MODULAR INVARIANTS FOR REAL QUADRATIC FIELDS AND KLOOSTERMAN SUMS

NICKOLAS ANDERSEN AND WILLIAM DUKE

Abstract. We investigate the asymptotic distribution of integrals of the j-function that are associated to ideal classes in a real quadratic field. To estimate the error term in our asymptotic formula, we prove a bound for sums of Kloosterman sums of half-integral weight that is uniform in every parameter. To establish this estimate we prove a variant of Kuznetsov's formula where the spectral data is restricted to half-integral weight forms in the Kohnen plus space, and we apply Young's hybrid subconvexity estimates for twisted modular L-functions.

1. Introduction

The relationship between modular forms and quadratic fields is exceedingly rich. For instance, the Hilbert class field of an imaginary quadratic field may be generated by adjoining to the quadratic field a special value of the modular j-function. The connection between class fields of real quadratic fields and invariants of the modular group is much less understood, although there has been striking progress lately by Darmon and Vonk [6]. Our aim in this paper is to study the asymptotic behavior of certain integrals of the modular j-function that are associated to ideal classes in a real quadratic field. Before turning to this, it is useful to make some definitions and to recall the corresponding problem in the imaginary quadratic case.

Let $\mathbb{K}$ be the quadratic field of discriminant d and let Cl_d^+ denote the narrow class group of $\mathbb{K}$. Let $h(d) = \#\mathrm{Cl}_d^+$ denote the class number. If $d < 0$ then each ideal class $A \in \mathrm{Cl}_d^+$ contains exactly one fractional ideal of the form $z_A\mathbb{Z} + \mathbb{Z}$, where

$$z_A = \frac{-b + i\sqrt{|d|}}{2a}$$

for some relatively prime integers a, b, c with $a > 0$ and $b^2 - 4ac = d$, and where z_A is in the fundamental domain

$$\mathcal{F} := \{z \in \mathcal{H} : -\tfrac{1}{2} < \mathrm{Re}\, z \le \tfrac{1}{2}, |z| \ge 1\}$$

for the action of the modular group $\Gamma_1 = \mathrm{PSL}_2(\mathbb{Z})$. Such z_A are called *reduced*. A beautiful result from the theory of complex multiplication states that the values $j_1(z_A)$, as A runs over ideal classes of discriminant d, are conjugate algebraic integers. Here $j_1 = j - 744$ is the normalized modular j-invariant

$$j_1(z) = q^{-1} + 196\,884q + 21\,493\,760q^2 + \dots,$$

where $q = e(z) = e^{2\pi i z}$. It follows that the trace

$$\mathrm{Tr}_d(j_1) := \frac{1}{\omega_d} \sum_{A \in \mathrm{Cl}_d^+} j_1(z_A), \tag{1.1}$$

Date: October 16, 2024.

The authors were supported by NSF grant DMS–1701638. The second author was also suppored by the Simons Foundation, award number 554649.

where $\omega_{-3} = 3$, $\omega_{-4} = 2$, and $\omega_d = 1$ otherwise, is a rational integer. For example,

$$\mathrm{Tr}_{-3}(j_1) = -248, \quad \mathrm{Tr}_{-4}(j_1) = 492, \quad \mathrm{Tr}_{-7}(j_1) = -4119, \quad \mathrm{Tr}_{-8}(j_1) = 7256.$$

It is natural to ask how these values are distributed as $|d| \to \infty$. As a first approximation, it is not too hard to show that $\mathrm{Tr}_d(j_1) \sim (-1)^d \exp(\pi\sqrt{|d|})$ for large d, but in fact much more is known. In [4] it was observed, and in [10] the second author proved, that

$$\mathrm{Tr}_d(j_1) - \sum_{\operatorname{Im} z_A > 1} e(-z_A) \sim -24h(d) \tag{1.2}$$

as $d \to -\infty$ through fundamental discriminants. The value -24 is a suitably defined "average of j_1" over the fundamental domain $\mathcal{F}$ (see [10]).

Now suppose that $\mathbb{K}$ is a real quadratic field, i.e. $d > 0$. Each ideal class $A \in \mathrm{Cl}_d^+$ contains a fractional ideal of the form $w\mathbb{Z} + \mathbb{Z} \in A$ where $w \in \mathbb{K}$ is such that

$$0 < w^\sigma < 1 < w,$$

where σ is the nontrivial Galois automorphism of $\mathbb{K}$. Such w are called *reduced* (in the sense of Zagier [46]); unlike in the imaginary quadratic case, a given ideal class may have many reduced representatives. Let $\mathcal{S}_w$ be the oriented hyperbolic geodesic in $\mathcal{H}$ from w^σ to w, and let $\mathcal{C}_A$ be the closed geodesic obtained by projecting $\mathcal{S}_w$ to $\Gamma_1\backslash\mathcal{H}$. The choice of reduced w does not affect $\mathcal{C}_A$. One can view $\mathcal{C}_A$ in $\mathcal{H}$ as the geodesic from some point z_0 on $\mathcal{S}_w$ to $\gamma_w(z_0)$, where γ_w is the hyperbolic element which generates the stabilizer of w in Γ_1. It is well-known that

$$\mathrm{length}(\mathcal{C}_A) = 2\log\varepsilon_d,$$

where ε_d is the fundamental unit of $\mathbb{K}$.

A real quadratic analogue of the trace (1.1) is the sum of integrals

$$\mathrm{Tr}_d(j_1) := \sum_{A \in \mathrm{Cl}_d^+} \int_{\mathcal{C}_A} j_1(z) \frac{|dz|}{y}, \tag{1.3}$$

and one might ask how these invariants are distributed as the discriminant d varies. Numerically, we have

$$\mathrm{Tr}_5(j_1) \approx -11.5417, \quad \mathrm{Tr}_8(j_1) \approx -19.1374, \quad \mathrm{Tr}_{13}(j_1) \approx -23.4094, \quad \mathrm{Tr}_{17}(j_1) \approx -43.9449.$$

Note that these values are quite small even though j_1 grows exponentially in the cusp. It was conjectured in [13] that

$$\mathrm{Tr}_d(j_1) \sim -24 \cdot 2\log\varepsilon_d\, h(d) \tag{1.4}$$

as $d \to \infty$ through fundamental discriminants. This was proved independently in [12] (for odd fundamental discriminants, with a power-saving of $d^{-\frac{1}{5325}}$) and in [32] (for all fundamental discriminants, with a power-saving of $d^{-\frac{1}{400}}$).

The real quadratic invariants $\mathrm{Tr}_d(j_1)$ were first studied in [13] in the context of harmonic Maass forms (nonholomorphic modular forms which are annihilated by the hyperbolic Laplacian). There is a family of harmonic Maass forms $\{f_{d'}\}$ of weight $\frac{1}{2}$, indexed by positive discriminants d', whose Fourier coefficients can be written in terms of the sums (1.3) twisted by genus characters. For each factorization $D = dd'$ of the fundamental discriminant D

into fundamental discriminants d, d', there is a real character $\chi_d = \chi_{d'}$ of Cl_D^+ called a genus character. The d-th Fourier coefficient of $f_{d'}$ is given by

$$\mathrm{Tr}_{d,d'}(j_1) := \sum_{A \in \mathrm{Cl}_D^+} \chi_d(A) \int_{\mathcal{C}_A} j_1(z) \frac{|dz|}{y}.$$

In particular, the d-th Fourier coefficient of f_1 is $\mathrm{Tr}_d(j_1)$. The remaining non-square-indexed coefficients can be described in terms of $\mathrm{Tr}_{d,d'}(j_m)$ for $m \geq 1$, where j_m is the unique modular function in $\mathbb{C}[j]$ of the form $j_m = q^{-m} + O(q)$. Our first result concerns the asymptotic distribution of the values of $\mathrm{Tr}_{d,d'}(j_m)$ as any of the parameters d, d', m tends to infinity. We define $\delta_1 = 1$ and $\delta_d = 0$ otherwise, and $\sigma_s(n) = \sum_{\ell | n} \ell^s$ for any $s \in \mathbb{C}$.

Theorem 1.1. *For each positive fundamental discriminant D, let d be any positive fundamental discriminant dividing D. Then for each $m \geq 1$ we have*

$$\sum_{A \in \mathrm{Cl}_D^+} \chi_d(A) \int_{\mathcal{C}_A} j_m(z) \frac{|dz|}{y} = -24\, \delta_d\, \sigma_1(m) \cdot 2h(D) \log \varepsilon_D + O\Big(m^{\frac{8}{9}} D^{\frac{13}{27}} (mD)^{\varepsilon}\Big). \tag{1.5}$$

Remarks. In the case $d = 1$, the power-saving of $D^{-\frac{1}{54}}$ in Theorem 1.1 improves on the results of [32, 12]. The generalizations to $d > 1$ and $m > 1$ are new, and the latter confirms the observation in [13] that $\mathrm{Tr}_D(j_m) \sim -24\sigma_1(m) \cdot 2 \log \varepsilon_D h(D)$ as $m \to \infty$.

When $D = dd'$ is a factorization of D into negative fundamental discriminants, the left-hand side of (1.5) is identically zero. To see this, let J denote the class of the different $(\sqrt{D})$ of $\mathbb{K}$. The closed geodesic associated to JA^{-1} has the same image in $\Gamma_1 \backslash \mathcal{H}$ as $\mathcal{C}_A$ but with the opposite orientation. Since $\chi_d(J) = \operatorname{sgn} d$, the left-hand side of (1.5) is forced to vanish whenever $d < 0$.

In order to give a better geometric interpretation when $D = dd'$ where d and d' are negative, Imamoḡlu, Tóth, and the second author [14] recently defined a new invariant $\mathcal{F}_A$, which is a finite area hyperbolic surface with boundary $\mathcal{C}_A$. We briefly describe the construction of $\mathcal{F}_A$; for details see [14]. Let w be one of the reduced quadratic irrationalities associated to A, and let $\gamma_w \in \Gamma_1$ be the hyperbolic element that fixes w and w^σ. Then γ_w can be written as

$$\gamma_w = T^{\lceil w \rceil} S T^{n_1} S T^{n_2} S \cdots T^{n_\ell} S \tag{1.6}$$

for some integers $n_i \geq 2$, where $T = \pm\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \pm\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ are generators of Γ_1. The cycle $(n_1, \ldots n_\ell)$ is the period of the minus continued fraction of w, and ℓ is the number of distinct reduced representatives of A. Let $S_k := T^{(n_1 + \cdots + n_k)} S T^{-(n_1 + \cdots + n_k)}$ and define

$$\Gamma_A := \langle S_1, \ldots, S_\ell, T^{(n_1 + \cdots + n_\ell)} \rangle.$$

This group is an infinite-index (i.e. thin) subgroup of Γ_1. Let $\mathcal{N}_A$ be the Nielsen region of Γ_A: the smallest non-empty Γ_A-invariant open convex subset of $\mathcal{H}$. Then the surface $\mathcal{F}_A$ is defined as $\Gamma_A \backslash \mathcal{N}_A$. A different choice of reduced w representing A yields a subgroup of Γ_1 conjugate to Γ_A by a translation, so the surface $\mathcal{F}_A$ is uniquely defined by A; see Theorem 1 of [14]. In that theorem we also find that the area of $\mathcal{F}_A$ is $\pi\ell$, with ℓ as in (1.6).

Our second result concerns the distribution of sums of the integrals of j_m over the surfaces $\mathcal{F}_A$ as the discriminant varies. The functions j_m grow exponentially in the cusp, so we regularize[1] the integrals using the function $\nu_A(z)$ defined in Section 2.

[1]The published version of this paper uses a different regularization that we erroneously claimed was equivalent to the one given here using $\nu_A(z)$. See Section 2 for an explanation of how these two regularizations differ.

Theorem 1.2. *For each positive fundamental discriminant D, let $D = dd'$ be any factorization into negative fundamental discriminants. Then for each $m \geq 1$ we have*

$$\frac{1}{4\pi} \sum_{A \in \mathrm{Cl}_D^+} \chi_d(A) \int_{\mathcal{F}} j_m(z) \nu_A(z) \frac{dxdy}{y^2} = -24\sigma_1(m) \frac{h(d)h(d')}{\omega_d \omega_{d'}} + O\Big(m^{\frac{8}{9}} D^{\frac{13}{27}} (mD)^{\varepsilon}\Big). \tag{1.7}$$

Remark. When $D = dd'$ is a factorization into positive discriminants, the left-hand side of (1.7) is identically zero because $A \mapsto JA^{-1}$ reverses the orientation of the surface $\mathcal{F}_A$.

An interesting special case occurs when $D = 4p$ where $p \equiv 3 \pmod 4$ is a prime. In this case the identity class $I = I_p$ is not equivalent to the class of the different $J = J_p$. The Cohen-Lenstra heuristics predict that approximately 75% of such fields have wide class number one, which would imply that the classes containing I and J are the only ideal classes. If this is the case, then there is a sequence of primes $p \equiv 3 \pmod 4$ for which

$$\int_{\mathcal{F}_{I_p}} j_1(z) \frac{dxdy}{y^2} \sim -2\pi h(-p) \quad \text{and} \quad \int_{\mathcal{F}_{J_p}} j_1(z) \frac{dxdy}{y^2} \sim 2\pi h(-p).$$

The method used in [10] to prove (1.2) and in [32] to prove (1.4) involves the equidistribution of CM points and closed geodesics originally developed by the second author in [9]. By contrast, here we employ a relation between the invariants in (1.5) and (1.7) and sums of Kloosterman sums (see Section 2). We then estimate the sums of Kloosterman sums directly via a Kuznetsov-type formula.

The Kloosterman sums in question are those which appear in the Fourier coefficients of Poincaré series of half-integral weight in the Kohnen plus space. In weight $k = \lambda + \frac{1}{2}$, the plus space consists of holomorphic or Maass cusp forms whose Fourier coefficients are supported on exponents n such that $(-1)^\lambda n \equiv 0, 1 \pmod 4$. For integers m, n satisfying the plus space condition and c a positive integer divisible by 4 we define

$$S_k^+(m,n,c) := e\Big(-\frac{k}{4}\Big) \sum_{d \bmod c} \Big(\frac{c}{d}\Big) \varepsilon_d^{2k} e\Big(\frac{m\bar{d} + nd}{c}\Big) \times \begin{cases} 1 & \text{if } 8 \mid c, \\ 2 & \text{if } 4 \,\|\, c, \end{cases} \tag{1.8}$$

where $d\bar{d} \equiv 1 \pmod c$ and $\varepsilon_d = 1$ or i according to $d \equiv 1$ or $3 \pmod 4$, respectively. The Kloosterman sums (1.8) are real-valued and satisfy the relation

$$S_k^+(m,n,c) = S_{-k}^+(-m,-n,c). \tag{1.9}$$

We prove a strong uniform bound for these sums which is of independent interest. We remark that similar (but weaker) estimates are hiding in the background of the methods of [10, 32].

Theorem 1.3. *Let $k = \pm\frac{1}{2} = \lambda + \frac{1}{2}$. Suppose that m, n are positive integers such that $(-1)^\lambda m = v^2 d'$ and $(-1)^\lambda n = w^2 d$, where d, d' are fundamental discriminants not both equal to 1. Then*

$$\sum_{4 \mid c \leq x} \frac{S_k^+(m,n,c)}{c} \ll \Big(x^{\frac{1}{6}} + (dd')^{\frac{2}{9}} (vw)^{\frac{1}{3}}\Big) (mnx)^{\varepsilon}. \tag{1.10}$$

Friedlander, Iwaniec, and the second author [12] proved an analogous estimate for smoothed sums of Kloosterman sums on $\Gamma_0(4q)$ with a power saving of $n^{-1/1330}$ when n is squarefree. Individually, the Kloosterman sums satisfy the Weil-type bound

$$|S_k^+(m,n,c)| \leq 2\sigma_0(c) \gcd(m,n,c)^{\frac{1}{2}} \sqrt{c}, \tag{1.11}$$

(see, e.g., Lemma 6.1 of [12]) so the sum in (1.10) is trivially bounded above by $(mnx)^{\varepsilon}\sqrt{x}$. Theorem 1.3 should be compared with the bound of Sarnak and Tsimerman [39] for the ordinary integral weight Kloosterman sums $S(m,n,c)$ which improves on the pivotal result of Kuznetsov in [28]. The main result of [39] is unconditional and depends on progress toward the Ramanujan conjecture for Maass cusp forms of weight 0. Assuming that conjecture, their theorem states that

$$\sum_{c\leq x}\frac{S(m,n,c)}{c}\ll\Big(x^{\frac16}+(mn)^{\frac16}\Big)(mnx)^{\varepsilon}.$$

Our method also yields an exponent of $\frac16$ for dd' in (1.10) if we assume the Lindelöf hypothesis for $L(\frac12,\chi)$ and $L(\frac12,f\times\chi)$, where χ is a quadratic Dirichlet character and f is an integral weight cusp form (holomorphic or Maass). Via the correspondence of Waldspurger, the Lindelöf hypothesis for all such $L(\frac12,f\times\chi)$ is equivalent to the Ramanujan conjecture for half-integral weight forms.

Recently Ahlgren and the first author used a similar approach to study the half-integral weight Kloosterman sums associated to the multiplier system for the Dedekind eta function. This was used in [1] to improve the error bounds of [29, 17] for the classical formula of Hardy, Ramanujan, and Rademacher for the partition function $p(n)$. In particular, it was shown that the discrepancy between $p(n)$ and the first $O(\sqrt{n})$ terms in the formula is at most $O(n^{-\frac12-\frac{1}{168}+\varepsilon})$.

The proof of Theorem 1.3 hinges on a version of Kuznetsov's formula which relates the Kloosterman sums (1.8) to the coefficients of holomorphic cusp forms, Maass cusp forms, and Eisenstein series of half-integral weight in the plus space. One advantage of the plus space is that the Waldspurger correspondence is completely explicit on that space via [25] and [2]; knowledge of the exact proportionality constant in the Waldspurger correspondence is crucial for us. Here we briefly define the relevant quantities and state a special case of our version of the Kuznetsov formula. Let $\mathcal{H}_k^+$ (resp. $\mathcal{V}_k^+$) denote an orthonormal Hecke basis for the plus space of holomorphic (resp. Maass) cusp forms of weight k for $\Gamma_0(4)$. For each $g\in\mathcal{H}_k^+$ (resp. $u_j\in\mathcal{V}_k^+$) let $\rho_g(n)$ (resp. $\rho_j(n)$) denote the suitably normalized n-th Fourier coefficient of g (resp. u_j). For each j, let $\lambda_j=\frac14+r_j^2$ denote the Laplace eigenvalue of u_j. The full statement with detailed definitions appears in Section 5 below.

