Fourier series of modular graph functions

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Abstract

Modular graph functions associate to a graph an \( SL(2,\mathbb{Z}) \)-invariant function on the upper half plane. We obtain the Fourier series of modular graph functions of arbitrary weight \( w \) and two-loop order. The motivation for this work is to develop a deeper understanding of the origin of the algebraic identities between modular graph functions which have been discovered recently, and of the relation between the existence of these identities and the occurrence of cusp forms. We show that the constant Fourier mode, as a function of the modulus \( \tau \), consists of a Laurent polynomial in \( y = \pi \text{Im} \tau \) of degree \( (w,1-w) \), plus a contribution which decays exponentially as \( y \to \infty \). The Laurent polynomial is a linear combination with rational coefficients of the top term \( y^w \), and lower order terms \( \zeta(2k+1)y^{w-2k-1} \) for \( 1 \leq k \leq w-1 \), as well as terms \( \zeta(2w-2\ell-3)\zeta(2\ell+1)y^{2-w} \) for \( 1 \leq \ell \leq w-3 \). The exponential contribution is a linear combination of exponentials of \( y \) and incomplete \( \Gamma \)-functions whose coefficients are Laurent polynomials in \( y \) with rational coefficients.


1 Introduction and statement of the main result

A modular graph function associates to a certain kind of graph an $SL(2,\mathbb{Z})$-invariant function of the upper half plane $\mathbb{H}$. Modular graph functions naturally arise in the low energy expansion of closed string amplitudes and govern the contributions to this expansion at genus one [1, 2]. For one-loop graphs they are non-holomorphic Eisenstein series, while for two-loop graphs they were found to obey a system of differential equations [3]. Connections between modular graph functions, multiple-zeta values and single-valued elliptic polylogarithms were put forward in [4]. Earlier relations between open and closed string amplitudes, multiple-zeta-functions, and polylogarithms were exhibited in [5, 6, 7]. The structure of the low energy expansion of genus-two string amplitudes, in as much as is known to date, may be found in [8] and references therein.

A number of algebraic identities between modular graph functions with two or more loops were conjectured in [3] by matching their asymptotic expansions near the cusp $\tau \to i\infty$. The simplest of these identities were proven by direct summation of the Eisenstein series in [9, 10]. More complicated identities and their various generalizations were obtained and proven by appealing to the rich system of differential equations they satisfy [11, 12, 13, 14]. In particular, it was shown in [11, 12] that the existence of these identities may be traced back to the existence of identities between holomorphic modular forms.

Better understanding the origin of these algebraic identities is the main motivation for the present work, as will be explained further below. Since the coefficients in the asymptotic expansion near the cusp $\tau \to i\infty$, the algebraic identities between them contain and generalize some of the well-known relations between multiple zeta-values (see for example [16, 17, 18, 19] and references therein) to the world of modular functions. A recent characterization of classes of modular graph functions, as single-valued projections of elliptic multiple zeta values introduced in [20], may be found in [21, 22].

A simple infinite family is given by modular graph functions depending on $a_1, \ldots, a_\ell \in \mathbb{N}$ with $\ell \geq 2$ and may be expressed for $\tau = \tau_1 + i\tau_2 \in \mathbb{H}$ with $\tau_1, \tau_2 \in \mathbb{R}$ in the form,

$$C_{a_1, \ldots, a_\ell}(\tau) = \sum_{(m_r, n_r) \in \mathbb{Z}^2} \delta_{m_0,0} \delta_{n_0,0} \prod_{r=1}^\ell \left( \frac{\tau_2}{\pi |m_r + n_r \tau|^2} \right)^{a_r}$$  \hspace{1cm} (1.1)

where $m = m_1 + m_2 + \cdots + m_\ell$, $n = n_1 + n_2 + \cdots + n_\ell$ and the Kronecker $\delta$-symbols force $m, n$ to vanish. Throughout, a prime over the summation symbol indicates that the summation is restricted to disallow division by zero, in this case to $(m_r, n_r) \neq (0, 0)$. The sum is absolutely convergent and $C_{a_1, \ldots, a_\ell}$ is smooth and modular,

$$C_{a_1, \ldots, a_\ell}(M \tau) = C_{a_1, \ldots, a_\ell}(\tau)$$  \hspace{1cm} (1.2)
for each \( M \in SL(2, \mathbb{Z}) \) acting on \( \tau \in \mathbb{H} \) by a linear fractional map.

The associated graph \( \Gamma = \Gamma_{a_1, \ldots, a_\ell} \) is planar and may be realized by first taking a graph \( \hat{\Gamma} \) in the plane with 2 vertices connected to each other by \( \ell \) edges, numbered 1, 2, \ldots, \( \ell \) and then adjoining \( a_r - 1 \) distinct new bivalent vertices to the \( r \)th edge for each \( r \). Thus \( \hat{\Gamma} \) has \( \ell \) edges, 2 vertices and \( \ell \) faces, while \( \Gamma \) has \( w = a_1 + \cdots + a_\ell \) edges, \( w - 1 \) vertices and \( \ell \) faces, or equivalently \( \ell - 1 \) loops. We will review in the next section how \( C_{a_1, \ldots, a_\ell} \), which is said to be a \((\ell - 1)\)-loop modular graph function of weight \( w \), arises from \( \Gamma \). Its birth is as a Feynman graph in quantum field theory and string theory, and in that context the combination \( p_r = m_r \tau + n_r \) represents the lattice momentum running through the edge \( r \) taking values in the lattice \( \Lambda = \mathbb{Z} + \mathbb{Z} \tau \).

In the most simple case of a one-loop graph we have \( \ell = 2 \) and \( C_{a_1, a_2} \) is given by a specialization of the classical Kronecker–Eisenstein series, since

\[
C_{a_1, a_2}(\tau) = E_w(\tau) = \sum_{(m,n) \in \mathbb{Z}^2} \frac{\tau_2^w}{\pi^w |m + n\tau|^{2w}}. \tag{1.3}
\]

It is well known that \( E_w(\tau) \) is an eigenfunction of the hyperbolic Laplacian,

\[
\Delta E_w = w(w - 1)E_w, \tag{1.4}
\]

where \( \Delta \) is normalized by \( \Delta = 4\tau^2 \partial_\tau \partial_{\bar{\tau}} \). The effect of the Laplacian on \( C_{a_1, \ldots, a_\ell}(\tau) \) was obtained in [3] where it was also shown that, as a result, a number of identities between various kind of modular graph functions are forced to exist. The simplest such identity was obtained in this way in [3] and states that,

\[
C_{1,1,1}(\tau) = E_3(\tau) + \zeta(3) \tag{1.5}
\]

where \( \zeta(s) \) is the Riemann zeta-function. A proof by direct summation of the Kronecker-Eisenstein series was given by Zagier [9]. In general, for \( \ell = 3 \) and for each odd value of \( w \) there is precisely one linear combination of those \( C_{a_1,a_2,a_3} \) with weight \( w \) that differs from \( E_w \) by a constant. For instance, for \( w = 5, 7, 9 \),

\[
\begin{align*}
30C_{2,2,1}(\tau) &= 12E_5(\tau) + \zeta(5) \\
252C_{3,3,1}(\tau) + 252C_{3,2,2}(\tau) &= 108E_7(\tau) + \zeta(7) \\
2160C_{4,4,1}(\tau) + 4320C_{4,3,2}(\tau) + 960C_{3,3,3}(\tau) &= 960E_9(\tau) + \zeta(9)
\end{align*} \tag{1.6}
\]

Note that, if we assign “weight \( s \)” to \( \zeta(s) \), then the above identities are all homogeneous of their respective weights. The case of even weight is more difficult. Here the identities involve the functions \( C \) with \( \ell > 3 \). For instance we have,

\[
C_{1,1,1,1}(\tau) = 24C_{2,1,1}(\tau) - 18E_4(\tau) + 3E_2^2(\tau) \tag{1.7}
\]
This identity was conjectured in [3] and proven in [10].

Identities such as (1.5), (1.6) and (1.7) are reminiscent of various identities from the classical theory of modular forms. Consider the classical holomorphic Eisenstein series of even modular weight\(^1\) \(w \geq 4\),

\[
G_w(\tau) = \frac{(w-1)!}{2(2\pi i)^w} \sum_{(m,n) \in \mathbb{Z}^2}^{'} \frac{1}{(m+n\tau)^w}.
\]  

(1.8)

We have the Fourier expansion,

\[
G_w(\tau) = -\frac{B_w}{2w} + \sum_{k \geq 1} \sigma_{w-1}(k) q^k,
\]

(1.9)

where \(B_w\) is the Bernoulli number, \(q = e^{2\pi i \tau}\) and \(\sigma_s(n) = \sum_{d|n} d^s\) is the divisor sum. The following identity is forced by the fact that \(G_8\) and \(G_{24}\) are in the same one-dimensional space,

\[
120 G_{24}^2 = G_8
\]

(1.10)

and implies the following arithmetic identity,

\[
\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(n-m) \sigma_3(m).
\]

(1.11)

At higher weight such identities will involve holomorphic cusp forms, a phenomenon which arises first at weight 12 where we have, for example,

\[
65 G_{12}(\tau) - 174132 G_6^2(\tau) = 756 \Delta(\tau),
\]

(1.12)

where \(\Delta(\tau) = q \prod_m (1-q^m)^{24}\) is the famous cusp form of modular weight 12. Knowledge of the dimensions of holomorphic modular forms for given modular weight, together with their Fourier series expansion, provides all the information needed to establish all such identities in the holomorphic case.

Returning to non-holomorphic modular graph functions, it is one of the ultimate goals of this and subsequent work to understand the relation between identities amongst modular graphs functions, such as (1.5), (1.6) and (1.7), their Fourier series expansion, and the existence of non-holomorphic cusp forms. The Fourier expansion of the non-holomorphic Eisenstein series \(E_w\) for integer \(w \geq 2\) is well-known and is given by,

\[
E_w(\tau) = -\frac{B_{2w}}{(2w)!} (-4y)^w + \frac{4(2w-3)!}{(w-2)! (w-1)!} \zeta(2w-1) (4y)^{1-w} + \frac{2}{(w-1)!} \sum_{k=1}^{\infty} k^{w-1} \sigma_{1-2w}(k) \left(q^k + \bar{q}^k\right) P_w(4ky).
\]

(1.13)

\(^1\)The modular weight is in general distinct from the weight \(w\) of the modular functions \(C\) defined earlier.
where we set $y = \pi \tau_2$, and $P_w(x)$ is the polynomial in $1/x$ defined by,
\[ P_w(x) = \sum_{m=0}^{w-1} \frac{(w + m - 1)!}{m! (w - m - 1)!} x^m \] (1.14)

For example, we have,
\[ E_3(\tau) = \frac{2}{945} y^3 + \frac{3}{4} \zeta(5) y^{-2} + \sum_{k=1}^{\infty} \sigma_3(k) \left( \frac{1}{k} + \frac{3}{2k^2 y} + \frac{3}{4k^3 y^2} \right) (q^k + q^k) \] (1.15)

We see that the constant Fourier mode of $E_w$ is a Laurent polynomial with only two terms.

In the present paper, we shall study the Fourier series expansion of two-loop modular graphs functions $C_{a_1,a_2,a_3}$. By combining (1.5) and (1.6) with (1.15) we readily get the Fourier expansions of $C_{1,1,1}$ and $C_{2,2,1}$. For functions of higher weight, the Laurent series have been evaluated only in special cases, tabulated below,

\[
\begin{align*}
C_{2,1,1}(\tau) & = \frac{2y^4}{14175} + \frac{\zeta(3)y}{45} + \frac{5\zeta(5)}{12y} - \frac{\zeta(3)^2}{4y^2} + \frac{9\zeta(7)}{16y^3} + O(e^{-2\pi\tau_2}) \\
C_{3,1,1}(\tau) & = \frac{2y^5}{155925} + \frac{2\zeta(3)y^2}{945} - \frac{\zeta(5)}{180} + \frac{7\zeta(7)}{16y^2} - \frac{\zeta(3)\zeta(5)}{2y^3} + \frac{43\zeta(9)}{64y^4} + O(e^{-2\pi\tau_2}) \\
C_{4,1,1}(\tau) & = \frac{808y^6}{638512875} + \frac{\zeta(3)y^3}{4725} - \frac{\zeta(5)y}{1890} + \frac{\zeta(7)}{720y} + \frac{23\zeta(9)}{64y^3} \\
& \quad - \frac{\zeta(5)^2 + 30\zeta(3)\zeta(7)}{64y^4} + \frac{167\zeta(11)}{256y^5} + O(e^{-2\pi\tau_2}) \\
C_{3,2,1}(\tau) & = \frac{43y^5}{58046625} + \frac{y\zeta(5)}{630} + \frac{\zeta(7)}{144y} + \frac{7\zeta(9)}{64y^2} - \frac{17\zeta(5)^2}{64y^4} + \frac{99\zeta(11)}{256y^5} + O(e^{-2\pi\tau_2}) \\
C_{2,2,2}(\tau) & = \frac{38y^6}{91216125} + \frac{\zeta(7)}{24y} - \frac{7\zeta(9)}{16y^3} + \frac{15\zeta(5)^2}{16y^4} - \frac{81\zeta(11)}{128y^5} + O(e^{-2\pi\tau_2}) \quad (1.16)
\end{align*}
\]

Along different lines, a systematic algorithm was developed to evaluate modular graph functions which have at most four vertices (including bivalent vertices) in [15].

