2017 August 26
2018 April 14 (revised)

Fourier series of modular graph functions

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Abstract

Modular graph functions associate to a graph an $SL(2,\mathbb{Z})$-invariant function on the upper half plane. We obtain the Fourier series of modular graph functions of arbitrary weight w and two-loop order. The motivation for this work is to develop a deeper understanding of the origin of the algebraic identities between modular graph functions which have been discovered recently, and of the relation between the existence of these identities and the occurrence of cusp forms. We show that the constant Fourier mode, as a function of the modulus τ, consists of a Laurent polynomial in $y = \pi \,\mathrm{Im}\, \tau$ of degree $(w, 1-w)$, plus a contribution which decays exponentially as $y \to \infty$. The Laurent polynomial is a linear combination with rational coefficients of the top term y^w, and lower order terms $\zeta(2k+1) y^{w-2k-1}$ for $1 \leq k \leq w-1$, as well as terms $\zeta(2w-2\ell-3)\zeta(2\ell+1) y^{2-w}$ for $1 \leq \ell \leq w-3$. The exponential contribution is a linear combination of exponentials of y and incomplete Γ-functions whose coefficients are Laurent polynomials in y with rational coefficients.

1 Introduction and statement of the main result

A modular graph function associates to a certain kind of graph an $SL(2,\mathbb{Z})$-invariant function of the upper half plane $\mathbb{H}$. Modular graph functions naturally arise in the low energy expansion of closed string amplitudes and govern the contributions to this expansion at genus one [1, 2]. For one-loop graphs they are non-holomorphic Eisenstein series, while for two-loop graphs they were found to obey a system of differential equations [3]. Connections between modular graph functions, multiple-zeta values and single-valued elliptic polylogarithms were put forward in [4]. Earlier relations between open and closed string amplitudes, multiple-zeta-functions, and polylogarithms were exhibited in [5, 6, 7]. The structure of the low energy expansion of genus-two string amplitudes, in as much as is known to date, may be found in [8] and references therein.

A number of algebraic identities between modular graph functions with two or more loops were conjectured in [3] by matching their asymptotic expansions near the cusp $\tau \to i\infty$. The simplest of these identities were proven by direct summation of the Eisenstein series in [9, 10]. More complicated identities and their various generalizations were obtained and proven by appealing to the rich system of differential equations they satisfy [11, 12, 13, 14]. In particular, it was shown in [11, 12] that the existence of these identities may be traced back to the existence of identities between holomorphic modular forms.

Better understanding the origin of these algebraic identities is the main motivation for the present work, as will be explained further below. Since the coefficients in the asymptotic expansion near the cusp of modular graph functions are known to involve general multiple zeta-values [4, 15], the algebraic identities between them contain and generalize some of the well-known relations between multiple zeta-values (see for example [16, 17, 18, 19] and references therein) to the world of modular functions. A recent characterization of classes of modular graph functions, as single-valued projections of elliptic multiple zeta values introduced in [20], may be found in [21, 22].

A simple infinite family is given by modular graph functions depending on $a_1, \dots a_\ell \in \mathbb{N}$ with $\ell \geq 2$ and may be expressed for $\tau = \tau_1 + i\tau_2 \in \mathbb{H}$ with $\tau_1, \tau_2 \in \mathbb{R}$ in the form,

$$C_{a_1,\cdots,a_\ell}(\tau) = \sum_{\substack{(m_r,n_r)\in\mathbb{Z}^2 \\ r=1,2,\dots\ell}}{}^{\prime} \delta_{m,0}\, \delta_{n,0} \prod_{r=1}^{\ell} \left(\frac{\tau_2}{\pi |m_r + n_r \tau|^2} \right)^{a_r} \tag{1.1}$$

where $m = m_1 + m_2 + \cdots m_\ell$, $n = n_1 + n_2 + \cdots n_\ell$ and the Kronecker δ-symbols force m, n to vanish. Throughout, a prime over the summation symbol indicates that the summation is restricted to disallow division by zero, in this case to $(m_r, n_r) \neq (0,0)$. The sum is absolutely convergent and $C_{a_1,\cdots,a_\ell}$ is smooth and modular,

$$C_{a_1,\cdots,a_\ell}(M\tau) = C_{a_1,\cdots,a_\ell}(\tau) \tag{1.2}$$

for each $M \in SL(2,\mathbb{Z})$ acting on $\tau \in \mathbb{H}$ by a linear fractional map.

The associated graph $\Gamma = \Gamma_{a_1,\dots,a_\ell}$ is planar and may be realized by first taking a graph $\hat{\Gamma}$ in the plane with 2 vertices connected to each other by ℓ edges, numbered $1, 2, \dots, \ell$ and then adjoining $a_r - 1$ distinct new bivalent vertices to the rth edge for each r. Thus $\hat{\Gamma}$ has ℓ edges, 2 vertices and ℓ faces, while Γ has $w = a_1 + \cdots + a_\ell$ edges, $w-1$ vertices and ℓ faces, or equivalently $\ell - 1$ loops. We will review in the next section how $C_{a_1,\cdots,a_\ell}$, which is said to be a $(\ell-1)$-loop modular graph function of weight w, arises from Γ. Its birth is as a Feynman graph in quantum field theory and string theory, and in that context the combination $p_r = m_r\tau + n_r$ represents the lattice momentum running through the edge r taking values in the lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$.

In the most simple case of a one-loop graph we have $\ell = 2$ and C_{a_1,a_2} is given by a specialization of the classical Kronecker–Eisenstein series, since

$$C_{a_1,a_2}(\tau) = E_w(\tau) = \sum_{(m,n)\in\mathbb{Z}^2}{}^{'} \frac{\tau_2^w}{\pi^w |m+n\tau|^{2w}}. \tag{1.3}$$

It is well known that $E_w(\tau)$ is an eigenfunction of the hyperbolic Laplacian,

$$\Delta E_w = w(w-1)E_w, \tag{1.4}$$

where Δ is normalized by $\Delta = 4\tau_2^2 \partial_{\bar\tau}\partial_\tau$. The effect of the Laplacian on $C_{a_1,\cdots,a_\ell}(\tau)$ was obtained in [3] where it was also shown that, as a result, a number of identities between various kind of modular graph functions are forced to exist. The simplest such identity was obtained in this way in [3] and states that,

$$C_{1,1,1}(\tau) = E_3(\tau) + \zeta(3) \tag{1.5}$$

where $\zeta(s)$ is the Riemann zeta-function. A proof by direct summation of the Kronecker-Eisenstein series was given by Zagier [9]. In general, for $\ell = 3$ and for each odd value of w there is precisely one linear combination of those C_{a_1,a_2,a_3} with weight w that differs from E_w by a constant. For instance, for $w = 5, 7, 9$,

$$\begin{aligned} 30C_{2,2,1}(\tau) &= 12E_5(\tau) + \zeta(5) \\ 252C_{3,3,1}(\tau) + 252C_{3,2,2}(\tau) &= 108E_7(\tau) + \zeta(7) \\ 2160C_{4,4,1}(\tau) + 4320C_{4,3,2}(\tau) + 960C_{3,3,3}(\tau) &= 960E_9(\tau) + \zeta(9) \end{aligned} \tag{1.6}$$

Note that, if we assign "weight s" to $\zeta(s)$, then the above identities are all homogeneous of their respective weights. The case of even weight is more difficult. Here the identities involve the functions C with $\ell > 3$. For instance we have,

$$C_{1,1,1,1}(\tau) = 24C_{2,1,1}(\tau) - 18E_4(\tau) + 3E_2^2(\tau) \tag{1.7}$$

This identity was conjectured in [3] and proven in [10].

Identities such as (1.5), (1.6) and (1.7) are reminiscent of various identities from the classical theory of modular forms. Consider the classical holomorphic Eisenstein series of even modular weight[1] $w \geq 4$,

$$\mathbb{G}_w(\tau) = \frac{(w-1)!}{2(2\pi i)^w} \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{1}{(m+n\tau)^w} \tag{1.8}$$

We have the Fourier expansion,

$$\mathbb{G}_w(\tau) = -\frac{B_w}{2w} + \sum_{k\geq 1} \sigma_{w-1}(k) q^k, \tag{1.9}$$

where B_w is the Bernoulli number, $q = e^{2\pi i\tau}$ and $\sigma_s(n) = \sum_{d|n} d^s$ is the divisor sum. The following identity is forced by the fact that $\mathbb{G}_8$ and $\mathbb{G}_4^2$ are in the same one-dimensional space,

$$120\, \mathbb{G}_4^2 = \mathbb{G}_8 \tag{1.10}$$

and implies the following arithmetic identity,

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(n-m)\sigma_3(m). \tag{1.11}$$

At higher weight such identities will involve holomorphic cusp forms, a phenomenon which arises first at weight 12 where we have, for example,

$$65\, \mathbb{G}_{12}(\tau) - 174132\, \mathbb{G}_6^2(\tau) = 756\, \Delta(\tau), \tag{1.12}$$

where $\Delta(\tau) = q\prod_m (1-q^m)^{24}$ is the famous cusp form of modular weight 12. Knowledge of the dimensions of holomorphic modular forms for given modular weight, together with their Fourier series expansion, provides all the information needed to establish all such identities in the holomorphic case.

Returning to non-holomorphic modular graph functions, it is one of the ultimate goals of this and subsequent work to understand the relation between identities amongst modular graphs functions, such as (1.5), (1.6) and (1.7), their Fourier series expansion, and the existence of non-holomorphic cusp forms. The Fourier expansion of the non-holomorphic Eisenstein series E_w for integer $w \geq 2$ is well-known and is given by,

$$\begin{aligned} E_w(\tau) \quad &= \quad -\frac{B_{2w}}{(2w)!}(-4y)^w + \frac{4(2w-3)!}{(w-2)!\,(w-1)!}\,\zeta(2w-1)\,(4y)^{1-w} \\ &\quad + \frac{2}{(w-1)!} \sum_{k=1}^{\infty} k^{w-1} \sigma_{1-2w}(k) \left(q^k + \bar q^k\right) P_w(4ky). \end{aligned} \tag{1.13}$$

[1] The *modular weight* is in general distinct from the weight w of the modular functions C defined earlier.

where we set $y = \pi\tau_2$, and $P_w(x)$ is the polynomial in $1/x$ defined by,

$$P_w(x) = \sum_{m=0}^{w-1} \frac{(w+m-1)!}{m!\,(w-m-1)!\; x^m} \tag{1.14}$$

For example, we have,

$$E_3(\tau) = \tfrac{2}{945} y^3 + \tfrac{3}{4}\zeta(5)\, y^{-2} + \sum_{k=1}^{\infty} \sigma_3(k) \Big(\tfrac{1}{k} + \tfrac{3}{2k^2 y} + \tfrac{3}{4k^3 y^2} \Big) \Big(q^k + \bar q^k \Big). \tag{1.15}$$

We see that the constant Fourier mode of E_w is a Laurent polynomial with only two terms.