Theorem 1.4. *Let $k=\pm\frac12=\lambda+\frac12$. Suppose that m,n are positive integers such that $(-1)^{\lambda}m$ and $(-1)^{\lambda}n$ are fundamental discriminants. Suppose that $\varphi:[0,\infty)\to\mathbb{R}$ is a smooth test function which satisfies* (5.1)*, and let $\widetilde{\varphi}$ and $\widehat{\varphi}$ denote the integral transforms in* (5.2)–(5.3)*. Then*

$$\begin{aligned}\sum_{4|c>0}&\frac{S_k^+(m,n,c)}{c}\varphi\left(\frac{4\pi\sqrt{mn}}{c}\right)\\&=6\sqrt{mn}\sum_{u_j\in\mathcal{V}_k^+}\frac{\overline{\rho_j(m)}\rho_j(n)}{\cosh\pi r_j}\widehat{\varphi}(r_j)+\frac32\sum_{\ell\equiv k\bmod 2}e\Big(\tfrac{\ell-k}{4}\Big)\widetilde{\varphi}(\ell)\Gamma(\ell)\sum_{g\in\mathcal{H}_\ell^+}\overline{\rho_g(m)}\rho_g(n)\\&\qquad+\int_{-\infty}^{\infty}\Big(\frac{n}{m}\Big)^{ir}\frac{L(\frac12-2ir,\chi_{(-1)^{\lambda}m})L(\frac12+2ir,\chi_{(-1)^{\lambda}n})}{2\cosh\pi r|\Gamma(\frac{k+1}{2}+ir)|^2|\zeta(1+4ir)|^2}\widehat{\varphi}(r)\,dr.\end{aligned}\tag{1.12}$$

Remark. This version of the Kuznetsov formula for Maass forms in the plus space for $\Gamma_0(4)$ with weight $\pm\frac12$ is precisely analogous to the original version of Kuznetsov's formula for the full modular group. To prove it we apply Biró's idea [3] of taking a linear combination of

Proskurin's Kuznetsov-type formula evaluated at various cusp-pairs in order to project the holomorphic and Maass cusp forms into the plus space. The main technical complication arises from the sum of Eisenstein series terms from Proskurin's formula, which we show simplifies to the integral of Dirichlet L-functions in (1.12). The simplicity of that integral is reminiscent of the corresponding term in Kuznetsov's original formula [28, Theorem 2] for the ordinary weight 0 Kloosterman sums; in that formula, the Eisenstein term is

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{n}{m}\right)^{ir}\frac{\sigma_{2ir}(m)\sigma_{-2ir}(n)}{|\zeta(1+2ir)|^2}\widehat{\varphi}(r)\,dr.$$

Note that if $k=0$ then $\cosh \pi r|\Gamma(\frac{k+1}{2}+ir)|^2=\pi$.

The most crucial input in the proof of Theorem 1.3 is Young's Weyl-type hybrid subconvexity estimates [45] for $L(\frac{1}{2}, f\times\chi_d)$ and $L(\frac{1}{2},\chi_d)$ which improve on the groundbreaking results of Conrey and Iwaniec [5]. Young proved that

$$\sum_f L(\tfrac{1}{2}, f\times\chi_d)^3 \ll (kd)^{1+\varepsilon} \tag{1.13}$$

for odd fundamental discriminants d, where the sum is over all holomorphic newforms of weight k and level dividing d. In Appendix A we sketch the details required to generalize Young's result to twists by χ_d for even fundamental discriminants d, where the sum is over f of level dividing the squarefree part of d. The uniformity of Young's result in both the level and weight directly influences the quality of the exponents in (1.10). There are corresponding results in [45] for twisted L-functions of Maass cusp forms and Dirichlet L-functions which we also use in the proof of Theorem 1.3.

Remark. The condition in (1.13) (and our extension in Appendix A) that f have level dividing the squarefree part of d (which is odd unless $d=4q$ with $q\equiv 2 \pmod 4$) is why we require a Kuznetsov formula that involves only coefficients of cusp forms in the plus space. Under the Shimura correspondence, the plus spaces of half-integral weight forms on $\Gamma_0(4)$ are isomorphic as Hecke modules to spaces of weight 0 cusp forms on $\Gamma_0(1)$, whereas the full spaces on $\Gamma_0(4)$ lift to $\Gamma_0(2)$.

The paper is organized as follows. In Section 2 we use the formulas of [14] to relate the geometric invariants to sums of Kloosterman sums, and we apply Theorem 1.3 to prove Theorems 1.1 and 1.2. The remainder of the paper is dedicated to the proof of Theorem 1.3. In Section 3 we give some background on the spectrum of the hyperbolic Laplacian in half-integral weight. In Section 4 we prove general estimates for the mean square of Fourier coefficients of Maass cusp forms of half-integral weight with arbitrary multiplier system. We prove Theorem 1.4 in Section 5 and Theorem 1.3 in Section 6. Finally, Appendix A contains a sketch of the proof of Young's subconvexity result extended to even discriminants.

Acknowledgement. The authors thank the referee for their thorough and careful reading of an earlier version of the manuscript, as well as their helpful comments and suggestions. We are also indebted to Vaibhav Kalia and Balesh Kumar, who discovered some mistakes in the published version of this paper, and who carefully corrected these mistakes in their work [22] (see the three footnotes in Sections 1 and 2 of this paper for details and corrections).

2. Geometric invariants and Kloosterman sums

In this section we relate the real quadratic invariants to Kloosterman sums and show how Theorems 1.1 and 1.2 follow from Theorem 1.3. Actually, we will prove more general forms of the main theorems which allow for non-fundamental discriminants. It is convenient to use binary quadratic forms

$$Q(x, y) = [a, b, c] = ax^2 + bxy + cy^2$$

in place of ideal classes, as this point of view makes the generalization to arbitrary discriminants straightforward. A discriminant is any integer $D \equiv 0, 1 \pmod 4$. A discriminant D is fundamental if it is either odd and squarefree or if $D/4$ is squarefree and congruent to $2, 3 \pmod 4$. Fix a discriminant $D > 1$ and a factorization $D = dd'$ into positive or negative discriminants d, d' such that d is fundamental. Let $\mathcal{Q}_D$ be the set of all integral binary quadratic forms $[a, b, c]$ with discriminant $b^2 - 4ac = D$. The modular group Γ_1 acts on $\mathcal{Q}_D$ in the usual way. When D is fundamental all forms in $\mathcal{Q}_D$ are primitive (i.e. $\gcd(a, b, c) = 1$) and there is a simple correspondence between $\Gamma_1\backslash\mathcal{Q}_D$ and Cl_D^+ via

$$[a, b, c] \mapsto w\mathbb{Z} + \mathbb{Z}, \qquad \text{where } w = \frac{-b + \sqrt{D}}{2a}, \tag{2.1}$$

assuming $[a, b, c]$ is chosen in its class to have $a > 0$. If D is fundamental and if Q corresponds to A via (2.1) then we define $\mathcal{C}_Q := \mathcal{C}_A$ and $\mathcal{F}_Q := \mathcal{F}_A$. We extend this to arbitrary discriminants via $\mathcal{C}_{\delta Q} := \mathcal{C}_Q$ and $\mathcal{F}_{\delta Q} := \mathcal{F}_Q$. There is a generalized genus character χ_d on $\Gamma_1\backslash\mathcal{Q}_D$ (see [18, I.2]) associated to the factorization $D = dd'$ defined by

$$\chi_d(Q) = \begin{cases} \left(\frac{d}{n}\right) & \text{if } (a, b, c, d) = 1 \text{ and } Q \text{ represents } n \text{ and } (d, n) = 1, \\ 0 & \text{if } (a, b, c, d) > 1. \end{cases}$$

If D is fundamental then $\chi_d = \chi_{d'}$ is the usual genus character, and there is exactly one such character for each such factorization.

As mentioned in the introduction, we need to regularize[2] the surface integrals of j_m. Following [15, (70)], for $z, \tau \in \mathcal{H}$ we define

$$K(z, \tau) := \frac{j'(\tau)}{j(z) - j(\tau)},$$

where $j' := \frac{1}{2\pi i}\frac{dj}{dz}$. This function transforms on Γ_1 with weight 0 in z and weight 2 in τ. For each indefinite quadratic form Q define

$$\nu_Q(z) := \int_{\mathcal{C}_Q} K(z, \tau) d\tau.$$

[2]The published version of this paper uses the regularization

$$\int_{\mathcal{F}_A} j_m(z)\frac{dxdy}{y^2} := \lim_{Y\to\infty} \int_{\mathcal{F}_{A,Y}} j_m(z)\frac{dxdy}{y^2},$$

where $\mathcal{F}_{A,Y}$ is the surface $\mathcal{F}_A$ truncated at height Y. In [22, Lemma 2.5], it is shown that this natural regularization is related to (2.2) by

$$\int_{\mathcal{F}} j_m(z)\nu_Q(z)\frac{dxdy}{y^2} = -\lim_{Y\to\infty} \int_{\mathcal{F}_{A,Y}} j_m(z)\frac{dxdy}{y^2} - 8\pi \mathrm{m}_Q \sigma_1(m),$$

where $\mathrm{m}_Q = \mathrm{m}_A = n_1 + \ldots + n_\ell$ (see the discussion following (1.6)).

As explained in [15], for $z \notin C_Q$ the value of $\nu_Q(z)$ is an integer which counts with signs the number of crossings that a path from $i\infty$ to z in $\mathcal{F}$ makes with $\mathcal{C}_Q$. Furthermore, $\nu_Q(z)$ is Γ_1-invariant and is identically zero for $\operatorname{Im} z$ sufficiently large. It follows that the integral

$$\int_{\mathcal{F}} j_m(z)\nu_Q(z)\frac{dxdy}{y^2} \tag{2.2}$$

converges, providing the desired regularization.

The following theorem generalizes Theorems 1.1 and 1.2 to more general discriminants.

Theorem 2.1. *For each positive nonsquare discriminant D, let $D = dd'$ be any factorization into discriminants such that d is fundamental. Let m be any positive integer. If d is positive, we have*

$$\sum_{Q\in\Gamma_1\backslash\mathcal{Q}_D} \chi_d(Q)\int_{\mathcal{C}_Q} j_m(z)\frac{|dz|}{y} = -24\,\delta_d\,\sigma_1(m)\cdot 2h(D)\log\varepsilon_D + O\Big(m^{\frac{8}{9}}D^{\frac{13}{27}}(mD)^{\varepsilon}\Big),$$

while if d is negative, we have

$$\frac{1}{4\pi}\sum_{Q\in\Gamma_1\backslash\mathcal{Q}_D} \chi_d(Q)\int_{\mathcal{F}} j_m(z)\nu_Q(z)\frac{dxdy}{y^2} = -24\sigma_1(m)\frac{h(d)h(d')}{\omega_d\omega_{d'}} + O\Big(m^{\frac{8}{9}}D^{\frac{13}{27}}(mD)^{\varepsilon}\Big).$$

To deduce Theorem 2.1 from Theorem 1.3 we require several results from [14, §8–9], which we borrow from freely here. For $m \geq 0$, let $F_{-m}(z,s)$ denote the index $-m$ nonholomorphic Poincaré series and let

$$j_m(z,s) := 2\pi m^{\frac{1}{2}}F_{-m}(z,s) - \frac{2\pi m^{1-s}\sigma_{2s-1}(m)}{\pi^{-(s+\frac{1}{2})}\Gamma(s+\frac{1}{2})\zeta(2s-1)}F_0(z,s).$$

For $m \geq 1$ the Fourier expansion of $F_{-m}(z,s)$ shows that it has an analytic continuation to $\operatorname{Re}(s) > \frac{3}{4}$. In particular, $F_{-m}(z,1)$ is holomorphic as a function of z. Furthermore, $F_0(z,s)$ is the nonholomorphic Eisenstein series of weight $\frac{1}{2}$, and we have

$$\lim_{s\to 1}\frac{2\pi m^{1-s}\sigma_{2s-1}(m)}{\pi^{-(s+\frac{1}{2})}\Gamma(s+\frac{1}{2})\zeta(2s-1)}F_0(z,s) = 24\sigma_1(m).$$

A computation then shows that $j_m(z) = j_m(z,1)$ for $m \geq 1$ (see [13, (4.11)]).

Since the length of $\mathcal{C}_Q$ is $2\log\varepsilon_D$ for every $Q \in \mathcal{Q}_D$, we have

$$\sum_{Q\in\Gamma_1\backslash\mathcal{Q}_D} \chi_d(Q)\int_{\mathcal{C}_Q}\frac{|dz|}{y} = 2\log\varepsilon_D\sum_{Q\in\Gamma_1\backslash\mathcal{Q}_D}\chi_d(Q) = 2\delta_d h(D)\log\varepsilon_D.$$

By Corollary 4 of [15], we have (note that $w_d = 2\omega_d$ in that paper)

$$\frac{1}{4\pi}\sum_{Q\in\Gamma_1\backslash\mathcal{Q}_D}\chi_d(Q)\int_{\mathcal{F}}\nu_Q(z)\frac{dxdy}{y^2} = \frac{h(d)h(d')}{\omega_d\omega_d'}.$$

So to prove Theorem 2.1 it suffices to show that

$$\sqrt{m}\sum_{Q\in\Gamma_1\backslash\mathcal{Q}_D}\chi_d(Q)\int_{\mathcal{C}_Q}F_{-m}(z,1)\frac{|dz|}{y} \ll m^{\frac{8}{9}}D^{\frac{13}{27}}(mD)^{\varepsilon} \tag{2.3}$$

and

$$\sqrt{m}\sum_{Q\in\Gamma_1\backslash\mathcal{Q}_D}\chi_d(Q)\int_{\mathcal{F}}F_{-m}(z,1)\nu_Q(z)\frac{dxdy}{y^2} \ll m^{\frac{8}{9}}D^{\frac{13}{27}}(mD)^{\varepsilon}. \tag{2.4}$$

We will prove (2.3)–(2.4) by relating the integrals of $F_{-m}(z,1)$ to the quadratic Weyl sums

$$T_m(d',d;c) := \sum_{\substack{b \bmod c \\ b^2\equiv D \bmod c}} \chi_d\Big(\Big[\tfrac{c}{4}, b, \tfrac{b^2-D}{c}\Big]\Big) e\Big(\frac{2mb}{c}\Big).$$

Here we are still assuming that $D = dd'$ with d fundamental. Note that $T_m(d',d;c) = S_m(d',d;c)$ in the notation of [14]; we have changed the notation here to avoid confusion with the Kloosterman sums. The Weyl sums are related to the plus space Kloosterman sums via Kohnen's identity

$$T_m(d,d';c) = \sum_{n|(m,\frac{c}{4})} \Big(\frac{d}{n}\Big) \sqrt{\frac{2n}{c}}\, S^+_{\frac12}\Big(d', \frac{m^2}{n^2}d; \frac{c}{n}\Big) \tag{2.5}$$

(see Lemma 8 of [14]). The Weil bound (1.11) for Kloosterman sums shows that

$$T_m(d,d';c) \ll \gcd(d', m^2 d, c)^{\frac12} c^{\varepsilon}.$$

A direct corollary of Theorem 1.3 is the following bound for the Weyl sums.

Theorem 2.2. *Suppose that $D = dd'$ is a positive nonsquare discriminant and that d is a fundamental discriminant. Then for any $m \geq 1$ we have*

$$\sum_{4|c\leq x} \frac{T_m(d,d';c)}{\sqrt{c}} \ll \Big(x^{\frac16} + D^{\frac29} m^{\frac13}\Big)(mDx)^{\varepsilon}. \tag{2.6}$$

Proof. When d, d' are positive this is immediate from (2.5) and the $k = \frac12$ case of Theorem 1.3. When d, d' are negative we apply (1.9) after (2.5). Then the estimate (2.6) follows from the $k = -\frac12$ case of Theorem 1.3. □

We are now ready to prove (2.3)–(2.4).

