{K}^+(G)$ with arbitrarily large discriminant for which

$$h \gg d^{\frac{1}{2}} (\log\log d/\log d)^{n-1}. \tag{8}$$

At this level of generality this is clearly difficult since it subsumes the inverse Galois problem. However, for certain G we have enough examples of $K \in \mathcal{K}^+(G)$ to make progress, as in the quadratic case. We remark that in such cases the main difficulty (and interest) of this problem is to show that the conditional upper bound (7) cannot be improved upon, apart from the constant.

3 Abelian cubic fields

The simplest case to consider after the quadratic is that of abelian cubic fields, that is $\mathcal{K}^+(G)$ where G is the cyclic group of order 3 in S_3. In this section we will prove the following unconditional result.

Theorem 1 *There is an absolute constant $c > 0$ such that there exist abelian cubic fields with arbitrarily large discriminant d for which*

$$h > c\, d^{\frac{1}{2}} (\log\log d/\log d)^2.$$

First we must construct a sequence of abelian cubic fields for which we have a good upper bound for the regulator. Consider the polynomial for $t \in \mathbb{Z}^+$ defined by

$$f_t(x) = x^3 - tx^2 - (t+3)x - 1,$$

easily checked to be irreducible over $\mathbb{Q}$. The discriminant of f_t is given by $\operatorname{disc}(f_t) = g(t)^2$ where

$$g(t) = t^2 + 3t + 9.$$

Thus f_t defines an abelian cubic field $K = K_t$ obtained by adjoining to $\mathbb{Q}$ any root of f_t. These fields have been studied extensively after Shanks' paper [19]. It can be checked that exactly one of these roots, say ε, is positive and satisfies

$$t + 1 < \varepsilon < t + 2. \tag{9}$$

The other roots are $-1/(1+\varepsilon)$ and $-1 - 1/\varepsilon$. As shown in [22], ε and $\varepsilon + 1$ give a basis for the units of the order $\mathbb{Z}[\varepsilon]$. The regulator R of this order thus satisfies

$$R = \log^2 \varepsilon + \log^2(\varepsilon + 1) - \log \varepsilon \log(\varepsilon + 1) \sim \log^2 t$$

as $t \to \infty$, as is easily seen from (9). It is shown in the Corollary to Proposition 1 of [23] that, if $g(t) = t^2 + 3t + 9$ is square-free, then $\mathbb{Z}[\varepsilon]$ is the maximal order in K (see also [8]). Thus we can state the following lemma.

Lemma 1 *For $t \in \mathbb{Z}^+$ let K be the splitting field of $f_t(x)$. Then K is an abelian cubic number field. If $g(t) = t^2 + 3t + 9$ is square-free then $\mathbb{Z}[\varepsilon]$ is the maximal order in K and we have that $d = \operatorname{disc}(K) = g(t)^2$ and*

$$R = \operatorname{reg}(K) = \frac{1}{16}(1 + \mathrm{o}(1)) \log^2 d.$$

For these K the class number formula (4) can be written

$$h = \frac{d^{\frac{1}{2}}}{4R} |L(1, \chi)|^2 \tag{10}$$

where now χ denotes a primitive Dirichlet character of order 3 and conductor $m = g(t)$. In order to force $|L(1,\chi)|^2$ to be large we will need to show that enough square-free values of $g(t) = t^2 + 3t + 9$ exist when t is restricted to arithmetic progressions. This will allow us to produce many primes p that split completely in K and hence for which $\chi(p) = 1$.

The following lemma is a modification of Lemma 1 of [16].

Lemma 2 *Suppose that $q, a \in \mathbb{Z}^+$ satisfy $6|q$ and $(g(a), q) = 1$ and let*

$$N(x; q, a) = \#\{1 \le t \le x : t \equiv a (\operatorname{mod} q) \text{ with } g(t) \text{ square-free}\}.$$

Then for $x \ge 2$

$$N(x; q, a) = \frac{x}{q} c_q + \mathrm{O}(x^{\frac{2}{3}} \log x),$$

where c_q is given in (13) below and satisfies $\frac{1}{2} < c_q < 1$. The implied constant is absolute.