In the present paper, we shall study the Fourier series expansion of two-loop modular graphs functions C_{a_1,a_2,a_3}. By combining (1.5) and (1.6) with (1.15) we readily get the Fourier expansions of $C_{1,1,1}$ and $C_{2,2,1}$. For functions of higher weight, the Laurent series have been evaluated only in special cases, tabulated below,

$$\begin{aligned}
C_{2,1,1}(\tau) &= \frac{2y^4}{14175} + \frac{\zeta(3)y}{45} + \frac{5\zeta(5)}{12y} - \frac{\zeta(3)^2}{4y^2} + \frac{9\zeta(7)}{16y^3} + \mathcal{O}(e^{-2\pi\tau_2}) \\
C_{3,1,1}(\tau) &= \frac{2y^5}{155925} + \frac{2\zeta(3)y^2}{945} - \frac{\zeta(5)}{180} + \frac{7\zeta(7)}{16y^2} - \frac{\zeta(3)\zeta(5)}{2y^3} + \frac{43\zeta(9)}{64y^4} + \mathcal{O}(e^{-2\pi\tau_2}) \\
C_{4,1,1}(\tau) &= \frac{808y^6}{638512875} + \frac{\zeta(3)y^3}{4725} - \frac{\zeta(5)y}{1890} + \frac{\zeta(7)}{720y} + \frac{23\zeta(9)}{64y^3} \\
&\quad - \frac{\zeta(5)^2 + 30\zeta(3)\zeta(7)}{64y^4} + \frac{167\zeta(11)}{256y^5} + \mathcal{O}(e^{-2\pi\tau_2}) \\
C_{3,2,1}(\tau) &= \frac{43y^6}{58046625} + \frac{y\zeta(5)}{630} + \frac{\zeta(7)}{144y} + \frac{7\zeta(9)}{64y^3} - \frac{17\zeta(5)^2}{64y^4} + \frac{99\zeta(11)}{256y^5} + \mathcal{O}(e^{-2\pi\tau_2}) \\
C_{2,2,2}(\tau) &= \frac{38y^6}{91216125} + \frac{\zeta(7)}{24y} - \frac{7\zeta(9)}{16y^3} + \frac{15\zeta(5)^2}{16y^4} - \frac{81\zeta(11)}{128y^5} + \mathcal{O}(e^{-2\pi\tau_2})
\end{aligned} \tag{1.16}$$

Along different lines, a systematic algorithm was developed to evaluate modular graph functions which have at most four vertices (including bivalent vertices) in [15].

Except in such cases where we may apply a known identity, it is an open problem to determine the Fourier coefficients of $C_{a_1,\cdots,a_\ell}(\tau)$ for a fixed $a_1,\cdots,a_\ell$ when $\ell > 2$. Its expansion has the form,

$$C_{a_1,\cdots,a_\ell}(\tau) = \sum_{k=-\infty}^{\infty} C^{(k)}_{a_1,\cdots,a_\ell}(\tau_2)\, e^{2\pi i k \tau_1}, \tag{1.17}$$

If we can determine $C^{(k)}_{a_1,\cdots,a_\ell}(\tau_2)$, independently of any knowledge of special identities between modular graph forms, then identities such as (1.5), (1.6) and (1.7) and their generalizations should emerge from identities between the Fourier coefficients of these modular graph functions. Understanding the structure of the Fourier series thus appears key to understanding the structural mechanism behind the existence of the identities between modular graph functions, and their relation with the existence of cusp forms. Clearly, one obstruction to the existence of identities is the presence of cusp forms. Therefore, an urgent question is whether we can find combinations of the C-functions of a fixed weight having zero constant mode in their Fourier expansions. Finally, it would be interesting to find out whether such cusp forms play a natural role in string theory.

As a step toward answering such questions, in this paper we will compute rather explicitly the constant mode $C^{(0)}_{a_1,a_2,a_3}(\tau_2)$ in the Fourier expansion of C_{a_1,a_2,a_3}. The main results of this paper may be summarized by three Theorems, and one conjectured decomposition formula.

Theorem 1.1 *The constant Fourier mode for C_{a_1,a_2,a_3} with fixed a_1, a_2, a_3 is given by,*

$$C^{(0)}_{a_1,a_2,a_3}(\tau_2) = \mathcal{L}(\tau_2) + \mathcal{E}(\tau_2) \tag{1.18}$$

where $\mathcal{L}(\tau_2)$ is a Laurent polynomial in τ_2 of degree $(w, 1-w)$ and $\mathcal{E}(\tau_2)$ is exponentially decaying as $\tau \to i\infty$. The Laurent polynomial is given by,

$$\mathcal{L}(\tau_2) = c_w(-4\pi\tau_2)^w + \sum_{k=1}^{w-1} c_{w-2k-1} \frac{\zeta(2k+1)}{(4\pi\tau_2)^{2k+1-w}} + \frac{c_{2-w}}{(4\pi\tau_2)^{w-2}} \tag{1.19}$$

(a) The coefficient c_w is a rational number given by,

$$c_w = \sum_{k=0}^{a_2} \frac{B_{2k}B_{2w-2k}}{(2k)!\,(2w-2k)!} \frac{\Gamma(2a_2+2a_3-2k)}{\Gamma(2a_3)\Gamma(2a_2-2k+1)} + (a_2 \leftrightarrow a_3) \tag{1.20}$$

(b) The coefficients c_{w-2k-1} for $1 \leq k \leq w-1$ are rational numbers given by,

$$\begin{aligned} c_{w-2k-1} \quad = \quad & \frac{2B_{2w-2k-2}}{(2w-2k-2)!} \sum_{\alpha=0}^{a_1-1} \sum_{\beta=0}^{a_1-1-\alpha} (-)^{a_1+a_3+\beta+1}\, \theta\left(a_3 + \left[\frac{a_2+\beta}{2}\right] - w + k + 1\right) \\ & \times g_{a_1,a_2}(\alpha,\beta) \binom{2k-2a_1+\alpha+\beta+1}{a_2+\alpha-1} + 5 \text{ permutations of } a_1, a_2, a_3 \end{aligned}$$

where $\theta(x)$ is the step function defined to equal 1 when $x \geq 0$ and to vanish otherwise. The function $g_{a_1,a_2}(\alpha,\beta)$ is integer-valued and given by,

$$g_{a_1,a_2}(\alpha,\beta) = (-)^{a_1} \binom{2a_1-2-\alpha-\beta}{a_1-1} \binom{a_2+\alpha-1}{a_2-1} \binom{a_2+\beta-1}{a_2-1} \tag{1.21}$$

(c) The coefficient c_{2-w} is given by the following expression,

$$c_{2-w} = c^0_{2-w}(-)^w \zeta(2w-2) + 2 \sum_{\sigma \in \mathfrak{S}_3} Z(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}) \tag{1.22}$$

Here, c^0_{2-w} is an integer given by,

$$c^0_{2-w} = \sum_{\alpha=0}^{a_1-1} \sum_{\beta=0}^{a_1-1-\alpha} (-)^{a_2+\beta} g_{a_1,a_2}(\alpha,\beta) \binom{2a_2+2a_3+\alpha+\beta}{a_2+\alpha-1} \tag{1.23}$$

while $Z(a_1,a_2,a_3)$ is a linear combination with integer coefficients of depth-two multiple zeta-functions of total weight $2w-2$ given by,

$$\begin{aligned} Z(a_1,a_2,a_3) \;\; = \;\; & \sum_{k=1}^{a_1} \sum_{\ell=1}^{a_1} \binom{a_1+a_2-k-1}{a_2-1} \binom{a_1+a_2-\ell-1}{a_2-1} \binom{k+\ell-2}{k-1} \\ & \times \binom{2w-k-\ell-2}{w-k-1} \zeta(2w-k-\ell-1, k+\ell-1) \end{aligned} \tag{1.24}$$

The normalization of the double ζ-functions is as follows,

$$\zeta(a,b) = \sum_{m,n=1}^{\infty} \frac{1}{(m+n)^a n^b} \tag{1.25}$$

Theorem 1.2 *The coefficient c_{2-w} is a linear combination, with integer coefficients, of products of two odd zeta-values whose weights add up to $2w-2$,*

$$c_{2-w} = \sum_{k=1}^{w-2} \frac{1}{2} \gamma_k \, \zeta(2k+1)\zeta(2w-2k-3) \tag{1.26}$$

with rational coefficients $\gamma_k \in \mathbb{Q}$.

Theorem 1.3 *The general structure of the exponential part (1.18) is given as follows,*

$$\mathcal{E}(\tau_2) = \sum_{s=1}^{w-2} \sum_{n=1}^{\infty} \left(\left(f_+(s;n)(4\pi\tau_2)^s + \frac{f_-(s;n)}{(4\pi\tau_2)^{s-1}} \right) \mathrm{Ei}(4\pi n\tau_2) + e^{-4\pi n\tau_2} \sum_{m=2-w}^{w-3} \frac{f(s;n,m)}{(4\pi\tau_2)^m} \right) \tag{1.27}$$

where Ei is the incomplete Γ-function and the coefficients $f_\pm(s,n)$, and $f(s,n,m)$ are rational numbers.

Conjecture 1.4 *(Decomposition Formula) The coefficients γ_k entering the decomposition of c_{2-w} in formula (1.26) of Theorem 1.2 are given by the following expression,*

$$
\begin{aligned}
\gamma_k &= 2Z_k(a_1,a_2,a_3)\theta(a_1-1-k) - Z_0(a_1,a_2,a_3) \\
&\quad + \sum_{\alpha=1}^{a_1-1} Z_\alpha(a_1,a_2,a_3) \sum_{n=0}^{2\alpha-1} E_n(0) \binom{2k}{2\alpha-n} \binom{2w-2\alpha+n-4}{n} \\
&\quad + \text{ 5 permutations of } a_1,a_2,a_3
\end{aligned} \tag{1.28}
$$

and are integers. Here, $E_n(x)$ are the Euler polynomials and the integer-valued function $Z_\alpha(a_1,a_2,a_3)$ is given by the following sum,

$$
Z_\alpha(a_1,a_2,a_3) = \sum_{k=k_-}^{k_+} \binom{a_1+a_3-k-1}{a_3-1} \binom{a_1+a_3-2\alpha+k-3}{a_3-1} \binom{2\alpha}{k-1} \binom{2w-2\alpha-4}{w-k-1} \tag{1.29}
$$

with $k_+ = \min(a_1, 2\alpha+1)$ and $k_- = \max(1, 2\alpha+2-a_1)$.

To obtain the decomposition formula, we make use of a conjectured relation (conjecture 6.2), which we have verified extensively using Maple calculations, but for which we have no analytical proof. Therefore, a full proof of the decomposition formula remains outstanding.

The explicit formulas for the Laurent polynomial part $\mathcal{L}$ of $C_{a_1,a_2,a_3}(\tau)$, obtained in Theorems 1.1 and 1.2, and the Decomposition Formula of 1.4, completely reproduce the Laurent polynomials of (1.16) which have been evaluated earlier in the literature.

2 Modular graph functions

Before turning to the proof of the Theorems, and the derivation of the Decomposition Formula, we will briefly review the general definition of a modular graph function as it comes from string theory and show that the function $C_{a_1,\cdots,a_\ell}(\tau)$, which was defined in (1.1), is one such modular graph function.