Proof of (2.3). Let $J_\nu(x)$ denote the J-Bessel function

$$J_\nu(2x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+\nu}}{k!\Gamma(\nu+k+1)}. \tag{2.7}$$

By Lemma 4 of [14] we have

$$\sum_{Q\in\Gamma\backslash\mathcal{Q}_D} \chi_d(Q) \int_{\mathcal{C}_Q} F_{-m}(z,s)\frac{|dz|}{y} = 2^{s-\frac12}\frac{\Gamma(\frac{s}{2})^2}{\Gamma(s)} D^{\frac14} \sum_{0<c\equiv 0(4)} \frac{T_m(d',d;c)}{\sqrt{c}} J_{s-\frac12}\Big(\frac{4\pi m\sqrt{D}}{c}\Big) \tag{2.8}$$

for $\mathrm{Re}(s) > 1$. By (2.7) we find that $J_\nu(1/x) \ll x^{-\nu}$ and $[J_\nu(1/x)]' \ll x^{-\nu-1}$ as $x \to \infty$ uniformly for $\nu \in [\frac12, 1]$. Let $a = 4\pi m\sqrt{D}$ and let $N \geq a$. Suppose that $s \in [1, \frac32]$. Then by partial summation and Theorem 2.2 we have

$$\sum_{4|c\geq N} \frac{T_m(d',d;c)}{\sqrt{c}} J_{s-\frac12}\Big(\frac{4\pi m\sqrt{D}}{c}\Big)$$
$$= \lim_{x\to\infty} S(x)J_{s-\frac12}\Big(\frac{a}{x}\Big) - S(N)J_{s-\frac12}\Big(\frac{a}{N}\Big) - \int_N^\infty S(t)\Big(J_{s-\frac12}\Big(\frac{a}{t}\Big)\Big)' dt \ll_a N^{-\frac13+\varepsilon},$$

where $S(x)$ denotes the partial sum on the left-hand side of (2.6). It follows that the sum on the right-hand side of (2.8) converges uniformly for $s \in [1, \frac{3}{2}]$. Since $J_{\frac{1}{2}}(x) = \sqrt{2/\pi x}\,\sin x$ we conclude that

$$\sqrt{m} \sum_{Q \in \Gamma \backslash \mathcal{Q}_D} \chi_d(Q) \int_{\mathcal{C}_Q} F_{-m}(z,1) \frac{|dz|}{y} = \sum_{0<c\equiv 0(4)} T_m(d',d;c) \sin\left(\frac{4\pi m\sqrt{D}}{c}\right). \tag{2.9}$$

We split the sum at $c = A$ with $A \ll m\sqrt{D}$. Estimating the initial segment $c \le A$ trivially, we obtain

$$\sum_{c \le A} T_m(d',d;c) \sin\left(\frac{4\pi m\sqrt{D}}{c}\right) \ll A(mDA)^{\varepsilon}. \tag{2.10}$$

Then by partial summation we have

$$\sum_{c>A} T_m(d',d;c) \sin\left(\frac{4\pi m\sqrt{D}}{c}\right) = -S(A)\sqrt{A}\,\sin\left(\frac{4\pi m\sqrt{D}}{A}\right) - \int_A^\infty S(t)\left(\sqrt{t}\,\sin\left(\frac{4\pi m\sqrt{D}}{t}\right)\right)' dt,$$

where $S(x)$ denotes the partial sum on the left-hand side of (2.6). Since

$$\left(\sqrt{t}\,\sin\left(\frac{4\pi m\sqrt{D}}{t}\right)\right)' \ll \frac{m\sqrt{D}}{t^{\frac{3}{2}}},$$

we conclude that

$$\sum_{c>A} T_m(d',d;c) \sin\left(\frac{4\pi m\sqrt{D}}{c}\right) \ll \left(mD^{\frac{1}{2}}A^{-\frac{1}{3}} + m^{\frac{4}{3}}D^{\frac{13}{18}}A^{-\frac{1}{2}}\right)(mDA)^{\varepsilon}. \tag{2.11}$$

Letting $A = m^a D^b$, we choose $a = \frac{8}{9}$ and $b = \frac{13}{27}$ to balance the exponents in (2.10) and (2.11). This, together with (2.9), yields (2.3). $\square$

Proof of (2.4). Define $\mathcal{F}_Y := \mathcal{F} \cap \{z : \operatorname{Im} z \le Y\}$. Let $Q \in \mathcal{Q}_D$, and let Y be sufficiently large so that $\nu_Q(z) = 0$ for $\operatorname{Im} z > Y$ and so that the image of $\mathcal{C}_Q$ in $\mathcal{F}$ is contained in $\mathcal{F}_Y$. Then for $\operatorname{Re} s > 1$ we have

$$\int_{\mathcal{F}} F_{-m}(z,s)\nu_Q(z)\frac{dxdy}{y^2} = \int_{\mathcal{C}_Q}\int_{\mathcal{F}_Y} F_{-m}(z,s)K(\tau,z)\frac{dxdy}{y^2}\,d\tau.$$

The function $F_{-m}(z,s)$ satisfies the relation

$$\Delta_0 F_{-m}(z,s) := -4y^2\partial_{\bar{z}}\partial_z F_{-m}(z,s) = s(1-s)F_{-m}(z,s)$$

(see §8 of [14]). So by the proof of Lemma 1 of [15] (essentially an application of Stokes' theorem), we find that[3]

$$\frac{s(1-s)}{2}\int_{\mathcal{F}_Y} F_{-m}(z,s)K(\tau,z)\frac{dxdy}{y^2} = i\partial_\tau F_{-m}(\tau,s) + g(\tau,s,Y) + h(\tau), \tag{2.12}$$

for some functions g and h with $g(\tau,s,Y) \to 0$ as $Y \to \infty$. It follows that

$$\frac{s(1-s)}{2}\int_{\mathcal{F}} F_{-m}(z,s)\nu_Q(z)\frac{dxdy}{y^2} = \int_{\mathcal{C}_Q} i\partial_z F_{-m}(z,s)dz + h(\tau).$$

[3]The published version of this paper contains an error: the functions g and h are omitted from the right-hand side of (2.12). The details of this corrected computation, including an explicit description of the functions g and h, can be found in the proof of Proposition 2.4 of [22].

Differentiating with respect to s and setting $s = 1$ we conclude that

$$\int_{\mathcal{F}} F_{-m}(z,1)\nu_Q(z)\frac{dxdy}{y^2} = -2\partial_s \int_{\mathcal{C}_Q} i\partial_z\, F_{-m}(z,s)\, dz\Big|_{s=1}.$$

By Lemma 5 of [14] we have

$$\sum_{Q\in\Gamma\backslash\mathcal{Q}_D} \chi_d(Q)\int_{\mathcal{C}_Q} i\partial_z F_{-m}(z,s)dz = 2^{s-\frac{1}{2}}\frac{\Gamma(\frac{s+1}{2})^2}{\Gamma(s)}D^{\frac{1}{4}}\sum_{0<c\equiv 0(4)}\frac{T_m(d',d;c)}{\sqrt{c}}J_{s-\frac{1}{2}}\left(\frac{4\pi m\sqrt{D}}{c}\right).$$

A straightforward computation involving (2.7) shows that, uniformly for $s \in [1, \frac{3}{2}]$, we have

$$\partial_s\left[2^{s-\frac{1}{2}}\frac{\Gamma(\frac{s+1}{2})^2}{\Gamma(s)}J_{s-\frac{1}{2}}(x)\right] \ll x^{s-\frac{1}{2}}\,|\log x| \quad \text{as} \quad x\to 0^+.$$

Thus, an argument involving partial summation, as in the proof of (2.3), shows that we are justified in setting $s = 1$, and we obtain

$$\sqrt{m}\sum_{Q\in\Gamma\backslash\mathcal{Q}_D}\chi_d(Q)\int_{\mathcal{F}} F_{-m}(z,1)\nu_Q(z)\frac{dxdy}{y^2} = -\frac{2}{\pi}\sum_{0<c\equiv 0(4)} T_m(d',d;c) f\left(\frac{4\pi m\sqrt{D}}{c}\right),$$

where

$$f(x) := \mathrm{Ci}(2x)\sin(x) - \mathrm{Si}(2x)\cos(x) + \log(2)\sin(x)$$

and Ci, Si are the cosine and sine integrals, respectively. The remainder of the proof is quite similar to the proof of (2.3) because we have

$$f(x) \ll \min\{1, x|\log x|\} \quad \text{and} \quad \left(\sqrt{t}\, f\left(\frac{4\pi m\sqrt{D}}{t}\right)\right)' \ll \frac{m\sqrt{D}}{t^{\frac{3}{2}}}(mDt)^{\varepsilon}.$$

We omit the details. □

3. Background

In this section we recall several facts about automorphic functions which transform according to multiplier systems of half-integral weight k, and the spectrum of the hyperbolic Laplacian Δ_k in this setting. For more details see [11, 38, 36, 1] along with the original papers of Maass [30, 31], Roelcke [37], and Selberg [40, 41].

Let $\Gamma = \Gamma_0(N)$ for some $N \geq 1$, and let k be a real number. We say that $\nu : \Gamma \to \mathbb{C}^\times$ is a multiplier system of weight k if

(i) $|\nu| = 1$,

(ii) $\nu(-I) = e^{-\pi i k}$, and

(iii) $\nu(\gamma_1\gamma_2) = w(\gamma_1,\gamma_2)\nu(\gamma_1)\nu(\gamma_2)$ for all $\gamma_1, \gamma_2 \in \Gamma$, where

$$w(\gamma_1,\gamma_2) = j(\gamma_2,z)^k j(\gamma_1,\gamma_2 z)^k j(\gamma_1\gamma_2,z)^{-k},$$

and $j(\gamma, z)$ is the automorphy factor

$$j(\gamma,z) := \frac{cz+d}{|cz+d|} = e^{i\arg(cz+d)}.$$

If ν is a multiplier system of weight k, then $\overline{\nu}$ is a multiplier system of weight $-k$.

The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H}$ via $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)z = \frac{az+b}{cz+d}$. The cusps of Γ are those points in the extended upper half-plane $\mathcal{H}^*$ which are fixed by parabolic elements of Γ. Given a cusp $\mathfrak{a}$ of Γ

let $\Gamma_{\mathfrak{a}} := \{\gamma \in \Gamma : \gamma\mathfrak{a} = \mathfrak{a}\}$ denote its stabilizer in Γ, and let $\sigma_{\mathfrak{a}}$ denote any element of $\mathrm{SL}_2(\mathbb{R})$ satisfying $\sigma_{\mathfrak{a}}\infty = \mathfrak{a}$ and $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \Gamma_\infty$. Define $\kappa_{\mathfrak{a}} = \kappa_{\nu,\mathfrak{a}}$ by the conditions

$$\nu\big(\sigma_{\mathfrak{a}}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\sigma_{\mathfrak{a}}^{-1}\big) = e(-\kappa_{\mathfrak{a}}) \quad \text{and} \quad 0 \le \kappa_{\mathfrak{a}} < 1.$$

We say that $\mathfrak{a}$ is singular with respect to ν if ν is trivial on $\Gamma_{\mathfrak{a}}$, that is, if $\kappa_{\nu,\mathfrak{a}} = 0$. Note that if $\kappa_{\nu,\mathfrak{a}} > 0$ then

$$\kappa_{\overline{\nu},\mathfrak{a}} = 1 - \kappa_{\nu,\mathfrak{a}}.$$

We are primarily interested in the multiplier system ν_θ of weight $\frac{1}{2}$ (and its conjugate $\overline{\nu}_\theta = \nu_\theta^{-1}$ of weight $-\frac{1}{2}$) on $\Gamma_0(4)$ defined by

$$\theta(\gamma z) = \nu_\theta(\gamma)\sqrt{cz+d}\,\theta(z),$$

where

$$\theta(z) := \sum_{n\in\mathbb{Z}} e(n^2 z).$$

Explicitly, we have

$$\nu_\theta\left(\begin{pmatrix} * & * \\ c & d\end{pmatrix}\right) = \left(\frac{c}{d}\right)\varepsilon_d^{-1},$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the extension of the Kronecker symbol given e.g. in [42] and

$$\varepsilon_d = \left(\frac{-1}{d}\right)^{\frac{1}{2}} = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

For $\gamma \in \mathrm{SL}_2(\mathbb{R})$ we define the weight k slash operator by

$$f\big|_k\gamma := j(\gamma,z)^{-k} f(\gamma z).$$

The weight k hyperbolic Laplacian

$$\Delta_k := y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - iky\frac{\partial}{\partial x}$$

commutes with the weight k slash operator for every $\gamma \in \mathrm{SL}_2(\mathbb{R})$. A real analytic function $f : \mathcal{H} \to \mathbb{C}$ is an eigenfunction of Δ_k with eigenvalue λ if

$$\Delta_k f + \lambda f = 0. \tag{3.1}$$

If f satisfies (3.1) then for notational convenience we write

$$\lambda = \frac{1}{4} + r^2,$$

and we refer to r as the spectral parameter of f.

A function $f : \mathcal{H} \to \mathbb{C}$ is automorphic of weight k and multiplier ν for Γ if

$$f\big|_k\gamma = \nu(\gamma) f \qquad \text{for all } \gamma \in \Gamma.$$

Let $\mathcal{A}_k(N,\nu)$ denote the space of all such functions. A smooth automorphic function which is also an eigenfunction of Δ_k and which has at most polynomial growth at the cusps of Γ is called a Maass form. We let $\mathcal{A}_k(N,\nu,r)$ denote the vector space of Maass forms with spectral parameter r. Complex conjugation $f \to \overline{f}$ gives a bijection $\mathcal{A}_k(N,\nu,r) \longleftrightarrow \mathcal{A}_{-k}(N,\overline{\nu},r)$.

If $f \in \mathcal{A}_k(n, \nu, r)$, then $f\big|_k \sigma_{\mathfrak{a}}$ satisfies $(f\big|_k \sigma_{\mathfrak{a}})(z+1) = e(-\kappa_{\mathfrak{a}})(f\big|_k \sigma_{\mathfrak{a}})(z)$. For $n \in \mathbb{Z}$ define

$$n_{\mathfrak{a}} := n - \kappa_{\mathfrak{a}}.$$

Then f has a Fourier expansion at the cusp $\mathfrak{a}$ of the form

$$(f\big|_k \sigma_{\mathfrak{a}})(z) = \rho_{f,\mathfrak{a}}(0) y^{\frac{1}{2}+ir} + \rho'_{f,\mathfrak{a}}(0) y^{\frac{1}{2}-ir} + \sum_{n_{\mathfrak{a}} \neq 0} \rho_{f,\mathfrak{a}}(n) W_{\frac{k}{2}\operatorname{sgn}(n), ir}(4\pi |n_{\mathfrak{a}}| y) e(n_{\mathfrak{a}} x),$$

where $W_{\kappa,\mu}(y)$ is the W-Whittaker function. When the weight is 0, many authors normalize the Fourier coefficients so that $\rho_{f,\mathfrak{a}}(n)$ is the coefficient of $\sqrt{y}\, K_{ir}(2\pi|n_{\mathfrak{a}}|y)$, where $K_\nu(y)$ is the K-Bessel function. Using the relation

$$W_{0,\mu}(y) = \frac{\sqrt{y}}{\sqrt{\pi}} K_\mu(y/2),$$

we see that this has the effect of multiplying $\rho_{f,\mathfrak{a}}(n)$ by $2|n_{\mathfrak{a}}|^{1/2}$.

Let $\mathcal{L}_k(\nu)$ denote the L^2-space of automorphic functions with respect to the Petersson inner product

$$\langle f, g\rangle := \int_{\Gamma\backslash\mathcal{H}} f(z)\overline{g(z)}\, d\mu, \qquad d\mu := \frac{dx\, dy}{y^2},$$

and let $\mathcal{L}_k(\nu, \lambda)$ denote the λ-eigenspace. The spectrum of Δ_k is real and contained in $[\lambda_0(k), \infty)$, where $\lambda_0(k) := \frac{|k|}{2}(1 - \frac{|k|}{2})$. The minimal eigenvalue $\lambda_0(k)$ occurs if and only if there is a holomorphic modular form F of weight $|k|$ and multiplier ν, in which case

$$f_0(z) = \begin{cases} y^{\frac{k}{2}}\, F(z) & \text{if } k \geq 0, \\ y^{-\frac{k}{2}}\, \overline{F}(z) & \text{if } k < 0, \end{cases}$$

is the corresponding eigenfunction. When $k = \pm\frac{1}{2}$ and $\nu = \nu_\theta^{2k}$, the eigenspace $\mathcal{L}_k(\nu, \frac{3}{16})$ is one-dimensional, spanned by $y^{\frac{1}{4}}\theta(z)$ if $k = \frac{1}{2}$ and $y^{-\frac{1}{4}}\overline{\theta}(z)$ if $k = -\frac{1}{2}$.

The spectrum of Δ_k on $\mathcal{L}_k(\nu)$ consists of an absolutely continuous spectrum of multiplicity equal to the number of singular cusps, and a discrete spectrum of finite multiplicity. The Eisenstein series, of which there is one for each singular cusp $\mathfrak{a}$, give rise to the continuous spectrum, which is bounded below by $1/4$. Let $\mathfrak{a}$ be a singular cusp. The Eisenstein series for the cusp $\mathfrak{a}$ is defined by

$$E_{\mathfrak{a}}(z, s) := \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma_\infty} \overline{\nu}(\gamma)\overline{w}(\sigma_{\mathfrak{a}}^{-1}, \gamma) j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k} \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s.$$

If $\mathfrak{b}$ is any cusp, the Fourier expansion for $E_{\mathfrak{a}}$ at the cusp $\mathfrak{b}$ is given by

$$j(\sigma_{\mathfrak{b}}, z)^{-k} E_{\mathfrak{a}}(z, s) = \delta_{\mathfrak{a}=\mathfrak{b}} y^s + \delta_{\kappa_{\mathfrak{b}}=0} \phi_{\mathfrak{ab}}(0, s) y^{1-s} + \sum_{n_{\mathfrak{b}} \neq 0} \phi_{\mathfrak{ab}}(n, s) W_{\frac{k}{2}\operatorname{sgn}(n), s-\frac{1}{2}}(4\pi|n_{\mathfrak{b}}|y) e(n_{\mathfrak{b}} x),$$

where

$$\phi_{\mathfrak{ab}}(n, s) = \begin{cases} \dfrac{e(-\frac{k}{4})\pi^s |n|^{s-1}}{\Gamma(s + \frac{k}{2}\operatorname{sgn}(n))} \displaystyle\sum_{c \in \mathcal{C}(\mathfrak{a},\mathfrak{b})} \frac{S_{\mathfrak{ab}}(0, n, c, \nu)}{c^{2s}} & \text{if } n_{\mathfrak{b}} \neq 0, \\ \dfrac{e(-\frac{k}{4})\pi 4^{1-s}\Gamma(2s-1)}{\Gamma(s+\frac{k}{2})\Gamma(s-\frac{k}{2})} \displaystyle\sum_{c \in \mathcal{C}(\mathfrak{a},\mathfrak{b})} \frac{S_{\mathfrak{ab}}(0, 0, c, \nu)}{c^{2s}} & \text{if } n_{\mathfrak{b}} = 0. \end{cases} \tag{3.2}$$

Here $\mathcal{C}(\mathfrak{a},\mathfrak{b}) = \{c > 0 : \left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right) \in \sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{b}\}$ is the set of allowed moduli and $S_{\mathfrak{a}\mathfrak{b}}(m,n,c,\nu)$ is the Kloosterman sum (defined for any cusp pair $\mathfrak{a}\mathfrak{b}$)

$$S_{\mathfrak{a}\mathfrak{b}}(m,n,c,\nu) := \sum_{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{b}/\Gamma_\infty} \overline{\nu}_{\mathfrak{a}\mathfrak{b}}(\gamma) e\left(\frac{m_\mathfrak{a} a + n_\mathfrak{b} d}{c}\right), \tag{3.3}$$

where

$$\nu_{\mathfrak{a}\mathfrak{b}}(\gamma) = \nu(\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1}) \frac{w(\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1}, \sigma_\mathfrak{b})}{w(\sigma_\mathfrak{a}, \gamma)}.$$

The coefficients $\phi_{\mathfrak{a}\mathfrak{b}}(n,s)$ can be meromorphically continued to the entire s-plane and, in particular, are well-defined on the line $\mathrm{Re}(s) = \frac{1}{2}$. In Section 5 we will evaluate certain linear combinations of the coefficients $\phi_{\mathfrak{a}\mathfrak{b}}(n, \frac{1}{2} \pm ir)$ in terms of Dirichlet L-functions in the cases $k = \pm\frac{1}{2}$ and $\nu = \nu_\theta^{2k}$.

Let $\mathcal{V}_k(\nu)$ denote the orthogonal complement in $\mathcal{L}_k(\nu)$ of the space generated by Eisenstein series. The spectrum of Δ_k on $\mathcal{V}_k(\nu)$ is countable and of finite multiplicity. The exceptional eigenvalues are those which lie in $(\lambda_0(k), 1/4)$ (conjecturally, the set of exceptional eigenvalues is empty). The subspace $\mathcal{V}_k(\nu)$ consists of functions f which decay exponentially at every cusp; equivalently, the zeroth Fourier coefficient of f at each singular cusp vanishes. Eigenfunctions of Δ_k in $\mathcal{V}_k(\nu)$ are called Maass cusp forms.