Except in such cases where we may apply a known identity, it is an open problem to determine the Fourier coefficients of $C_{a_1,\ldots,a_\ell}(\tau)$ for a fixed $a_1,\ldots,a_\ell$ when $\ell > 2$. Its expansion has the form,
\[
C_{a_1,\ldots,a_\ell}(\tau) = \sum_{k=-\infty}^{\infty} C_{a_1,\ldots,a_\ell}(\tau_2) e^{2\pi i k \tau_1}, \quad (1.17)
\]

If we can determine $C_{a_1,\ldots,a_\ell}(\tau_2)$, independently of any knowledge of special identities between modular graph forms, then identities such as (1.5), (1.6) and (1.7) and their generalizations
should emerge from identities between the Fourier coefficients of these modular graph functions. Understanding the structure of the Fourier series thus appears key to understanding the structural mechanism behind the existence of the identities between modular graph functions, and their relation with the existence of cusp forms. Clearly, one obstruction to the existence of identities is the presence of cusp forms. Therefore, an urgent question is whether we can find combinations of the \( C \)-functions of a fixed weight having zero constant mode in their Fourier expansions. Finally, it would be interesting to find out whether such cusp forms play a natural role in string theory.

As a step toward answering such questions, in this paper we will compute rather explicitly the constant mode \( C^{(0)}_{a_1,a_2,a_3}(\tau_2) \) in the Fourier expansion of \( C_{a_1,a_2,a_3} \). The main results of this paper may be summarized by three Theorems, and one conjectured decomposition formula.

**Theorem 1.1** The constant Fourier mode for \( C_{a_1,a_2,a_3} \) with fixed \( a_1, a_2, a_3 \) is given by,

\[
C^{(0)}_{a_1,a_2,a_3}(\tau_2) = L(\tau_2) + E(\tau_2) \tag{1.18}
\]

where \( L(\tau_2) \) is a Laurent polynomial in \( \tau_2 \) of degree \((w, 1 - w)\) and \( E(\tau_2) \) is exponentially decaying as \( \tau \to i\infty \). The Laurent polynomial is given by,

\[
L(\tau_2) = c_w(-4\pi \tau_2)^w + \sum_{k=1}^{w-1} c_{w-2k-1}(\tau_2) \tag{1.19}
\]

(a) The coefficient \( c_w \) is a rational number given by,

\[
c_w = \sum_{k=0}^{a_2} \frac{B_k B_{2w-2k}}{(2k)! (2w-2k)!} \frac{\Gamma(2a_2 + 2a_3 - 2k)}{\Gamma(2a_3) \Gamma(2a_2 - 2k + 1)} + (a_2 \leftrightarrow a_3) \tag{1.20}
\]

(b) The coefficients \( c_{w-2k-1} \) for \( 1 \leq k \leq w - 1 \) are rational numbers given by,

\[
c_{w-2k-1} = \frac{2B_{2w-2k-2}}{(2w-2k-2)!} \sum_{\alpha=0}^{a_1-1} \sum_{\beta=0}^{a_3-1} (-)^{a_1+a_3+\beta+1} \theta \left( a_3 + \left[ \frac{a_2 + \beta}{2} \right] - w + k + 1 \right)
\]

\[
\times g_{a_1,a_2}(\alpha, \beta) \left( \frac{2k - 2a_1 + \alpha + \beta + 1}{a_2 + \alpha - 1} \right) + 5 \text{ permutations of } a_1, a_2, a_3
\]

where \( \theta(x) \) is the step function defined to equal 1 when \( x \geq 0 \) and to vanish otherwise. The function \( g_{a_1,a_2}(\alpha, \beta) \) is integer-valued and given by,

\[
g_{a_1,a_2}(\alpha, \beta) = (-)^{a_1} \left( \frac{2a_1 - 2 - \alpha - \beta}{a_1 - 1} \right) \left( \frac{a_2 + \alpha - 1}{a_2 - 1} \right) \left( \frac{a_2 + \beta - 1}{a_2 - 1} \right) \tag{1.21}
\]
(c) The coefficient \( c_{2-w} \) is given by the following expression,
\[
c_{2-w} = c^0_{2-w} \zeta(2w - 2) + 2 \sum_{\sigma \in \mathfrak{S}_3} Z(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})
\] (1.22)

Here, \( c^0_{2-w} \) is an integer given by,
\[
c^0_{2-w} = \sum_{\alpha=0}^{a_1-1} \sum_{\beta=0}^{a_2-1} (-)^{a_2+\beta} g_{a_1, a_2}(\alpha, \beta) \left( \frac{2a_2 + 2a_3 + \alpha + \beta}{a_2 + \alpha - 1} \right)
\] (1.23)

while \( Z(a_1, a_2, a_3) \) is a linear combination with integer coefficients of depth-two multiple zeta-functions of total weight \( 2w - 2 \) given by,
\[
Z(a_1, a_2, a_3) = \sum_{k=1}^{a_1} \sum_{\ell=1}^{a_2} \left( a_1 + a_2 - k - 1 \right) \left( a_1 + a_2 - \ell - 1 \right) \left( k + \ell - 2 \right)
\times \left( \frac{2w - k - \ell - 2}{w - k - 1} \right) \zeta(2w - k - \ell - 1, k + \ell - 1)
\] (1.24)

The normalization of the double \( \zeta \)-functions is as follows,
\[
\zeta(a, b) = \sum_{m,n=1}^{\infty} \frac{1}{(m + n)^a n^b}
\] (1.25)

**Theorem 1.2** The coefficient \( c_{2-w} \) is a linear combination, with integer coefficients, of products of two odd zeta-values whose weights add up to \( 2w - 2 \),
\[
c_{2-w} = \sum_{k=1}^{w-2} \frac{1}{2} \gamma_k \zeta(2k + 1) \zeta(2w - 2k - 3)
\] (1.26)

with rational coefficients \( \gamma_k \in \mathbb{Q} \).

**Theorem 1.3** The general structure of the exponential part (1.18) is given as follows,
\[
E(\tau_2) = \sum_{s=1}^{w-2} \sum_{n=1}^{\infty} \left( f_+ (s; n) (4\pi \tau_2)^s + f_- (s; n) (4\pi \tau_2)^s \right) \operatorname{Ei}(4\pi n \tau_2) + e^{-4\pi n \tau_2} \sum_{m=2-w}^{w-3} \frac{f(s; n, m)}{(4\pi \tau_2)^m}
\] (1.27)

where \( \operatorname{Ei} \) is the incomplete \( \Gamma \)-function and the coefficients \( f_\pm (s, n) \), and \( f(s, n, m) \) are rational numbers.
Conjecture 1.4 (Decomposition Formula) The coefficients $\gamma_k$ entering the decomposition of $c_{2-w}$ in formula (1.26) of Theorem 1.2 are given by the following expression,

$$
\gamma_k = 2Z_k(a_1, a_2, a_3)\theta(a_1 - 1 - k) - Z_0(a_1, a_2, a_3)
$$

$$
+ \sum_{a_1=1}^{a_1-1} \sum_{a_2=1}^{2a_1-1} \sum_{a_3=1}^{2a_1-a_3-1} E_n(0) \binom{2k}{2a_1-n} \binom{2w-2a_3+n-4}{n}
$$

$$
+ 5 \text{ permutations of } a_1, a_2, a_3
$$

(1.28)

and are integers. Here, $E_n(x)$ are the Euler polynomials and the integer-valued function $Z_\alpha(a_1, a_2, a_3)$ is given by the following sum,

$$
Z_\alpha(a_1, a_2, a_3) = \sum_{k=k_+}^{k_+} \left( \frac{a_1 + a_3 - k - 1}{a_3 - 1} \frac{a_1 + a_3 - 2\alpha + k - 3}{a_3 - 1} \right) \binom{2\alpha}{k-1} \binom{2w-2\alpha-4}{w-k-1}
$$

(1.29)

with $k_+ = \min(a_1, 2\alpha + 1)$ and $k_- = \max(1, 2\alpha + 2 - a_1)$.

To obtain the decomposition formula, we make use of a conjectured relation (conjecture 6.2), which we have verified extensively using Maple calculations, but for which we have no analytical proof. Therefore, a full proof of the decomposition formula remains outstanding.

The explicit formulas for the Laurent polynomial part $L$ of $C_{a_1, a_2, a_3}(\tau)$, obtained in Theorems 1.1 and 1.2, and the Decomposition Formula of 1.4, completely reproduce the Laurent polynomials of (1.16) which have been evaluated earlier in the literature.

2 Modular graph functions

Before turning to the proof of the Theorems, and the derivation of the Decomposition Formula, we will briefly review the general definition of a modular graph function as it comes from string theory and show that the function $C_{a_1, \ldots, a_\ell}(\tau)$, which was defined in (1.1), is one such modular graph function.

Modular graph functions arise as follows. The torus $\Sigma$ with modulus $\tau$ may be represented in the complex plane by the quotient $\Sigma = \mathbb{C}/\Lambda$ for the lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$. We choose local complex coordinates $(z, \bar{z})$ on $\Sigma$ in which the metric is given by $|dz|^2/\tau_2$. The volume form of this metric $d\mu(z) = idz \wedge d\bar{z}/(2\tau_2)$ has unit area and the Dirac $\delta$-function $\delta(z - w)$ is normalized by $\int_\Sigma d\mu(z) \delta(z - w) = 1$. The scalar Green function $G(z - w|\tau)$ is defined by,

$$
\tau_2\partial_z \partial_{\bar{z}} G(z - w|\tau) = -\pi \delta(z - w) + \pi
$$

(2.1)

along with the normalization condition,

$$
\int_\Sigma d\mu(z) G(z - w|\tau) = 0
$$

(2.2)
To a graph \( \Gamma \) with \( v \geq 2 \) vertices and \( w \) edges we associate \( v \) points \( z_i \) on the torus \( \Sigma \) labelled by the index \( i = 1, \ldots, v \). We denote by \( \nu_{ij} \) the number of edges connecting the pair of vertices \( i, j = 1, \ldots, v \). The number \( \nu_{ij} \) is allowed to be a positive or zero integer for any pair of distinct vertices \( i, j \). We set \( \nu_{ii} = 0 \) for all vertices \( i = 1, \ldots, v \), and thus restrict the type of graphs on which we can define modular graph functions.\(^2\) The total number of edges is \( w = \sum_{1 \leq i < j \leq v} \nu_{ij} \).

The modular graph function \( C_\Gamma(\tau) \) is defined in terms of absolutely convergent integrals over the torus by

\[
C_\Gamma(\tau) = \left( \prod_{k=1}^{v} \int_{\Sigma} d\mu(z_k) \right) \prod_{1 \leq i < j \leq v} G(z_i - z_j | \tau)^{\nu_{ij}}
\] (2.3)

\( C_\Gamma(\tau) \) is clearly modular.

We may assume that \( \Gamma \) is connected and remains connected after the removal of a vertex (and its adjoining edges), since otherwise \( C_\Gamma = C_{\Gamma_1} C_{\Gamma_2} \) for subgraphs \( \Gamma_1 \) and \( \Gamma_2 \). We may also assume that \( \Gamma \) remains connected when any single edge is omitted, for otherwise \( C_\Gamma = 0 \). In particular we may assume that \( \Gamma \) contains no vertices with valence 1. Suppose that \( \hat{\nu}_v \) vertices of valence at least 3. As in the case of the graphs \( \Gamma_{a_1, \ldots, a_r} \), which were introduced in the third paragraph of section 1, it is convenient to build up \( \Gamma \) using an auxiliary graph \( \hat{\Gamma} \) having \( \hat{\nu}_v \) vertices with the same valences \( \geq 3 \), no bivalent vertices and \( \ell \) edges. Note that, unlike \( \Gamma \), \( \hat{\Gamma} \) can be a single edge with no vertices. This happens when \( \hat{\nu}_v = 0 \). To recover \( \Gamma \) we adjoin \( a_r - 1 \) bivalent vertices to the \( r \)th edge of \( \hat{\Gamma} \) for \( r = 1, \ldots, \ell \). If \( \hat{\nu}_v > 0 \) consider the \( \hat{\nu}_v \times \ell \) incidence matrix of the graph \( \hat{\Gamma} \), when it is given some orientation. The entry \( a_{ir} \) of the incidence matrix equals \( \pm 1 \) if edge \( r \) starts or ends on vertex \( i \) (the sign is determined by the choice of orientation through the graph), and equals 0 otherwise.

**Proposition 2.1** Under the assumptions and notation introduced above, we have,

\[
C_\Gamma(\tau) = \sum_{(m, n) \in \mathbb{Z}^2} \prod_{r=1}^{\ell} \left( \frac{\tau}{|m_r + n_r \tau|} \right)^{a_r} \prod_{i=1}^{\hat{\nu}_v} \delta \left( \sum_{r=1}^{\ell} a_{ir} m_r \right) \delta \left( \sum_{r=1}^{\ell} a_{ir} n_r \right)
\] (2.4)

When \( \hat{\nu}_v = 0 \) we must evaluate the second product to 1.