Modular graph functions arise as follows. The torus Σ with modulus τ may be represented in the complex plane by the quotient $\Sigma = \mathbb{C}/\Lambda$ for the lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$. We choose local complex coordinates $(z, \bar{z})$ on Σ in which the metric is given by $|dz|^2/\tau_2$. The volume form of this metric $d\mu(z) = idz \wedge d\bar{z}/(2\tau_2)$ has unit area and the Dirac δ-function $\delta(z-w)$ is normalized by $\int_\Sigma d\mu(z)\delta(z-w) = 1$. The scalar Green function $G(z-w|\tau)$ is defined by,

$$
\tau_2 \partial_{\bar{z}} \partial_z G(z-w|\tau) = -\pi\delta(z-w) + \pi \tag{2.1}
$$

along with the normalization condition,

$$
\int_\Sigma d\mu(z)\, G(z-w|\tau) = 0 \tag{2.2}
$$

To a graph Γ with $v \geq 2$ vertices and w edges we associate v points z_i on the torus Σ labelled by the index $i = 1, \cdots, v$. We denote by ν_{ij} the number of edges connecting the pair of vertices $i, j = 1, \cdots, v$. The number ν_{ij} is allowed to be a positive or zero integer for any pair of distinct vertices i, j. We set $\nu_{ii} = 0$ for all vertices $i = 1, \cdots, v$, and thus restrict the type of graphs on which we can define modular graph functions.[2] The total number of edges is $w = \sum_{1 \leq i < j \leq v} \nu_{ij}$. The modular graph function $\mathcal{C}_\Gamma(\tau)$ is defined in terms of absolutely convergent integrals over the torus by

$$\mathcal{C}_\Gamma(\tau) = \left(\prod_{k=1}^{v} \int_\Sigma d\mu(z_k) \right) \prod_{1 \leq i < j \leq v} G(z_i - z_j|\tau)^{\nu_{ij}} \tag{2.3}$$

$\mathcal{C}_\Gamma(\tau)$ is clearly modular.

We may assume that Γ is connected and remains connected after the removal of a vertex (and its adjoining edges), since otherwise $C_\Gamma = C_{\Gamma_1} C_{\Gamma_2}$ for subgraphs Γ_1 and Γ_2. We may also assume that Γ remains connected when any single edge is omitted, for otherwise $C_\Gamma = 0$. In particular we may assume that Γ contains no vertices with valence 1. Suppose that Γ has $\hat{v}$ vertices of valence at least 3. As in the case of the graphs $\Gamma_{a_1, \cdots, a_\ell}$, which were introduced in the third paragraph of section 1, it is convenient to build up Γ using an auxiliary graph $\hat{\Gamma}$ having $\hat{v}$ vertices with the same valences ≥ 3, no bivalent vertices and ℓ edges. Note that, unlike Γ, $\hat{\Gamma}$ can be a single edge with no vertices. This happens when $\hat{v} = 0$. To recover Γ we adjoin $a_r - 1$ bivalent vertices to the rth edge of $\hat{\Gamma}$ for $r = 1, \ldots \ell$. If $\hat{v} > 0$ consider the $\hat{v} \times \ell$ incidence matrix of the graph $\hat{\Gamma}$, when it is given some orientation. The entry a_{ir} of the incidence matrix equals ± 1 if edge r starts or ends on vertex i (the sign is determined by the choice of orientation through the graph), and equals 0 otherwise.

Proposition 2.1 *Under the assumptions and notation introduced above, we have,*

$$\mathcal{C}_\Gamma(\tau) = \sideset{}{'}\sum_{\substack{(m_r, n_r) \in \mathbb{Z}^2 \\ r=1, \cdots, \ell}} \prod_{r=1}^{\ell} \left(\frac{\tau_2}{\pi |m_r + n_r \tau|^2} \right)^{a_r} \prod_{i=1}^{\hat{v}} \delta \left(\sum_{r=1}^{\ell} a_{ir} m_r \right) \delta \left(\sum_{r=1}^{\ell} a_{ir} n_r \right) \tag{2.4}$$

When $\hat{v} = 0$ we must evaluate the second product to 1.

Proof: The Green function $G(z|\tau)$, defined in (2.1) and (2.2), is given by a Fourier sum on the torus, parametrized in terms of real coordinates x, y by $z = x + y\tau$ and $x, y \in \mathbb{R}/\mathbb{Z}$,

$$G(z|\tau) = \sideset{}{'}\sum_{(m,n) \in \mathbb{Z}^2} \frac{\tau_2}{\pi |m + n\tau|^2} \, e^{2\pi i (my - nx)} \tag{2.5}$$

[2] An immediate justification for this restriction on the graphs is that without it divergent contributions involving the Green function at coincident points $G(0|\tau)$ would arise. In quantum field theory such graphs do arise in un-renormalized correlation functions, but are eliminated by the process of renormalization.

Bivalent vertices play a special role, as they produce a convolution of concatenated Green functions. We parametrize their effect by introducing the functions $G_a(z|\tau)$, defined recursively in the index a by setting $G_1(z|\tau) = G(z|\tau)$ for $a = 1$ and,

$$G_a(z|\tau) = \int_\Sigma d\mu(w)\, G(z-w|\tau)\, G_{a-1}(w|\tau) \tag{2.6}$$

for $a \geq 2$. The Fourier series for G_a on the torus is readily obtained using $d\mu(z) = dx \wedge dy$,

$$G_a(z|\tau) = \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{\tau_2^a}{\pi^a |m+n\tau|^{2a}}\, e^{2\pi i(my-nx)}. \tag{2.7}$$

Applying this to (2.3) and carrying out the integrals over the $\hat{v}$ vertex positions z_i corresponding to vertices of valence ≥ 3, we obtain (2.4).

Corollary 2.2 *For $C_{a_1,\cdots,a_\ell}$ and $\Gamma_{a_1,\cdots,a_\ell}$ defined in and below (1.1) we have,*

$$C_{\Gamma_{a_1,\cdots,a_\ell}} = C_{a_1,\cdots,a_\ell} \tag{2.8}$$

3 Fourier series of two-loop modular graph functions

Now we turn to the proof of the Theorems. In this section we shall introduce a Mellin-transform formulation of two-loop modular graph functions C_{a_1,a_2,a_3} and a partial Poisson resummation to obtain the Fourier series expansion of (1.17). The method naturally generalizes to the case of higher modular graph functions, but we shall treat here only the case of two-loop modular graph functions of arbitrary weight $w = a_1 + a_2 + a_3$.

3.1 Mellin-transform representation

We begin with the elementary integral representation,

$$\frac{\tau_2^{a_r}}{\pi^{a_r}|m_r + n_r\tau|^{2a_r}} = \int_0^\infty dt_r\, \frac{t_r^{a_r-1}}{\Gamma(a_r)} \exp\left\{-\frac{\pi}{\tau_2} t_r |m_r + n_r\tau|^2\right\} \tag{3.1}$$

Collecting the sum over the product of three such factors, and taking care of omitting the zero mode from the summation over each edge, we find,

$$\begin{aligned} C_{a_1,a_2,a_3}(\tau) &= \left(\prod_{r=1}^3 \int_0^\infty dt_r\, \frac{t_r^{a_r-1}}{\Gamma(a_r)}\right) S(t_1,t_2,t_3|\tau) \\ S(t_1,t_2,t_3|\tau) &= \sum_{\substack{(m_r,n_r)\in\mathbb{Z}^2 \\ r=1,2,3}} \delta_{m,0}\,\delta_{n,0} \prod_{r=1}^3 \left(\exp\left\{-\frac{\pi}{\tau_2} t_r |m_r+n_r\tau|^2\right\} - \delta_{m_r,0}\delta_{n_r,0}\right) \end{aligned} \tag{3.2}$$

The function $S(t_1, t_2, t_3|\tau)$ is invariant under $SL(2, \mathbb{Z})$ acting on τ, as well as under permutations of the t_r. Next, we decompose S by expanding the triple product into a sum of eight terms. The three contributions for which two pairs (m_r, n_r) are set to zero must also have the third pair equal to zero in view of overall momentum conservation, and therefore combine with the terms in which all three pairs are zero. The result is as follows,

$$S(t_1, t_2, t_3|\tau) = A(t_1, t_2, t_3|\tau) - B(t_1 + t_2|\tau) - B(t_2 + t_3|\tau) - B(t_3 + t_1|\tau) + 2 \tag{3.3}$$

The last term arises from the contribution with all pairs (m_r, n_r) equal to $(0, 0)$, and the functions A and B are given by,

$$\begin{aligned} A(t_1, t_2, t_3|\tau) &= \sum_{\substack{(m_r,n_r)\in\mathbb{Z}^2 \\ r=1,2,3}} \delta_{m,0}\, \delta_{n,0}\, \exp\left\{ -\frac{\pi}{\tau_2} \sum_{r=1}^{3} t_r |m_r + n_r\tau|^2 \right\} \\ B(t|\tau) &= \sum_{(m_1,n_1)\in\mathbb{Z}^2} \exp\left\{ -\frac{\pi}{\tau_2} t |m_1 + n_1\tau|^2 \right\} \end{aligned} \tag{3.4}$$

Note that the summation over the three pairs (m_r, n_r) in A includes the contribution from all the zero pairs, and is constrained only by the requirement that their sum (m, n) vanishes. The summation over pairs (m_1, n_1) in B is unconstrained.

It will be convenient to solve the constraint $m = n = 0$ in the summation which defines the function A by setting $m_3 = -m_1 - m_2$ and $n_3 = -n_1 - n_2$, with m_1, m_2, n_1, n_2 taking values in $\mathbb{Z}$ unconstrained. Furthermore, we introduce the matrix notation,

$$M = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \qquad N = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \qquad T = \begin{pmatrix} t_1 + t_3 & t_3 \\ t_3 & t_2 + t_3 \end{pmatrix} \tag{3.5}$$

The function A then takes the form,

$$A(t_1, t_2, t_3|\tau) = \sum_{M,N\in\mathbb{Z}^2} \exp\left\{ -\frac{\pi}{\tau_2} (M + \tau N)^\dagger T (M + \tau N) \right\} \tag{3.6}$$

One may think of this expression as defining a ϑ-function.

3.2 Partial Poisson resummation

To compute the Fourier series of $C_{a_1,a_2,a_3}(\tau)$ as a function of τ_1, we perform a Poisson resummation on the sum in the expression for the function A on the matrix M, but *not* on N. To do so, we evaluate the Fourier transform of the M-dependent part as follows,

$$\int_{\mathbb{R}^2} d^2M\, e^{-2\pi i M^t X} e^{-\pi(M+\tau_1 N)^t T (M+\tau_1 N)/\tau_2} = \frac{\tau_2}{(\det T)^{\frac{1}{2}}}\, e^{2\pi i \tau_1 N^t X - \pi\tau_2 X^t T^{-1} X} \tag{3.7}$$

It will be convenient to express the inverse of T as follows,

$$T^{-1} = \frac{1}{\det T}\, \varepsilon^t\, T\, \varepsilon \qquad\qquad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.8}$$

Upon the change of summation variables $M \to -\varepsilon M$, the Fourier series takes the form,

$$A(t_1,t_2,t_3|\tau) = \frac{\tau_2}{(\det T)^{\frac{1}{2}}} \sum_{M,N\in\mathbb{Z}^2} e^{2\pi i M^t \varepsilon N \tau_1} \exp\left\{ -\pi\tau_2 \frac{M^tTM}{\det T} - \pi\tau_2 N^tTN \right\} \tag{3.9}$$

To obtain the Fourier series of the function $C_{a_1,a_2,a_3}(\tau)$ we shall need to integrate S over t_1,t_2,t_3, which requires combining the contributions of A to the integral with those from B. To simplify this recombination, we perform a Poisson resummation in m_1 of $B(t)$,

$$B(t|\tau) = \sqrt{\frac{\tau_2}{t}} \sum_{m_1,n_1\in\mathbb{Z}} e^{2\pi i m_1 n_1 \tau_1}\, e^{-\pi\tau_2 m_1^2/t - \pi\tau_2 n_1^2 t} \tag{3.10}$$

which exhibits the Fourier series in τ_1 of B.