Proof: We have that

$$N(x; q, a) = \sum_{\substack{1 \le t \le x \\ t \equiv a (\operatorname{mod} q)}} \sum_{\substack{r^2 | g(t) \\ (r,q)=1}} \mu(r),$$

where we may assume that $(r, q) = 1$ in the second sum since $(g(t), q) = 1$ for $t \equiv a(\operatorname{mod} q)$. Thus rearranging gives

$$N(x; q, a) = \sum_{\substack{r \le g(x) \\ (r,q)=1}} \mu(r) \, \#\{1 \le t \le x : t \equiv a(\operatorname{mod} q) \text{ and } g(t) \equiv 0(\operatorname{mod} r^2)\}.$$

For a value of y with $1 \le y \le x$ to be chosen later the above sum over $r > y$ is

$$\le \sum_{s \le g(x) y^{-2}} \#\{(t, r) : 1 \le t \le x \text{ and } g(t) = r^2 s\}$$

$$\le \sum_{s \le 14 x^2 y^{-2}} \#\{(u, v) : 1 \le u \le 2x + 3 \text{ and } u^2 - 4sv^2 = 27\},$$

on using that $g(t) = ((2t + 3)^2 + 27)/4$. For $x \ge 2$ this is $\ll (x/y)^2 \log x$ since, by Hilfssatz 2 p.660. of [10], we have that

$$\#\{(u, v) : 1 \le u \le 2x + 3 \text{ and } u^2 - 4sv^2 = 27\} \ll \log x,$$

with an absolute constant (note that in case s is a square this is trivial). Thus we have for $x \ge 2$

$$N(x; q, a) = \sum_{\substack{r \le y \\ (r,q)=1}} \mu(r) \, \#\{1 \le t \le x : t \equiv a(q) \text{ and } g(t) \equiv 0(r^2)\} \qquad (11)$$

$$+ \mathrm{O}(x^2 y^{-2} \log x).$$

For $(m, 6) = 1$ the number of solutions to the congruence $g(t) \equiv 0(\operatorname{mod} m)$ is given by

$$c(m) = \prod_{p|m} \left(1 + \left(\frac{-3}{p}\right)\right), \qquad (12)$$

which is multiplicative and satisfies $c(m) = c(m^2)$. Thus from (11)

$$N(x; q, a) = \sum_{\substack{r \le y \\ (r,q)=1}} \mu(r)\, c(r)(\frac{x}{qr^2} + \mathrm{O}(1)) + \mathrm{O}(x^2 y^{-2} \log x)$$

since by assumption $6|q$ and so $(r, 6) = 1$. Clearly $c(r) \le d(r)$, where $d(r)$ is the number of divisors of r, so

$$N(x; q, a) = \frac{x}{q} \sum_{(r,q)=1} \mu(r)\, c(r) r^{-2} + \mathrm{O}(x^2 y^{-2} \log x) + \mathrm{O}(y \log y) + \mathrm{O}(x q^{-1} y^{-1} \log y).$$

Choosing $y = x^{\frac{2}{3}}$ gives the result since by (12)

$$c_q = \sum_{(r,q)=1} \mu(r)\, c(r) r^{-2} = \prod_{p \nmid q} \left(1 - \left(1 + \left(\frac{-3}{p}\right)\right) p^{-2}\right) \ge \prod_{p \ge 7} \left(1 - 2p^{-2}\right) > 1/2, \tag{13}$$

completing the proof of Lemma 2. □

Proof of Theorem 1: We shall apply Lemma 2 with

$$q = \prod_{p \le y} p,$$

where p runs over primes and $y = \alpha \log x$ for an appropriately chosen α. We want to choose a so that all p with $5 \le p \le y$ split completely in K when $t \equiv a (\operatorname{mod} q)$. To force this, observe that

$$f_t(x) = x^3 - tx^2 - (t+3)x - 1 = 0$$

defines a rational (genus 0) curve over $\mathbb{Q}$ and that for $p \ge 5$ there will always be points (x, t) on this curve over $\mathbb{F}_p$ with $g(t) = t^2 + 3t + 9 \ne 0$. In fact, one computes that the number of $t (\operatorname{mod} p)$ for which $f_t(x)$ factors into 3 distinct linear factors $(\operatorname{mod} p)$ is

$$\begin{cases} \frac{p-4}{3} & \text{if } p \equiv 1 (\operatorname{mod} 3), \\ \frac{p-2}{3} & \text{if } p \equiv 2 (\operatorname{mod} 3). \end{cases}$$