**Proof:** The Green function \( G(z | \tau) \), defined in (2.1) and (2.2), is given by a Fourier sum on the torus, parametrized in terms of real coordinates \( x, y \) by \( z = x + y \tau \) and \( x, y \in \mathbb{R}/\mathbb{Z} \),

\[
G(z | \tau) = \sum_{(m, n) \in \mathbb{Z}^2} \frac{\tau}{|m + n \tau|} e^{2\pi i (my - nx)}
\] (2.5)

\(^2\)An immediate justification for this restriction on the graphs is that without it divergent contributions involving the Green function at coincident points \( G(0 | \tau) \) would arise. In quantum field theory such graphs do arise in un-renormalized correlation functions, but are eliminated by the process of renormalization.
Bivalent vertices play a special role, as they produce a convolution of concatenated Green functions. We parametrize their effect by introducing the functions \( G_a(z|\tau) \), defined recursively in the index \( a \) by setting \( G_1(z|\tau) = G(z|\tau) \) for \( a = 1 \) and,

\[
G_a(z|\tau) = \int \Sigma d\mu(w) G(z - w|\tau) G_{a-1}(w|\tau)
\]

for \( a \geq 2 \). The Fourier series for \( G_a \) on the torus is readily obtained using \( d\mu(z) = dx \wedge dy \),

\[
G_a(z|\tau) = \sum_{(m,n) \in \mathbb{Z}^2} \frac{\tau_2^a}{\pi^a|m + n\tau|^{2a}} e^{2\pi i(my-nx)}.
\]

Applying this to (2.3) and carrying out the integrals over the \( \hat{v} \) vertex positions \( z_i \) corresponding to vertices of valence \( \geq 3 \), we obtain (2.4).

**Corollary 2.2** For \( C_{a_1,\cdots,a_\ell} \) and \( \Gamma_{a_1,\cdots,a_\ell} \) defined in and below (1.1) we have,

\[
C_{\Gamma_{a_1,\cdots,a_\ell}} = C_{a_1,\cdots,a_\ell}
\]

### 3 Fourier series of two-loop modular graph functions

Now we turn to the proof of the Theorems. In this section we shall introduce a Mellin-transform formulation of two-loop modular graph functions \( C_{a_1,a_2,a_3} \) and a partial Poisson resummation to obtain the Fourier series expansion of (1.17). The method naturally generalizes to the case of higher modular graph functions, but we shall treat here only the case of two-loop modular graph functions of arbitrary weight \( w = a_1 + a_2 + a_3 \).

#### 3.1 Mellin-transform representation

We begin with the elementary integral representation,

\[
\frac{\tau_2^a}{\pi^a|m_r + n_r\tau|^{2a_r}} = \int_0^\infty dt_r \frac{t_r^{a_r-1}}{\Gamma(a_r)} \exp \left\{ -\frac{\pi}{\tau_2} t_r|m_r + n_r\tau|^2 \right\}
\]

Collecting the sum over the product of three such factors, and taking care of omitting the zero mode from the summation over each edge, we find,

\[
C_{a_1,a_2,a_3}(\tau) = \left( \prod_{r=1}^3 \int_0^\infty dt_r \frac{t_r^{a_r-1}}{\Gamma(a_r)} \right) S(t_1,t_2,t_3|\tau)
\]

\[
S(t_1,t_2,t_3|\tau) = \sum_{(m_r,n_r) \in \mathbb{Z}^2} \delta_{m,0} \delta_{n,0} \prod_{r=1}^3 \left( \exp \left\{ -\frac{\pi}{\tau_2} t_r|m_r + n_r\tau|^2 \right\} - \delta_{m_r,0} \delta_{n_r,0} \right)
\]
The function $S(t_1, t_2, t_3 | \tau)$ is invariant under $SL(2, \mathbb{Z})$ acting on $\tau$, as well as under permutations of the $t_r$. Next, we decompose $S$ by expanding the triple product into a sum of eight terms. The three contributions for which two pairs $(m_r, n_r)$ are set to zero must also have the third pair equal to zero in view of overall momentum conservation, and therefore combine with the terms in which all three pairs are zero. The result is as follows,

$$S(t_1, t_2, t_3 | \tau) = A(t_1, t_2, t_3 | \tau) - B(t_1 + t_2 | \tau) - B(t_2 + t_3 | \tau) - B(t_3 + t_1 | \tau) + 2 \quad (3.3)$$

The last term arises from the contribution with all pairs $(m_r, n_r)$ equal to $(0, 0)$, and the functions $A$ and $B$ are given by,

$$A(t_1, t_2, t_3 | \tau) = \sum_{(m_r,n_r) \in \mathbb{Z}^2} \delta_{m,0} \delta_{n,0} \exp \left\{ -\frac{\pi}{\tau_2} \sum_{r=1}^{3} t_r | m_r + n_r \tau |^2 \right\}$$

$$B(t | \tau) = \sum_{(m_1,n_1) \in \mathbb{Z}^2} \exp \left\{ -\frac{\pi}{\tau_2} t | m_1 + n_1 \tau |^2 \right\} \quad (3.4)$$

Note that the summation over the three pairs $(m_r, n_r)$ in $A$ includes the contribution from all the zero pairs, and is constrained only by the requirement that their sum $(m, n)$ vanishes. The summation over pairs $(m_1, n_1)$ in $B$ is unconstrained.

It will be convenient to solve the constraint $m = n = 0$ in the summation which defines the function $A$ by setting $m_3 = -m_1 - m_2$ and $n_3 = -n_1 - n_2$, with $m_1, m_2, n_1, n_2$ taking values in $\mathbb{Z}$ unconstrained. Furthermore, we introduce the matrix notation,

$$M = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \quad N = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \quad T = \begin{pmatrix} t_1 + t_3 & t_3 \\ t_3 & t_2 + t_3 \end{pmatrix} \quad (3.5)$$

The function $A$ then takes the form,

$$A(t_1, t_2, t_3 | \tau) = \sum_{M,N \in \mathbb{Z}^2} \exp \left\{ -\frac{\pi}{\tau_2} (M + \tau N)^\dagger T(M + \tau N) \right\} \quad (3.6)$$

One may think of this expression as defining a $\vartheta$-function.

### 3.2 Partial Poisson resummation

To compute the Fourier series of $C_{a_1,a_2,a_3}(\tau)$ as a function of $\tau_1$, we perform a Poisson resummation on the sum in the expression for the function $A$ on the matrix $M$, but not on $N$. To do so, we evaluate the Fourier transform of the $M$-dependent part as follows,

$$\int_{\mathbb{R}^2} d^2 M \ e^{-2\pi i M^\dagger X} e^{-\pi (M + \tau_1 N)^\dagger T(M + \tau_1 N)/\tau_2} = \frac{\tau_2}{(\det T)^{\tau}} e^{2\pi i \tau_1 N^\dagger X - \pi \tau_2 X^\dagger T^{-1} X} \quad (3.7)$$
It will be convenient to express the inverse of $T$ as follows,

$$T^{-1} = \frac{1}{\det T} \varepsilon^t T \varepsilon \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(3.8)

Upon the change of summation variables $M \rightarrow -\varepsilon M$, the Fourier series takes the form,

$$A(t_1, t_2, t_3|\tau) = \frac{\tau_2}{(\det T)^{\frac{1}{2}}} \sum_{M,N \in \mathbb{Z}^2} e^{2\pi iT \varepsilon N \tau} \exp \left\{ -\pi \tau_2 \frac{M' TM}{\det T} - \pi \tau_2 N'TN \right\}$$

(3.9)

To obtain the Fourier series of the function $C_{a_1,a_2,a_3}(\tau)$ we shall need to integrate $S$ over $t_1, t_2, t_3$, which requires combining the contributions of $A$ to the integral with those from $B$. To simplify this recombination, we perform a Poisson resummation in $m_1$ of $B(t)$,

$$B(t|\tau) = \sqrt{\frac{\tau_2}{t}} \sum_{m_1,n_1 \in \mathbb{Z}} e^{2\pi im_1 t_1} e^{-\pi \tau_2 m_1^2 / t - \pi \tau_2 n_1^2 t}$$

(3.10)

which exhibits the Fourier series in $\tau_1$ of $B$.

### 3.3 Fourier series expansions of $A$ and $B$

The Fourier modes $S_k(t_1, t_2, t_3|\tau_2)$ of $S(t_1, t_2, t_3|\tau)$ as a function of $\tau_1$ are given by,

$$S(t_1, t_2, t_3|\tau) = \sum_{k \in \mathbb{Z}} e^{2\pi ik\tau_1} S_k(t_1, t_2, t_3|\tau_2)$$

(3.11)

The Fourier modes $A_k(t_1, t_2, t_3|\tau_2)$ of $A(t_1, t_2, t_3|\tau)$, and the Fourier modes $B_k(t|\tau_2)$ of $B(t|\tau)$ as functions of $\tau_1$ are defined analogously. They are related to one another by,

$$S_k(t_1, t_2, t_3|\tau_2) = A_k(t_1, t_2, t_3|\tau_2) - B_k(t_1 + t_2|\tau_2) - B_k(t_2 + t_3|\tau_2) - B_k(t_3 + t_1|\tau_2) + 2\delta_{k,0}$$

(3.12)

The expressions for the Fourier modes are obtained from (3.9) and (3.10) and are given by,

$$A_k(t_1, t_2, t_3|\tau_2) = \frac{\tau_2}{(\det T)^{\frac{1}{2}}} \sum_{M,N \in \mathbb{Z}^2} \delta_{M',\varepsilon N,k} \exp \left\{ -\pi \tau_2 \frac{M' TM}{\det T} - \pi \tau_2 N'TN \right\}$$

$$B_k(t|\tau_2) = \sqrt{\frac{\tau_2}{t}} \sum_{m,n \in \mathbb{Z}} \delta_{mn,k} e^{-\pi \tau_2 m^2 / t - \pi \tau_2 n^2 t}$$

(3.13)

The Fourier modes $B_k$ for $k \neq 0$ are exponentially decaying as $t \rightarrow \infty$. Thus, the term $B_k(t_1 + t_2|\tau)$ decays exponentially as $t_1 \rightarrow \infty$ or $t_2 \rightarrow \infty$ or both, but not when $t_3 \rightarrow \infty$. We will show in the subsequent subsection that uniform exponential decay is recovered upon
combining the contributions of the Fourier modes $A_k$ and $B_k$ into the modes $S_k$ given by (3.3). With exponential decay secured, the Fourier modes of $C_{a_1,a_2,a_3}(\tau)$, expressed in the notation of (1.17) with the help of (3.2), are then obtained by the following integrals,

$$C^{(k)}_{a_1,a_2,a_3}(\tau) = \left( \prod_{r=1}^{3} \int_{0}^{\infty} dt_r \frac{t_r^{a_r-1}}{\Gamma(a_r)} \right) S_k(t_1,t_2,t_3|\tau)$$  \hspace{1cm} (3.14)

which are absolutely convergent for large $t_r$, and may be analytically continued in $a_r$ for small $t_r$ if necessary.

### 3.4 Partitioning the sum over $N$

To expose uniform exponential decay in $t_1, t_2, t_3$, we partition the summation over $N \in \mathbb{Z}^2$,

$$\mathbb{Z}^2 = \mathcal{N}^{(0)} \cup \mathcal{N}^{(1)} \cup \mathcal{N}^{(2)} \cup \mathcal{N}^{(3)} \cup \mathcal{N}^{(4)}$$  \hspace{1cm} (3.15)

into the following (disjoint) parts,

- $\mathcal{N}^{(0)} = \{(0,0)\}$
- $\mathcal{N}^{(i)} = \{(n_1,n_2) \text{ such that } n_1,n_2 \in \mathbb{Z}, n_i = 0, n_j \neq 0 \text{ for } j \neq i\}$,  \hspace{1cm} $i,j = 1,2,3$
- $\mathcal{N}^{(4)} = \{(n_1,n_2) \text{ such that } n_1,n_2 \in \mathbb{Z}, n_1,n_2,n_3 \neq 0\}$  \hspace{1cm} (3.16)

where we enforce the constraint $n = n_1 + n_2 + n_3 = 0$ throughout. The quadratic form $N^T N$ vanishes for $N \in \mathcal{N}^{(0)}$; is uniformly non-degenerate in $t_1, t_2, t_3$ for $N \in \mathcal{N}^{(4)}$; while for $i = 1,2,3$ and $N \in \mathcal{N}^{(i)}$ the quadratic form decays exponentially, but non-uniformly, in all directions except $t_i \to \infty$ where it remains bounded. Therefore, the above partition accurately governs the asymptotic behavior as $t_1, t_2, t_3 \to \infty$. We arrange the contributions to $S_k$ arising from the partitions $\mathcal{N}^{(i)}$ for $i = 0,1,2,3,4$ to $A_k$ and from $B_k$ as follows,

$$S_k(t_1,t_2,t_3|\tau_2) = \sum_{i=0}^{4} S_k^{(i)}(t_1,t_2,t_3|\tau_2)$$  \hspace{1cm} (3.17)

We shall spell out their precise contributions in the subsequent subsections.

#### 3.4.1 Contributions to $S_k^{(0)}$

The term $S_k^{(0)}$ arises from the contribution to $A_k$ of $N = 0$, and is non-zero only for the constant Fourier mode $k = 0$. Performing a Poisson resummation in $M$ gives,

$$S_k^{(0)}(t_1,t_2,t_3|\tau_2) = \delta_{k,0} \sum_{M \in \mathbb{Z}^2} \exp \left\{-\frac{\pi}{\tau_2} M^T M \right\}$$  \hspace{1cm} (3.18)
Partitioning the summation over $M$ according to (3.15), we have,

$$S_k^{(0)}(t_1, t_2, t_3 | \tau_2) = \delta_{k,0} \left( 1 + \sum_{i=1}^{3} L(t'_i | \tau_2) + \tilde{S}^{(0)}_0(t_1, t_2, t_3 | \tau_2) \right)$$

(3.19)

where we have defined $t'_i = t_1 + t_2 + t_3 - t_i$ while the functions $L$ and $\tilde{S}^{(0)}_0$ are defined by,

$$L(t'_i | \tau_2) = \sum_{m \in \mathbb{Z}} e^{-\pi t'_m^2/\tau_2}$$

$$\tilde{S}^{(0)}_0(t_1, t_2, t_3 | \tau_2) = \sum_{M \in \mathbb{Z}^3} \exp \left\{ -\frac{\pi}{\tau_2} M' TM \right\}$$

(3.20)

The contributions of the functions $L(t)$ naturally combine with those from $B_k(t)$.