3.3 Fourier series expansions of A and B

The Fourier modes $S_k(t_1,t_2,t_3|\tau_2)$ of $S(t_1,t_2,t_3|\tau)$ as a function of τ_1 are given by,

$$S(t_1,t_2,t_3|\tau) = \sum_{k\in\mathbb{Z}} e^{2\pi i k\tau_1} S_k(t_1,t_2,t_3|\tau_2) \tag{3.11}$$

The Fourier modes $A_k(t_1,t_2,t_3|\tau_2)$ of $A(t_1,t_2,t_3|\tau)$, and the Fourier modes $B_k(t|\tau_2)$ of $B(t|\tau)$ as functions of τ_1 are defined analogously. They are related to one another by,

$$\begin{aligned} S_k(t_1,t_2,t_3|\tau_2) &= A_k(t_1,t_2,t_3|\tau_2) - B_k(t_1+t_2|\tau_2) \\ &\quad - B_k(t_2+t_3|\tau_2) - B_k(t_3+t_1|\tau_2) + 2\delta_{k,0} \end{aligned} \tag{3.12}$$

The expressions for the Fourier modes are obtained from (3.9) and (3.10) and are given by,

$$\begin{aligned} A_k(t_1,t_2,t_3|\tau_2) &= \frac{\tau_2}{(\det T)^{\frac{1}{2}}} \sum_{M,N\in\mathbb{Z}^2} \delta_{M^t\varepsilon N,k}\, \exp\left\{ -\pi\tau_2 \frac{M^tTM}{\det T} - \pi\tau_2 N^tTN \right\} \\ B_k(t|\tau_2) &= \sqrt{\frac{\tau_2}{t}} \sum_{m,n\in\mathbb{Z}} \delta_{mn,k}\, e^{-\pi\tau_2 m^2/t - \pi\tau_2 n^2 t} \end{aligned} \tag{3.13}$$

The Fourier modes B_k for $k \neq 0$ are exponentially decaying as $t \to \infty$. Thus, the term $B_k(t_1+t_2|\tau)$ decays exponentially as $t_1 \to \infty$ or $t_2 \to \infty$ or both, but not when $t_3 \to \infty$. We will show in the subsequent subsection that uniform exponential decay is recovered upon combining the contributions of the Fourier modes A_k and B_k into the modes S_k given by (3.3). With exponential decay secured, the Fourier modes of $C_{a_1,a_2,a_3}(\tau)$, expressed in the notation of (1.17) with the help of (3.2), are then obtained by the following integrals,

$$C^{(k)}_{a_1,a_2,a_3}(\tau_2) = \left(\prod_{r=1}^{3} \int_0^\infty dt_r \, \frac{t_r^{a_r-1}}{\Gamma(a_r)} \right) S_k(t_1,t_2,t_3|\tau) \tag{3.14}$$

which are absolutely convergent for large t_r, and may be analytically continued in a_r for small t_r if necessary.

3.4 Partitioning the sum over N

To expose uniform exponential decay in t_1, t_2, t_3, we partition the summation over $N \in \mathbb{Z}^2$,

$$\mathbb{Z}^2 \;=\; \mathfrak{N}^{(0)} \cup \mathfrak{N}^{(1)} \cup \mathfrak{N}^{(2)} \cup \mathfrak{N}^{(3)} \cup \mathfrak{N}^{(4)} \tag{3.15}$$

into the following (disjoint) parts,

$$\begin{aligned} \mathfrak{N}^{(0)} &= \{(0,0)\} \\ \mathfrak{N}^{(i)} &= \{(n_1,n_2) \text{ such that } n_1,n_2 \in \mathbb{Z}, n_i = 0, n_j \neq 0 \text{ for } j \neq i\} \qquad i,j = 1,2,3 \\ \mathfrak{N}^{(4)} &= \{(n_1,n_2) \text{ such that } n_1,n_2 \in \mathbb{Z}, n_1,n_2,n_3 \neq 0\} \end{aligned} \tag{3.16}$$

where we enforce the constraint $n = n_1 + n_2 + n_3 = 0$ throughout. The quadratic form N^tTN vanishes for $N \in \mathfrak{N}^{(0)}$; is uniformly non-degenerate in t_1, t_2, t_3 for $N \in \mathfrak{N}^{(4)}$; while for $i = 1,2,3$ and $N \in \mathfrak{N}^{(i)}$ the quadratic form decays exponentially, but non-uniformly, in all directions except $t_i \to \infty$ where it remains bounded. Therefore, the above partition accurately governs the asymptotic behavior as $t_1, t_2, t_3 \to \infty$. We arrange the contributions to S_k arising from the partitions $\mathfrak{N}^{(i)}$ for $i = 0,1,2,3,4$ to A_k and from B_k as follows,

$$S_k(t_1,t_2,t_3|\tau_2) = \sum_{i=0}^{4} S_k^{(i)}(t_1,t_2,t_3|\tau_2) \tag{3.17}$$

We shall spell out their precise contributions in the subsequent subsections.

3.4.1 Contributions to $S_k^{(0)}$

The term $S_k^{(0)}$ arises from the contribution to A_k of $N = 0$, and is non-zero only for the constant Fourier mode $k = 0$. Performing a Poisson resummation in M gives,

$$S_k^{(0)}(t_1,t_2,t_3|\tau_2) = \delta_{k,0} \sum_{M \in \mathbb{Z}^2} \exp\left\{ -\frac{\pi}{\tau_2} M^t T M \right\} \tag{3.18}$$

Partitioning the summation over M according to (3.15), we have,

$$S_k^{(0)}(t_1,t_2,t_3|\tau_2) = \delta_{k,0}\Big(1+\sum_{i=1}^{3} L(t'_i|\tau_2) + \tilde S_0^{(0)}(t_1,t_2,t_3|\tau_2)\Big) \tag{3.19}$$

where we have defined $t'_i = t_1+t_2+t_3-t_i$ while the functions L and $\tilde S_0^{(0)}$ are defined by,

$$\begin{aligned} L(t'_i|\tau_2) &= \sideset{}{'}\sum_{m\in\mathbb{Z}} e^{-\pi t'_i m^2/\tau_2} \\ \tilde S_0^{(0)}(t_1,t_2,t_3|\tau_2) &= \sum_{M\in\mathfrak{N}^{(4)}} \exp\left\{-\frac{\pi}{\tau_2} M^t T M\right\} \end{aligned} \tag{3.20}$$

The contributions of the functions $L(t)$ naturally combine with those from $B_k(t)$.

3.4.2 Contributions to $S_k^{(i)}$ for $i=1,2,3$

For $i=1,2,3$, the term $S_k^{(i)}$ arises from the contribution to A_k of the partition with $N\in\mathfrak{N}^{(i)}$ and from $B_k(t'_i|\tau_2)$. It will be convenient to exhibit it as follows,

$$S_k^{(i)}(t_1,t_2,t_3|\tau_2) = \tilde S_k^{(i)}(t_1,t_2,t_3|\tau_2) - \delta_{k,0} - \delta_{k,0} L(t'_i|\tau_2) \tag{3.21}$$

where the reduced Fourier mode $\tilde S_k^{(i)}(t_1,t_2,t_3|\tau_2)$ is given by,

$$\begin{aligned} \tilde S_k^{(i)}(t_1,t_2,t_3|\tau_2) &= \frac{\tau_2}{(\det T)^{\frac{1}{2}}} \sum_{N\in\mathfrak{N}^{(i)}} \sum_{M\in\mathbb{Z}^2} \delta_{M^t\varepsilon N,k} \exp\left\{-\pi\tau_2 \frac{M^t T M}{\det T} - \pi\tau_2 N^t T N\right\} \\ & \quad - B_k(t'_i) + \delta_{k,0}\Big(1+L(t'_i|\tau_2)\Big) \end{aligned} \tag{3.22}$$

The contributions individually fail to exponentially decay in the direction $t_i\to\infty$ but the sum of the two lines above produces a function $S_k^{(i)}$ which exponentially decays in all directions t_1,t_2,t_3 at infinity. To establish uniform exponential decay, it will be convenient to treat the cases $k=0$ and $k\neq 0$ separately.

For $k=0$, consider the case $i=1$, the other cases being obtained by cyclic permutations of t_1,t_2,t_3. For $i=1$, the partition $\mathfrak{N}^{(i)}$ may be parametrized explicitly by $N=(0,n_2)$ with $n_1=0$ and $n_2=-n_3\neq 0$. The constraint $M^t\varepsilon N=0$ reduces to $m_1 n_2=0$ so that we must have $m_1=0$, while $m_2\in\mathbb{Z}$. Furthermore, the summation over m,n in the function $B_0(t_2+t_3|\tau_2)$ is constrained by $mn=0$ and simplifies as follows,

$$B_0(t_2+t_3|\tau_2) = \sqrt{\frac{\tau_2}{t_2+t_3}} \sum_{m\in\mathbb{Z}} e^{-\pi\tau_2 m^2/(t_2+t_3)} + \sqrt{\frac{\tau_2}{t_2+t_3}} \sum_{n\neq 0} e^{-\pi\tau_2(t_2+t_3)n^2} \tag{3.23}$$

Poisson resummation over m in the first term on the right side gives,

$$B_0(t_2+t_3|\tau_2) = 1 + L(t_2+t_3|\tau_2) + \sqrt{\frac{\tau_2}{t_2+t_3}} \sum_{n\neq 0} e^{-\pi\tau_2(t_2+t_3)n^2} \tag{3.24}$$

Using the parametrization of M and N for $k=0$ and $i=1$ given above, the sums over m_2 and n_2 on the first line of the right side of (3.22) factorize, and the expression takes the form,

$$\frac{\tau_2}{(\det T)^{\frac{1}{2}}} \sum_{m_2} \sum_{n_2\neq 0} \exp\left\{ -\pi\tau_2 \frac{t_2+t_3}{\det T}\, m_2^2 - \pi\tau_2(t_2+t_3)\, n_2^2 \right\} \tag{3.25}$$

Poisson resumming over m_2, and combining the expression with the result obtained for B_0 shows that the $m_2=0$ mode of the Poisson resummation is cancelled by the terms from B_0 and L, and gives the following expression for $\tilde S_0^{(1)}$,

$$\tilde S_0^{(1)}(t_1,t_2,t_3|\tau_2) = \sqrt{\frac{\tau_2}{t_2+t_3}} \sum_{m_2,n_2\neq 0} \exp\left\{ -\pi \frac{\det T}{\tau_2\,(t_2+t_3)}\, m_2^2 - \pi\tau_2(t_2+t_3)\, n_2^2 \right\} \tag{3.26}$$

The functions $\tilde S_0^{(2)}$ and $\tilde S_0^{(3)}$ are obtained by cyclic permutations in the variables t_1,t_2,t_3 of (3.26), and the resulting functions $\tilde S_0^{(i)}(t_1,t_2,t_3|\tau_2)$ have uniform exponential decay in all directions of t_1,t_2,t_3. The case $k\neq 0$ may be handled similarly, but will not be needed to prove the Theorems, and we shall not discuss it further.