Hence, for such t, p will split completely in the Galois extension K. For each p with $5 \le p \le y$ fix such a t_p. By the Chinese remainder theorem we can find a with $a \equiv 1 (\operatorname{mod} 6)$ and $a \equiv t_p (\operatorname{mod} p)$ for each such p. Then $(g(a), q) = 1$ and so by Lemma 2 we conclude that there will exist $\gg x/q$ values of $t \le x$ for which all primes p with $5 \le p \le y$ split completely in K and for which $d = \operatorname{disc}(K) = g(t)^2$ with $g(t)$ square-free, provided that $q \ll x^{\frac{1}{4}}$, say. This can be ensured by taking $y = \frac{1}{5} \log x$. By Lemma 1 we conclude the following.

Lemma 3 *There exists an absolute constant $c > 0$ so that there are at least $cx^{\frac{3}{8}}$ distinct cubic abelian fields K, each with discriminant $d = m^2$ and conductor m for some m with $1 < m \leq x$, and in which all primes p with $5 \leq p \leq \frac{1}{5}\log x$ split completely and whose regulator R satisfies $R \ll \log^2 d$.*

Returning to (10) we next show that the proof of Theorem 1 follows from Lemma 3 and the following result, which is Lemma 2 of [16].

Lemma 4 *(Montgomery-Weinberger) Suppose that $0 < \delta < 1$. Then for $(\log m)^{\delta} \leq y \leq \log m$, and χ a primitive Dirichlet character $(\operatorname{mod} m)$ with $m > 1$, we have*

$$\log L(1,\chi) = \sum_{p \leq y} \chi(p)p^{-1} + \mathrm{O}_{\delta}(1),$$

unless χ lies in an exceptional set $\mathcal{E}(\delta)$. The set $\mathcal{E}(\delta)$ contains $\ll x^{\delta}$ primitive characters χ with conductor m such that $1 < m \leq x$.

Take $\delta < 3/8$ and $y = \frac{1}{5}\log x$ in Lemma 4 to conclude that there is an absolute constant $c_1 > 0$ so that for at least $c_1 x^{\frac{3}{8}}$ of the fields in Lemma 3 the associated Dirichlet L value satisfies

$$|L(1,\chi)|^2 \gg \log\log d.$$

Theorem 1 now follows from (10). $\square$

4 Non-abelian fields and L-functions

For number fields whose normal closure is non-abelian much less is known about our problem. The easiest non-abelian situation in which to produce fields with class numbers as large as predicted in (8) is when the Galois group is the full symmetric group, this being the "generic" case in view of well known results of probabilistic Galois theory (see [11]). Let $\mathcal{K}_n = \mathcal{K}^+(S_n)$, the set of all totally real number fields K of degree n whose normal closure $\hat{K}$ has Galois group S_n. As in §2, let $L(s,\chi) = \zeta_K(s)/\zeta(s)$ be the associated Artin L-function. The following result is proven in [9] and shows (conditionally) that for $G = S_n$ the upper bound (7) is sharp up to the constant.