**3.4.2 Contributions to $S_k^{(i)}$ for $i = 1, 2, 3$**

For $i = 1, 2, 3$, the term $S_k^{(i)}$ arises from the contribution to $A_k$ of the partition with $N \in \mathcal{N}^{(i)}$ and from $B_k(t'_i | \tau_2)$. It will be convenient to exhibit it as follows,

$$S_k^{(i)}(t_1, t_2, t_3 | \tau_2) = \tilde{S}_k^{(i)}(t_1, t_2, t_3 | \tau_2) - \delta_{k,0} - \delta_{k,0} L(t'_i | \tau_2)$$

(3.21)

where the reduced Fourier mode $\tilde{S}_k^{(i)}(t_1, t_2, t_3 | \tau_2)$ is given by,

$$\tilde{S}_k^{(i)}(t_1, t_2, t_3 | \tau_2) = \frac{\tau_2}{(\det T)^{1/2}} \sum_{N \in \mathcal{N}^{(i)}} \sum_{M \in \mathbb{Z}^3} \delta_{M' \in N,k} \exp \left\{ -\frac{\pi}{\tau_2} M' TM - \pi \tau_2 N'TN \right\}$$

$$- B_k(t'_i) + \delta_{k,0} \left( 1 + L(t'_i | \tau_2) \right)$$

(3.22)

The contributions individually fail to exponentially decay in the direction $t_i \to \infty$ but the sum of the two lines above produces a function $S_k^{(i)}$ which exponentially decays in all directions $t_1, t_2, t_3$ at infinity. To establish uniform exponential decay, it will be convenient to treat the cases $k = 0$ and $k \neq 0$ separately.

For $k = 0$, consider the case $i = 1$, the other cases being obtained by cyclic permutations of $t_1, t_2, t_3$. For $i = 1$, the partition $\mathcal{N}^{(i)}$ may be parametrized explicitly by $N = (0, n_2)$ with $n_1 = 0$ and $n_2 = -n_3 \neq 0$. The constraint $M' \in N = 0$ reduces to $m_1 n_2 = 0$ so that we must have $m_1 = 0$, while $m_2 \in \mathbb{Z}$. Furthermore, the summation over $m, n$ in the function $B_0(t_2 + t_3 | \tau_2)$ is constrained by $mn = 0$ and simplifies as follows,

$$B_0(t_2 + t_3 | \tau_2) = \sqrt{\frac{\tau_2}{t_2 + t_3}} \sum_{m \in \mathbb{Z}} e^{-\pi \tau_2 m^2/(t_2 + t_3)} + \sqrt{\frac{\tau_2}{t_2 + t_3}} \sum_{n \neq 0} e^{-\pi \tau_2 (t_2 + t_3)n^2}$$

(3.23)
Poisson resummation over \( m \) in the first term on the right side gives,

\[
B_0(t_2 + t_3|\tau) = 1 + L(t_2 + t_3|\tau) + \sqrt{\frac{\tau_2}{t_2 + t_3}} \sum_{n \neq 0} e^{-\pi \tau_2 (t_2 + t_3)n^2}
\]  

(3.24)

Using the parametrization of \( M \) and \( N \) for \( k = 0 \) and \( i = 1 \) given above, the sums over \( m \) and \( n \) on the first line of the right side of (3.22) factorize, and the expression takes the form,

\[
\frac{\tau_2}{(\det T)^{\frac{1}{2}}} \sum_{m_2} \sum_{n_2 \neq 0} \exp \left\{ -\pi \frac{\tau_2}{\tau_2 + t_3} \frac{m_2^2}{2} - \pi \tau_2(t_2 + t_3)n_2^2 \right\}
\]  

(3.25)

Poisson resumming over \( m_2 \), and combining the expression with the result obtained for \( B_0 \) shows that the \( m_2 = 0 \) mode of the Poisson resummation is cancelled by the terms from \( B_0 \) and \( L \), and gives the following expression for \( \tilde{S}_0^{(1)} \),

\[
\tilde{S}_0^{(1)}(t_1, t_2, t_3|\tau_2) = \sqrt{\frac{\tau_2}{t_2 + t_3}} \sum_{m_2, n_2 \neq 0} \exp \left\{ -\pi \frac{\det T}{\tau_2} \frac{m_2^2}{2} - \pi \tau_2(t_2 + t_3)n_2^2 \right\}
\]  

(3.26)

The functions \( \tilde{S}_0^{(2)} \) and \( \tilde{S}_0^{(3)} \) are obtained by cyclic permutations in the variables \( t_1, t_2, t_3 \) of (3.26), and the resulting functions \( \tilde{S}_0^{(i)}(t_1, t_2, t_3|\tau_2) \) have uniform exponential decay in all directions of \( t_1, t_2, t_3 \). The case \( k \neq 0 \) may be handled similarly, but will not be needed to prove the Theorems, and we shall not discuss it further.

### 3.4.3 Contributions to \( S_k^{(4)} \)

The term \( S_k^{(4)} \) arises solely from the contribution to \( A_k \) of the partition with \( N \in \mathfrak{N}^{(4)} \), and is given as follows,

\[
S_k^{(4)}(t_1, t_2, t_3|\tau_2) = \frac{\tau_2}{(\det T)^{\frac{1}{2}}} \sum_{N \in \mathfrak{N}^{(4)}} \sum_{M \in \mathbb{Z}^2} \delta_{M^t \in N,k} \exp \left\{ -\pi \tau_2 \frac{M^t TM}{\det T} - \pi \tau_2 N^t TN \right\}
\]  

(3.27)

Clearly, \( S_k^{(4)} \) is uniformly exponentially decaying in \( t_1, t_2, t_3 \) at infinity. For \( k = 0 \) the constraint \( M^t \in N = 0 \) forces \( M \) to either vanish, or to belong to \( \mathfrak{N}^{(4)} \), excluding the cases \( M \in \mathfrak{N}^{(i)} \) for \( i = 1, 2, 3 \). Therefore, it will be convenient to split the sum accordingly,

\[
S_k^{(4)}(t_1, t_2, t_3|\tau_2) = \tilde{S}_k^{(4)}(t_1, t_2, t_3|\tau_2) + \tilde{S}_k^{(5)}(t_1, t_2, t_3|\tau_2)
\]  

(3.28)

where

\[
\tilde{S}_k^{(4)}(t_1, t_2, t_3|\tau_2) = \frac{\tau_2 \delta_{k,0}}{(\det T)^{\frac{1}{2}}} \sum_{N \in \mathfrak{N}^{(4)}} \exp \left\{ -\pi \tau_2 N^t TN \right\}
\]  

(3.29)

\[
\tilde{S}_k^{(5)}(t_1, t_2, t_3|\tau_2) = \frac{\tau_2}{(\det T)^{\frac{1}{2}}} \sum_{N \in \mathfrak{N}^{(4)}} \sum_{M \neq 0} \delta_{M^t \in N,k} \exp \left\{ -\pi \tau_2 \frac{M^t TM}{\det T} - \pi \tau_2 N^t TN \right\}
\]

where \( \tilde{S}_k^{(4)} \) arises from \( M = 0 \) while \( \tilde{S}_k^{(5)} \) arises from \( M \in \mathfrak{N}^{(4)} \).
3.5 Summary of contributions to the constant Fourier mode

Collecting the contributions obtained in (3.20), (3.26) and (3.29), we find that the constant Fourier mode is given by the sum of six terms,

\[ C_{a_1, a_2, a_3}(\tau_2) = \sum_{i=0}^{5} C_{0}^{(i)}(\tau_2) \]  

(3.30)

each of which is given by the following integrals over \( t_1, t_2, t_3 \) of the corresponding functions \( \tilde{S}_{0}^{(i)} \) evaluated in the previous section,

\[ C_{0}^{(i)}(\tau_2) = \left( \prod_{r=1}^{3} \frac{1}{\Gamma(a_r)} \int_0^\infty dt_r t_r^{a_r-1} \right) \tilde{S}_{0}^{(i)}(t_1, t_2, t_3|\tau_2) \]  

(3.31)

Each integrand uniformly decays to zero exponentially fast in any direction as \( t_r \to \infty \).

When no confusion is expected to arise, we shall often suppress the dependence on the parameters \( a_1, a_2, a_3 \) to save notation.

4 The Laurent polynomial

In this section, we shall obtain the Laurent polynomial \( \mathcal{L}(\tau_2) \) in the constant Fourier mode \( C_{a_1, a_2, a_3}(\tau_2) \) of the modular functions \( C_{a_1, a_2, a_3}(\tau) \), and prove Theorem 1.1. To this end, we evaluate the contributions \( C_{0}^{(i)}(\tau_2) \) for \( i = 0, 1, \ldots, 5 \) in the subsections below. The remaining exponential contributions \( \mathcal{E}(\tau_2) \) to the constant Fourier mode will be evaluated in the subsequent section.

4.1 Evaluating \( C_{0}^{(0)} \)

The integral over \( S_{0}^{(0)} \) evaluates to a sum over \( M \in \mathcal{M}^{(4)} \) which may be parametrized by,

\[ C_{0}^{(0)}(\tau_2) = \frac{\tau_2^{w}}{\pi^{w}} \sum_{m_1 \neq 0} \frac{1}{m_1^{2a_1}} \sum_{m_2 \neq 0, -m_1} \frac{1}{m_2^{2a_2}(m_1 + m_2)^{2a_3}} \]  

(4.1)

To compute the infinite sum over \( m_2 \), we proceed by decomposing the summand into partial fractions in \( m_2 \), using the general partial fraction decomposition formulas, valid for \( a, b \in \mathbb{N} \),

\[ \frac{1}{(z + x)^a(z + y)^b} = \sum_{k=1}^{a} \frac{A_k(a, b)}{(z + x)^k(y - x)^{a+b-k}} + \sum_{k=1}^{b} \frac{B_k(a, b)}{(z + y)^k(y - x)^{a+b-k}} \]  

(4.2)
where $A_k(a,b)$ and $B_k(a,b)$ are given by binomial coefficients,

$$
A_k(a,b) = (-)^{a+k} \binom{a+b-k-1}{a-k}, \\
B_k(a,b) = (-)^{a} \binom{a+b-k-1}{b-k}
$$

(4.3)

For the case at hand, we set $x = 0$, $y = m_1$, $z = m_2$, for the positive integers exponents $a = 2a_2$ and $b = 2a_3$, and we find,

$$
\sum_{m_2 \neq 0, -m_1} \frac{1}{m_2^{2a_2} (m_1 + m_2)^{2a_3}} = \sum_{k=1}^{a_2} \frac{2\zeta(2k)}{m_1^{2a_2+2a_3-2k}} \left( A_{2k}(2a_2, 2a_3) + B_{2k}(2a_2, 2a_3) \right)
\quad - \frac{1}{m_1^{2a_2+2a_3}} \frac{\Gamma(2a_2 + 2a_3 + 1)}{\Gamma(2a_2 + 1)\Gamma(2a_3 + 1)}
$$

(4.4)

Evaluating next the sum over $m_1$, and expressing the resulting even $\zeta$-values in terms of Bernoulli numbers using,

$$
\zeta(2k) = \frac{1}{2} (2\pi)^{2k} (-)^{k+1} \frac{B_{2k}}{(2k)!}
$$

(4.5)

we find,

$$
C_0^{(0)}(\tau_2) = (-4\pi\tau_2)^w \sum_{k=0}^{a_2} \frac{B_{2k}B_{2w-2k}}{(2k!)(2w-2k)!} \frac{\Gamma(2a_2 + 2a_3 - 2k)}{\Gamma(2a_3)\Gamma(2a_2 - 2k + 1)} + (a_2 \leftrightarrow a_3)
$$

(4.6)

This formula reproduces correctly the top terms previously evaluated in (1.16).