3.4.3 Contributions to $S_k^{(4)}$

The term $S_k^{(4)}$ arises solely from the contribution to A_k of the partition with $N\in\mathfrak{N}^{(4)}$, and is given as follows,

$$S_k^{(4)}(t_1,t_2,t_3|\tau_2) = \frac{\tau_2}{(\det T)^{\frac{1}{2}}} \sum_{N\in\mathfrak{N}^{(4)}} \sum_{M\in\mathbb{Z}^2} \delta_{M^t\varepsilon N,k} \exp\left\{ -\pi\tau_2 \frac{M^tTM}{\det T} - \pi\tau_2 N^tTN \right\} \tag{3.27}$$

Clearly, $S_k^{(4)}$ is uniformly exponentially decaying in t_1,t_2,t_3 at infinity. For $k=0$ the constraint $M^t\varepsilon N=0$ forces M to either vanish, or to belong to $\mathfrak{N}^{(4)}$, excluding the cases $M\in\mathfrak{N}^{(i)}$ for $i=1,2,3$. Therefore, it will be convenient to split the sum accordingly,

$$S_k^{(4)}(t_1,t_2,t_3|\tau_2) = \tilde S_k^{(4)}(t_1,t_2,t_3|\tau_2) + \tilde S_k^{(5)}(t_1,t_2,t_3|\tau_2) \tag{3.28}$$

where

$$\begin{aligned} \tilde S_k^{(4)}(t_1,t_2,t_3|\tau_2) &= \frac{\tau_2\delta_{k,0}}{(\det T)^{\frac{1}{2}}} \sum_{N\in\mathfrak{N}^{(4)}} \exp\left\{-\pi\tau_2 N^tTN\right\} \qquad (3.29)\\ \tilde S_k^{(5)}(t_1,t_2,t_3|\tau_2) &= \frac{\tau_2}{(\det T)^{\frac{1}{2}}} \sum_{N\in\mathfrak{N}^{(4)}} \sum_{M\neq 0} \delta_{M^t\varepsilon N,k} \exp\left\{ -\pi\tau_2 \frac{M^tTM}{\det T} - \pi\tau_2 N^tTN \right\} \end{aligned}$$

where $\tilde S_k^{(4)}$ arises from $M=0$ while $\tilde S_k^{(5)}$ arises from $M\in\mathfrak{N}^{(4)}$.

3.5 Summary of contributions to the constant Fourier mode

Collecting the contributions obtained in (3.20), (3.26) and (3.29), we find that the constant Fourier mode is given by the sum of six terms,

$$C^{(0)}_{a_1,a_2,a_3}(\tau_2) = \sum_{i=0}^{5} \mathcal{C}^{(i)}_0(\tau_2) \tag{3.30}$$

each of which is given by the following integrals over t_1, t_2, t_3 of the corresponding functions $\tilde{S}^{(i)}_0$ evaluated in the previous section,

$$\mathcal{C}^{(i)}_0(\tau_2) = \left(\prod_{r=1}^{3} \frac{1}{\Gamma(a_r)} \int_0^\infty dt_r \, t_r^{a_r-1} \right) \tilde{S}^{(i)}_0(t_1,t_2,t_3|\tau_2) \tag{3.31}$$

Each integrand uniformly decays to zero exponentially fast in any direction as $t_r \to \infty$. When no confusion is expected to arise, we shall often suppress the dependence on the parameters a_1, a_2, a_3 to save notation.

4 The Laurent polynomial

In this section, we shall obtain the Laurent polynomial $\mathcal{L}(\tau_2)$ in the constant Fourier mode $C^{(0)}_{a_1,a_2,a_3}(\tau_2)$ of the modular functions $C_{a_1,a_2,a_3}(\tau)$, and prove Theorem 1.1. To this end, we evaluate the contributions $\mathcal{C}^{(i)}_0(\tau_2)$ for $i = 0, 1, \cdots, 5$ in the subsections below. The remaining exponential contributions $\mathcal{E}(\tau_2)$ to the constant Fourier mode will be evaluated in the subsequent section.

4.1 Evaluating $\mathcal{C}^{(0)}_0$

The integral over $S^{(0)}_0$ evaluates to a sum over $M \in \mathfrak{N}^{(4)}$ which may be parametrized by,

$$\mathcal{C}^{(0)}_0(\tau_2) = \frac{\tau_2^w}{\pi^w} \sum_{m_1 \neq 0} \frac{1}{m_1^{2a_1}} \sum_{m_2 \neq 0, -m_1} \frac{1}{m_2^{2a_2}(m_1+m_2)^{2a_3}} \tag{4.1}$$

To compute the infinite sum over m_2, we proceed by decomposing the summand into partial fractions in m_2, using the general partial fraction decomposition formulas, valid for $a, b \in \mathbb{N}$,

$$\frac{1}{(z+x)^a (z+y)^b} = \sum_{k=1}^{a} \frac{\mathcal{A}_k(a,b)}{(z+x)^k \, (y-x)^{a+b-k}} + \sum_{k=1}^{b} \frac{\mathcal{B}_k(a,b)}{(z+y)^k \, (y-x)^{a+b-k}} \tag{4.2}$$

where $\mathcal{A}_k(a,b)$ and $\mathcal{B}_k(a,b)$ are given by binomial coefficients,

$$\begin{aligned}\mathcal{A}_k(a,b) &= (-)^{a+k}\binom{a+b-k-1}{a-k} \\ \mathcal{B}_k(a,b) &= (-)^{a}\binom{a+b-k-1}{b-k}\end{aligned} \tag{4.3}$$

For the case at hand, we set $x=0$, $y=m_1$, $z=m_2$, for the positive integers exponents $a=2a_2$ and $b=2a_3$, and we find,

$$\begin{aligned}\sum_{m_2\neq 0,-m_1}\frac{1}{m_2^{2a_2}(m_1+m_2)^{2a_3}} &= \sum_{k=1}^{a_2}\frac{2\zeta(2k)}{m_1^{2a_2+2a_3-2k}}\Big(\mathcal{A}_{2k}(2a_2,2a_3)+\mathcal{B}_{2k}(2a_2,2a_3)\Big) \\ &\quad -\frac{1}{m_1^{2a_2+2a_3}}\frac{\Gamma(2a_2+2a_3+1)}{\Gamma(2a_2+1)\Gamma(2a_3+1)}\end{aligned} \tag{4.4}$$

Evaluating next the sum over m_1, and expressing the resulting even ζ-values in terms of Bernoulli numbers using,

$$\zeta(2k)=\frac{1}{2}(2\pi)^{2k}(-)^{k+1}\frac{B_{2k}}{(2k)!} \tag{4.5}$$

we find,

$$\mathcal{C}_0^{(0)}(\tau_2)=(-4\pi\tau_2)^w\sum_{k=0}^{a_2}\frac{B_{2k}B_{2w-2k}}{(2k)!\,(2w-2k)!}\,\frac{\Gamma(2a_2+2a_3-2k)}{\Gamma(2a_3)\Gamma(2a_2-2k+1)}+(a_2\leftrightarrow a_3) \tag{4.6}$$

This formula reproduces correctly the top terms previously evaluated in (1.16).

4.2 Evaluating $\mathcal{C}_0^{(i)}$ for $i=1,2,3$

We evaluate the case $i=3$, the cases $i=1,2$ being obtained by cyclic permutations of a_1,a_2,a_3. The integral over t_3 in $S_0^{(3)}$ may be readily carried since the dependence of $S_0^{(3)}$ on t_3 is entirely contained in $\det T$, whose dependence on t_3 is as follows,

$$\frac{\det T}{t_1+t_2}=t_3+\frac{t_1t_2}{t_1+t_2} \tag{4.7}$$

Carrying out the integral over t_3 gives,

$$\begin{aligned}\mathcal{C}_0^{(3)}(\tau_2) &= \sum_{m,n\neq 0}\left(\frac{\tau_2}{\pi m^2}\right)^{a_3}\prod_{r=1}^{2}\int_0^\infty\frac{dt_r\,t_r^{a_r-1}}{\Gamma(a_r)}\sqrt{\frac{\tau_2}{t_1+t_2}} \\ &\qquad\times\exp\left\{-\pi\tau_2 n^2(t_1+t_2)-\pi\frac{m^2}{\tau_2}\,\frac{t_1t_2}{t_1+t_2}\right\}\end{aligned} \tag{4.8}$$

Parametrizing the integration variables by $t_1 = xt$ and $t_2 = t(1-x)$ with $t \geq 0$ and $0 \leq x \leq 1$, and carrying out the integral in t produces the following result,

$$\mathcal{C}_0^{(3)}(\tau_2) = \frac{\tau_2^w}{\pi^w} \sum_{m,n\neq 0} \frac{1}{|m|^{2w-1}} \, G_{a_1,a_2}\left(\frac{\tau_2 n}{m}\right) \tag{4.9}$$

where the function $G_{a_1,a_2}(\mu)$ is given by the integral representation,

$$G_{a_1,a_2}(\mu) = \frac{\sqrt{\pi}\,\Gamma(a_1+a_2-\frac{1}{2})}{\Gamma(a_1)\Gamma(a_2)} \int_0^1 dx \, \frac{x^{a_1-1}(1-x)^{a_2-1}}{\big(\mu^2 + x(1-x)\big)^{a_1+a_2-\frac{1}{2}}} \tag{4.10}$$

The function $G_{a_1,a_2}(\mu)$ is even in μ, and invariant under interchanging a_1 and a_2. To evaluate it, we shall make use of the following Lemma.

Lemma 4.1 *The function $G_{a_1,a_2}(\mu)$ defined in (4.10) admits the equivalent representations.*

(a) For $\mu \in \mathbb{R}$, and $a_1, a_2 \in \mathbb{C}$ with $\mathrm{Re}\,(a_1), \mathrm{Re}\,(a_2) \geq 1$,

$$G_{a_1,a_2}(\mu) = \int_{\mathbb{R}} du \, \frac{1}{(u^2+\mu^2)^{a_1}((u+1)^2+\mu^2)^{a_2}} \tag{4.11}$$

(b) For $a_1, a_2 \in \mathbb{N}$,

$$G_{a_1,a_2}(\mu) = \left(\sum_{\alpha=0}^{a_1-1} \sum_{\beta=0}^{a_1-1-\alpha} \frac{-i\pi \, g_{a_1,a_2}(\alpha,\beta)}{(2i\mu)^{2a_1-1-\alpha-\beta}(1+2i\mu)^{a_2+\alpha}} + c.c. \right) + (a_1 \leftrightarrow a_2) \tag{4.12}$$

The coefficients $g_{a_1,a_2}(\alpha,\beta)$ are given by the product of binomial coefficients of (1.21).

To prove part *(a)* of Lemma 1, we start from expression (4.11) and use standard techniques for the evaluation of Feynman diagrams in quantum field theory to derive its expression given in (4.10). One makes use of an integral representation formula for a product of denominators,

$$\frac{1}{A_1^{a_1} A_2^{a_2}} = \frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)} \int_0^1 dx \frac{x^{a_1-1}(1-x)^{a_2-1}}{(xA_1+(1-x)A_2)^{a_1+a_2}} \tag{4.13}$$

valid for $A_1, A_2 > 0$ and $\mathrm{Re}\,(a_1), \mathrm{Re}\,(a_2) > 0$, and the evaluation of the resulting u-integral,

$$\int_{\mathbb{R}} du \frac{1}{(u^2+\mu^2)^a} = \frac{\sqrt{\pi}\,\Gamma(a-\frac{1}{2})}{\Gamma(a)\,|\mu|^{2a-1}} \tag{4.14}$$

valid for $\mathrm{Re}\,(a) > \frac{1}{2}$. To prove part *(b)* of Lemma 1, we make use of the fact that, for $a_1, a_2 \in \mathbb{N}$, the integrand in (4.11) is a rational function of u, with poles at $u = \pm i\mu$ and $u = -1 \pm i\mu$. The integral may then be evaluated by standard residue methods.