Let $\{f_j\}$ be an orthonormal basis of $\mathcal{V}_k(\nu)$, and for each j let $\lambda_j = \frac{1}{4} + r_j^2$ denote the Laplace eigenvalue and $\{\rho_{j,\mathfrak{a}}(n)\}$ the Fourier coefficients. Weyl's law describes the distribution of the spectral parameters r_j. Theorem 2.28 of [19] shows that

$$\sum_{0 \le r_j \le T} 1 - \frac{1}{4\pi}\int_{-T}^{T} \frac{\varphi'}{\varphi}\Big(\frac{1}{2} + it\Big)\, dt = \frac{\mathrm{vol}(\Gamma\backslash\mathcal{H})}{4\pi} T^2 - \frac{K_0}{\pi} T\log T + O(T),$$

where $\varphi(s)$ and K_0 are the determinant (see [19, p. 298]) and dimension (see [19, p. 281]), respectively, of the scattering matrix $\Phi(s)$ whose entries are given in terms of constant terms of Eisenstein series.

4. An estimate for coefficients of Maass cusp forms

In this section we prove a general theorem which applies to the Fourier coefficients at the cusp $\mathfrak{a}$ of weight $\pm\frac{1}{2}$ Maass cusp forms with multiplier ν for $\Gamma = \Gamma_0(N)$. We assume that the bound

$$\sum_{c>0} \frac{|S_{\mathfrak{a}\mathfrak{a}}(n,n,c,\nu)|}{c^{1+\beta}} \ll_\nu n^\varepsilon \tag{4.1}$$

holds for some $\beta = \beta_{\nu,\mathfrak{a}} \in (1/2, 1)$. A similar estimate was proved in [1, Theorem 3.1], but the following theorem improves the bound in the x-aspect when $k = \frac{1}{2}$. The proof given here is also considerably shorter.

Theorem 4.1. *Suppose that $k = \pm\frac{1}{2}$ and that ν is a multiplier system of weight k which satisfies* (4.1)*. Fix an orthonormal basis of cusp forms $\{u_j\}$ for $\mathcal{V}_k(\nu)$. For each j, let $\rho_{j,\mathfrak{a}}(n)$ denote the n-th Fourier coefficient of u_j at $\mathfrak{a}$ and let r_j denote the spectral parameter. Then for all $n \ge 1$ we have*

$$n_\mathfrak{a} \sum_{x \le r_j \le 2x} |\rho_{j,\mathfrak{a}}(n)|^2 e^{-\pi r_j} \ll x^{-k}\big(x^2 + n^{\beta+\varepsilon} x^{1-2\beta} \log^\beta x\big).$$

We begin with an auxiliary version of Kuznetsov's formula ([28, §5]) which is Lemma 3 of [36] with $m = n$, $t \mapsto 2t$, and $\sigma = 1$ (see [1, Section 3] for justification of the latter). While Proskurin assumes that $k > 0$ throughout his paper, this lemma is still valid for $k < 0$ by the same proof, and straightforward modifications give the result for an arbitrary cusp $\mathfrak{a}$.

Lemma 4.2. *With the assumptions of Theorem 4.1, and for any $t \in \mathbb{R}$ we have*

$$\frac{2\pi^2 n_{\mathfrak{a}}}{|\Gamma(1-\frac{k}{2}+it)|^2}\left[\sum_{r_j}\frac{|\rho_{j,\mathfrak{a}}(n)|^2}{\cosh 2\pi r_j + \cosh 2\pi t} + \frac{1}{4}\sum_{\mathfrak{c}}\int_{-\infty}^{\infty}\frac{\left|\phi_{\mathfrak{c}\mathfrak{a}}(n,\frac{1}{2}+ir)\right|^2}{(\cosh 2\pi r + \cosh 2\pi t)|\Gamma(\frac{k+1}{2}+ir)|^2}\,dr\right] = \frac{1}{4\pi} + \frac{2n_{\mathfrak{a}}}{i^{k+1}}\sum_{c>0}\frac{S_{\mathfrak{a}\mathfrak{a}}(n,n,c,\nu)}{c^2}\int_L K_{2it}\Big(\frac{4\pi n_{\mathfrak{a}}}{c}\,q\Big)q^{k-1}\,dq, \quad (4.2)$$

where $\sum_{\mathfrak{c}}$ is a sum over singular cusps, and L is the semicircular contour $|q| = 1$ with $\mathrm{Re}(q) > 0$, from $-i$ to i.

To prove Theorem 4.1 we follow the method of Motohashi [33, Section 2]. We begin by evaluating the integral on the right-hand side of (4.2) via the following lemma. For the remainder of this section we frequently use the notation $\int_{(\xi)}$ to denote $\int_{\xi-i\infty}^{\xi+i\infty}$.

Lemma 4.3. *Let $k = \pm\frac{1}{2}$. Suppose that $a > 0$, $\xi > \frac{k}{2}$. Then*

$$2\int_L K_{2it}(2aq)q^{k-1}\,dq = \frac{1}{2\pi}\int_{(\xi)}\frac{\sin(\pi s - \frac{\pi k}{2})}{s-\frac{k}{2}}\Gamma(s+it)\Gamma(s-it)a^{-2s}\,ds.$$

Proof. For any $\xi > 0$ we have the Mellin-Barnes integral representation [8, (10.32.13)]

$$2K_{2it}(2z) = \frac{1}{2\pi i}\int_{(\xi)}\Gamma(s)\Gamma(s-2it)z^{2it-2s}\,ds,$$

which is valid for $|\arg z| < \frac{\pi}{2}$. It follows that

$$\begin{aligned}2\int_L K_{2it}(2aq)q^{k-1}\,dq &= \frac{1}{2\pi i}\int_{(\xi)}\Gamma(s)\Gamma(s-2it)a^{2it-2s}\int_L q^{2it-2s+k-1}\,dq\,ds\\ &= \frac{1}{2\pi}\int_{(\xi)}\Gamma(s)\Gamma(s-2it)a^{2it-2s}\frac{\sin(\pi(s-it-\frac{k}{2}))}{s-it-\frac{k}{2}}\,ds.\end{aligned}$$

The lemma follows after replacing s by $s+it$. ☐

Let K be a large positive real number. In (4.2) we multiply by the positive weight

$$e^{-(t/K)^2} - e^{-(2t/K)^2}$$

and integrate on t over $\mathbb{R}$. Applying Lemma 4.3 to the result (and noting that all terms on the left-hand side are positive), we obtain

$$n_{\mathfrak{a}}\sum_{r_j}|a_j(n)|^2 h_K(r_j) \ll K + \sum_{c>0}\frac{|S(n,n,c,\nu)|}{c}\Big|M_k\Big(K,\frac{2\pi n_{\mathfrak{a}}}{c}\Big)\Big|, \quad (4.3)$$

where
$$h_K(r) := \int_{-\infty}^{\infty} \frac{e^{-(t/K)^2} - e^{-(2t/K)^2}}{|\Gamma(1-\frac{k}{2}+it)|^2(\cosh 2\pi r + \cosh 2\pi t)}\,dt \tag{4.4}$$
and
$$M(K,a) = \int_{-\infty}^{\infty} \Big(e^{-(t/K)^2} - e^{-(2t/K)^2}\Big) \int_{(\xi)} \frac{\sin(\pi s - \frac{\pi k}{2})}{s-\frac{k}{2}} \Gamma(s+it)\Gamma(s-it) a^{1-2s}\,ds\,dt.$$

We will make use of the following well-known estimate for oscillatory integrals (see, for instance, [44, Chapter IV]).

Lemma 4.4. *Suppose that F and G are real-valued functions on $[a,b]$ with F differentiable, such that $G(x)/F'(x)$ is monotonic. If $|F'(x)/G(x)| \geq m > 0$ then*
$$\int_a^b G(x) e(F(x))\,dx \ll \frac{1}{m}.$$

Proposition 4.5. *Let K be a large positive real number. Suppose that $k = \pm\frac{1}{2}$ and let $M(K,a)$ be as above. For $a > 0$ we have*
$$M(K,a) \ll \min\Big(1, \frac{a \log K}{K^2}\Big). \tag{4.5}$$

Proof. Starting with the integral representation [8, (5.12.1)]
$$\Gamma(s+it)\Gamma(s-it) = \Gamma(2s) \int_0^1 y^{s+it-1}(1-y)^{s-it-1}\,dy,$$
we interchange the order of integration, putting the integral on t inside, and find that the integral on t equals
$$\begin{aligned} T(K,y) &= \int_{-\infty}^{\infty} \Big(\frac{y}{1-y}\Big)^{it} \Big(e^{-(t/K)^2} - e^{-(2t/K)^2}\Big)\,dt \\ &= Ke^{-\frac{1}{4}K^2 \log^2\left(\frac{y}{1-y}\right)} - \frac{1}{2} K e^{-\frac{1}{16}K^2\log^2\left(\frac{y}{1-y}\right)}. \end{aligned}$$
Hence
$$M(K,a) = \int_0^1 \frac{T(K,y)}{y(1-y)} \int_{(\xi)} \frac{\sin(\pi s - \frac{\pi k}{2})}{s - \frac{k}{2}} \Gamma(2s)[y(1-y)]^s a^{1-2s}\,ds\,dy. \tag{4.6}$$
To evaluate the inner integral, we use that
$$\frac{u^{k-2s}}{s-\frac{k}{2}} = 2\int_u^{\infty} t^{-2s+k-1}\,dt.$$
Setting $u = a\,[y(1-y)]^{-\frac{1}{2}}$, the integral on s in (4.6) equals
$$2au^{-k} \int_u^{\infty} t^{k-1} \int_{(\xi)} \sin(\pi s - \tfrac{\pi k}{2})\Gamma(2s) t^{-2s}\,ds\,dt = a f_k(u),$$
where
$$f_k(u) = \cos\Big(\frac{\pi k}{2}\Big) u^{-k} \int_u^{\infty} t^{k-1} \sin t\,dt - \sin\Big(\frac{\pi k}{2}\Big) u^{-k} \int_u^{\infty} t^{k-1}\cos t\,dt.$$
Finally, we set $z = K \log \frac{y}{1-y}$ to obtain
$$M(K,a) = a \int_{-\infty}^{\infty} \Big(e^{-z^2/4} - \frac{1}{2} e^{-z^2/16}\Big) f_k\Big(2a \cosh\Big(\frac{z}{2K}\Big)\Big)\,dz.$$

We claim that $f_k(u) \ll \min(1, 1/u)$. For $u \geq 1$ this follows from Lemma 4.4. Suppose that $u \leq 1$. In the case $k = -\frac{1}{2}$, we have $f_k(u) \ll 1$ by estimating the integrals trivially. When $k = \frac{1}{2}$ a computation shows that

$$f_{\frac{1}{2}}(u) = \frac{\sqrt{\pi}\, C\big(\sqrt{2u/\pi}\big) - \sqrt{\pi}\, S\big(\sqrt{2u/\pi}\big)}{\sqrt{u}},$$

where $C(x)$ and $S(x)$ are the Fresnel integrals [8, §7.2]. It follows that $f_{\frac{1}{2}}(u) \ll 1$.

From the estimate $f_k(u) \ll \min(1, 1/u)$ it follows that

$$M(K, a) \ll 1.$$

Now suppose that $a \ll K^2$. In this case we add and subtract $f_k(2a)$ from the integrand and notice that

$$\int_{-\infty}^{\infty} \Big(e^{-z^2/4} - \frac{1}{2} e^{-z^2/16}\Big)\, dz = 0,$$

so

$$M(K, a) \ll a \int_0^{\infty} e^{-z^2/16} \Big| f_k(2a) - f_k\Big(2a \cosh\Big(\frac{z}{2K}\Big)\Big)\Big|\, dz.$$

Let $T = c\sqrt{\log K}$ with c a large constant, and let $F(z) = f_k(2a) - f_k(2a \cosh z)$. Then $F(0) = F'(0) = 0$, so for $|z| \leq T/K$ we have

$$F(z) \ll z^2 \max_{|w| \leq T/K} |F''(w)|. \tag{4.7}$$

Since

$$F''(w) \ll a \cosh w |f_k'(2a \cosh w)| + a^2 \sinh^2 w |f_k''(2a \cosh w)|$$

and, by Lemma 4.4,

$$f_k'(u), f_k''(u) \ll u^{-1},$$

we conclude that

$$F''(w) \ll a \sinh(T/K) \tanh(T/K) \ll \frac{aT^2}{K^2} \ll T^2. \tag{4.8}$$

By (4.7) and (4.8) we have

$$a \int_0^{T} e^{-z^2/16} \Big| F\Big(\frac{z}{2K}\Big)\Big|\, dz \ll \frac{aT^2}{K^2} \int_0^{\infty} z^2 e^{-z^2/16}\, dz \ll \frac{aT^2}{K^2}$$

and by $f_k(u) \ll 1$ we have

$$a \int_T^{\infty} e^{-z^2/16} \Big| f_k(2a) - f_k\Big(2a \cosh\Big(\frac{z}{2K}\Big)\Big)\Big|\, dz \ll a \int_T^{\infty} e^{-z^2/16}\, dz \ll a\, e^{-T^2/16}.$$

With our choice of T this yields (4.5). □

Proof of Theorem 4.1. First note that when $r \sim x$ we have $h_x(r) \gg e^{-\pi r} x^{k-1}$, where $h_x(r)$ is defined in (4.4), so by (4.3) and positivity we have

$$n_{\mathfrak{a}} x^k \sum_{x \leq r_j \leq 2x} |\rho_{j,\mathfrak{a}}(n)| e^{-\pi r_j} \ll x^2 + x \sum_{c > 0} \frac{|S_{\mathfrak{a}\mathfrak{a}}(n, n, c, \nu)|}{c} \Big| M_k\Big(x, \frac{2\pi n_{\mathfrak{a}}}{c}\Big)\Big|.$$

Let β be as in (4.1). By Proposition 4.5 we have

$$M_k(x, a) \ll \min\Big(1, \frac{a \log x}{x^2}\Big) \ll \frac{a^{\beta} \log^{\beta} x}{x^{2\beta}},$$

from which it follows that

$$x\sum_{c>0}\frac{|S_{\mathfrak{a}\mathfrak{a}}(n,n,c,\nu)|}{c}\left|M_k\left(x,\frac{2\pi n_{\mathfrak{a}}}{c}\right)\right| \ll n_{\mathfrak{a}}^{\beta}x^{1-2\beta}\log^{\beta}x\sum_{c>0}\frac{|S_{\mathfrak{a}\mathfrak{a}}(n,n,c,\nu)|}{c^{1+\beta}}$$
$$\ll n_{\mathfrak{a}}^{\beta+\varepsilon}x^{1-2\beta}\log^{\beta}x.$$

The theorem follows. □

5. The Kuznetsov formula for Kohnen's plus space

In this section we define the plus spaces of holomorphic and Maass cusp forms, and we prove an analogue of Kuznetsov's formula relating the Kloosterman sums $S_k^+(m,n,c)$ to the Fourier coefficients of such forms. For the remainder of the paper we specialize to the case $\Gamma=\Gamma_0(4)$ with $(k,\nu)=(\frac12,\nu_\theta)$ or $(-\frac12,\overline{\nu}_\theta)$. We will often write $k=\lambda+\frac12$, and to simplify notation, we write $\mathcal{V}_k=\mathcal{V}_k(\nu)$ and $\mathcal{S}_\ell=\mathcal{S}_\ell(\nu)$, where $S_\ell(\nu)$ is the space of holomorphic cusp forms of weight ℓ and multiplier ν. We fix once and for all a set of inequivalent representatives for the cusps of Γ, namely ∞, 0, and $\frac12$, with associated scaling matrices

$$\sigma_\infty=\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad \sigma_0=\begin{pmatrix}0&-\frac12\\2&0\end{pmatrix},\qquad \sigma_{\frac12}=\begin{pmatrix}1&-\frac12\\2&0\end{pmatrix}.$$

Then

$$\kappa_\infty=\kappa_0=0\quad\text{ and }\quad \kappa_{\frac12}=\frac{(-1)^\lambda 3}{4}.$$

Following Kohnen [26, 27] we define an operator L on automorphic functions as follows. If f satisfies $f\big|_k\gamma=\nu(\gamma)f$ for all $\gamma\in\Gamma_0(4)$ then we define

$$Lf:=\frac{1}{2(1+i^{2k})}\sum_{w=0}^{3}f\big|_k\begin{pmatrix}1+w&1/4\\4w&1\end{pmatrix}.$$

It is not difficult to show that L maps Maass cusp forms to Maass cusp forms. It follows from [26] (see also [23]) that L is self-adjoint, that it commutes with the Hecke operators T_{p^2}, and that it satisfies the equation

$$(L-1)(L+\tfrac12)=0$$

(Kohnen proves this in the holomorphic case, but the necessary modifications are simple). The space $\mathcal{V}_k$ decomposes as $\mathcal{V}_k=\mathcal{V}_k^+\oplus\mathcal{V}_k^-$ where $\mathcal{V}_k^+$ is the eigenspace with eigenvalue 1, and $\mathcal{V}_k^-$ is the eigenspace with eigenvalue $-\frac12$. For each $f\in\mathcal{V}_k$, we have $f\in\mathcal{V}_k^+$ if and only if $\rho_{f,\infty}(n)=0$ for $(-1)^\lambda n\equiv 2,3\pmod 4$. The following lemma describes the action of L on Fourier expansions.

Lemma 5.1. *Let $k=\pm\frac12=\lambda+\frac12$ and $\nu=\nu_\theta^{2k}$. Suppose that $f\big|_k\gamma=\nu(\gamma)f$ for all $\gamma\in\Gamma$. For each cusp $\mathfrak{a}$ of Γ write the Fourier expansion of f as*

$$(f\big|_k\sigma_{\mathfrak{a}})(z)=\sum_{n\in\mathbb{Z}}c_{f,\mathfrak{a}}(n,y)e(n_{\mathfrak{a}}x).$$

Then

$$c_{Lf,\infty}(n,y)=\begin{cases}\frac12 c_{f,\infty}(n,y)+\frac{1}{2(1-i^{2k})}\,c_{f,\mathfrak{a}}\left(\frac n4+\kappa_{\mathfrak{a}},4y\right) & \text{if } (-1)^\lambda n\equiv 0,1\pmod 4,\\ -\frac12 c_{f,\infty}(n,y) & \text{if } (-1)^\lambda n\equiv 2,3\pmod 4,\end{cases}$$

where $\mathfrak{a}=0$ if $n\equiv 0\pmod 4$ and $\mathfrak{a}=\frac12$ if $n\equiv(-1)^\lambda\pmod 4$.