Theorem 2 *Fix $n \geq 2$ and assume that each $L(s,\chi)$ is entire and satisfies the GRH. Then there is a constant $c > 0$ depending only on n such that there exist $K \in \mathcal{K}_n$ with arbitrarily large discriminant d for which*

$$h > c\, d^{\frac{1}{2}} (\log\log d/\log d)^{n-1}.$$

A previously known result in this direction, due to Ankeny-Brauer-Chowla [1], states that for any $\epsilon > 0$ there are infinitely many totally real fields K of degree n with $h > d^{\frac{1}{2}-\epsilon}$. It imposes no condition on $\mathrm{Gal}(\hat{K}/\mathbb{Q})$ and is not sharp, but it is unconditional. Later Sprindžuk [21][2] strengthened this result by giving an upper bound for the density (when measured in terms of the regulator) of totally real number fields of degree n with $h \leq d^{\frac{1}{2}-\epsilon}$. It should be observed, however, that the upper bound (7) likely does not hold in general for all totally real fields of degree $n > 3$. In Lemma 3 of [8] an infinite family of totally real biquadratic fields is constructed whose regulators satisfy $R \ll \log^2 d$ and in this family there are likely infinitely many that violate (7) with $n = 4$.

As in the proof of Theorem 1, in order to prove Theorem 2 we construct the needed fields as specializations of a function field. Specifically, we define K by adjoining to $\mathbb{Q}$ a root α of

$$f_t(x) = (x-t)(x-2^2t)\dots(x-n^2t) - t.$$

If $t > 1$ is square-free then $f_t(x)$ is an Eisenstein polynomial. This allows us to use the fact that p with $p \mid t$ ramifies completely in K to bound from below the discriminant d of K in terms of $\mathrm{disc}(f_t)$, and hence t. A frequent difficulty in doing so in such problems is due to the index of $\mathbb{Z}[\alpha]$ in the ring of integers in K. This family has another needed feature; if t is sufficiently large, then K is totally real and

$$R \ll_n (\log d)^{n-1}$$

since it is possible to write down a system of $n-1$ multiplicatively independent units with a determined asymptotic behavior as $t \to \infty$, a property reminiscent of the cyclic cubic example discussed above. A difficulty not present in the cyclic cubic case, since it is automatic, is the need to control the Galois group of $\hat{K}/\mathbb{Q}$. Here we show that $f(t,x) = f_t(x)$ is irreducible over $\mathbb{C}$ and that the monodromy group of the Riemann surface defined by $f(t,x) = 0$ is S_n, using old techniques of Jordan and Hilbert. Then we apply a result of S.D Cohen [7], which is a quantitative form of Hilbert's Irreducibility Theorem, to show that for most integral specializations of t, the Galois group of $\hat{K}/\mathbb{Q}$ is maximal, that is S_n. The proof of this uses the multi-dimensional large sieve inequality in a way first done by Gallagher [11].

The technique used in the abelian cubic case to produce many split primes by restricting the parameter t to certain arithmetic progressions continues to work but the genus of the needed curve is no longer zero and the Riemann Hypothesis for curves must be invoked. The relatively mild condition that t be square-free may be imposed without difficulty. [3] This ingredient in the proof of Theorem 2 is an analogue of a classical result of Bohr and Landau [5], as refined by Littlewood [13], which is used to find small values of $\zeta(1+it)$:

[2] I thank C. Levesque for pointing out this reference.

[3] The construction of Theorem 2 in case $n = 2$ provides a simplification of that of [16] since it is unnecessary to force d to be square-free, only t.

Classical Bohr-Landau: *If y is sufficiently large then there is a positive integer t with $\log t \ll y$ so that p^{it} is uniformly close to -1 for each prime $p \leq y$, the implied constants depending on how good the approximation is required to be.*

This part of the proof is generalized in [9] to families of continuous Galois representations

$$\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL(n, \mathbb{C})$$

whose image is a fixed finite group G with property Gal_T. To say that G has property Gal_T means that G is the Galois group of a finite regular extension E of $\mathbb{Q}(t)$, so $\bar{Q} \cap E = \mathbb{Q}$. [4] Here as usual $\bar{Q}$ is a fixed algebraic closure of $\mathbb{Q}$. The "size" of such a representation is measured by its conductor. Choose $a(p)$ to be, for each prime p, the trace of an arbitrary element of G. Then we have

Bohr-Landau for Galois: *If y is sufficiently large, then there is a Galois representation of conductor N whose image is G and such that $\log N \ll y$ and $\chi(\sigma_p) = a(p)$ for $1 \ll p \leq y$, where σ_p is any Frobenius element over p.*