4.2.1 Calculating the infinite sums for $\mathcal{C}_0^{(3)}$

Formula (4.9) expresses $\mathcal{C}_0^{(3)}$ as a double sum over the function G_{a_1,a_2}. We use the expression for G_{a_1,a_2} obtained in (4.12), interchange the order of the sums over m,n with the sums over α,β, and combine the factors of i with factors of the absolute values $|m|$,

$$\mathcal{C}_0^{(3)} = \frac{\tau_2^w}{\pi^w}\left(\sum_{\alpha=0}^{a_1-1}\sum_{\beta=0}^{a_1-1-\alpha}\sum_{m,n\neq 0}\frac{(-)^w(-i|m|)^{1-2w}\pi\, g_{a_1,a_2}(\alpha,\beta)}{(i|\frac{2\tau_2 n}{m}|)^{2a_1-1-\alpha-\beta}(1+i|\frac{2\tau_2 n}{m}|)^{a_2+\alpha}} + \text{c.c.}\right) + (a_1 \leftrightarrow a_2) \tag{4.15}$$

The role of the complex conjugate contribution is to reverse the sign of $-i|m|$, so that we may omit the absolute value symbol on m and include a factor of 2 to account for the addition of the complex conjugate term. Restricting the sum over n to $n>0$ gives another factor of 2, and rearranging the factors of i, we express $\mathcal{C}_0^{(3)}$ as follows,

$$\mathcal{C}_0^{(3)} = \mathcal{K}(a_1,a_2,a_3) + \mathcal{K}(a_2,a_1,a_3) \tag{4.16}$$

where $\mathcal{K}$ is given by,

$$\mathcal{K}(a_1,a_2,a_3) = \frac{\tau_2^w}{\pi^w}\sum_{\alpha=0}^{a_1-1}\sum_{\beta=0}^{a_1-1-\alpha}\sum_{n=1}^{\infty}\sum_{m\neq 0}\frac{-4\pi i\, g_{a_1,a_2}(\alpha,\beta)}{m^A(m+2i\tau_2 n)^B(2i\tau_2 n)^C} \tag{4.17}$$

where we have used the following abbreviations,

$$\begin{aligned} a &= a_2+2a_3+\beta \\ b &= a_2+\alpha \\ c &= 2a_1-\alpha-\beta-1 \end{aligned} \tag{4.18}$$

The sum over m may be carried out using the partial fraction decomposition formulas of (4.2) and (4.3) for the parameters $x=0$, $y=2i\tau_2 n$ with $n>0$, $z=m$, and the exponents a, b defined in (4.18), and we find,

$$\begin{aligned} \sum_{m\neq 0}\frac{1}{m^a(m+2i\tau_2 n)^b} &= \sum_{k=1}^{[a/2]}\frac{2\zeta(2k)\,\mathcal{A}_{2k}(a,b)}{(2i\tau_2 n)^{a+b-2k}} - \sum_{k=1}^{b}\frac{\mathcal{B}_k(a,b)}{(2i\tau_2 n)^{a+b}} - \frac{i\pi\mathcal{B}_1(a,b)}{(2i\tau_2 n)^{a+b-1}} \\ &\quad + \sum_{k=1}^{b}\frac{\mathcal{B}_k(a,b)}{(2i\tau_2 n)^{a+b-k}}\frac{(-2\pi i)^k}{\Gamma(k)}\sum_{p=1}^{\infty}p^{k-1}e^{-4\pi p n\tau_2} \end{aligned} \tag{4.19}$$

The sum over n is carried out by multiplying the above relation by $-4i(2i\tau_2 n)^{-c}$ with c given in (4.18), and separating the resulting sum into the contributions to the Laurent polynomial part and the contributions to the exponential part,

$$\mathcal{K}(a_1,a_2,a_3) = \mathcal{K}_L(a_1,a_2,a_3) + \mathcal{K}_E(a_1,a_2,a_3) \tag{4.20}$$

The Laurent polynomial part is given by,

$$\begin{aligned}\mathcal{K}_L(a_1,a_2,a_3) &= \sum_{\alpha=0}^{a_1-1}\sum_{\beta=0}^{a_1-1-\alpha} g_{a_1,a_2}(\alpha,\beta)\left[\frac{\zeta(2w-2)\mathcal{B}_1(a,b)}{(-4\pi\tau_2)^{w-2}}\right.\\ &\quad +\frac{2\zeta(2w-1)}{(-4\pi\tau_2)^{w-1}}\frac{(-)^{a_2+\beta}\,\Gamma(2a_2+2a_3+\alpha+\beta)}{\Gamma(a_2+2a_3+\beta+1)\Gamma(a_2+\alpha)}\\ &\quad \left.-\sum_{k=1}^{a_3+[(a_2+\beta)/2]}\frac{4\,(-)^k\zeta(2k)\,\zeta(2w-1-2k)\mathcal{A}_{2k}(a,b)}{(-4\pi\tau_2)^{w-1-2k}\,(2\pi)^{2k}}\right]\end{aligned} \tag{4.21}$$

while the purely exponential part is given by,

$$\mathcal{K}_E(a_1,a_2,a_3)=\sum_{\alpha=0}^{a_1-1}\sum_{\beta=0}^{a_1-1-\alpha} g_{a_1,a_2}(\alpha,\beta)\sum_{k=1}^{a_2+\alpha}\frac{2\,(-)^w\,\mathcal{B}_k(a,b)}{(4\pi\tau_2)^{w-1-k}\,\Gamma(k)}\sum_{n=1}^{\infty}n^{k-1}\sigma_{2-2w}(n)q^n\bar{q}^n \tag{4.22}$$

To simplify the purely exponential part, we have used the standard rearrangement formula,

$$\sum_{n=1}^{\infty}\frac{1}{n^{2w-1-k}}\sum_{p=1}^{\infty}p^{k-1}e^{-4\pi pn\tau_2}=\sum_{n=1}^{\infty}n^{k-1}\sigma_{2-2w}(n)q^n\bar{q}^n \tag{4.23}$$

4.3 Evaluating $\mathcal{C}_0^{(4)}$

We shall use the following integral representation for the factor $\det T$ in $\tilde{S}_0^{(4)}$,

$$\frac{1}{\tau_2(\det T)^{\frac{1}{2}}}=\int_{\mathbb{R}^2}dudv\,e^{-\pi\tau_2\left(t_1u^2+t_2v^2+t_3(u+v)^2\right)} \tag{4.24}$$

in order to decouple the t-integrals in (3.31) for this function, and we obtain,

$$\mathcal{C}_0^{(4)}=\frac{\tau_2^2}{(\pi\tau_2)^w}\sum_{N\in\mathfrak{N}^{4)}}\int_{\mathbb{R}^2}\frac{dudv}{(u^2+n_1^2)^{a_1}(v^2+n_2^2)^{a_2}((u+v)^2+n_3^2)^{a_3}} \tag{4.25}$$

The integral is independent of τ_2 so that $\mathcal{C}_0^{(4)}$ contributes exclusively to the order τ_2^{2-w} in the Laurent polynomial. To evaluate the integral over $\mathbb{R}^2$ and the summation over $\mathfrak{N}^{(4)}$, we shall proceed as follows.

We begin by simplifying the summation over $\mathfrak{N}^{(4)}$. Invariance of the set $\mathfrak{N}^{(4)}$ under permutations of n_1,n_2,n_3 guarantees invariance of $\mathcal{C}_0^{(4)}$ under permutations of a_1,a_2,a_3. We partition $\mathfrak{N}^{(4)}$ into three disjoint subsets, $n_1n_2>0$, $n_2n_3>0$ and $n_1n_3>0$, and we may restrict the summation over $\mathfrak{N}^{(4)}$ to any single one of these subsets provided we add the contribution of the two cyclic permutations of a_1,a_2,a_3. We shall choose the subset

$n_1 n_3 > 0$. Within this subset, the sectors $n_1, n_3 > 0$ and $n_1, n_3 < 0$ contribute equally; we shall restrict to $n_1, n_3 \geq 1$ upon including a factor of 2.

Since the exponents a_1, a_2, a_3 are positive integers, the integrals are over a rational function $f(u,v)$, and may be evaluated by residue methods. Taking the above preparations into account, we obtain the following expression,

$$\begin{aligned}
\mathcal{C}_0^{(4)} &= \frac{2\tau_2^2}{(\pi\tau_2)^w} \sum_{n_1,n_3=1}^{\infty} \int_{\mathbb{R}^2} du\, dv\, |f(u,v)|^2 + 2 \text{ cyclic permutations of } a_1, a_2, a_3 \\
f(u,v) &= \frac{1}{(u+in_1)^{a_1}(v+in_2)^{a_2}(u+v+in_1+in_2)^{a_3}}
\end{aligned} \tag{4.26}$$

The function f is not unique and is chosen such that the last denominator argument is the sum of the preceding two. This choice guarantees that $f(u,v)$ will have a convenient partial fraction expansion in the variable u, the integral over which we shall carry out first,

$$f(u,v) = \sum_{k_1=1}^{a_1} \frac{(-)^{a_1-k_1}\binom{a_1+a_3-k_1-1}{a_3-1}}{(u+in_1)^{k_1}(v+in_2)^{w-k_1}} + \sum_{k_3=1}^{a_3} \frac{(-)^{a_1}\binom{a_1+a_3-k_3-1}{a_1-1}}{(u+v-in_3)^{k_3}(v+in_2)^{w-k_3}} \tag{4.27}$$

Because the sum is restricted to $n_1, n_3 > 0$, the u-integrals in the cross terms of the product $f(u,v)\,\overline{f(u,v)}$ vanish identically since in each case the poles are all either in the upper half plane or all in the lower half plane. Hence only the integrals in the direct terms contribute,

$$\begin{aligned}
\mathfrak{I}_1(n_1,n_3) &= \frac{2^{2w}}{16\pi^2} \int_{\mathbb{R}^2} \frac{(-)^{k_1+\ell_1}\, du\, dv}{(u+in_1)^{k_1}(u-in_1)^{\ell_1}(v+in_2)^{w-k_1}(v-in_2)^{w-\ell_1}} \\
\mathfrak{I}_2(n_1,n_3) &= \frac{2^{2w}}{16\pi^2} \int_{\mathbb{R}^2} \frac{du\, dv}{(u+v-in_3)^{k_3}(u+v+in_3)^{\ell_3}(v+in_2)^{w-k_3}(v-in_2)^{w-\ell_3}}
\end{aligned} \tag{4.28}$$

In terms of these integrals, $\mathcal{C}_0^{(4)}$ is given by,

$$\begin{aligned}
\mathcal{C}_0^{(4)} = \; & \frac{2}{(4\pi\tau_2)^{w-2}} \sum_{k_1=1}^{a_1}\sum_{\ell_1=1}^{a_1} \binom{a_1+a_3-k_1-1}{a_3-1}\binom{a_1+a_3-\ell_1-1}{a_3-1} \sum_{n_1,n_3=1}^{\infty} \mathfrak{I}_1(n_1,n_3) \\
& + \frac{2}{(4\pi\tau_2)^{w-2}} \sum_{k_3=1}^{a_3}\sum_{\ell_3=1}^{a_3} \binom{a_1+a_3-k_3-1}{a_1-1}\binom{a_1+a_3-\ell_3-1}{a_1-1} \sum_{n_1,n_3=1}^{\infty} \mathfrak{I}_2(n_1,n_3) \\
& + 2 \text{ cyclic permutations of } a_1, a_2, a_3
\end{aligned} \tag{4.29}$$

The integrals over u and v are manifestly decoupled from one another in $\mathfrak{I}_1$, as well as in $\mathfrak{I}_2$ after performing the shift $u \to u - v$. Using residue methods, the integrals evaluate to,

$$\begin{aligned}
\mathfrak{I}_1(n_1,n_3) &= \binom{k_1+\ell_1-2}{k_1-1}\binom{2w-k_1-\ell_1-2}{w-k_1-1} \frac{\theta(-n_1n_2) + (-)^{k_1+\ell_1}\theta(n_1n_2)}{|n_1|^{k_1+\ell_1-1}\,|n_2|^{2w-k_1-\ell_1-1}} \\
\mathfrak{I}_2(n_1,n_3) &= \binom{k_3+\ell_3-2}{k_3-1}\binom{2w-k_3-\ell_3-2}{w-k_3-1} \frac{\theta(-n_2n_3) + (-)^{k_3+\ell_3}\theta(n_2n_3)}{|n_3|^{k_3+\ell_3-1}\,|n_2|^{2w-k_3-\ell_3-1}}
\end{aligned} \tag{4.30}$$

The contributions of the integrals $\mathfrak{I}_1$ and $\mathfrak{I}_2$ to $\mathcal{C}_0^{(4)}$ are related by permuting a_1 and a_3. Hence we may retain only the sum involving $\mathfrak{I}_1$ provided we then include all five permutations of a_1, a_2, a_3. Putting all together we have,

$$\mathcal{C}_0^{(4)} = \frac{2}{(4\pi\tau_2)^{w-2}} \sum_{\sigma\in\mathfrak{S}_3} Z(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}) \tag{4.31}$$

where the function $Z(a_1, a_2, a_3)$ was defined in (1.24).