Proof. Let $A_w = \left(\begin{smallmatrix} 1+w & 1/4 \\ 4w & 1 \end{smallmatrix}\right)$. Since $A_2 = \left(\begin{smallmatrix} 3 & -2 \\ 8 & -5 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 3/4 \\ 0 & 1 \end{smallmatrix}\right)$ and $\nu(\left(\begin{smallmatrix} 3 & -2 \\ 8 & -5 \end{smallmatrix}\right)) = i^{2k}$ we have

$$\begin{aligned} f\big|_k A_0 + f\big|_k A_2 &= f(z+\tfrac14) + i^{2k} f(z+\tfrac34) \\ &= (1+i^{2k}) \sum_{(-1)^\lambda n \equiv 0,1(4)} c_{f,\infty}(n,y)e(nx) - (1+i^{2k}) \sum_{(-1)^\lambda n \equiv 2,3(4)} c_{f,\infty}(n,y)e(nx). \end{aligned}$$

For $w=1,3$ we have $A_1 = \left(\begin{smallmatrix} -1 & 1 \\ -4 & 3 \end{smallmatrix}\right)\sigma_{\frac12}\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1/2 \end{smallmatrix}\right)$ and $A_3 = \left(\begin{smallmatrix} -1 & 1 \\ -4 & 3 \end{smallmatrix}\right)\sigma_0\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1/2 \end{smallmatrix}\right)$. Since $\nu(\left(\begin{smallmatrix} -1 & 1 \\ -4 & 3 \end{smallmatrix}\right)) = i^{2k}$, we have

$$\begin{aligned} f\big|_k A_1 + f\big|_k A_3 &= i^{2k}(f\big|_k \sigma_0)(4z) + i^{2k}(f\big|_k \sigma_{\frac12}(4z)) \\ &= i^{2k} \sum_{n\equiv 0(4)} c_{f,0}(\tfrac{n}{4}, 4y)e(nx) + i^{2k} \sum_{n \equiv (-1)^\lambda (4)} c_{f,\frac12}(\tfrac{n}{4} + \kappa_{\frac12}, 4y)e(nx). \end{aligned}$$

The lemma follows. □

The analogue of L for holomorphic cusp forms is defined as follows. If for some ℓ we have $F(\gamma z) = \nu(\gamma)(cz+d)^\ell F(z)$ for all $\gamma \in \Gamma_0(4)$ then $f(z) = y^{\ell/2}F(z)$ satisfies $f\big|_\ell \gamma = \nu(\gamma) f$, and we define

$$L^*F := y^{-\frac{\ell}{2}} Lf.$$

The plus space $\mathcal{S}_\ell^+$ of holomorphic cusp forms is defined as the subspace of $\mathcal{S}_\ell$ consisting of forms F satisfying $L^*F = F$. If $\rho_{F,\mathfrak{a}}(n)$ is the n-th coefficient of F at the cusp $\mathfrak{a}$ then, in the notation of the previous lemma, we have $\rho_{F,\mathfrak{a}}(\frac{n}{4} + \kappa_{\mathfrak{a}}, 4y) = \frac12 c_{f,\mathfrak{a}}(\frac{n}{4}+\kappa_{\mathfrak{a}}, 4y)$. Therefore we have the following analogue of Lemma 5.1.

Lemma 5.2. *Let $k = \pm\frac12 = \lambda + \frac12$ and $\nu = \nu_\theta^{2k}$. Suppose that $\ell \equiv k \pmod 2$ and that $F \in \mathcal{S}_\ell(\nu)$. Then*

$$\rho_{L^*F,\infty}(n,y) = \begin{cases} \frac12 \rho_{F,\infty}(n,y) + \frac{1}{(1-i^{2k})}\,\rho_{F,\mathfrak{a}}\Big(\frac{n}{4} + \kappa_{\mathfrak{a}}, 4y\Big) & \text{if } (-1)^\lambda n \equiv 0,1 \pmod 4, \\ -\frac12 \rho_{F,\infty}(n,y) & \text{if } (-1)^\lambda n \equiv 2,3 \pmod 4, \end{cases}$$

where $\mathfrak{a} = 0$ if $n \equiv 0 \pmod 4$ and $\mathfrak{a} = \frac12$ if $n \equiv (-1)^\lambda \pmod 4$.

To state the plus space version of the Kuznetsov trace formula, we first fix some notation. Recall that $S_k^+(m,n,c)$ is the plus space Kloosterman sum

$$S_k^+(m,n,c) = e\Big(-\frac{k}{4}\Big) \sum_{d \bmod c} \Big(\frac{c}{d}\Big)\varepsilon_d^{2k} e\Big(\frac{m\overline{d} + nd}{c}\Big) \times \begin{cases} 1 & \text{if } 8 \mid c, \\ 2 & \text{if } 4 \mid\mid c. \end{cases}$$

Let $\varphi : [0,\infty) \to \mathbb{R}$ be a smooth test function which satisfies

$$\varphi(0) = \varphi'(0) = 0 \quad \text{and} \quad \varphi^{(j)}(x) \ll x^{-2-\varepsilon} \quad \text{for } j = 0,1,2,3. \tag{5.1}$$

Define the integral transforms

$$\widetilde{\varphi}(\ell) := \frac{1}{\pi}\int_0^\infty J_{\ell-1}(x)\varphi(x)\,\frac{dx}{x}, \tag{5.2}$$

$$\widehat{\varphi}(r) := \frac{-i\,\xi_k(r)}{\cosh 2\pi r}\int_0^\infty \Big(\cos(\tfrac{\pi k}{2} + \pi i r)J_{2ir}(x) - \cos(\tfrac{\pi k}{2} - \pi i r)J_{-2ir}(x)\Big)\phi(x)\frac{dx}{x}, \tag{5.3}$$

where

$$\xi_k(r) := \frac{\pi^2}{\sinh \pi r \Gamma(\frac{1-k}{2}+ir)\Gamma(\frac{1-k}{2}-ir)} \sim \tfrac{1}{2}\pi r^k \quad \text{as } r\to\infty.$$

Note that $\widehat{\varphi}(r)$ is real-valued when $r \geq 0$ and when $ir \in (-\frac14, \frac14)$. If d is a fundamental discriminant, let $\chi_d = \left(\frac{d}{\cdot}\right)$ and let $L(s,\chi_d)$ denote the Dirichlet L-function with Dirichlet series

$$L(s,\chi_d) := \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s}.$$

Finally, we define

$$\mathfrak{S}_d(w,s) = \sum_{\ell \mid w} \mu(\ell)\chi_d(\ell)\frac{\tau_s(w/\ell)}{\sqrt{\ell}},$$

where τ_s is the normalized sum of divisors function

$$\tau_s(\ell) = \sum_{ab=\ell}\left(\frac{a}{b}\right)^s = \frac{\sigma_{2s}(\ell)}{\ell^s}.$$

Theorem 5.3. *Let $\varphi : [0,\infty) \to \mathbb{R}$ be a smooth test function satisfying* (5.1)*. Let $k = \pm\frac12 = \lambda + \frac12$ and $\nu = \nu_\theta^{2k}$. Suppose that $m, n \geq 1$ with $(-1)^\lambda m, (-1)^\lambda n \equiv 0, 1 \pmod 4$, and write*

$$(-1)^\lambda m = v^2 d', \quad (-1)^\lambda n = w^2 d, \qquad \text{with } d, d' \text{ fundamental discriminants.}$$

Fix an orthonormal basis of Maass cusp forms $\{u_j\} \subset \mathcal{V}_k^+$ with associated spectral parameters r_j and coefficients $\rho_j(n)$. For each $\ell \equiv k \pmod 2$ with $\ell > 2$, fix an orthonormal basis of holomorphic cusp forms $\mathcal{H}_\ell^+ \subset \mathcal{S}_\ell^+$ with normalized coefficients given by

$$g(z) = \sum_{n=1}^{\infty} (4\pi n)^{\frac{\ell-1}{2}} \rho_g(n) e(nz) \quad \text{for } g \in \mathcal{H}_\ell^+. \tag{5.4}$$

Then

$$\sum_{0<c\equiv 0(4)} \frac{S_k^+(m,n,c)}{c}\varphi\left(\frac{4\pi\sqrt{mn}}{c}\right)$$
$$= 6\sqrt{mn}\sum_{j\geq 0}\frac{\overline{\rho_j(m)}\rho_j(n)}{\cosh \pi r_j}\widehat{\varphi}(r_j) + \frac32 \sum_{\ell \equiv k \bmod 2} e\left(\tfrac{\ell-k}{4}\right)\widetilde{\varphi}(\ell)\Gamma(\ell)\sum_{g\in\mathcal{H}_\ell^+}\overline{\rho_g(m)}\rho_g(n)$$
$$+ \frac12 \int_{-\infty}^{\infty}\left(\frac{d}{d'}\right)^{ir}\frac{L(\frac12 - 2ir,\chi_{d'})L(\frac12+2ir,\chi_d)\mathfrak{S}_{d'}(v,2ir)\mathfrak{S}_d(w,2ir)}{|\zeta(1+4ir)|^2\cosh \pi r|\Gamma(\frac{k+1}{2}+ir)|^2}\widehat{\varphi}(r)\,dr.$$

Biró [3, Theorem B] stated a version of Theorem 5.3 for $\Gamma_0(4N)$ in the case $k = \frac12$ under the added assumption that $\widetilde{\varphi}(\ell) = 0$ for all ℓ. His theorem involves coefficients of half-integral weight Eisenstein series at cusps instead of Dirichlet L-functions.

To prove Theorem 5.3, we start with Proskurin's version of the Kuznetsov formula [36] which is valid for arbitrary weight k and for the cusp-pair $\infty\infty$. The necessary modifications for an arbitrary cusp-pair are straightforward (see [7] for details in the $k = 0$ case). Recall the definitions of the generalized Kloosterman sum $S_{\mathfrak{ab}}(m,n,c,\nu)$ in (3.3) and the Eisenstein series coefficients $\phi_{\mathfrak{ab}}(m,s)$ in (3.2).

Proposition 5.4. *Suppose that φ satisfies* (5.1)*. Suppose that $m, n \geq 1$ and that $k = \pm\frac{1}{2}$. Let $\nu = \nu_\theta^{2k}$ and $\Gamma = \Gamma_0(4)$ and let $\mathfrak{a}, \mathfrak{b}$ be cusps of Γ. Let $\{u_j\}$ denote an orthonormal basis of Maass cusp forms of weight k with spectral parameters r_j. For each $2 < \ell \equiv k \pmod 2$, let $\mathcal{H}_\ell$ denote an orthonormal basis of holomorphic cusp forms of weight ℓ with coefficients normalized as in* (5.4)*. Then*

$$\begin{aligned} e(-\tfrac{k}{4}) &\sum_{c \in \mathcal{C}(\mathfrak{a},\mathfrak{b})} \frac{S_{\mathfrak{ab}}(m,n,c,\nu)}{c} \varphi\left(\frac{4\pi\sqrt{m_\mathfrak{a} n_\mathfrak{b}}}{c}\right) \\ &= 4\sqrt{m_\mathfrak{a} n_\mathfrak{b}} \sum_{j \geq 0} \frac{\overline{\rho_{j\mathfrak{a}}(m)}\rho_{j\mathfrak{b}}(n)}{\cosh \pi r_j} \widehat{\varphi}(r_j) + \sum_{\ell \equiv k \bmod 2} e\left(\tfrac{\ell-k}{4}\right)\widetilde{\varphi}(\ell)\Gamma(\ell) \sum_{g \in \mathcal{H}_\ell} \overline{\rho_{g\mathfrak{a}}(m)}\rho_{g\mathfrak{b}}(n) \\ &\qquad + \sum_{\mathfrak{c} \in \{0,\infty\}} \int_{-\infty}^{\infty} \left(\frac{n_\mathfrak{b}}{m_\mathfrak{a}}\right)^{ir} \frac{\overline{\phi_{\mathfrak{ca}}(m, \frac{1}{2}+ir)}\, \phi_{\mathfrak{cb}}(n, \frac{1}{2}+ir)}{\cosh \pi r |\Gamma(\frac{k+1}{2}+ir)|^2} \widehat{\varphi}(r)\, dr. \end{aligned} \tag{5.5}$$

We will apply (5.5) for the cusp-pairs $\infty\infty$, $\infty 0$, and $\infty\frac{1}{2}$, and take a certain linear combination which annihilates all but the plus space coefficients. The following lemma is essential to make this work.

Lemma 5.5. *Suppose that $4 \,||\, c$. Let $k = \pm\frac{1}{2} = \lambda + \frac{1}{2}$ and $\nu = \nu_\theta^{2k}$. Let $\mathfrak{a} = 0$ or $\frac{1}{2}$ according to $(-1)^\lambda n \equiv 0, 1 \pmod 4$, respectively. Then*

$$S_{\infty\infty}(m,n,c,\nu) = (1 + i^{2k}) S_{\infty\mathfrak{a}}(m, \tfrac{n}{4} + \kappa_\mathfrak{a}, \tfrac{c}{2}, \nu).$$

Proof. Since $\overline{S_{\mathfrak{ab}}(m,n,c,\nu)} = S_{\mathfrak{ab}}(-m,-n,c,\overline{\nu})$, it is enough to show that

$$S_{\infty\infty}(m,n,c,\nu_\theta) = (1+i) \times \begin{cases} S_{\infty 0}(m, \frac{n}{4}, \frac{c}{2}, \nu_\theta) & \text{if } n \equiv 0 \pmod 4, \\ S_{\infty \frac{1}{2}}(m, \frac{n+3}{4}, \frac{c}{2}, \nu_\theta) & \text{if } n \equiv 1 \pmod 4. \end{cases}$$

This is proved in [3, Lemma A.7]. Note that Biro chooses different representatives and scaling matrices for the cusps 0 and $\frac{1}{2}$, which has the effect of changing the factor $(1-i)$ to $(1+i)$. □

Proof of Theorem 5.3. Let k, ν, and $\mathfrak{a}$ be as in Lemma 5.5. From that lemma and the definition (1.8) it follows that

$$S_k^+(m,n,c) = e(-\tfrac{k}{4}) S_{\infty\infty}(m,n,c,\nu) + \delta_{4||c}\sqrt{2}\, S_{\infty\mathfrak{a}}(m, \tfrac{n}{4} + \kappa_\mathfrak{a}, \tfrac{c}{2}, \nu).$$

Therefore

$$\begin{aligned} \sum_{4|c>0} \frac{S_k^+(m,n,c)}{c} \varphi\left(\frac{4\pi\sqrt{mn}}{c}\right) &= e(-\tfrac{k}{4}) \sum_{4|c>0} \frac{S_{\infty\infty}(m,n,c,\nu)}{c} \varphi\left(\frac{4\pi\sqrt{mn}}{c}\right) \\ &\quad + \frac{1}{\sqrt{2}} \sum_{2||c>0} \frac{S_{\infty\mathfrak{a}}(m, \frac{n}{4}+\kappa_\mathfrak{a}, c, \nu)}{c} \varphi\left(\frac{4\pi\sqrt{m(\frac{n}{4}+\kappa_\mathfrak{a})_\mathfrak{a}}}{c}\right). \end{aligned} \tag{5.6}$$

Note that $\mathcal{C}(\infty, \mathfrak{a}) = \{c \in \mathbb{Z}_+ : c \equiv 2 \pmod 4\}$ for $\mathfrak{a} = 0, \frac{1}{2}$. We apply Proposition 5.4 for each of the cusp-pairs $\infty\infty$ and $\infty\mathfrak{a}$ on the right-hand side of (5.6). We fix an orthonormal basis $\{u_j^+\}$ for $\mathcal{V}_k^+$ and we choose an orthonormal basis $\{u_j\}$ for $\mathcal{V}_k$ such that $\{u_j^+\} \subseteq \{u_j\}$. Then we do the same for $\mathcal{H}_k^+ \subseteq \mathcal{H}_k$. The Maass form contribution is

$$4\sqrt{mn} \sum_{u_j \in \mathcal{V}_k} \frac{\overline{\rho}_j(m)}{\cosh \pi r_j} \widehat{\varphi}(r_j) \Big(\rho_{j,\infty}(n) + \frac{1}{2(1-i^{2k})} \rho_{j,\mathfrak{a}}(\tfrac{n}{4} + \kappa_\mathfrak{a}) \Big).$$

Let $\rho_j^{(L)}$ denote the coefficients of Lu_j. Then by Lemma 5.1 we have

$$\rho_{j,\infty}(n) + \frac{1}{2(1-i^{2k})}\rho_{j,\mathfrak{a}}(\tfrac{n}{4}+\kappa_{\mathfrak{a}}) = \tfrac{1}{2}\rho_{j,\infty}(n) + \rho_{j,\infty}^{(L)}(n) = \rho_j(n) \times \begin{cases} \frac{3}{2} & \text{if } u_j \in \mathcal{V}_k^+, \\ 0 & \text{if } u_j \in \mathcal{V}_k^-. \end{cases}$$

We compute the contribution from the holomorphic forms similarly. For the Eisenstein series contribution we apply the following proposition, together with the relation $S_{\mathfrak{ab}}(m,n,c,\nu) = \overline{S_{\mathfrak{ab}}(-m,-n,c,\overline{\nu})}$. □

Proposition 5.6. *Let $k=\frac{1}{2}$ and $\nu=\nu_\theta$ and suppose that $m,n \equiv 0,1 \pmod 4$. Write $m=v^2d'$ and $n=w^2d$, where d',d are fundamental discriminants. Let $\mathfrak{a}=0$ or $\frac{1}{2}$ according to $n\equiv 0,1 \pmod 4$, respectively. Then*

$$\sum_{\mathfrak{c}\in\{\infty,0\}} \overline{\phi}_{\mathfrak{c}\infty}(m,\tfrac{1}{2}+ir)\Big(\phi_{\mathfrak{c}\infty}(n,\tfrac{1}{2}+ir) + \frac{1+i}{2\cdot 4^{ir}}\phi_{\mathfrak{c}\mathfrak{a}}\Big(\frac{n}{4}+\kappa_{\mathfrak{a}},\tfrac{1}{2}+ir\Big)\Big)$$
$$= \frac{L(\frac{1}{2}-2ir,\chi_{d'})L(\frac{1}{2}+2ir,\chi_d)}{2|\zeta(1+4ir)|^2}\Big(\frac{v}{w}\Big)^{2ir}\mathfrak{S}_{d'}(v,2ir)\mathfrak{S}_d(w,2ir). \quad (5.7)$$

The proof of this proposition is quite technical, and we will proceed in several steps. In order to work in the region of absolute convergence, we will evalute the sum

$$\sum_{\mathfrak{c}\in\{\infty,0\}} \overline{\phi}_{\mathfrak{c}\infty}(m,s)\Big(\phi_{\mathfrak{c}\infty}(n,s) + \frac{1+i}{4^s}\phi_{\mathfrak{c}\mathfrak{a}}\Big(\frac{n}{4}+\kappa_{\mathfrak{a}},s\Big)\Big),$$

for $\mathrm{Re}(s)$ sufficiently large. Then, by analytic continuation, we can set $s=\frac{1}{2}+ir$ to obtain (5.7). First, for the term $\mathfrak{c}=\infty$, by Lemma 5.5 we have

$$\phi_{\infty\infty}(n,s) + \frac{1+i}{4^s}\phi_{\infty\mathfrak{a}}\Big(\frac{n}{4}+\kappa_{\mathfrak{a}},s\Big) = e\Big(\frac{1}{8}\Big)\phi^+(n,s), \quad (5.8)$$

where

$$\phi^+(n,s) = \sum_{4|c>0}\frac{S^+(0,n,c)}{c^{2s}}. \quad (5.9)$$

Here we have written $S^+(m,n,c)=S^+_{1/2}(m,n,c)$ for convenience. The following proposition evaluates $\phi^+(n,s)$. It is proved in [21] and applied in [13, Lemma 4]; here we give an alternative proof which uses Kohnen's identity (2.5).