### 4.2 Evaluating $C_0^{(i)}$ for $i = 1, 2, 3$

We evaluate the case $i = 3$, the cases $i = 1, 2$ being obtained by cyclic permutations of $a_1, a_2, a_3$. The integral over $t_3$ in $S_0^{(3)}$ may be readily carried since the dependence of $S_0^{(3)}$ on $t_3$ is entirely contained in $\det T$, whose dependence on $t_3$ is as follows,

$$
\frac{\det T}{t_1 + t_2} = t_3 + \frac{t_1 t_2}{t_1 + t_2}
$$

(4.7)

Carrying out the integral over $t_3$ gives,

$$
C_0^{(3)}(\tau_2) = \sum_{m,n \neq 0} \left( \frac{\tau_2}{\pi m^2} \right)^{a_2} \prod_{r=1}^{2} \int_0^{\infty} dt_r \frac{t_r^{a_r-1}}{\Gamma(a_r)} \sqrt{\frac{\tau_2}{t_1 + t_2}}
\quad \times \exp \left\{ -\pi \tau_2 n^2 (t_1 + t_2) - \frac{m^2}{\tau_2} \frac{t_1 t_2}{t_1 + t_2} \right\}
$$

(4.8)
Parametrizing the integration variables by $t_1 = xt$ and $t_2 = t(1-x)$ with $t \geq 0$ and $0 \leq x \leq 1$, and carrying out the integral in $t$ produces the following result,

$$C_0^{(3)}(\tau_2) = \frac{\tau_2 w}{\pi^w} \sum_{m,n \neq 0} \frac{1}{|m|^{2w-1}} G_{a_1,a_2} \left( \frac{\tau_2 n}{m} \right)$$

where the function $G_{a_1,a_2}(\mu)$ is given by the integral representation,

$$G_{a_1,a_2}(\mu) = \frac{\sqrt{\pi} \Gamma(a_1 + a_2 - \frac{1}{2})}{\Gamma(a_1) \Gamma(a_2)} \int_0^1 dx \frac{x^{a_1-1}(1-x)^{a_2-1}}{(\mu^2 + x(1-x))^{a_1+a_2-\frac{1}{2}}} \quad (4.10)$$

The function $G_{a_1,a_2}(\mu)$ is even in $\mu$, and invariant under interchanging $a_1$ and $a_2$. To evaluate it, we shall make use of the following Lemma.

**Lemma 4.1** The function $G_{a_1,a_2}(\mu)$ defined in (4.10) admits the equivalent representations.

(a) For $\mu \in \mathbb{R}$, and $a_1, a_2 \in \mathbb{C}$ with $\text{Re}(a_1), \text{Re}(a_2) \geq 1$,

$$G_{a_1,a_2}(\mu) = \int_\mathbb{R} du \frac{1}{(u^2 + \mu^2)^{a_1}((u+1)^2 + \mu^2)^{a_2}} \quad (4.11)$$

(b) For $a_1, a_2 \in \mathbb{N}$,

$$G_{a_1,a_2}(\mu) = \left( \sum_{\alpha=0}^{a_1-1} \sum_{\beta=0}^{a_2-1} \frac{-i\pi}{(2i\mu)^{2a_1-1-a-\beta}(1+2i\mu)^{a_2+a-\alpha}} + \text{c.c.} \right) + (a_1 \leftrightarrow a_2) \quad (4.12)$$

The coefficients $g_{a_1,a_2}(\alpha, \beta)$ are given by the product of binomial coefficients of (1.21).

To prove part (a) of Lemma 1, we start from expression (4.11) and use standard techniques for the evaluation of Feynman diagrams in quantum field theory to derive its expression given in (4.10). One makes use of an integral representation formula for a product of denominators,

$$\frac{1}{A_1^n A_2^n} = \frac{\Gamma(a_1 + a_2)}{\Gamma(a_1) \Gamma(a_2)} \int_0^1 dx \frac{x^{a_1-1}(1-x)^{a_2-1}}{(xA_1 + (1-x)A_2)^{a_1+a_2}} \quad (4.13)$$

valid for $A_1, A_2 > 0$ and $\text{Re}(a_1), \text{Re}(a_2) > 0$, and the evaluation of the resulting $u$-integral,

$$\int_\mathbb{R} du \frac{1}{(u^2 + \mu^2)^a} = \frac{\sqrt{\pi} \Gamma(a - \frac{1}{2})}{\Gamma(a) |\mu|^{2a-1}} \quad (4.14)$$

valid for $\text{Re}(a) > \frac{1}{2}$. To prove part (b) of Lemma 1, we make use of the fact that, for $a_1, a_2 \in \mathbb{N}$, the integrand in (4.11) is a rational function of $u$, with poles at $u = \pm i\mu$ and $u = -1 \pm i\mu$. The integral may then be evaluated by standard residue methods.
4.2.1 Calculating the infinite sums for $C_0^{(3)}$

Formula (4.9) expresses $C_0^{(3)}$ as a double sum over the function $G_{a_1,a_2}$. We use the expression for $G_{a_1,a_2}$ obtained in (4.12), interchange the order of the sums over $m, n$ with the sums over $\alpha, \beta$, and combine the factors of $i$ with factors of the absolute values $|m|$,\[
C_0^{(3)} = \frac{\tau_{w}^{w}}{\pi^{w}} \sum_{\alpha=0}^{a-1} \sum_{\beta=0}^{a-1-\alpha} \sum_{m,n \neq 0} (-i)^{m} (-1-2 \pi \nu g_{a_1,a_2}(\alpha, \beta)) + (a_1 \leftrightarrow a_2) \tag{4.15}\]
The role of the complex conjugate contribution is to reverse the sign of $-i|m|$, so that we may omit the absolute value symbol on $m$ and include a factor of 2 to account for the addition of the complex conjugate term. Restricting the sum over $n$ to $n > 0$ gives another factor of 2, and rearranging the factors of $i$, we express $C_0^{(3)}$ as follows,\[
C_0^{(3)} = K(a_1, a_2, a_3) + K(a_2, a_1, a_3) \tag{4.16}\]
where $K$ is given by,\[
K(a_1, a_2, a_3) = \frac{\tau_{w}^{w}}{\pi^{w}} \sum_{\alpha=0}^{a-1} \sum_{\beta=0}^{a-1-\alpha} \sum_{n=1}^{\infty} \sum_{m \neq 0} \frac{-4i \nu g_{a_1,a_2}(\alpha, \beta)}{m^{A}(m + 2 i \tau_{2} n)^{B}(2i \tau_{2} n)^{C}} \tag{4.17}\]
where we have used the following abbreviations,\[
a = a_2 + 2a_3 + \beta \\
b = a_2 + \alpha \\
c = 2a_1 - \alpha - \beta - 1 \tag{4.18}\]
The sum over $m$ may be carried out using the partial fraction decomposition formulas of (4.2) and (4.3) for the parameters $x = 0, y = 2i \tau_{2} n$ with $n > 0, z = m$, and the exponents $a, b$ defined in (4.18), and we find,\[
\sum_{m \neq 0} \frac{1}{m^{a}(m + 2 i \tau_{2} n)^{b}} = \frac{[a/2]}{2} \zeta(2k) A_{2k}(a, b) - \sum_{k=1}^{b} \frac{B_{k}(a, b)}{(2i \tau_{2} n)^{a+b-2k}} - \frac{i \pi B_{1}(a, b)}{(2i \tau_{2} n)^{a+b-1}} + \sum_{k=1}^{b} \frac{B_{k}(a, b)}{(2i \tau_{2} n)^{a+b-2k}} \frac{(-2 \pi i)^{k}}{\Gamma(k)} \sum_{p=1}^{\infty} p^{k-1} e^{-4 \pi p m \tau_{2}} \tag{4.19}\]
The sum over $n$ is carried out by multiplying the above relation by $-4i(2i \tau_{2} n)^{-c}$ with $c$ given in (4.18), and separating the resulting sum into the contributions to the Laurent polynomial part and the contributions to the exponential part,\[
K(a_1, a_2, a_3) = K_{L}(a_1, a_2, a_3) + K_{E}(a_1, a_2, a_3) \tag{4.20}\]
The Laurent polynomial part is given by,

\[ \mathcal{K}_L(a_1, a_2, a_3) = \sum_{a=0}^{a_1-1} \sum_{\beta=0}^{a_1-1} g_{a_1, a_2}(\alpha, \beta) \left[ \frac{\zeta(2w - 2) B_1(a, b)}{(-4\pi \tau_2)^{w-2}} \right. \right.
\]
\[ \left. \quad + \frac{2\zeta(2w - 1)}{(-4\pi \tau_2)^{w-1}} \frac{(-)^{a_2+\beta} \Gamma(2a_2 + 2a_3 + \alpha + \beta)}{\Gamma(a_2 + 2a_3 + \beta + 1) \Gamma(a_2 + \alpha)} \right. \]
\[ \left. \quad - \sum_{k=1}^{a_3 + [(a_2 + \beta)/2]} \frac{4(-)^k \zeta(2k) \zeta(2w - 1 - 2k) A_{2k}(a, b)}{(-4\pi \tau_2)^{w-1-2k} (2\pi)^{2k}} \right] \quad (4.21) \]

while the purely exponential part is given by,

\[ \mathcal{K}_E(a_1, a_2, a_3) = \sum_{a=0}^{a_1-1} \sum_{\beta=0}^{a_1-1} g_{a_1, a_2}(\alpha, \beta) \left[ \frac{2(-)^w \mathcal{B}_k(a, b)}{(4\pi \tau_2)^{w-1-k} \Gamma(k)} \right. \]
\[ \left. \quad \sum_{n=1}^{\infty} n^{k-1} \sigma_{2-2w}(n) q^n \bar{q}^n \right] \quad (4.22) \]

To simplify the purely exponential part, we have used the standard rearrangement formula,

\[ \sum_{n=1}^{\infty} \frac{1}{n^{2w-1-k}} \sum_{p=1}^{\infty} p^{k-1} e^{-4\pi \tau_2^2} = \sum_{n=1}^{\infty} n^{k-1} \sigma_{2-2w}(n) q^n \bar{q}^n \quad (4.23) \]

### 4.3 Evaluating \( C_0^{(4)} \)

We shall use the following integral representation for the factor \( \det T \) in \( \tilde{\mathcal{E}}_0^{(4)} \),

\[ \frac{1}{\tau_2 (\det T)^{\frac{1}{2}}} = \int_{\mathbb{R}^2} du dv e^{-\pi \tau_2 (t_1 u^2 + t_2 v^2 + t_3 u v)^2} \quad (4.24) \]

in order to decouple the \( t \)-integrals in (3.31) for this function, and we obtain,

\[ C_0^{(4)} = \frac{\tau_2^2}{(\pi \tau_2)^w} \sum_{N \in \mathcal{N}^{(4)}} \int_{\mathbb{R}^2} \frac{du dv}{(u^2 + n_1^2)^{a_1} (v^2 + n_2^2)^{a_2} (u + v)^2 + n_3^2} \quad (4.25) \]

The integral is independent of \( \tau_2 \) so that \( C_0^{(4)} \) contributes exclusively to the order \( \tau_2^{2-w} \) in the Laurent polynomial. To evaluate the integral over \( \mathbb{R}^2 \) and the summation over \( \mathcal{N}^{(4)} \), we shall proceed as follows.

We begin by simplifying the summation over \( \mathcal{N}^{(4)} \). Invariance of the set \( \mathcal{N}^{(4)} \) under permutations of \( n_1, n_2, n_3 \) guarantees invariance of \( C_0^{(4)} \) under permutations of \( a_1, a_2, a_3 \). We partition \( \mathcal{N}^{(4)} \) into three disjoint subsets, \( n_1 n_2 > 0, n_2 n_3 > 0 \) and \( n_1 n_3 > 0 \), and we may restrict the summation over \( \mathcal{N}^{(4)} \) to any single one of these subsets provided we add the contribution of the two cyclic permutations of \( a_1, a_2, a_3 \). We shall choose the subset
The integrals over plane or all in the lower half plane. Hence only the integrals in the direct terms contribute, after performing the shift $u,v$ after including a factor of 2.

Since the exponents $a_1, a_2, a_3$ are positive integers, the integrals are over a rational function $f(u,v)$, and may be evaluated by residue methods. Taking the above preparations into account, we obtain the following expression,

$$C_0^{(4)} = \frac{2\tau_2^2}{(\pi \tau_2)^w} \sum_{n_1,n_3=1}^{\infty} \int_{\mathbb{R}^2} du \, dv \, |f(u,v)|^2 + 2 \text{ cyclic permutations of } a_1, a_2, a_3$$

$$f(u,v) = \frac{1}{(u + in_1)^{a_1} (v + in_2)^{a_2} (u + v + in_1 + in_2)^{a_3}}$$

(4.26)

The function $f$ is not unique and is chosen such that the last denominator argument is the sum of the preceding two. This choice guarantees that $f(u,v)$ will have a convenient partial fraction expansion in the variable $u$, the integral over which we shall carry out first,

$$f(u,v) = \sum_{k_1=1}^{a_1} \frac{(-)^{a_1-k_1}}{(u + in_1)^{k_1}} \frac{(a_1+a_3-k_1-1)}{a_3-1} \frac{(u + v + in_1 + in_2)^{w-k_1}}{a_1-1}$$

$$+ \sum_{k_3=1}^{a_3} \frac{(-)^{a_1+a_3-k_3-1}}{(u + v - in_3)^{k_3}} \frac{(u + v + in_2)^{w-k_3}}{(u + v - in_3)^{k_3}}$$

(4.27)

Because the sum is restricted to $n_1, n_3 > 0$, the $u$-integrals in the cross terms of the product $f(u,v)$ vanish identically since in each case the poles are all either in the upper half plane or all in the lower half plane. Hence only the integrals in the direct terms contribute,

$$J_1(n_1,n_3) = \frac{2^{2w}}{16\pi^2} \int_{\mathbb{R}^2} (-)^{k_1+\ell_1} \frac{du \, dv}{(u + in_1)^{k_1} (v - in_1)^{\ell_1} (v + in_2)^{w-k_1} (v - in_2)^{w-\ell_1}}$$

$$J_2(n_1,n_3) = \frac{2^{2w}}{16\pi^2} \int_{\mathbb{R}^2} (-)^{k_3+\ell_3} \frac{du \, dv}{(u + v - in_3)^{k_3} (u + v + in_3)^{\ell_3} (v + in_2)^{w-k_3} (v - in_2)^{w-\ell_3}}$$

(4.28)