4.4 Evaluating $\mathcal{C}_0^{(5)}$

Combining the second equation of (3.29) with (3.31) we see that $\mathcal{C}_0^{(5)}$ is given by,

$$\mathcal{C}_0^{(5)} = \prod_{r=1}^{3} \int_0^\infty dt_r \, \frac{t_r^{a_r-1}}{\Gamma(a_r)} \sum_{M,N\in\mathfrak{N}^{(4)}} \frac{\tau_2\, \delta_{M^t\varepsilon N,0}}{(\det T)^{\frac{1}{2}}} \exp\left\{-\pi\tau_2 \frac{M^t T M}{\det T} - \pi\tau_2 N^t T N\right\} \tag{4.32}$$

We begin by parametrizing the space of matrices M, N. To satisfy the condition $M^t\varepsilon N = 0$, the column matrices M, N must be proportional to one another, and thus proportional to a common matrix K with integer entries k_1, k_2. Since $M, N \in \mathfrak{N}^{(4)}$, the numbers m_i, n_i, k_i with $i = 1, 2, 3$ are also non-vanishing. The complete solution is given by,

$$M = \mu K \qquad\qquad N = \nu K \qquad\qquad K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \tag{4.33}$$

where we choose k_1 and k_2 relatively prime, $k_1 > 0$, and $\mu, \nu \in \mathbb{Z}$ with $\mu, \nu \neq 0$. Since the summand in $\mathcal{C}_0^{(5)}$ depends only on μ^2, ν^2, k_1^2, k_2^2 and k_3^2, we may include in the sum over k_1 and k_2 pairs with arbitrary signs along with an overall factor of $\frac{1}{2}$, and restrict μ, ν to be positive upon including an overall factor of 4. Finally, we may choose a particular ordering of k_1^2, k_2^2, k_3^2 upon including symmetrization under all six permutations of the variables a_1, a_2, a_3. We shall denote the space of such pairs k_1, k_2 of relatively prime, ordered, integers by $\mathfrak{K}$.

Next, we change integration variables from (t_1, t_2, t_3) to (x_1, x_2, t_3) with $t_1 = x_1 t_3$ and $t_2 = x_2 t_3$ for $x_1, x_2 \geq 0$, and subsequently change variables from (x_1, x_2, t_3) to (x_1, x_2, t) with,

$$t_3 = \left|\frac{\mu}{\nu}\right| \frac{t}{\sqrt{x_1 + x_2 + x_1 x_2}} \tag{4.34}$$

In terms of these new variables, the integrals become,

$$\begin{aligned} \mathcal{C}_0^{(5)} &= 2\tau_2 \sum_{(k_1,k_2)\in\mathfrak{K}} \sum_{\mu,\nu=1}^{\infty} \left(\frac{\mu}{\nu}\right)^{w-1} \int_0^\infty dt \, \frac{t^{w-2}}{\Gamma(a_3)} \prod_{r=1}^{2} \int_0^\infty dx_r \, \frac{x_r^{a_r-1}}{\Gamma(a_r)} \frac{1}{(x_1 + x_2 + x_1 x_2)^{w/2}} \\ &\quad \times \exp\left\{-\pi\tau_2\mu\nu\left(t + \frac{1}{t}\right) \frac{x_1 k_1^2 + x_2 k_2^2 + k_3^2}{\sqrt{x_1 + x_2 + x_1 x_2}}\right\} + \text{ 5 perms of } a_1, a_2, a_3 \end{aligned} \tag{4.35}$$

The Mellin transform of $\mathcal{C}_0^{(5)}$ with respect to τ_2 is defined by,

$$\mathcal{M}\mathcal{C}_0^{(5)}(s) = \int_0^\infty d\tau_2 \, \tau_2^{s-1} \mathcal{C}_0^{(5)}(\tau_2) \tag{4.36}$$

Carrying out the integration over t, and performing the sums over μ and ν we obtain,

$$\mathcal{M}\mathcal{C}_0^{(5)}(s) = \frac{\xi(s+w)\xi(s-w+2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \sum_{(k_1,k_2)\in\mathfrak{K}} \prod_{r=1}^{2} \int_0^\infty dx_r \, x_r^{a_r-1} \frac{(x_1+x_2+x_1x_2)^{\frac{s-w+2}{2}}}{(x_1k_1^2+x_2k_2^2+k_3^2)^{s+1}} \tag{4.37}$$

The function $\xi(s) = \Gamma(s/2)\zeta(s)$ has single poles at $s=0$ and $s=1$ and is analytic elsewhere. We now prove the following Lemma.

Lemma 4.2 *Near the cusp $\tau_2 \to \infty$, the function $\mathcal{C}_0^{(5)}$ decays exponentially,*

$$\mathcal{C}_0^{(5)}(\tau_2) = \mathcal{O}(e^{-4\pi\tau_2}) \tag{4.38}$$

up to factors which are power-behaved in τ_2, controlled by the poles in the Mellin transform.

To prove the Lemma, we shall extract out of the integrand a factor of $e^{-4\pi\tau_2}$. First, we have the standard sharp bound $t + t^{-1} \geq 2$. Secondly, we shall show that the last factor of the exponential has a uniform sharp lower bound by $2k_1k_2$. To prove it, we set $y_1 = x_1 + 1$ and $y_2 = x_2 + 1$, with $y_1, y_2 \geq 1$, in terms of which the expression becomes,

$$\frac{x_1k_1^2 + x_2k_2^2 + k_3^2}{\sqrt{x_1+x_2+x_1x_2}} = \frac{y_1k_1^2 + y_2k_2^2 + 2k_1k_2}{\sqrt{y_1y_2-1}} \tag{4.39}$$

Since $(k_1,k_2) \in \mathfrak{K}$ we have $k_1, k_2 > 0$. Keeping the product y_1y_2 fixed, the combination $y_1k_1^2 + y_2k_2^2$ is bounded from below by $2k_1k_2\sqrt{y_1y_2}$. The resulting lower bound equals $2k_1k_2$ times a function of y_1y_2 which is independent of k_1,k_2 and whose lower bound is 1. Thus, for $(k_1,k_2) \in \mathfrak{K}$, we may extract out of the integrand a factor of $e^{-4\pi\tau_2\mu\nu k_1k_2}$. The Lemma follows from $\mu, \nu, k_1k_2 \geq 1$, and the observation that the remaining integrations converge. Their power behavior is governed by the positions of the poles of the Mellin transform.

4.5 Proof of Theorem 1.1

Having shown that $\mathcal{C}_0^{(5)}$ does not contribute to the Laurent polynomial $\mathcal{L}$, Theorem 1.1 may now be proven by collecting the contributions to $\mathcal{L}$ from the calculations of $\mathcal{C}_0^{(i)}$ for $i = 0,1,2,3,4$ performed in the preceding subsections. From the explicit results in (4.6), (4.16), (4.21) and (4.31), we deduce the following results.

1. c_w receives contributions exclusively from $\mathcal{C}_0^{(0)}$ which was evaluated explicitly in (4.6). The coefficient c_w given by (4.6) is manifestly a rational number. This proves part (a).

2. c_{w-2k-1} for $1 \leq k \leq w-1$ receives contributions exclusively from $\mathcal{C}_0^{(i)}$ with $i=1,2,3$, obtained from the second and third terms under the sum in (4.21). The coefficients c_{w-2k-1} given by (4.6) are manifestly rational numbers. This proves part (b).

3. c_{2-w} receives contributions exclusively from the first term under the sum in (4.21) which gives rise to the term in $\zeta(2w-2)$ in (1.22), as well as from the entire contribution of $\mathcal{C}_0^{(4)}$ in (4.31). Putting both together proves part (c).

5 Differential equations and exponential terms

For given weight $w = a_1 + a_2 + a_3$ the functions $C_{a_1,a_2,a_3}(\tau)$ with $a_1, a_2, a_3 \in \mathbb{N}$ satisfy a system of inhomogeneous linear differential equations whose inhomogeneous part is a linear combination of the non-holomorphic Eisenstein series E_w and products of the form $E_{w-\ell}E_\ell$ with $2 \leq \ell \leq w-2$. Analyzing the general structure of these differential equations and their solutions provides convenient paths towards proving Theorems 1.2 and 1.3, which we shall carry out in the present section.

5.1 Inhomogeneous Laplace-eigenvalue equations

The two-loop modular graph functions satisfy a system of differential equations,

$$\begin{aligned}\left(\Delta - \sum_{i=1}^{3} a_i(a_i-1)\right) C_{a_1,a_2,a_3} \quad &= \quad a_1 a_2 \Big(C_{a_1-1,a_2+1,a_3} + \frac{1}{2} C_{a_1+1,a_2+1,a_3-2} - 2C_{a_1,a_2+1,a_3-1} \Big) \\ &\quad + 5 \text{ permutations of } a_1, a_2, a_3 \end{aligned} \tag{5.1}$$

valid for $a_r \geq 3$ for $r = 1,2,3$. The validity of these equations may be extended to $a_r \geq 1$ by supplementing them with the following degenerate cases,

$$\begin{aligned} C_{a_1,a_2,0} &= E_{a_1} E_{a_2} - E_{a_1+a_2} && a_1 + a_2 \geq 3 \\ C_{a_1,a_2,-1} &= E_{a_1-1} E_{a_2} + E_{a_1} E_{a_2-1} && a_1, a_2 \geq 2 \end{aligned} \tag{5.2}$$

The right side of (5.2) may involve the symbol E_1 (formally corresponding to a divergent series), but its contribution systematically cancel out of the right side of (5.1). For example, the lowest weight cases are as follows,

$$\begin{aligned} \Delta\, C_{1,1,1} &= 6\, E_3 \\ \Delta\, C_{2,1,1} &= 2\, C_{2,1,1} + 9\, E_4 - E_2^2 \\ \Delta\, C_{2,2,1} &= 8\, E_5 \end{aligned} \tag{5.3}$$

The Laplacian preserves the weight $w = a + b + c$ of the modular graph functions, and the equations of (5.1) and (5.2) may be viewed as acting on the space of modular graph functions of given weight w, provided we assign the weight s to the Eisenstein series E_s.