Proposition 5.7. *Let $w\in\mathbb{Z}_+$ and let d be a fundamental discriminant. Then*

$$\phi^+(w^2d,s) = 2^{\frac{3}{2}-4s}w^{1-2s}\frac{L(2s-\frac{1}{2},\chi_d)}{\zeta(4s-1)}\mathfrak{S}_d(w,2s-1).$$

Proof. By Möbius inversion, it suffices to prove that

$$\sum_{\ell|w}\chi_d(\ell)\ell^{\frac{1}{2}-2s}\phi^+\Big(\frac{w^2}{\ell^2}d,s\Big) = 2^{\frac{3}{2}-4s}w^{1-2s}\tau_{2s-1}(w)\frac{L(2s-\frac{1}{2},\chi_d)}{\zeta(4s-1)}.$$

Writing ϕ^+ as the Dirichlet series (5.9), reversing the order of summation, and applying the identity (2.5), we find that

$$\sum_{\ell|w}\chi_d(\ell)\ell^{\frac12-2s}\phi^+\Big(\frac{w^2}{\ell^2}d,s\Big)=\frac{1}{\sqrt2}\sum_{4|c>0}\frac{1}{c^{2s-\frac12}}\sum_{\ell|(w,\frac{c}{4})}\chi_d(\ell)\sqrt{\frac{2\ell}{c}}\,S^+\Big(0,\frac{w^2}{\ell^2}d;\frac{c}{\ell}\Big)$$
$$=2^{\frac12-4s}\sum_{c=1}^{\infty}\frac{T_w(0,d;4c)}{c^{2s-\frac12}}.$$

To evaluate $T_w(0,d;4c)$ for a given c, we write $4c=tu$, where

$$u=\prod_{p^a||4c}p^{\lceil\frac a2\rceil}\quad\text{and}\quad t=\prod_{p^a||4c}p^{\lfloor\frac a2\rfloor}.$$

Then $b^2\equiv 0 \pmod{4c}$ if and only if $b=xu$ for some x modulo t. For each such b, let $g=(x,\frac t2)$ and choose $\lambda\in\mathbb{Z}$ such that

$$\gamma=\begin{pmatrix}t/2g & x/g\\ \lambda & \frac{1+\lambda x/g}{t/2g}\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z}).$$

Then $\gamma[c,b,b^2/4c]=[ug^2/t,0,0]$ and $\chi_d([c,b,b^2/4c])=\chi_d(ug^2/t)$. It follows that

$$T_w(0,d;4c)=2\chi_d(u/t)\sum_{\substack{x\bmod t/2\\(x,t/2,d)=1}}e\Big(\frac{mx}{t/2}\Big)=:2f(c).$$

It is straightforward to verify that $f(c)$ is a multiplicative function and that for each prime p we have

$$\text{if } p\mid d \text{ then } f(p^a)=\begin{cases}c_{p^{\frac a2}}(w) & \text{if } a \text{ is even,}\\ 0 & \text{if } a \text{ is odd,}\end{cases}$$
$$\text{if } p\nmid d \text{ then } f(p^a)=\chi_d(p)^a\times\begin{cases}p^{\lfloor\frac a2\rfloor} & \text{if } p^{\lfloor\frac a2\rfloor}\mid w,\\ 0 & \text{otherwise.}\end{cases}$$

Here $c_q(w)$ is the Ramanujan sum which satisfies

$$\frac{w^{1-s}\sigma_{s-1}(w)}{\zeta(s)}=\sum_{q=1}^{\infty}\frac{c_q(w)}{q^s}=\prod_p\sum_{a=0}^{\infty}\frac{c_{p^a}(w)}{p^s}.$$

It follows that

$$\sum_{c=1}^{\infty}\frac{f(c)}{c^{2s-\frac12}}=w^{2-4s}\sigma_{4s-2}(w)\frac{L(2s-\frac12,\chi_d)}{\zeta(4s-1)}.$$

The proposition follows. □

Next we evaluate the term in (5.7) corresponding to the cusp $\mathfrak{c}=0$. The following lemma will be useful.

Lemma 5.8. *Let $k=\frac12$ and $\nu=\nu_\theta$ and suppose that $n\equiv 0,1 \pmod 4$. Suppose that $4\mid c$ and $\mathfrak{a}=0$ or $2\,||\,c$ and $\mathfrak{a}=\frac12$ according to whether $n\equiv 0$ or $1 \pmod 4$, respectively. Then*

$$S_{0\mathfrak{a}}(0,\tfrac n4+\kappa_{\mathfrak{a}},c,\nu_\theta)=\tfrac14 S_{\infty\infty}(0,n,4c,\nu_\theta).\tag{5.10}$$

Proof. For each cusp $\mathfrak{a}$ we have $(\frac{n}{4}+\kappa_{\mathfrak{a}})_{\mathfrak{a}} = \frac{n}{4}$. Suppose first that $n \equiv 0 \pmod 4$ and $\mathfrak{a} = 0$. A straightforward computation shows that $S_{00}(m,n,c,\nu) = S_{\infty\infty}(m,n,c,\nu)$ for all $m,n \in \mathbb{Z}$. From the definition of $S_{\infty\infty}(m,n,c,\nu)$ it follows that, for $c \equiv 0 \pmod 4$, we have $S_{00}(0,\frac{n}{4},c,\nu) = \frac{1}{4}S_{\infty\infty}(0,n,4c,\nu)$.

Now suppose that $n \equiv 1 \pmod 4$ and $\mathfrak{a} = \frac{1}{2}$. We will prove (5.10) directly from the definition of $S_{0\frac{1}{2}}(m,n,c,\nu)$. Let $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) = \sigma_0^{-1}\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)\sigma_{\frac{1}{2}}$, where $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \Gamma_0(4)$. Then $2 \mid\mid c$ and a,d are odd, so (after shifting by $\left(\begin{smallmatrix} 1 & * \\ 0 & 1\end{smallmatrix}\right)$ on the right) we can assume that $4 \mid b$. Then $\varepsilon_D = \varepsilon_{a+2b} = \varepsilon_a = \varepsilon_d$ since $ad \equiv 1 \pmod 4$. We also have

$$\left(\frac{C}{D}\right) = \left(\frac{-4b}{a+2b}\right) = \left(\frac{2a}{a+2b}\right) = (-1)^{\frac{a-1}{2}}\left(\frac{2}{a}\right)\left(\frac{2b}{a}\right) = \left(\frac{4c}{a}\right) = \left(\frac{4c}{d}\right)$$

since $bc \equiv -1 \pmod a$ and $ad \equiv 1 \pmod{4c}$. It follows that

$$S_{0\mathfrak{a}}(0,\tfrac{n}{4}+\kappa_{\mathfrak{a}},c,\nu_\theta) = \sum_{d \bmod c}\left(\frac{4c}{d}\right)\varepsilon_d\, e\left(\frac{nd}{4c}\right).$$

Note that replacing d by $d+c$ has no net effect since $\varepsilon_{d+c} = \varepsilon_{-d}$ and $\left(\frac{4c}{d+c}\right) = -\left(\frac{-1}{d}\right)\left(\frac{c}{d}\right)$, so

$$\left(\frac{4c}{d+c}\right)\varepsilon_{d+c}\, e\left(\frac{n(d+c)}{4c}\right) = \left(\frac{4c}{d}\right)e\left(\frac{nd}{4c}\right)\left[-\varepsilon_{-d}\left(\frac{-1}{d}\right)e\left(\frac{n}{4}\right)\right] = \left(\frac{4c}{d}\right)\varepsilon_d\, e\left(\frac{nd}{4c}\right)$$

since $n \equiv 1 \pmod 4$. The relation (5.10) follows. □

Proposition 5.9. *Let $k = \frac{1}{2}$ and $\nu = \nu_\theta$ and suppose that $n \equiv 0,1 \pmod 4$. Write $n = w^2 d$ with d a fundamental discriminant. Let $\mathfrak{a} = 0$ or $\frac{1}{2}$ according to $n \equiv 0,1 \pmod 4$, respectively. Then*

$$\phi_{0\infty}(n,s) + \frac{1+i}{4^s}\,\phi_{0\mathfrak{a}}\left(\frac{n}{4}+\kappa_{\mathfrak{a}},s\right) = \frac{i}{4^s}w^{1-2s}\frac{L(2s-\frac{1}{2},\chi_d)}{\zeta(4s-1)}\mathfrak{S}_d(w,2s-1). \tag{5.11}$$

Proof. We will prove that

$$\phi_{0\infty}(n,s) + \frac{1+i}{4^s}\,\phi_{0\mathfrak{a}}\left(\frac{n}{4}+\kappa_{\mathfrak{a}},s\right) = i\cdot 2^{2s-\frac{3}{2}}\phi^+(n,s);$$

then equation (5.11) will follow from Proposition 5.7. A straightforward computation gives the relation $S_{0\infty}(m,n,c,\nu_\theta) = iS_{\infty 0}(m,n,c,\nu_\theta)$. This, together with Lemma 5.5, shows that

$$\phi_{0\infty}(n,s) = \frac{2^{2s}}{1-i}\sum_{4||c>0}\frac{S_{\infty\infty}(0,n,c,\nu_\theta)}{c^{2s}}. \tag{5.12}$$

Next, by Lemma 5.8 we find that

$$\frac{1+i}{4^s}\,\phi_{0\mathfrak{a}}\left(\frac{n}{4}+\kappa_{\mathfrak{a}},s\right) = \frac{2^{2s}}{2(1-i)}\sum\frac{S_{\infty\infty}(0,n,c,\nu_\theta)}{c^{2s}}, \tag{5.13}$$

where the sum is over $c \equiv 0 \pmod{16}$ if $\mathfrak{a} = 0$, or $c \equiv 8 \pmod{16}$ if $\mathfrak{a} = \frac{1}{2}$. We claim that we can let the sum run over all $c \equiv 0 \pmod 8$ in either case. Equivalently,

$$S_{\infty\infty}(0,n,c,\nu_\theta) = 0 \quad \text{ when } \quad \begin{cases} c \equiv 8 \pmod{16} & \text{if } n \equiv 0 \pmod 4, \\ c \equiv 0 \pmod{16} & \text{if } n \equiv 1 \pmod 4. \end{cases} \tag{5.14}$$

To see this, we decompose the Kloosterman sum as follows (see Lemma 1 of [43]): if $c = 2^t c'$ with c' odd, then

$$S_{\infty\infty}(0,n,c,\nu_\theta) = \varepsilon_{c'}^{-1}G(n,c')\sum_{r \bmod 2^t}\left(\frac{2^t}{r}\right)\varepsilon_r e\left(\frac{nr}{2^t}\right),$$

where $G(n,c')$ is a Gauss sum. In the case $n \equiv 0 \pmod 4$ and $c \equiv 8 \pmod{16}$ it is easy to see that

$$\sum_{r \bmod 8} \left(\frac{8}{r}\right) \varepsilon_r e\left(\frac{nr}{8}\right) = 0.$$

If $n \equiv 1 \pmod 4$ then, by replacing r by $r + 2^{t-2}$, we see that

$$\sum_{r \bmod 2^t} \left(\frac{2^t}{r}\right) \varepsilon_r e\left(\frac{nr}{2^t}\right) = e\left(\frac{n}{4}\right) \sum_{r \bmod 2^t} \left(\frac{2^t}{r}\right) \varepsilon_r e\left(\frac{nr}{2^t}\right)$$

as long as $t \geq 4$, from which it follows that the sum modulo 2^t is zero.

By (5.12), (5.13), and (5.14), we conclude that

$$\begin{aligned}\phi_{0\infty}(n,s) + \frac{1+i}{4^s}\,\phi_{0\mathfrak{a}}\left(\frac{n}{4} + \kappa_{\mathfrak{a}}, s\right) &= \frac{2^{2s-1}}{1-i}\left(2\sum_{4||c>0} \frac{S_{\infty\infty}(0,n,c,\nu_\theta)}{c^{2s}} + \sum_{8|c>0} \frac{S_{\infty\infty}(0,n,c,\nu_\theta)}{c^{2s}}\right) \\ &= i \cdot 2^{2s-\frac{3}{2}} \phi^+(n,s),\end{aligned}$$

which completes the proof of the proposition. □

Proof of Proposition 5.6. By equation (5.8) and Propositions 5.7 and 5.9 we have

$$\begin{aligned}&\sum_{\mathfrak{c}\in\{\infty,0\}} \overline{\phi}_{\mathfrak{c}\infty}(m,s)\left(\phi_{\mathfrak{c}\infty}(n,s) + \frac{1+i}{4^s}\phi_{\mathfrak{c}\mathfrak{a}}\left(\frac{n}{4} + \kappa_{\mathfrak{a}}, s\right)\right) \\ &\qquad = \left(e\left(\frac{1}{8}\right)2^{\frac{3}{2}-4s}\overline{\phi}_{\infty\infty}(m,s) + i \cdot 2^{-2s}\overline{\phi}_{0\infty}(m,s)\right) w^{1-2s} \frac{L(2s-\frac{1}{2},\chi_d)}{\zeta(4s-1)} \mathfrak{S}(w^2 d, 2s-1).\end{aligned}$$

Then by (5.12) we have (writing $s = \sigma + ir$)

$$\begin{aligned}&e\left(\frac{1}{8}\right)2^{\frac{3}{2}-4s}\overline{\phi}_{\infty\infty}(m,s) + i \cdot 2^{-2s}\overline{\phi}_{0\infty}(m,s) \\ &\qquad = (1+i)2^{1-4s} \sum_{4|c>0} \frac{S_{\infty\infty}(0,m,c,\nu_\theta)}{c^{2\overline{s}}} + \frac{4^{\overline{s}-s}}{1-i} \sum_{4||c>0} \frac{S_{\infty\infty}(0,m,c,\nu_\theta)}{c^{2\overline{s}}} \\ &\qquad = \frac{2^{-4ir}}{1-i}\left(\phi^+(m,\overline{s}) + (4^{1-2\sigma}-1)\phi_{\infty\infty}(m,\overline{s})\right).\end{aligned}$$

The proposition follows after applying Proposition 5.7 and setting $s = \frac{1}{2} + ir$, noting that the factor $4^{1-2\sigma} - 1$ in the second term vanishes. □

6. Proof of Theorem 1.3

Let $a = 4\pi\sqrt{mn}$ and $x > 0$ and let $x^{\frac{1}{3}} \ll T \ll x^{\frac{2}{3}}$ be a free parameter to be chosen later. We choose a test function $\varphi = \varphi_{a,x,T} : [0,\infty) \to [0,1]$ satisfying

(i) $\varphi(t) = 1$ for $\frac{a}{2x} \leq t \leq \frac{a}{x}$,

(ii) $\varphi(t) = 0$ for $t \leq \frac{a}{2x+2T}$ and $t \geq \frac{a}{x-T}$,

(iii) $\varphi'(t) \ll \left(\frac{a}{x-T} - \frac{a}{x}\right)^{-1} \ll \frac{x^2}{aT}$, and

(iv) φ and φ' are piecewise monotonic on a fixed number of intervals (whose number is independent of a, x, T).

We apply the plus space Kuznetsov formula in Theorem 5.3 with this test function and we estimate each of the terms on the right-hand side.