In terms of these integrals, $C_0^{(4)}$ is given by,

$$C_0^{(4)} = \frac{2}{(4\pi \tau_2)^{w-2}} \sum_{k_1=1}^{a_1} \sum_{\ell_1=1}^{a_1} \left( a_1 + a_3 - k_1 - 1 \right) \left( a_1 + a_3 - \ell_1 - 1 \right) \frac{(u + v + in_1 + in_2)^{w-k_1}}{(u + v - in_3)^{k_3}} \frac{(u + v + in_2)^{w-k_3}}{(u + v - in_3)^{k_3}} + 2 \text{ cyclic permutations of } a_1, a_2, a_3$$

(4.29)

The integrals over $u$ and $v$ are manifestly decoupled from another in $J_1$, as well as in $J_2$ after performing the shift $u \rightarrow u - v$. Using residue methods, the integrals evaluate to,

$$J_1(n_1,n_3) = \frac{k_1 + \ell_1 - 2}{k_1 - 1} \left( \frac{2w - k_1 - \ell_1 - 2}{w - k_1 - 1} \right) \frac{\theta(-n_1 n_2)}{|n_1|^{k_1+\ell_1-1}} \frac{\theta(-n_1 n_2)}{|n_1|^{2w-k_1-\ell_1-1}}$$

$$J_2(n_1,n_3) = \frac{k_3 + \ell_3 - 2}{k_3 - 1} \left( \frac{2w - k_3 - \ell_3 - 2}{w - k_3 - 1} \right) \frac{\theta(-n_2 n_3)}{|n_2|^{k_3+\ell_3-1}} \frac{\theta(-n_2 n_3)}{|n_2|^{2w-k_3-\ell_3-1}}$$

(4.30)
The contributions of the integrals \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) to \( C_0^{(4)} \) are related by permuting \( a_1 \) and \( a_3 \). Hence we may retain only the sum involving \( \mathcal{I}_1 \) provided we then include all five permutations of \( a_1, a_2, a_3 \). Putting all together we have,

\[
C_0^{(4)} = \frac{2}{(4\pi^2)^w} \sum_{\sigma \in S_3} Z(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})
\]

where the function \( Z(a_1, a_2, a_3) \) was defined in (1.24).

4.4 Evaluating \( C_0^{(5)} \)

Combining the second equation of (3.29) with (3.31) we see that \( C_0^{(5)} \) is given by,

\[
C_0^{(5)} = \prod_{r=1}^{3} \int_0^\infty dt_r \frac{T_r^{a_r-1}}{\Gamma(a_r)} \sum_{M,N \in \mathcal{A}(4)} \frac{\tau_2 \delta_{M' \in N,0} (\text{det} T)^2}{(\text{det} T)^2} \exp \left\{ -\pi \tau_2 M'TM - \pi \tau_2 N'TN \right\}
\]

We begin by parametrizing the space of matrices \( M, N \). To satisfy the condition \( M' \in N = 0 \), the column matrices \( M, N \) must be proportional to one another, and thus proportional to a common matrix \( K \) with integer entries \( k_1, k_2 \). Since \( M, N \in \mathcal{A}(4) \), the numbers \( m_i, n_i, k_i \) with \( i = 1, 2, 3 \) are also non-vanishing. The complete solution is given by,

\[
M = \mu K \quad \quad N = \nu K \quad \quad K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}
\]

where we choose \( k_1 \) and \( k_2 \) relatively prime, \( k_1 > 0 \), and \( \mu, \nu \in \mathbb{Z} \) with \( \mu, \nu \neq 0 \). Since the summand in \( C_0^{(5)} \) depends only on \( \mu^2, \nu^2, k_1^2, k_2^2 \), we may include in the sum over \( k_1 \) and \( k_2 \) pairs with arbitrary signs along with an overall factor of \( \frac{1}{2} \), and restrict \( \mu, \nu \) to be positive upon including an overall factor of 4. Finally, we may choose a particular ordering of \( k_1^2, k_2^2, k_3^2 \) upon including symmetrization under all six permutations of the variables \( a_1, a_2, a_3 \). We shall denote the space of such pairs \( k_1, k_2 \) of relatively prime, ordered, integers by \( \mathcal{R} \).

Next, we change integration variables from \( (t_1, t_2, t_3) \) to \( (x_1, x_2, t_3) \) with \( t_1 = x_1 t_3 \) and \( t_2 = x_2 t_3 \) for \( x_1, x_2 \geq 0 \), and subsequently change variables from \( (x_1, x_2, t_3) \) to \( (x_1, x_2, t) \) with,

\[
t_3 = \left| \frac{\mu}{\nu} \right| \frac{t}{\sqrt{x_1 + x_2 + x_1 x_2}}
\]

In terms of these new variables, the integrals become,

\[
C_0^{(5)} = 2\tau_2 \sum_{(k_1,k_2) \in \mathcal{R}} \sum_{\mu,\nu=1}^\infty \left( \frac{\mu}{\nu} \right)^{w-1} \int_0^\infty dt \frac{t^{w-2}}{\Gamma(a_3)} \prod_{r=1}^{2} \int_0^\infty dx_r \frac{x_r^{a_r-1}}{\Gamma(a_r)} \frac{1}{(x_1 + x_2 + x_1 x_2)^{w/2}}
\]

\[
\times \exp \left\{ -\pi \tau_2 \mu \nu \left( t + \frac{1}{t} \right) \frac{x_1 k_1^2 + x_2 k_2^2 + k_3^2}{\sqrt{x_1 + x_2 + x_1 x_2}} \right\} + 5 \text{ perms of } a_1, a_2, a_3
\]

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The Mellin transform of $C^{(5)}_0$ with respect to $\tau_2$ is defined by,

$$\mathcal{M}C^{(5)}_0(s) = \int_0^\infty d\tau_2 \tau_2^{s-1}C^{(5)}_0(\tau_2)$$  \hspace{1cm} (4.36)

Carrying out the integration over $t$, and performing the sums over $\mu$ and $\nu$ we obtain,

$$\mathcal{M}C^{(5)}_0(s) = \xi(s+w)\xi(s-w+2)\sum_{(k_1,k_2)\in\mathcal{R}} \prod_{r=1}^2 \int_0^\infty dx_r x_r^{a_r-1} \frac{(x_1+x_2+x_1x_2)^{s-w+2}}{(x_1k_1^2+x_2k_2^2+k_3^2)^{s+1}}$$  \hspace{1cm} (4.37)

The function $\xi(s) = \Gamma(s/2)\zeta(s)$ has single poles at $s = 0$ and $s = 1$ and is analytic elsewhere. We now prove the following Lemma.

**Lemma 4.2** Near the cusp $\tau_2 \to \infty$, the function $C^{(5)}_0$ decays exponentially,

$$C^{(5)}_0(\tau_2) = \mathcal{O}(e^{-4\pi\tau_2})$$  \hspace{1cm} (4.38)

up to factors which are power-behaved in $\tau_2$, controlled by the poles in the Mellin transform.

To prove the Lemma, we shall extract out of the integrand a factor of $e^{-4\pi\tau_2}$. First, we have the standard sharp bound $t + t^{-1} \geq 2$. Secondly, we shall show that the last factor of the exponential has a uniform sharp lower bound by $2k_1k_2$. To prove it, we set $y_1 = x_1 + 1$ and $y_2 = x_2 + 1$, with $y_1, y_2 \geq 1$, in terms of which the expression becomes,

$$\frac{x_1k_1^2 + x_2k_2^2 + k_3^2}{\sqrt{x_1+x_2+x_1x_2}} = \frac{y_1k_1^2 + y_2k_2^2 + 2k_1k_2}{\sqrt{y_1y_2-1}}$$  \hspace{1cm} (4.39)

Since $(k_1,k_2) \in \mathcal{R}$ we have $k_1, k_2 > 0$. Keeping the product $y_1y_2$ fixed, the combination $y_1k_1^2 + y_2k_2^2$ is bounded from below by $2k_1k_2\sqrt{y_1y_2}$. The resulting lower bound equals $2k_1k_2$ times a function of $y_1y_2$ which is independent of $k_1, k_2$ and whose lower bound is 1. Thus, for $(k_1,k_2) \in \mathcal{R}$, we may extract out of the integrand a factor of $e^{-4\pi\tau_2\mu k_1k_2}$. The Lemma follows from $\mu, \nu, k_1k_2 \geq 1$, and the observation that the remaining integrations converge. Their power behavior is governed by the positions of the poles of the Mellin transform.

### 4.5 Proof of Theorem 1.1

Having shown that $C^{(5)}_0$ does not contribute to the Laurent polynomial $\mathcal{L}$, Theorem 1.1 may now be proven by collecting the contributions to $\mathcal{L}$ from the calculations of $C^{(i)}_0$ for $i = 0, 1, 2, 3, 4$ performed in the preceding subsections. From the explicit results in (4.6), (4.16), (4.21) and (4.31), we deduce the following results.

1. $c_w$ receives contributions exclusively from $C^{(0)}_0$ which was evaluated explicitly in (4.6).

The coefficient $c_w$ given by (4.6) is manifestly a rational number. This proves part (a).
2. $c_{w-2k-1}$ for $1 \leq k \leq w - 1$ receives contributions exclusively from $C^{(i)}_{0}$ with $i = 1, 2, 3$, obtained from the second and third terms under the sum in (4.21). The coefficients $c_{w-2k-1}$ given by (4.6) are manifestly rational numbers. This proves part (b).

3. $c_{2-w}$ receives contributions exclusively from the first term under the sum in (4.21) which gives rise to the term in $\zeta(2w - 2)$ in (1.22), as well as from the entire contribution of $C^{(4)}_{0}$ in (4.31). Putting both together proves part (c).

5 Differential equations and exponential terms

For given weight $w = a_1 + a_2 + a_3$ the functions $C_{a_1,a_2,a_3}(\tau)$ with $a_1, a_2, a_3 \in \mathbb{N}$ satisfy a system of inhomogeneous linear differential equations whose inhomogeneous part is a linear combination of the non-holomorphic Eisenstein series $E_w$ and products of the form $E_{w-\ell}E_\ell$ with $2 \leq \ell \leq w - 2$. Analyzing the general structure of these differential equations and their solutions provides convenient paths towards proving Theorems 1.2 and 1.3, which we shall carry out in the present section.

5.1 Inhomogeneous Laplace-eigenvalue equations

The two-loop modular graph functions satisfy a system of differential equations,

\[
\left( \Delta - \sum_{i=1}^{3} a_i(a_i - 1) \right) C_{a_1,a_2,a_3} = a_1a_2 \left( C_{a_1-1,a_2+1,a_3} + \frac{1}{2} C_{a_1+1,a_2+1,a_3-2} - 2C_{a_1,a_2+1,a_3-1} \right) + 5 \text{ permutations of } a_1, a_2, a_3
\]

valid for $a_r \geq 3$ for $r = 1, 2, 3$. The validity of these equations may be extended to $a_r \geq 1$ by supplementing them with the following degenerate cases,

\[
C_{a_1,a_2,0} = E_{a_1} E_{a_2} - E_{a_1+a_2} \quad a_1 + a_2 \geq 3
\]
\[
C_{a_1,a_2,-1} = E_{a_1-1} E_{a_2} + E_{a_1} E_{a_2-1} \quad a_1, a_2 \geq 2
\]

The right side of (5.2) may involve the symbol $E_1$ (formally corresponding to a divergent series), but its contribution systematically cancel out of the right side of (5.1). For example, the lowest weight cases are as follows,

\[
\Delta C_{1,1,1} = 6 E_3
\]
\[
\Delta C_{2,1,1} = 2 C_{2,1,1} + 9 E_4 - E_2^2
\]
\[
\Delta C_{2,2,1} = 8 E_5
\]

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The Laplacian preserves the weight \( w = a + b + c \) of the modular graph functions, and the equations of (5.1) and (5.2) may be viewed as acting on the space of modular graph functions of given weight \( w \), provided we assign the weight \( s \) to the Eisenstein series \( E_s \).