The final ingredient needed to prove Theorem 2 is an approximation of $\log L(1, \chi)$ as a short sum over primes. This can be done for any Artin L-function associated to ρ. Denoting the character of ρ by χ, this is given for $\mathrm{Re}(s) > 1$ by the formula

$$\log L(s, \chi) = \sum_p \sum_{m=1}^{\infty} \frac{1}{m} \chi(p^m) p^{-ms}$$

where

$$\chi(p^m) = \frac{1}{|I_p|} \sum_{\iota \in I_p} \chi(\sigma_p^m \iota)$$

where I_p is the inertia group of any of the primes lying over p. In general, $L(s, \chi)$ is known to be meromorphic [6] and to satisfy the functional equation [2]

$$\Lambda(1 - s, \bar{\chi}) = \varepsilon_\chi N^{s - \frac{1}{2}} \Lambda(s, \chi)$$

where $|\varepsilon_\chi| = 1$ and

$$\Lambda(s, \chi) = \pi^{\frac{ns}{2}} \Gamma(s/2)^{\frac{n+\ell}{2}} \Gamma((s+1)/2)^{\frac{n-\ell}{2}} L(s, \chi)$$

where ℓ is the value of χ on complex conjugation and N is the conductor of ρ. Artin conjectured that $L(s, \chi)$ is entire unless ρ contains trivial components, in which case it should only have a pole at $s = 1$. Assuming this and GRH for $L(s, \chi)$, it is shown in [9] that (3) generalizes in a straightforward way to give

$$\log L(1, \chi) = \sum_{p \leq (\log N)^{\frac{1}{2}}} \chi(p) p^{-1} + \mathrm{O}_n(1),$$

[4] It has been conjectured (see p.35. of [18]) that every finite group G has property Gal_T.

which then allows the completion of the proof of Theorem 2.

One is led naturally to consider the extreme values of such a family of Artin L-functions at $s = 1$ and this can be done, at least conditionally. We immediately get the upper bound (under GRH)

$$L(1, \chi) \ll_n (\log \log N)^n \tag{14}$$

and prove in [9] the following result showing that this is best possible, up to the constant.

Theorem 3 *Suppose that $G \subset GL(n, \mathbb{C})$ is non-trivial, irreducible and has property* Gal_T*. Assume that every Artin L-function $L(s, \chi)$ whose Galois representation has image G is entire and satisfies GRH. Then there is a constant $c > 0$, depending only on G, so that there exist such L-functions with arbitrarily large conductor N that satisfy*

$$L(1, \chi) > c(\log \log N)^n. \tag{15}$$

5 Further prospects

It would obviously be desirable to prove Theorems 2 and 3 unconditionally, at least in some non-abelian cases. Clearly what is needed is a generalization of Lemma 4 to families of Artin L-functions. The proof of Lemma 4 is based on zero density estimates for Dirichlet L-functions. Proofs of these estimates involve orthogonality properties of distinct Dirichlet characters. This aspect can sometimes be found on the side of automorphic representations. In particular, it would be interesting to establish zero density estimates for families of automorphic L-functions chosen to include, at least conjecturally, the Artin L-functions. In some cases, for example non-abelian cubic extensions, the degree two Artin L-functions that arise in (5) are known to come from classical automorphic forms of weight 1 after Hecke, and the type of orthogonality needed would be over varying levels. Similarly, for certain quartic extensions, the associated degree three Artin L-functions that arise are known after Jacquet, Piatetski-Shapiro and Shalika [12] to come from $GL(3)$ automorphic forms. Langlands has conjectured that every Artin L-function should come from an automorphic representation on $\mathrm{GL}(n)$ for some n.

Another direction of interest is to consider other groups G than the symmetric group for the problem outlined in §2. For such generalizations the construction of fields with small regulators seems to be quite an interesting problem. As we have seen, the existence of Artin L-functions in rather general families with extremal values can be proven conditionally. There might also be some interest in obtaining good constants in (14) and (15) as was done in case $n = 2$ by Littlewood, and in studying the finer behavior of these L-values along the lines of [15]

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