We begin by estimating the contribution from the holomorphic cusp forms

$$\mathcal{K}^h := \sum_{\ell \equiv k \bmod 2} e\big(\tfrac{\ell-k}{4}\big)\widetilde{\varphi}(\ell)\Gamma(\ell) \sum_{g\in\mathcal{H}_\ell^+} \overline{\rho_g(m)}\rho_g(n). \tag{6.1}$$

Since the operator L commutes with the Hecke operators we may assume that the orthonormal basis $\mathcal{H}_\ell^+$ is also a basis consisting of Hecke eigenforms. We will estimate $\mathcal{K}^h$ by applying the Kohnen-Zagier formula [25] and Young's hybrid subconvexity bound [45]. Let $g \in \mathcal{H}_\ell^+$ and recall that the coefficients of g are normalized so that

$$g(z) = \sum_{n=1}^\infty (4\pi n)^{\frac{\ell-1}{2}} \rho_g(n) e(nz).$$

Since we are working in the plus space, the Shimura correspondence is an isomorphism between $\mathcal{S}_\ell^+(\nu)$ and the space $\mathcal{S}_{2\ell-1}$ of (even) weight $2\ell-1$ cusp forms on Γ_1. So g lifts to a unique normalized $f \in S_{2\ell-1}$ with Fourier expansion

$$f(z) = \sum_{n=1}^\infty n^{\ell-1} a_f(n) e(nz), \qquad \text{where } a_f(1) = 1.$$

The coefficients ρ_g and a_f are related via

$$\rho_g(v^2|d|) = \rho_g(|d|) \sum_{u|v} \mu(u)\Big(\frac{d}{u}\Big) u^{-\frac12} a_f(v/u),$$

where d is a fundamental discriminant with $(-1)^\lambda d > 0$. Using Deligne's bound $|a_f(n)| \le \sigma_0(n)$, it follows that

$$|\rho_g(v^2|d|)| \le |\rho_g(|d|)| \sigma_0^2(v). \tag{6.2}$$

Suppose that g is normalized so that $\langle g, g\rangle = 1$. If d is a fundamental discriminant satisfying $(-1)^\lambda d > 0$ then the Kohnen-Zagier formula [25, Theorem 1] can be written as

$$\Gamma(\ell)|\rho_g(|d|)|^2 = 4\pi \frac{\Gamma(2\ell-1)}{(4\pi)^{2\ell-1}\langle f, f\rangle} L(\tfrac12, f\times\chi_d),$$

where $L(s, f\times\chi_d)$ is the twisted L-function with Dirichlet series

$$L(s, f\times\chi_d) = \sum_{m=1}^\infty \frac{a_f(m)\chi_d(m)}{m^s}. \tag{6.3}$$

By a result of Hoffstein and Lockhart (see [20, Corollary 0.3] and the second remark that follows it, and note that their normalization differs from ours) we have the bound

$$\frac{\Gamma(2\ell-1)}{(4\pi)^{2\ell-1}\langle f, f\rangle} \ll \ell^\varepsilon,$$

so we conclude that

$$\Gamma(\ell)|\rho_g(|d|)|^2 \ll L(\tfrac12, f\times\chi_d)\ell^\varepsilon.$$

Let $\mathcal{H}_{2\ell-1}$ be the image in $\mathcal{S}_{2\ell-1}$ of the Shimura lift of $\mathcal{H}_\ell^+(\nu)$. Young's hybrid subconvexity bound [45, Theorem 1.1] yields

$$\sum_{f\in\mathcal{H}_{2\ell-1}} L(\tfrac12, f\times\chi_d)^3 \ll (\ell d)^{1+\varepsilon}$$

for odd fundamental d. See Appendix A for the case of even fundamental discriminants d. Applying Hölder's inequality in the case $\frac{1}{6}+\frac{1}{6}+\frac{2}{3}=1$, together with the fact that $\#\mathcal{H}_{2\ell-1} \asymp \ell$, we obtain the following theorem for d, d' fundamental discriminants. It is extended to all m, n using (6.2).

Theorem 6.1. *Let $\ell \equiv k \pmod 2$ with $k = \pm\frac{1}{2} = \lambda + \frac{1}{2}$ and suppose that $\mathcal{H}_\ell^+$ is an orthonormal basis for $\mathcal{S}_\ell^+$ consisting of Hecke eigenforms. Suppose that m, n are integers with $(-1)^\lambda m, (-1)^\lambda n > 0$, and write $(-1)^\lambda m = v^2 d'$ and $(-1)^\lambda n = w^2 d$ with d, d' fundamental discriminants. Then*

$$\Gamma(\ell) \sum_{g \in \mathcal{H}_\ell^+} |\rho_g(|m|)\rho_g(|n|)| \ll \ell |dd'|^{\frac{1}{6}+\varepsilon}(vw)^\varepsilon.$$

Applying Theorem 6.1 to the sum (6.1) we find that

$$\mathcal{K}^h \ll |dd'|^{\frac{1}{6}+\varepsilon}(vw)^\varepsilon \sum_{\ell \equiv k(2)} \ell\, \widetilde{\varphi}(\ell).$$

The latter sum was estimated in [39] (see the discussion following (50); see also Lemma 5.1 of [16]) where the authors found that $\sum_\ell \ell\, \widetilde{\varphi}(\ell) \ll \sqrt{mn}/x$. We conclude that

$$\mathcal{K}^h \ll \frac{vw|dd'|^{\frac{2}{3}}}{x}(mn)^\varepsilon. \tag{6.4}$$

Next, we estimate the contribution from the Maass cusp forms

$$\mathcal{K}^m := \sqrt{mn} \sum_{j \geq 0} \frac{\overline{\rho_j(m)}\rho_j(n)}{\cosh \pi r_j} \widehat{\varphi}(r_j).$$

We follow the same general idea as in the holomorphic case, but instead of the Kohnen-Zagier formula we apply a formula of Baruch and Mao [2]. As in the holomorphic case, we may assume that the orthonormal basis $\{u_j\}$ of $\mathcal{V}_k^+$ consists of eigenforms for the Hecke operators. Suppose that $u_j \in \mathcal{V}_k^+$ has spectral parameter r_j. The lowest eigenvalue is $\lambda_0 = \frac{3}{16}$ which corresponds to $u_0 = y^{1/4}\theta(z)$ or its conjugate. Since the coefficients $\rho_0(n)$ are supported on squares and since m, n are not both squares, we find that the term in $\mathcal{K}^m$ corresponding to $j = 0$ does not appear. In what follows we assume that $j \geq 1$.

Theorem 1.2 of [2] shows that there is a unique normalized Maass cusp form v_j of weight 0 with spectral parameter $2r_j$ which is even if $k = \frac{1}{2}$ and odd if $k = -\frac{1}{2}$, and such that the Hecke eigenvalues of u_j and v_j agree. Since there are no exceptional eigenvalues for weight 0 on $\mathrm{SL}_2(\mathbb{Z})$ this lift implies that there are no exceptional eigenvalues in weights $\pm\frac{1}{2}$ in the plus space. It follows that $r_j \geq 0$ for each $j \geq 1$ (in fact $r_1 \approx 1.5$). If $a_j(n)$ is the n-th coefficient of v_j (with respect to the Whittaker function, not the K-Bessel function) then for d a fundamental discriminant we have

$$w\, \rho_j(dw^2) = \rho_j(d) \sum_{\ell | w} \ell^{-1}\mu(\ell)\chi_d(\ell) a_j(w/\ell).$$

Let θ denote an admissible exponent toward the Ramanujan conjecture in weight 0; we have $\theta \leq \frac{7}{64}$ by work of Kim and Sarnak [24]. Then $a_j(w) \ll w^{\theta+\varepsilon}$ since v_j is normalized so that $a_j(1) = 1$. It follows that

$$w|\rho_j(dw^2)| \ll w^{\theta+\varepsilon}|\rho_j(d)|.$$

Suppose that d is a fundamental discriminant and that $\langle u_j, u_j\rangle = 1$. Then Theorem 1.4 of [2] implies that

$$|\rho_j(d)|^2 = \frac{L(\frac12, v_j \times \chi_d)}{\pi|d|\langle v_j, v_j\rangle}\Big|\Gamma\Big(\frac{1-k\operatorname{sgn} d}{2} - ir_j\Big)\Big|^2,$$

where $L(\frac12, v_j\times\chi_d)$ is defined in a similar way as (6.3). Hoffstein and Lockhart [20, Corollary 0.3] proved that $\langle v_j, v_j\rangle^{-1} \ll (1+r_j)^{\varepsilon} e^{2\pi r_j}$ (again, note that the Fourier coefficients are normalized differently in that paper). It follows that

$$|d|\sum_{r_j\le x}\frac{|\rho_j(d)|^2}{\cosh \pi r_j} \ll \sum_{2r_j\le 2x}(1+r_j)^{-k\operatorname{sgn}(d)+\varepsilon}L(\tfrac12, v_j\times\chi_d).$$

Young's subconvexity result [45, Theorem 1.1] in this case shows that

$$\sum_{T\le r_j\le T+1} L(\tfrac12, v_j\times\chi_d)^3 \ll (|d|(1+T))^{1+\varepsilon}.$$

After applying Hölder's inequality as above, we obtain the following.

Theorem 6.2. *Let $k = \pm\frac12 = \lambda + \frac12$. Suppose that $\{u_j\}$ is an orthonormal basis for $\mathcal{V}_k^+$ consisting of Hecke eigenforms with spectral parameters r_j and coefficients ρ_j. Suppose that m, n are integers with $(-1)^\lambda m, (-1)^\lambda n > 0$, and write $(-1)^\lambda m = v^2 d'$ and $(-1)^\lambda n = w^2 d$ with d, d' fundamental discriminants not both equal to 1. Then*

$$\sqrt{|mn|}\sum_{r_j\le x}\frac{|\rho_j(m)\rho_j(n)|}{\cosh \pi r_j} \ll |dd'|^{\frac16}(vw)^{\theta}x^{2-\frac12 k(\operatorname{sgn} m+\operatorname{sgn} n)}(mnx)^{\varepsilon}.$$

To estimate $\mathcal{K}^m$ we consider the dyadic sums

$$\mathcal{K}^m(A) := \sqrt{mn}\sum_{A\le r_j<2A}\frac{\overline{\rho_j(m)}\rho_j(n)}{\cosh \pi r_j}\widehat{\varphi}(r_j)$$

for $A \ge 1$. Theorem 6.2 gives one estimate for the coefficients $|\rho_j(m)\rho_j(n)|$. Applying Cauchy-Schwarz and Theorem 4.1 with $\beta = \frac12 + \varepsilon$ we obtain a second estimate:

$$\sqrt{mn}\sum_{r_j\le A}\frac{|\rho_j(m)\rho_j(n)|}{\cosh \pi r_j} \ll A^{-k}\Big(A^2 + (m+n)^{\frac14}A + (mn)^{\frac14}\Big)(mnA)^{\varepsilon}.$$

These theorems together imply that

$$\sqrt{mn}\sum_{A\le r_j<2A}\frac{|\rho_j(m)\rho_j(n)|}{\cosh \pi r_j} \ll A^{-k}\min\Big((dd')^{\frac16}(vw)^{\theta}A^2, A^2 + (m+n)^{\frac14}A + (mn)^{\frac14}\Big)(mnA)^{\varepsilon}.$$

The following lemma gives an estimate for $\widehat{\varphi}(r)$.

Lemma 6.3. *If $r \ge 1$ then with $\varphi = \varphi_{a,x,T}$ as above we have*

$$\widehat{\varphi}(r) \ll r^k \min\Big(r^{-\frac32}, r^{-\frac52}\frac{x}{T}\Big).$$

If $|r| \le 1$ then $\widehat{\varphi}(r) \ll |r|^{-2}$.

Proof. Recall that
$$\widehat{\varphi}(r) = \frac{-i\,\xi_k(r)}{\cosh 2\pi r}\int_0^\infty \big(\cos(\tfrac{\pi k}{2}+\pi i r)J_{2ir}(x) - \cos(\tfrac{\pi k}{2}-\pi i r)J_{-2ir}(x)\big)\varphi(x)\frac{dx}{x},$$
where $\xi_k(r)\asymp r^k$ as $r\to\infty$. Sarnak and Tsimerman [39, (47)–(48)] proved that
$$e^{-\pi|r|}\int_0^\infty J_{2ir}(x)\varphi(x)\frac{dx}{x} \ll \min\Big(|r|^{-\frac{3}{2}}, |r|^{-\frac{5}{2}}\frac{x}{T}\Big)$$
for $|r|\geq 1$. The first statement of the lemma follows. The second is similar, using [39, (43)]. □

Since $\min(x,y)\ll x^a y^{1-a}$ for any $a\in[0,1]$, we have
$$\mathcal{K}^m(A) \ll \min\Big(1, \frac{x}{AT}\Big)\Big(\sqrt{A} + (dd')^{\frac{1}{12}}(vw)^{\frac{\theta}{2}}(m+n)^{\frac{1}{8}} + (dd')^{\frac{3}{16}}(vw)^{\frac{1}{8}+\frac{3}{4}\theta}\Big)(mnA)^\varepsilon,$$
where we used $a=\frac{1}{2}$ in the second term and $a=\frac{3}{4}$ in the third term. Summing over A we conclude that
$$\mathcal{K}^m \ll \bigg(\sqrt{\frac{x}{T}} + (dd')^{\frac{1}{12}}(vw)^{\frac{\theta}{2}}(m+n)^{\frac{1}{8}} + (dd')^{\frac{3}{16}}(vw)^{\frac{1}{8}+\frac{3}{4}\theta}\bigg)(mnx)^\varepsilon. \tag{6.5}$$

We turn to the estimate of the integral
$$\mathcal{K}^e := \int_{\mathbb{R}}\Big(\frac{d}{d'}\Big)^{ir}\frac{L(\frac{1}{2}-2ir,\chi_{d'})L(\frac{1}{2}+2ir,\chi_d)\mathfrak{S}_{d'}(v,2ir)\mathfrak{S}_d(w,2ir)}{|\zeta(1+4ir)|^2\cosh\pi r|\Gamma(\frac{k+1}{2}+ir)|^2}\widehat{\varphi}(r)\,dr.$$
By symmetry it suffices to estimate the integrals $\mathcal{K}^e_0 = \int_0^1$ and $\mathcal{K}^e_1 = \int_1^\infty$. Estimating the divisor sums trivially we find that
$$|\mathfrak{S}_d(w,s)| \leq \sigma_0(w)^2.$$
For $|r|\leq 1$ we have $|\zeta(1+4ir)|^2 \gg r^{-2}$ and $\cosh\pi r|\Gamma(\frac{k+1}{2}+ir)|^2\gg 1$, so by Lemma 6.3 we have the estimate
$$\mathcal{K}^e_0 \ll (vw)^\varepsilon\int_0^1 \big|L(\tfrac{1}{2}-2ir,\chi_{d'})L(\tfrac{1}{2}+2ir,\chi_d)\big|\,dr.$$
Since $\cosh\pi r|\Gamma(\frac{k+1}{2}+ir)|^2\sim\pi r^k$ for large r and since $|\zeta(1+4ir)|^{-1}\ll r^\varepsilon$ for all r we have by Lemma 6.3 that
$$\mathcal{K}^e_1 \ll (vw)^\varepsilon\int_1^\infty \big|L(\tfrac{1}{2}-2ir,\chi_{d'})L(\tfrac{1}{2}+2ir,\chi_d)\big|\frac{dr}{r^{3/2-\varepsilon}}.$$
We multiply each Dirichlet L-function by $r^{-3/8}$ and the last factor by $r^{3/4}$, then apply Hölder's inequality in the case $\frac{1}{6}+\frac{1}{6}+\frac{2}{3}=1$. We obtain
$$\mathcal{K}^e_1 \ll (vw)^\varepsilon\bigg(\int_1^\infty |L(\tfrac{1}{2}+ir,\chi_{d'})|^6\frac{dr}{r^{9/4}}\bigg)^{\frac{1}{6}}\bigg(\int_1^\infty |L(\tfrac{1}{2}+ir,\chi_{d})|^6\frac{dr}{r^{9/4}}\bigg)^{\frac{1}{6}}\bigg(\int_1^\infty \frac{dr}{r^{9/8-\varepsilon}}\bigg)^{\frac{2}{3}}. \tag{6.6}$$
Young [45] proved that
$$\int_T^{T+1}|L(\tfrac{1}{2}+ir,\chi_d)|^6\,dr \ll (|d|(1+T))^{1+\varepsilon},$$
from which it follows that $K^e_0 \ll (vw)^\varepsilon|dd'|^{\frac{1}{6}+\varepsilon}$ and
$$\int_1^\infty |L(\tfrac{1}{2}+ir,\chi_d)|^6\frac{dr}{r^{9/4}} \leq \sum_{T=1}^\infty\frac{1}{T^{9/4}}\int_T^{T+1}|L(\tfrac{1}{2}+ir,\chi_d)|^6\,dr \ll |d|^{1+\varepsilon}.$$

This, together with (6.6) proves that

$$\mathcal{K}^e \ll (vw)^\varepsilon |dd'|^{\frac{1}{6}+\varepsilon}. \tag{6.7}$$

Putting (6.4), (6.5), and (6.7) together, we find that

$$\sum_{4|c>0} \frac{S_k^+(m,n,c)}{c} \varphi\left(\frac{4\pi\sqrt{mn}}{c}\right) \ll \left(\sqrt{\frac{x}{T}} + \frac{vw|dd'|^{\frac{2}{3}}}{x} + (dd')^{\frac{1}{12}}(vw)^{\frac{\theta}{2}}(m+n)^{\frac{1}{8}} + (dd')^{\frac{3}{16}}(vw)^{\frac{1}{8}+\frac{3}{4}\theta}\right)(mnx)^\varepsilon.$$

To unsmooth the sum of Kloosterman sums, we argue as in [39, 1] to obtain

$$\sum_{4|c>0} \frac{S_k^+(m,n,c)}{c} \varphi\left(\frac{4\pi\sqrt{mn}}{c}\right) - \sum_{x\le c<2x} \frac{S_k^+(m,n,c)}{c} \ll \frac{T\log x}{\sqrt{x}}(mn)^\varepsilon.$$

Choosing $T = x^{\frac{2}{3}}$ and using that $m+n \le mn$ we obtain

$$\sum_{x\le c<2x} \frac{S_k^+(m,n,c)}{c} \ll \left(x^{\frac{1}{6}} + \frac{vw|dd'|^{\frac{2}{3}}}{x} + (dd')^{\frac{5}{24}}(vw)^{\frac{1}{4}+\frac{\theta}{2}}\right)(mnx)^\varepsilon. \tag{6.8}$$

To prove (1.10) we sum the inital segment $c \le (dd')^a(vw)^b$ and apply the Weil bound (1.11), then sum the dyadic pieces for $c \ge (dd')^a(vw)^b$ using (6.8). To balance the resulting terms we take $a = \frac{4}{9}$ and $b = \frac{2}{3}$, which gives the bound

$$\sum_{c\le x} \frac{S_k^+(m,n,c)}{c} \ll \left(x^{\frac{1}{6}} + (dd')^{\frac{2}{9}}(vw)^{\frac{1}{3}}\right)(mnx)^\varepsilon.$$

This completes the proof. □

Appendix A. Young's theorem for even discriminants

Let D be a fundamental discriminant. Then $|D| = q$ or $4q$, where q is squarefree (but not necessarily odd). For a positive even integer k, let $\mathcal{B}_k(q)$ denote the set of weight k holomorphic Hecke newforms of level dividing q. Our goal in this appendix is to prove the following generalization of Young's hybrid subconvexity result [45].

Theorem A.1. *Notation as above, we have*

$$\sum_{f\in B_k(q)} L(\tfrac{1}{2}, f\times\chi_D)^3 \ll (k|D|)^{1+\varepsilon}.$$

A corresponding generalization also holds for Maass cusp forms and Eisenstein series; for simplicity we only deal with the holomorphic case here.

For ease of comparison with [45], we have adopted the notation of that paper for this section only. We will indicate the changes that need to be made and refer the reader to [5] and [45] for the remaining details. Starting in Section 4 of [45], our goal is to show that

$$\sum_{k\equiv a(4)} w\left(\frac{k-1-2T}{\Delta}\right) \sum_{f\in B_k(q)} \omega_f^* L(\tfrac{1}{2}, f\times\chi_D)^3 \ll \Delta(T|D|)^{1+\varepsilon},$$

where w is a smooth nonnegative function with support in $[\frac{1}{2}, 3]$ which equals 1 on the interval $[1,2]$, and a is determined by $i^k = \chi_D(-1)$. Here ω_f^* is a Petersson weight satisfying

$\omega_f^* \gg (kq)^{-\varepsilon}$. Applying the approximate functional equation and the Petersson formula as in Section 5 of [45], we find that it suffices to show the following.