The Fourier series for a non-holomorphic Eisenstein series \( E_n(\tau) \) with integer \( n \geq 2 \), was given in (1.13). It was shown in [3] that the Laplace operator \( \Delta = 4\tau^2 \partial_\tau \partial_{\bar{\tau}} \) on the space of functions \( C_{a_1,a_2,a_3}(\tau) \) for weight \( w = a_1 + a_2 + a_3 \geq 3 \) may be “diagonalized” resulting in “eigenfunctions” \( C_{w,s,p}(\tau) \) which are linear combinations of the functions \( C_{a_1,a_2,a_3}(\tau) \) of weight \( w \) which obey the following type of equation,

\[
(\Delta - s(s-1))C_{w,s,p}(\tau) = \mathcal{H}_0(w,s,p)E_w(\tau) + \sum_{\ell=2}^{w-2} \mathcal{H}_\ell(w,s,p)E_{w-\ell}(\tau)E_\ell(\tau)
\]

(5.4)

where \( s \) and \( p \) are integers running over the following ranges,

\[
s = w - 2m \quad 1 \leq m \leq \left[ \frac{w-1}{2} \right] \quad 0 \leq p \leq \left[ \frac{s-1}{3} \right]
\]

(5.5)

The coefficients \( \mathcal{H}_\ell \) are combinatorial rational numbers which depend on \( a, b, c, s, p, \ell \). Decomposing \( C_{w,s,p}(\tau) \) into Fourier modes,

\[
C_{w,s,p}(\tau) = \sum_{k \in \mathbb{Z}} C_{w,s,p}^{(k)}(\tau_2) e^{2\pi i k \tau_1}
\]

(5.6)

each Fourier mode satisfies a separate ODE in \( \tau_2 \),

\[
\left( \tau_2^2 \partial_{\tau_2}^2 - 4\pi^2 k^2 \tau_2^2 - s(s-1) \right) C_{w,s,p}^{(k)}(\tau_2) = \mathcal{H}_0(w,s,p) \int_0^1 d\tau_1 e^{-2\pi i k \tau_1} E_w(\tau) + \sum_{\ell=2}^{w-2} \mathcal{H}_\ell(w,s,p) \int_0^1 d\tau_1 e^{-2\pi i k \tau_1} E_{w-\ell}(\tau)E_\ell(\tau)
\]

(5.7)

5.2 Proof of Theorem 1.2

To prove Theorem 1.2, we concentrate on the constant Fourier mode for \( k = 0 \) and in particular on its contribution proportional to the functional behavior \( c_{2-w}(4\pi \tau_2)^{2-w} \). The Fourier transform on \( E_w \) makes vanishing contribution to this power of \( \tau_2 \), and the contributions of the bilinears \( E_{w-\ell}E_\ell \) are readily evaluated using (1.13), and we find,

\[
(w - s - 1)(w + s - 2)c_{2-w} = \sum_{\ell=2}^{w-2} 16 \mathcal{H}_\ell(w,s,p) \binom{2w-2\ell-3}{w-\ell-1} \binom{2\ell-3}{\ell-1} \zeta(2w-2\ell-1)\zeta(2\ell-1)
\]

(5.8)

Given that the allowed values of \( s \) in the spectral decomposition of the functions \( C_{a_1,a_2,a_3} \) onto the basis of eigenfunctions \( C_{w;s,p} \) is given by (5.5), and that this decomposition is
with rational coefficients, it is clear that the kernel of the operator on the left side of the
above equation must vanish. Therefore, \( c_{2-w} \) must be a linear combination, with rational
coefficients, of products of pairs of odd \( \zeta \)-values whose weights sum to \( 2w - 2 \), thus proving
Theorem 1.2. In particular, any contributions valued in \( \pi^{2w-2}Q \) must vanish.

5.3 Proof of Theorem 1.3

We shall now concentrate on the constant Fourier mode \( C_{w,s,p}^{(0)}(\tau_2) \). Having already determined
the Laurent polynomial contribution to \( C_{a_1,a_2,a_3}(\tau) \) at the cusp we shall consider here its
exponential contribution, which we denote by \( C_{w,s,p}^{(e)}(\tau_2) \). The contribution to \( C_{w,s,p}^{(e)}(\tau_2) \) from
the function \( E_w \) vanishes, so we are left with the contributions from the products
\( E_{w-\ell}E_{\ell} \) with \( 2 \leq \ell \leq w - 2 \), which obeys the following differential equation,

\[
\left( \tau_2^2 \partial^2_{\tau_2} - s(s-1) \right) C_{w,s,p}^{(e)}(\tau_2) = \sum_{\ell=2}^{w-2} \mathcal{H}_\ell(w,s,p) \int_0^1 d\tau_1 E_{w-\ell}(\tau) E_{\ell}(\tau) \bigg|_{\exp} \tag{5.9}
\]

The right side is readily evaluated and we find,

\[
\left( \tau_2^2 \partial^2_{\tau_2} - s(s-1) \right) C_{w,s,p}^{(e)}(\tau_2) = \sum_{\ell=2}^{w-2} \mathcal{H}_\ell(w,s,p) \left( \sum_{n=1}^{\infty} \frac{\sigma_{1-2w+2\ell}(n)\sigma_{1-2\ell}(n)}{(w-\ell+1)!\ell!(\ell-1)!} \right. \\
\left. \times 8 n^{w-2} e^{-4\pi \tau_2 n} P_{w-\ell}(4\pi \tau_2 n) P_{\ell}(4\pi \tau_2 n) \right) \tag{5.10}
\]

where the polynomials \( P_n \) are given by (1.14). The general structure of this equation, in
terms of the variable \( y = 4\pi \tau_2 \) is as follows,

\[
\left( y^2 \partial_y^2 - s(s-1) \right) f_s(y) = \sum_{n=1}^{\infty} \sum_{m=0}^{w-2} f_{m,n} \frac{e^{-ny}}{(ny)^m} \tag{5.11}
\]

To solve this equation, we solve for each power of \( m \) on the right side of the above equation,

\[
\left( y^2 \partial_y^2 - s(s-1) \right) \varphi_{s,m}(y) = \frac{e^{-y}}{y^m} \tag{5.12}
\]

so that \( f_s \) is given by,

\[
f_s(y) = \sum_{n=1}^{\infty} \sum_{m=0}^{w-2} f_{m,n} \varphi_{s,m}(ny) \tag{5.13}
\]

The solution for \( \varphi_{s,m}(y) \) is given in terms of the incomplete \( \Gamma \)-function, defined by,

\[
\Gamma(a,x) = \int_x^{\infty} dt t^{a-1} e^{-t} \tag{5.14}
\]
and we find,

$$\varphi_{s,m}(y) = \frac{y^s}{2s-1} \Gamma(-s - m, y) + \frac{y^{1-s}}{1 - 2s} \Gamma(s - 1 - m, y)$$  \hspace{1cm} (5.15)$$

Since we are interested in solutions with exponential decay, we have set the homogeneous part of the solution to zero. Using the recursion relation for the incomplete \(\Gamma\)-function,

$$\Gamma(a + 1, y) = a \Gamma(a, y) + \frac{y}{a} e^{-y}$$  \hspace{1cm} (5.16)$$

we may recast the result solely in terms of elementary functions and the exponential integral \(\text{Ei}_1(y) = \Gamma(0, y)\). Simplifying the \(y^s\) term, we find,

$$\frac{y^s}{2s-1} \Gamma(-s - m, y) = \gamma_0 y^s \text{Ei}(y) + \sum_{k=1}^{s+m} \gamma_k y^{s-k} e^{-y}$$  \hspace{1cm} (5.17)$$

for rational coefficients \(\gamma_0, \gamma_k\) whose value will not concern us here. Simplifying the \(y^{1-s}\) term, the cases \(s \geq m + 2\) and \(s \leq m + 1\) must be distinguished, and we have,

$$s \leq m + 1 \quad \frac{y^{1-s}}{1 - 2s} \Gamma(s - 1 - m, y) = \gamma_1 y^{1-s} \text{Ei}(y) + \sum_{k=1}^{1-s+m} \tilde{\gamma}_k y^{1-s+k} e^{-y}$$

$$s \geq m + 1 \quad \frac{y^{1-s}}{1 - 2s} \Gamma(s - 1 - m, y) = \sum_{k=0}^{s-m-2} \tilde{\gamma}_k y^{1-s+k} e^{-y}$$  \hspace{1cm} (5.18)$$

Thus, the solution \(\varphi_{s,m}(y)\) takes the form,

$$\varphi_{s,m}(y) = \gamma_0 y^s \text{Ei}(y) + \gamma_1 y^{1-s} \text{Ei}(y) + \sum_{k=k_-}^{s-1} \gamma_k y^k e^{-y}$$  \hspace{1cm} (5.19)$$

where \(k_- = \min(-m, 1-s)\) and the constant \(\gamma_1\) vanishes whenever \(s \geq m + 1\). Summing now over the range of \(m\) in (5.8), and taking into account that \(s \leq w - 2\), we obtain,

$$f_s(y) = f_+ y^s \text{Ei}(y) + f_- y^{1-s} \text{Ei}(y) + \sum_{k=2-w}^{w-3} f_k y^k e^{-y}$$  \hspace{1cm} (5.20)$$

When \(s = w - 2\) then we have \(m \leq s\) and we must have \(f_- = 0\). In other cases, \(f_\pm\) will generically be non-zero. This completes the proof of Theorem 1.3.
5.4 The example of $C_{2,1,1}$

We may work things out explicitly for the simplest case where the right side of the differential equation has a term which is non-linear in the Eisenstein series, and satisfies,

$$(\Delta - 2)C_{2,1,1} = 9E_4 - E_2^2$$  \hspace{1cm} (5.21)$$

The reduced differential equation for the exponential part of the constant Fourier mode is,

$$\left(\tau_2^2 \partial^2_\tau - 2\right)C_{2,1,1}^{(e)}(\tau_2) = -\frac{4}{3} \sum\limits_{n=1}^{\infty} n^2 \sigma_{-3}(n)^2 e^{-4\pi\tau_2 n} P_2(4\pi\tau_2 n)^2$$  \hspace{1cm} (5.22)$$

where the polynomials $P_n$ were defined in (1.14). Collecting the solution to the homogeneous part, and then summing over the particular solutions for each order in $N$ of the inhomogeneous part we find,

$$C_{2,1,1}^{(e)}(\tau_2) = -\frac{4}{3} \sum\limits_{n=1}^{\infty} n^2 \sigma_{-3}(n)^2 \varphi(4\pi\tau_2 n)$$  \hspace{1cm} (5.23)$$

where the function $\varphi$ satisfies the $n$-independent differential equation,

$$\left(y^2 \partial^2_y - 2\right)\varphi(y) = e^{-y} P_2(y)^2$$  \hspace{1cm} (5.24)$$

whose solution is as follows,

$$\varphi(y) = \frac{e^{-y}}{y^2}$$  \hspace{1cm} (5.25)$$

This result is consistent with the result of Theorem 1.3 for $s = 2$.

6 The conjectured decomposition formula

Theorem 1.2 states that the coefficient $c_{2-w}$ is given by a linear combination of products of pairs of odd $\zeta$-values whose weights add up to $2w - 2$, but the theorem does not provide the coefficients $\gamma_\ell$ in (1.26). In this section we shall obtain an explicit formula for $\gamma_\ell$ in the decomposition of $c_{2-w}$. Using Maple, we have accumulated extensive numerical evidence for the validity of this formula, but we have no complete analytical proof. The decomposition formula reproduces all the special cases obtained in earlier work and given in (1.16). To develop the required decomposition formula, we begin by proving the following Lemma.
Lemma 6.1 The linear combinations of depth-two $\zeta$-values,

\[ S(M,N) = \zeta(2M - 1, 2N + 1) + \sum_{\ell=0}^{N-1} \frac{\varphi_\ell(M, N) \Gamma(2M + 2\ell)}{(2\ell + 2)! \Gamma(2M - 1)} \zeta(2M + 2\ell, 2N - 2\ell) \]

\[ T(M,N) = \zeta(2N + 1, 2M - 1) + \sum_{\ell=0}^{N-1} \frac{\varphi_\ell(M, N) \Gamma(2M + 2\ell)}{(2\ell + 2)! \Gamma(2M - 1)} \zeta(2N - 2\ell, 2M + 2\ell) \] (6.1)

obey the following relations.

(a) For $N = 0$, the function $S(M,0)$ is given by,

\[ S(M,0) = \frac{1}{2}(2M - 1)\zeta(2M) - \frac{1}{2} \sum_{j=1}^{2M-3} \zeta(j + 1)\zeta(2M - 1 - j) \] (6.2)

(b) For $N \geq 1$, the functions $S(M,N)$ and $T(M,N)$ are related as follows,

\[ S(M,N) = -T(M,N) + \zeta(2M - 1)\zeta(2N + 1) - \zeta(2M + 2N) \]

\[ + \sum_{\ell=0}^{N-1} \frac{\varphi_\ell(M, N) \Gamma(2M + 2\ell)}{(2\ell + 2)! \Gamma(2M - 1)} \left( \zeta(2M + 2\ell)\zeta(2N - 2\ell) - \zeta(2M + 2N) \right) \] (6.3)

(c) The functions $S(M,N)$ and $T(M,N)$ reduce to a linear combination of products of two odd $\zeta$-values, plus a term valued in $\pi^{2w-2}Q$, provided the coefficients $\varphi_\ell(M, N)$ are given in terms of the Euler polynomial $E_{2\ell+1}$ by,

\[ \varphi_\ell(M, N) = -(2\ell + 2)E_{2\ell+1}(0) \] (6.4)

The coefficients $\varphi_\ell$ are integer-valued.

(d) For $N \geq 1$, the function $T(M,N)$ then evaluates as follows,

\[ T(M,N) = \sum_{\alpha=2}^{2M+2N-2} (-)^{\alpha+1} \zeta(\alpha)\zeta(2M + 2N - \alpha) \]

\[ \times \sum_{n=0}^{2N-1} \frac{1}{2} E_n(0) \left( \frac{\alpha - 1}{2N - n} \right) \left( \frac{2M + n - 2}{n} \right) \] (6.5)

While there have been many investigations into the decomposition of linear combinations of double $\zeta$-values onto products of single $\zeta$-values, such as for example in [23, 24, 25], the authors have not been able to find the precise statements of Lemma 6.1 in the literature.\footnote{We thank the referee for pointing out that the fact that the sum is totally reducible to a polynomial in the odd weight zeta values shows that the coefficient is a single-valued multiple zeta value.}
6.1 Proof of Lemma 6.1

The proof of part (a) follows from a well-known formula of Euler, valid for integer \( s \geq 2 \),

\[
\zeta(s, 1) = \frac{s}{2} \zeta(s + 1) - \frac{1}{2} \sum_{j=1}^{s-2} \zeta(j+1)\zeta(s-j) \tag{6.6}
\]

The proof of part (b) is obtained by adding the expressions for \( S(M, N) \) and \( T(M, N) \) of (6.1) and using Euler’s reflection formula, valid for \( s, t \in \mathbb{C} \),

\[
\zeta(s, t) + \zeta(t, s) = \zeta(s)\zeta(t) - \zeta(s + t) \tag{6.7}
\]

The proof of part (c) is more involved. Since the sum \( S(M, N) + T(M, N) \) reduces to the product of two odd \( \zeta \)-values plus a term valued in \( \pi^{2\omega-2}\mathbb{Q} \) by part (b), it will suffice to prove reducibility for \( T(M, N) \). To do so we express the depth-two \( \zeta \)-functions in the definition of \( T(M, N) \) in (6.1) in terms of a double sum using (1.25),

\[
T(M, N) = \frac{1}{2} \sum_{m,n=1}^{\infty} F(m, n) \tag{6.8}
\]

The function \( F(m, n) \) is chosen to be symmetric in \( m, n \), and is given by,

\[
F(m, n) = \frac{1}{(m+n)^{2N+1}m^{2M-1} + (m+n)^{2N+1}n^{2M-1}} + \sum_{\ell=0}^{N-1} \varphi_{\ell}(M, N) \frac{1}{(2\ell + 2)! \Gamma(2M-1)} \left( \frac{1}{(m+n)^{2N-2\ell}m^{2M+2\ell}} + \frac{1}{(m+n)^{2N-2\ell}n^{2M+2\ell}} \right) \tag{6.9}
\]

Next, we obtain the conditions on the coefficients \( \varphi_{\ell}(M, N) \) required for reducibility of \( T(M, N) \), and show that \( \varphi_{\ell}(M, N) \) is independent of \( M, N \). To do so, we proceed by partial fraction decomposition of \( F \) in the variable \( m \) (or equivalently in \( n \)). Homogeneity of \( F \) in the variables \( m, n \) of degree \( 2M + 2N \) restricts the decomposition to the following form,

\[
F(m, n) = \sum_{\alpha=0}^{2N} \tilde{f}_\alpha(M, N) \frac{1}{(m+n)^{2N+1-\alpha}n^{2M+\alpha-1}} + \sum_{\alpha=1}^{2M+2N-2} f_\alpha(M, N) \frac{1}{m^{\alpha} n^{2M+2N-\alpha}} \tag{6.10}
\]

We determine the coefficients \( \varphi_{\ell}(M, N) \) by requiring \( \tilde{f}_\alpha(M, N) = 0 \) for all \( 0 \leq \alpha \leq 2N \), namely the absence of double \( \zeta \)-values in the sum. Although the system appears over-determined with \( 2N + 1 \) conditions on \( N \) variables \( \varphi_{\ell} \) with \( 0 \leq \ell \leq N - 1 \), the symmetry in \( m, n \) guarantees that it is uniquely solvable. To set \( \tilde{f}_\alpha(M, N) = 0 \), we require that
\((m + n)^{2N+1}F(m, n)\) and its \(2N\) derivatives in \(m\) vanish at \(m = -n\). The vanishing of \(\tilde{f}_0(M, N)\) is automatic. For \(1 \leq \alpha \leq 2N\), the condition \(\tilde{f}_\alpha = 0\) is given by,

\[
\sum_{\ell=0}^{N-1} \frac{\varphi_{\ell}(M, N)}{2\ell + 2} \left( \delta_{2\ell+1, \alpha} + \left( \frac{\alpha}{2\ell + 1} \right) \right) = 1
\]  

(6.11)

To solve these conditions, we proceed as follows. The sum over \(\ell\) receives no contributions from \(2\ell + 1 > \alpha\). Therefore, the upper range of the sums may be replaced by \([\alpha/2]\), and the conditions on \(\varphi_{\ell}(M, N)\) are now independent of \(M\) and \(N\), and so are their solutions. Separating the equations for even and odd \(\alpha\), we set respectively \(\alpha = 2p + 1\) and \(\alpha = 2p + 2\). The relations determining \(\varphi_{\ell} = \varphi_{\ell}(M, N)\) are thus given as follows,

\[
\sum_{\ell=0}^{p} \frac{\varphi_{\ell}}{2\ell + 2} \left( \delta_{\ell, p} + \left( \frac{2p + 1}{2\ell + 1} \right) \right) = \sum_{\ell=0}^{p} \frac{\varphi_{\ell}}{2\ell + 2} \left( \frac{2p + 2}{2\ell + 1} \right) = 1
\]  

(6.12)

These equations are equivalent to one another. We solve the second equation by multiplying by \(x^{2p+1}/\Gamma(2p+2)\) and summing over all \(p \geq 0\). The sum over \(p\) may be carried out explicitly in terms of hyperbolic functions. Introducing the generating function,

\[
\varphi(x) = \sum_{\ell=0}^{\infty} \frac{\varphi_{\ell} x^{2\ell}}{\Gamma(2\ell + 3)}
\]  

(6.13)

it is found that \(\varphi(x)\) must satisfy \(\varphi(0) = \frac{1}{2}\) and the following linear differential equation,

\[
1 = x\varphi'(x) + \left( 1 + \frac{x}{\tanh(x)} \right) \varphi(x)
\]  

(6.14)

Its integral is given by \(\varphi(x) = (e^x - 1)x^{-1}(e^x + 1)^{-1}\). In terms of the Euler polynomials \(E_n(t)\) which are defined by,

\[
\frac{2e^{xt}}{e^x + 1} = \sum_{n=0}^{\infty} E_n(t) \frac{x^n}{n!}
\]  

(6.15)

we find the explicit expression \(\varphi_{\ell} = -(2\ell + 2)E_{2\ell+1}(0).\) This completes the proof of part (c). While \(E_{2\ell+1}(0)\) are generally rational numbers, the coefficients \(\varphi_{\ell}\) are integers.

To prove part (d), we determine the coefficients \(f_{\alpha}(M, N)\) in (6.10). Using the expression we have obtained for \(\varphi_{\ell}\) in (6.4), we readily find \(f_1(M, N) = 0\). Furthermore, symmetry of \(F\) in \(m, n\) implies the reflection property, \(f_{2M+2N-\alpha}(M, N) = f_{\alpha}(M, N)\). Combining (6.8) and (6.10) with \(\tilde{f}_\alpha(M, N) = 0\), we obtain \(T(M, N)\) by summing over \(m, n\),

\[
T(M, N) = \sum_{\alpha=2}^{2M+2N-2} \frac{1}{2f_{\alpha}(M, N)} \zeta(\alpha) \zeta(2M + 2N - \alpha)
\]  

(6.16)
To compute the expressions for the remaining coefficients \( f_\alpha(M, N) \), we match the poles in \( n \) at \( n = 0 \) between the expressions (6.9) and (6.10) with \( \tilde{f}_\alpha(M, N) = 0 \), and we obtain,

\[
f_\alpha(M, N) = (-)^{\alpha+1} \sum_{n=0}^{2N-1} E_n(0) \left( \frac{\alpha - 1}{2N - n} \right) \left( 2M + n - 2 \right)
\] (6.17)

Note that \( E_0(0) = 1 \), and \( E_{2\ell}(0) = 0 \) for \( \ell \in \mathbb{N} \), which simplifies the above sum to ranging only over \( n = 0 \) and the odd positive integers. Substituting the expression for \( f_\alpha(M, N) \) into (6.16) gives (6.5) and thereby proves part (d).

### 6.2 Derivation of the decomposition formula

The starting point for the decomposition formula is the expression for \( c_{2-w} \) in terms of double \( \zeta \)-values, given in (1.22), (1.23) and (1.24) of Theorem 1.1. We change summation variable in the definition of \( Z(a_1, a_2, a_3) \) in (1.24) from \( \ell \) to \( m = k + \ell - 1 \), so that we have,

\[
Z(a_1, a_2, a_3) = \sum_{m=1}^{k_-} \sum_{k=k_-}^{k_+} \left( a_1 + a_2 - k - 1 \right) \left( a_1 + a_2 - m + k - 2 \right) \left( a_2 - 1 \right)
\times \left( \frac{m - 1}{k - 1} \right) \left( \frac{2w - m - 3}{w - k - 1} \right) \zeta(2w - m - 2, m)
\] (6.18)

where \( k_\pm \) are defined by \( k_+ = \min(a_1, m) \) and \( k_- = \max(1, m - a_1 + 1) \). Next, we eliminate the contributions of the odd-odd double \( \zeta \)-values, which arise when \( m \) is odd, in favor of the function \( S \) plus even-even \( \zeta \)-values. The most interesting formula is obtained upon symmetrizing in \( a_2 \) and \( a_3 \), and we find,

\[
Z(a_1, a_2, a_3) + Z(a_1, a_3, a_2) = \sum_{n=0}^{a_1-1} \left( Z_n(a_1, a_2, a_3) + Z_n(a_1, a_3, a_2) \right) S(w - 1 - n, n)
+ \sum_{n=1}^{a_1-1} X_n(a_1, a_2, a_3) \zeta(2w - 2n - 2, 2n)
\] (6.19)

where \( Z_n(a_1, a_2, a_3) \) is given in (1.29), and the function \( X_n(a_1, a_2, a_3) \) is defined by,

\[
X_n(a_1, a_2, a_3) = \sum_{\ell=n}^{a_1-1} E_{2\ell-2n+1}(0) Z_{\ell}(a_1, a_2, a_3) \left( \frac{2w - 2n - 3}{2w - 2\ell - 4} \right)
+ \sum_{k=k_-}^{k_+} \left( a_1 + a_2 - k - 1 \right) \left( a_1 + a_2 - 2n + k - 2 \right) \left( a_2 - 1 \right)
\times \left( \frac{2n - 1}{k - 1} \right) \left( \frac{2w - 2n - 3}{w - k - 1} \right) + (a_2 \leftrightarrow a_3)
\] (6.20)
where $k_+^\prime$ are defined by $k_+^\prime = \min(a_1, 2n)$ and $k_-^\prime = \max(1, 2n - a_1 + 1)$. The first sum on the right arises from the elimination in (1.24) of the odd-odd double $\zeta$-values in favor of $S$, while the second term arises from the contribution of the even-even $\zeta$-values to (1.24).

The symmetrization in $a_2, a_3$ applies to the entire expression. The formula for $X_n$ may be rendered more explicit by expressing $Z_\ell$ as the sum given in (1.29), changing variables $n \to a_1 - n$ and $\ell \to a_1 - \ell$, and recognizing that the second sum may be combined with the first corresponding to the Euler polynomial with index zero, $E_0(0) = 1$, and we find,

$$X_{a_1-n}(a_1, a_2, a_3) = \sum_{k=0}^{a_1-2} \sum_{\ell=0}^{a_1-2} \theta(2n - k - \ell - 1) E_{2n-k-\ell-1}(0) \left(\frac{a_2 - 1 + k}{k} \left(\frac{a_2 - 1 + \ell}{\ell}\right) \right)$$

$$\times \left(\frac{2a_1 - k - \ell - 2}{a_1 - k - 1} \right) \left(\frac{2a_2 + 2a_3 + k + \ell - 2}{a_2 + a_3 + k - 1} \right) \left(\frac{2a_2 + 2a_3 + 2n - 3}{2n - k - \ell - 1}\right)$$

$$+ (a_2 \leftrightarrow a_3) \quad (6.21)$$

Note that the index of the Euler polynomial is now allowed to run over even and odd integers though, of course, we have $E_{2k}(0) = 1$ for all $k \geq 1$.

**Conjecture 6.2** For $a_1, a_2, a_3 \in \mathbb{N}$ and for $1 \leq n \leq a_1 - 1$ the following identities hold,

$$X_n(a_1, a_2, a_3) = 0 \quad (6.22)$$

Using MAPLE, the conjecture has been verified by showing that $X_n(a_1, a_2, a_3) = 0$ as functions of $a_2$ and $a_3$ for all $n$ in the range $1 \leq n \leq a_1 - 1$ and for $1 \leq a_1 \leq 65$. An analytical proof of the conjecture is, however, outstanding.

Assuming the validity of Conjecture 6.2, we prove the **decomposition formula** of Conjecture 1.4 by expressing $c_{2-w}$ in terms of $S$. To this end, we combine the formulas (1.22), (1.23), (6.19) and the result of Conjecture 6.2, to obtain,

$$c_{2-w} = c_{2-w}^0 \zeta(2w - 2) + 2 \sum_{\sigma \in \mathcal{E}_3} \sum_{n=0}^{a_1-1} Z_n(a_1, a_2, a_3) S(w - 1 - n, n) \quad (6.23)$$

Evaluating $S$ with the help of formulas (6.2), (6.3) and (6.5) of Lemma 6.1 allows us to express $S$ in terms of a sum over pairs of odd $\zeta$-values whose weights add up to $2w - 2$, as well as terms taking values in $\pi^{2w-2}Q$ (arising from the contributions of even $\zeta$-values in (6.2), (6.3) and (6.5), the latter for even $\alpha$). By Theorem 1.2, the contributions valued in $\pi^{2w-2}Q$ cancel. The remaining contributions give $\gamma_k$ of (1.28) as follows. The first term on the right in (1.28) arises from $S(w - 1, 0)$ given by (6.2), the second term from the second term on the right in the sum of $S$ and $T$ in (6.3), and the remaining terms from the terms for odd values of $\alpha$ in the decomposition of $T$ in (6.5). This completes the proof of the decomposition formula of Conjecture 1.4) assuming the validity of Conjecture 6.2.
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