Proposition A.2. *For $i = 1, 2, 3$, let w_i be a smooth weight function supported on $x \asymp N_i$, with $1 \ll N_i \ll (qT)^{1+\varepsilon}$ and with $w_i^{(k)} \ll N_i^{-k}$. Then*

$$\sum_{n_1,n_2,n_3} w_1(n_1)w_2(n_2)w_3(n_3)\chi_D(n_1n_2n_3) \sum_{c\equiv 0(q)} \frac{S(n_1n_2, n_3; c)}{c} B\left(\frac{4\pi\sqrt{n_1n_2n_3}}{c}\right) \ll (N_1N_2N_3)^{1/2}\Delta T(qT)^{\varepsilon},$$

where $S(m, n; c)$ is the ordinary Kloosterman sum,

$$B(x) = B^{holo}(x) = \sum_{k\equiv a(4)} (k-1)w\left(\frac{k-1-2T}{\Delta}\right) J_{k-1}(x)$$

and $J_{k-1}(x)$ is the J-Bessel function.

With w_1, w_2, and w_3 as in Proposition A.2, let

$$S(N_1, N_2, N_3; C; B) = \sum_{\substack{c\asymp C\\ c\equiv 0(q)}} \mathcal{S}(N_1, N_2, N_3; c),$$

where

$$\mathcal{S}(N_1, N_2, N_3; c) = \sum_{n_1,n_2,n_3} \chi_D(n_1n_2n_3)S(n_1n_2, n_3; c)w_1(n_1)w_2(n_2)w_3(n_3)B\left(\frac{4\pi\sqrt{n_1n_2n_3}}{c}\right).$$

We now follow Section 8 of [45], where the main difference is that we must keep track of the dependence on $\operatorname{lcm}(c, |D|)$, which we write as cs, with $s \in \{1, 2, 4, 8\}$. Applying Poisson summation modulo c to the sum over the lattice $\mathbb{Z}^3$ we find that

$$\mathcal{S}(N_1, N_2, N_3; c) = \sum_{m_1,m_2,m_3} G(m_1, m_2, m_3; c)K(m_1, m_2, m_3; c),$$

where

$$G(m_1, m_2, m_3; c) = \frac{1}{(cs)^3} \sum_{a_1,a_2,a_3 \bmod c} \chi_D(a_1a_2a_3)S(a_1a_2, a_3; c)e\left(\frac{a_1m_1 + a_2m_2 + a_3m_3}{cs}\right)$$

and

$$K(m_1, m_2, m_3; c) = \int_{\mathbb{R}^3} w_1(t_1)w_2(t_2)w_3(t_3)B\left(\frac{4\pi\sqrt{t_1t_2t_3}}{c}\right)e\left(\frac{-m_1t_1 - m_2t_2 - m_3t_3}{cs}\right)dt_1dt_2dt_3.$$

The analysis of the analytic piece $K(m_1, m_2, m_3; c)$ is almost exactly the same as in [45, Section 8]; simply replace t_i by $t_i/s^{2/3}$ and apply Lemma 8.1. The only difference is that the phase $e\left(\frac{-m_1m_2m_3}{c}\right)$ in (8.4) is replaced by

$$e\left(\frac{-m_1m_2m_3}{s^3c}\right). \tag{A.1}$$

For the remainder of this section we will focus on the arithmetic piece $G(m_1, m_2, m_3; c)$. We begin by fixing notation. Let $D = tq'$, where t and q' are fundamental discriminants with $t \mid 2^\infty$ and q' odd, so that $\chi_D = \chi_t\chi_{q'}$. With $q \mid c$ and $cs = \operatorname{lcm}(c, D)$ as before, we have $s = t/(c, t)$. Finally, write $c = c_oc_e$, with $c_e \mid 2^\infty$ and c_o odd. Then cs factors as $cs = c_0 \cdot sc_e$ into odd and even parts. From the twisted multiplicativity of the Kloosterman sums, a straightforward computation gives the factorization

$$G(m_1, m_2, m_3; c) = G(m_1, m_2, \overline{c}_o m_3; c_e) G(m_1, m_2, \overline{c}_e \overline{s}^3 m_3; c_o), \tag{A.2}$$

where we choose the inverse $\overline{c}_o$ such that

$$c_o \overline{c}_o \equiv 1 \pmod{s^3 c_e}.$$

The second term on the right-hand side of (A.2) was evaluated in Lemma 10.2 of [5], which we record here in the following lemma (see also (9.2) of [45]). Note that Young's definition of $G(m_1, m_2, m_3; c)$, which we are using here, is slightly different from that of Conrey-Iwaniec. Let $R_k(m) = S(0, m; k)$ denote the Ramanujan sum and let

$$H(w; q) = \sum_{u,v(q)} \chi_q(uv(u+1)(v+1)) e\left(\frac{(uv-1)w}{q}\right).$$

Lemma A.3. *Let $c_o = qr$ with c_o odd and q squarefree. Suppose m_1, m_2, m_3 are integers with*

$$(m_3, r) = 1 \quad \text{and} \quad (m_1 m_2, q, r) = 1. \tag{A.3}$$

Then we have

$$e\left(\frac{-m_1 m_2 m_3}{c_o}\right) G(m_1, m_2, m_3; c_o) = \frac{\chi_{k\ell}(-1) h}{r q^2 \varphi(k)} R_k(m_1) R_k(m_2) R_k(m_3) H(\overline{rhk} m_1 m_2 m_3; \ell),$$

where $h = (r, q)$, $k = (m_1 m_2 m_3, q)$, and $\ell = q/hk$. If the coprimality conditions above are not satisfied, then $G(m_1, m_2, m_3; c_o)$ vanishes.

Petrow and Young [35, Lemma 9.4] evaluated $G(m_1, m_2, m_3; c_e)$ when c_e is a power of 2.

Lemma A.4. *Suppose that $c_e \mid 2^\infty$ and factor m_i into even and odd parts as $m_i = m_i^e m_i^o$. Then*

$$e\left(\frac{-m_1 m_2 m_3}{s^3 c_e}\right) G(m_1, m_2, m_3; c_e) = \frac{s^3 c_e^2}{t} \sum_{\Delta | 64} \frac{1}{\varphi(\Delta)} \sum_{\chi \bmod \Delta} \mathfrak{g}_\chi \chi(m_1^o m_2^o m_3^o),$$

where $\mathfrak{g}_\chi$ depends on $m_1^e, m_2^e, m_3^e, t, c_e, \chi$ and is bounded by an absolute constant.

Note that the phase terms in Lemmas A.3 and A.4 combine to give

$$e\left(\frac{m_1 m_2 m_3}{s^3 c}\right),$$

which exactly matches the phase term (A.1) coming from $K(m_1, m_2, m_3; c)$.

The last result we require is the following analogue of Lemma 9.3 of [45]. The remainder of the proof of A.2 follows the proof of Proposition 7.3 of [45].

Lemma A.5. *Let $c = q'r$ with q' odd and squarefree. Let α_{m_1}, β_{m_2}, and γ_{m_3} be sequences of complex numbers satisfying $\alpha_{m_1} = \alpha_{m_1^e} \alpha_{m_1^o}$, $\beta_{m_2} = \beta_{m_2^e} \beta_{m_2^o}$, $\gamma_{m_3} = \gamma_{m_3^e} \gamma_{m_3^o}$, and $|\alpha_{m_1^e}| = |\beta_{m_2^e}| = |\gamma_{m_3^e}| = 1$, and let δ_r be an arbitrary sequence of complex numbers. Then for $U \geq 1$ we* *have*

$$\int_{|u|\le U}\left|\sum_{\substack{m_1,m_2,m_3\\ m_i\asymp M_i}}\sum_{r\asymp R}\alpha_{m_1}\beta_{m_2}\gamma_{m_3}\delta_r G(m_1,m_2,m_3;c)e\left(\frac{-m_1m_2m_3}{s^3c}\right)\left(\frac{m_1m_2m_3}{c}\right)^{iu}\right|du$$
$$\ll \frac{q^{\frac12+\varepsilon}}{Rq^2}(qU+M_1M_2)^{\frac12}(qU+M_3R)^{\frac12}\left(\sum_{d,m_1,m_2,m_3,r} d^{1+\varepsilon}|\alpha_{m_1}\beta_{m_2}\gamma_{dm_3}\delta_{dr}|^2\right)^{\frac12}. \quad \text{(A.4)}$$

Remark: as in [45, Lemma 9.3], when $\gamma_{m_3},\delta_r\ll 1$ the sum over d does not change the bound which arises from $d=1$.

Proof. Using Lemmas A.3 and A.4, the left-hand side of (A.4) is

$$\ll \sum_{hk\ell=q'}\frac{h}{Rq^2\varphi(k)}\sum_{\Delta|64}\frac{1}{\varphi(\Delta)}\int_{|u|\le U}\Big|\sum_{m_1,m_2,m_3}^{*}\sum_{r\asymp R}^{*}\alpha_{m_1}\beta_{m_2}\gamma_{m_3}\delta_r\mathfrak{g}_\chi\chi(m_1^om_2^om_3^o)$$
$$\times R_k(m_1)R_k(m_2)R_k(m_3)H(\overline{rhk}m_1m_2m_3;\ell)\left(\frac{m_1^om_2^om_3^o}{q'}\right)^{iu}\Big|du,$$

where the star indicates that the sum is restricted by the coprimality conditions (A.3). Using that $R_k(m_i)=R_k(m_i^e)R_k(m_i^o)$ and $|R_k(m)|\le(k,m)$ we bound the above by

$$\ll \sum_{hk\ell=q'}\frac{h}{Rq^2\varphi(k)}\sum_{\Delta|64}\frac{1}{\varphi(\Delta)}\sum_{\substack{j_1,j_2,j_3\\ j_i\ll\log_2(M_i)}}\int_{|u|\le U}\Big|\sum_{\substack{m_1^o,m_2^o,m_3^o\\ m_i^o\asymp M_i/2^{j_i},m_i^e=2^{j_i}}}^{*}\sum_{r\asymp R}^{*}\alpha_{m_1^o}\beta_{m_2^o}\gamma_{m_3^o}\delta_r\chi(m_1^om_2^om_3^o)$$
$$\times R_k(m_1^o)R_k(m_2^o)R_k(m_3^o)H(\overline{rhk}bm_1^om_2^om_3^o;\ell)\left(\frac{m_1^om_2^om_3^o}{q'}\right)^{iu}\Big|du,$$

where $b=m_1^em_2^em_3^e$. Now following the proof of Lemma 9.3 of [45] almost exactly, we obtain the desired bound. □

References

[1] Scott Ahlgren and Nickolas Andersen. Kloosterman sums and Maass cusp forms of half integral weight for the modular group. *Int. Math. Res. Not. IMRN*, (2):492–570, 2018.

[2] Ehud Moshe Baruch and Zhengyu Mao. A generalized Kohnen-Zagier formula for Maass forms. *J. Lond. Math. Soc. (2)*, 82(1):1–16, 2010.

[3] A. Biró. Cycle integrals of Maass forms of weight 0 and Fourier coefficients of Maass forms of weight 1/2. *Acta Arith.*, 94(2):103–152, 2000.

[4] Jan Hendrik Bruinier, Paul Jenkins, and Ken Ono. Hilbert class polynomials and traces of singular moduli. *Math. Ann.*, 334(2):373–393, 2006.

[5] J. B. Conrey and H. Iwaniec. The cubic moment of central values of automorphic L-functions. *Ann. of Math. (2)*, 151(3):1175–1216, 2000.

[6] Henri Darmon and Jan Vonk. Singular moduli for real quadratic fields: a rigid analytic approach. *preprint*.

[7] J.-M. Deshouillers and H. Iwaniec. Kloosterman sums and Fourier coefficients of cusp forms. *Invent. Math.*, 70(2):219–288, 1982/83.

[8] NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/, Release 1.0.10 of 2015-08-07. Online companion to [34].

[9] W. Duke. Hyperbolic distribution problems and half-integral weight Maass forms. *Invent. Math.*, 92(1):73–90, 1988.

[10] W. Duke. Modular functions and the uniform distribution of CM points. *Math. Ann.*, 334(2):241–252, 2006.

[11] W. Duke, J. B. Friedlander, and H. Iwaniec. The subconvexity problem for Artin L-functions. *Invent. Math.*, 149(3):489–577, 2002.

[12] W. Duke, J. B. Friedlander, and H. Iwaniec. Weyl sums for quadratic roots. *Int. Math. Res. Not. IMRN*, (11):2493–2549, 2012.

[13] W. Duke, Ö. Imamoḡlu, and Á. Tóth. Cycle integrals of the j-function and mock modular forms. *Ann. of Math. (2)*, 173(2):947–981, 2011.

[14] W. Duke, Ö. Imamoḡlu, and Á. Tóth. Geometric invariants for real quadratic fields. *Ann. of Math. (2)*, 184(3):949–990, 2016.

[15] W. Duke, Ö. Imamoḡlu, and Á. Tóth. Kronecker's first limit formula, revisited. *Res. Math. Sci.*, 5(2):Paper No. 20, 21, 2018.

[16] Alexander Dunn. Uniform bounds for sums of Kloosterman sums of half integral weight. *Res. Number Theory*, 4(4):Art. 45, 21, 2018.

[17] Amanda Folsom and Riad Masri. Equidistribution of Heegner points and the partition function. *Math. Ann.*, 348(2):289–317, 2010.

[18] B. Gross, W. Kohnen, and D. Zagier. Heegner points and derivatives of L-series. II. *Math. Ann.*, 278(1-4):497–562, 1987.

[19] Dennis A. Hejhal. *The Selberg trace formula for* $\mathrm{PSL}(2, \mathbb{R})$*. Vol. 2.* Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1983.

[20] Jeffrey Hoffstein and Paul Lockhart. Coefficients of Maass forms and the Siegel zero. *Ann. of Math. (2)*, 140(1):161–181, 1994. With an appendix by D. Goldfeld, Hoffstein and D. Lieman.

[21] Tomoyoshi Ibukiyama and Hiroshi Saito. On zeta functions associated to symmetric matrices, II: Functional equations and special values. *Nagoya Math. J.*, 208:265–316, 2012.

[22] Vaibhav Kalia and Balesh Kumar. Regularized inner products and modular invariants for real quadratic fields. *Preprint*, 2024.

[23] Svetlana Katok and Peter Sarnak. Heegner points, cycles and Maass forms. *Israel J. Math.*, 84(1-2):193–227, 1993.

[24] Henry H. Kim. Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2. *J. Amer. Math. Soc.*, 16(1):139–183, 2003. With appendix 1 by D. Ramakrishnan and appendix 2 by Kim and P. Sarnak.

[25] W. Kohnen and D. Zagier. Values of L-series of modular forms at the center of the critical strip. *Invent. Math.*, 64(2):175–198, 1981.

[26] Winfried Kohnen. Modular forms of half-integral weight on $\Gamma_0(4)$. *Math. Ann.*, 248(3):249–266, 1980.

[27] Winfried Kohnen. Newforms of half-integral weight. *J. Reine Angew. Math.*, 333:32–72, 1982.

[28] N. V. Kuznetsov. The Petersson conjecture for cusp forms of weight zero and the Linnik conjecture. Sums of Kloosterman sums. *Mat. Sb. (N.S.)*, 111(153):334–383, 1980.

[29] D. H. Lehmer. On the remainders and convergence of the series for the partition function. *Trans. Amer. Math. Soc.*, 46:362–373, 1939.

[30] Hans Maass. Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen. *Math. Ann.*, 121:141–183, 1949.

[31] Hans Maass. Die Differentialgleichungen in der Theorie der elliptischen Modulfunktionen. *Math. Ann.*, 125:235–263, 1952.

[32] Riad Masri. The asymptotic distribution of traces of cycle integrals of the j-function. *Duke Math. J.*, 161(10):1971–2000, 2012.

[33] Y. Motohashi. A functional equation for the spectral fourth moment of modular Hecke L-functions. In *Proceedings of the Session in Analytic Number Theory and Diophantine Equations*, volume 360 of *Bonner Math. Schriften*, page 19. Univ. Bonn, Bonn, 2003.

[34] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors. *NIST Handbook of Mathematical Functions*. Cambridge University Press, New York, NY, 2010. Print companion to [8].

[35] Ian Petrow and Matthew P. Young. A generalized cubic moment and the Petersson formula for newforms. *Mathematische Annalen*, 373(1-2):287–353, 2019.

[36] N. V. Proskurin. On general Kloosterman sums. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 302(19):107–134, 2003.

[37] Walter Roelcke. Das Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene, II. *Math. Ann.*, 168:261–324, 1966.

[38] Peter Sarnak. Additive number theory and Maass forms. In *Number theory (New York, 1982)*, volume 1052 of *Lecture Notes in Math.*, pages 286–309. Springer, Berlin, 1984.
[39] Peter Sarnak and Jacob Tsimerman. On Linnik and Selberg's conjecture about sums of Kloosterman sums. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, volume 270 of *Progr. Math.*, pages 619–635. Birkhäuser Boston, Inc., Boston, MA, 2009.
[40] A. Selberg. Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. *J. Indian Math. Soc. (N.S.)*, 20:47–87, 1956.
[41] Atle Selberg. On the estimation of Fourier coefficients of modular forms. In *Proc. Sympos. Pure Math., Vol. VIII*, pages 1–15. Amer. Math. Soc., Providence, R.I., 1965.
[42] Goro Shimura. On modular forms of half integral weight. *Ann. of Math. (2)*, 97:440–481, 1973.
[43] Jacob Sturm. Special values of zeta functions, and Eisenstein series of half integral weight. *Amer. J. Math.*, 102(2):219–240, 1980.
[44] E. C. Titchmarsh. *The Theory of the Riemann Zeta-Function.* Oxford, at the Clarendon Press, 1951.
[45] Matthew P. Young. Weyl-type hybrid subconvexity bounds for twisted L-functions and Heegner points on shrinking sets. *J. Eur. Math. Soc. (JEMS)*, 19(5):1545–1576, 2017.
[46] Don Zagier. A Kronecker limit formula for real quadratic fields. *Math. Ann.*, 213:153–184, 1975.

Brigham Young University Department of Mathematics, Provo, UT 84602
Email address: nick@math.byu.edu

UCLA Mathematics Department, Los Angeles, CA 90095
Email address: wdduke@ucla.edu