The Fourier series for a non-holomorphic Eisenstein series $E_n(\tau)$ with integer $n \geq 2$, was given in (1.13). It was shown in [3] that the Laplace operator $\Delta = 4\tau_2^2 \partial_\tau \partial_{\bar\tau}$ on the space of functions $C_{a_1,a_2,a_3}(\tau)$ for weight $w = a_1 + a_2 + a_3 \geq 3$ may be "diagonalized" resulting in "eigenfunctions" $\mathcal{C}_{w,s,\mathfrak{p}}(\tau)$ which are linear combinations of the functions $C_{a_1,a_2,a_3}(\tau)$ of weight w which obey the following type of equation,

$$(\Delta - s(s-1))\mathcal{C}_{w,s,\mathfrak{p}}(\tau) = \mathcal{H}_0(w,s,\mathfrak{p}) E_w(\tau) + \sum_{\ell=2}^{w-2} \mathcal{H}_\ell(w,s,\mathfrak{p}) E_{w-\ell}(\tau) E_\ell(\tau) \tag{5.4}$$

where s and $\mathfrak{p}$ are integers running over the following ranges,

$$s = w - 2\mathfrak{m} \qquad\qquad 1 \leq \mathfrak{m} \leq \left[\frac{w-1}{2}\right] \qquad\qquad 0 \leq \mathfrak{p} \leq \left[\frac{s-1}{3}\right] \tag{5.5}$$

The coefficients $\mathcal{H}_\ell$ are combinatorial rational numbers which depend on $a, b, c, s, \mathfrak{p}, \ell$. Decomposing $\mathcal{C}_{w,s,\mathfrak{p}}(\tau)$ into Fourier modes,

$$\mathcal{C}_{w,s,p}(\tau) = \sum_{k \in \mathbb{Z}} \mathcal{C}^{(k)}_{w,s,\mathfrak{p}}(\tau_2)\, e^{2\pi i k \tau_1} \tag{5.6}$$

each Fourier mode satisfies a separate ODE in τ_2,

$$\begin{aligned}
&\Big(\tau_2^2 \partial_{\tau_2}^2 - 4\pi^2 k^2 \tau_2^2 - s(s-1)\Big) \mathcal{C}^{(k)}_{w,s,\mathfrak{p}}(\tau_2) \qquad\qquad (5.7)\\
&\quad = \mathcal{H}_0(w,s,\mathfrak{p}) \int_0^1 d\tau_1 e^{-2\pi i k \tau_1} E_w(\tau) + \sum_{\ell=2}^{w-2} \mathcal{H}_\ell(w,s,\mathfrak{p}) \int_0^1 d\tau_1 e^{-2\pi i k \tau_1} E_{w-\ell}(\tau) E_\ell(\tau)
\end{aligned}$$

5.2 Proof of Theorem 1.2

To prove Theorem 1.2, we concentrate on the constant Fourier mode for $k = 0$ and in particular on its contribution proportional to the functional behavior $c_{2-w}(4\pi\tau_2)^{2-w}$. The Fourier transform on E_w makes vanishing contribution to this power of τ_2, and the contributions of the bilinears $E_{w-\ell} E_\ell$ are readily evaluated using (1.13), and we find,

$$\begin{aligned}
&(w - s - 1)(w + s - 2) c_{2-w} \\
&\qquad = \sum_{\ell=2}^{w-2} 16 \mathcal{H}_\ell(w,s,\mathfrak{p}) \binom{2w-2\ell-3}{w-\ell-1} \binom{2\ell-3}{\ell-1} \zeta(2w-2\ell-1)\zeta(2\ell-1)
\end{aligned} \tag{5.8}$$

Given that the allowed values of s in the spectral decomposition of the functions C_{a_1,a_2,a_3} onto the basis of eigenfunctions $\mathfrak{C}_{w;s;\mathfrak{p}}$ is given by (5.5), and that this decomposition is with rational coefficients, it is clear that the kernel of the operator on the left side of the above equation must vanish. Therefore, c_{2-w} must be a linear combination, with rational coefficients, of products of pairs of odd ζ-values whose weights sum to $2w-2.$, thus proving Theorem 1.2. In particular, any contributions valued in $\pi^{2w-2}\mathbb{Q}$ must vanish.

5.3 Proof of Theorem 1.3

We shall now concentrate on the constant Fourier mode $\mathcal{C}^{(0)}_{w,s,\mathfrak{p}}(\tau_2)$. Having already determined the Laurent polynomial contribution to $C_{a_1,a_2,a_3}(\tau)$ at the cusp we shall consider here its exponential contribution, which we denote by $\mathcal{C}^{(e)}_{w,s,\mathfrak{p}}(\tau_2)$. The contribution to $\mathcal{C}^{(e)}_{w,s,\mathfrak{p}}(\tau_2)$ from the function E_w vanishes, so we are left with the contributions from the products $E_{w-\ell}E_\ell$ with $2 \leq \ell \leq w-2$, which obeys the following differential equation,

$$\Big(\tau_2^2\partial_{\tau_2}^2 - s(s-1)\Big)\mathcal{C}^{(e)}_{w,s,\mathfrak{p}}(\tau_2) = \sum_{\ell=2}^{w-2}\mathcal{H}_\ell(w,s,\mathfrak{p})\int_0^1 d\tau_1 E_{w-\ell}(\tau)E_\ell(\tau)\Big|_{\text{exp}} \tag{5.9}$$

The right side is readily evaluated and we find,

$$\begin{aligned}\Big(\tau_2^2\partial_{\tau_2}^2 - s(s-1)\Big)\mathcal{C}^{(e)}_{w,s,\mathfrak{p}}(\tau_2) \; &= \; \sum_{\ell=2}^{w-2}\mathcal{H}_\ell(w,s,\mathfrak{p})\sum_{n=1}^{\infty}\frac{\sigma_{1-2w+2\ell}(n)\sigma_{1-2\ell}(n)}{(w-\ell+1)!(\ell-1)!} \\ &\qquad \times 8\, n^{w-2}\, e^{-4\pi\tau_2 n}P_{w-\ell}(4\pi\tau_2 n)P_\ell(4\pi\tau_2 n)\end{aligned} \tag{5.10}$$

where the polynomials P_n are given by (1.14). The general structure of this equation, in terms of the variable $y = 4\pi\tau_2$ is as follows,

$$\Big(y^2\partial_y^2 - s(s-1)\Big)f_s(y) = \sum_{n=1}^{\infty}\sum_{m=0}^{w-2} f_{m,n}\frac{e^{-ny}}{(ny)^m} \tag{5.11}$$

To solve this equation, we solve for each power of m on the right side of the above equation,

$$\Big(y^2\partial_y^2 - s(s-1)\Big)\varphi_{s,m}(y) = \frac{e^{-y}}{y^m} \tag{5.12}$$

so that f_s is given by,

$$f_s(y) = \sum_{n=1}^{\infty}\sum_{m=0}^{w-2} f_{m,n}\varphi_{s,m}(ny) \tag{5.13}$$

The solution for $\varphi_{s,m}(y)$ is given in terms of the incomplete Γ-function, defined by,

$$\Gamma(a,x) = \int_x^\infty dt\, t^{a-1}\, e^{-t} \tag{5.14}$$

and we find,

$$\varphi_{s,m}(y) = \frac{y^s}{2s-1}\Gamma(-s-m,y) + \frac{y^{1-s}}{1-2s}\Gamma(s-1-m,y) \tag{5.15}$$

Since we are interested in solutions with exponential decay, we have set the homogeneous part of the solution to zero. Using the recursion relation for the incomplete Γ-function,

$$\Gamma(a+1,y) = a\Gamma(a,y) + y^a\, e^{-y} \tag{5.16}$$

we may recast the result solely in terms of elementary functions and the exponential integral $\mathrm{Ei}_1(y) = \Gamma(0,y)$. Simplifying the y^s term, we find,

$$\frac{y^s}{2s-1}\Gamma(-s-m,y) = \gamma^0\, y^s\, \mathrm{Ei(y)} + \sum_{k=1}^{s+m} \gamma_k\, y^{s-k}\, e^{-y} \tag{5.17}$$

for rational coefficients γ_0, γ_k whose value will not concern us here. Simplifying the y^{1-s} term, the cases $s \geq m+2$ and $s \leq m+1$ must be distinguished, and we have,

$$\begin{aligned}
&s \leq m+1 \qquad && \frac{y^{1-s}}{1-2s}\,\Gamma(s-1-m,y) = \gamma^1 y^{1-s}\,\mathrm{Ei}(y) + \sum_{k=1}^{1-s+m} \tilde\gamma_k\, y^{1-s+k}\, e^{-y} \\
&s \geq m+1 \qquad && \frac{y^{1-s}}{1-2s}\,\Gamma(s-1-m,y) = \sum_{k=0}^{s-m-2} \tilde\gamma_k\, y^{1-s+k}\, e^{-y}
\end{aligned} \tag{5.18}$$

Thus, the solution $\varphi_{s,m}(y)$ takes the form,

$$\varphi_{s,m}(y) = \gamma^0\, y^s\, \mathrm{Ei}(y) + \gamma^1 y^{1-s}\,\mathrm{Ei}(y) + \sum_{k=k_-}^{s-1} \gamma_k\, y^k\, e^{-y} \tag{5.19}$$

where $k_- = \min(-m, 1-s)$ and the constant γ^1 vanishes whenever $s \geq m+1$. Summing now over the range of m in (5.8), and taking into account that $s \leq w-2$, we obtain,

$$f_s(y) = f_+\, y^s\, \mathrm{Ei}(y) + f_-\, y^{1-s}\,\mathrm{Ei}(y) + \sum_{k=2-w}^{w-3} f_k\, y^k\, e^{-y} \tag{5.20}$$

When $s = w-2$ then we have $m \leq s$ and we must have $f_- = 0$. In other cases, $f\pm$ will generically be non-zero. This completes the proof of Theorem 1.3.

5.4 The example of $C_{2,1,1}$

We may work things out explicitly for the simplest case where the right side of the differential equation has a term which is non-linear in the Eisenstein series, and satisfies,

$$(\Delta - 2)C_{2,1,1} = 9E_4 - E_2^2 \tag{5.21}$$

The reduced differential equation for the exponential part of the constant Fourier mode is,

$$\Big(\tau_2^2 \partial_{\tau_2}^2 - 2\Big)\mathcal{C}^{(e)}_{2,1,1}(\tau_2) = -\frac{4}{3}\sum_{n=1}^{\infty} n^2 \sigma_{-3}(n)^2 e^{-4\pi\tau_2 n} P_2(4\pi\tau_2 n)^2 \tag{5.22}$$

where the polynomials P_n were defined in (1.14). Collecting the solution to the homogeneous part, and then summing over the particular solutions for each order in N of the inhomogeneous part we find,

$$\mathcal{C}^{(e)}_{2,1,1}(\tau_2) = -\frac{4}{3}\sum_{n=1}^{\infty} n^2 \sigma_{-3}(n)^2 \varphi(4\pi\tau_2 n) \tag{5.23}$$

where the function φ satisfies the n-independent differential equation,

$$\Big(y^2 \partial_y^2 - 2\Big)\varphi(y) = e^{-y} P_2(y)^2 \tag{5.24}$$

whose solution is as follows,

$$\varphi(y) = \frac{e^{-y}}{y^2} \tag{5.25}$$

This result is consistent with the result of Theorem 1.3 for $s = 2$.

6 The conjectured decomposition formula

Theorem 1.2 states that the coefficient c_{2-w} is given by a linear combination of products of pairs of odd ζ-values whose weights add up to $2w-2$, but the theorem does not provide the coefficients γ_ℓ in (1.26). In this section we shall obtain an explicit formula for γ_ℓ in the decomposition of c_{2-w}. Using Maple, we have accumulated extensive numerical evidence for the validity of this formula, but we have no complete analytical proof. The decomposition formula reproduces all the special cases obtained in earlier work and given in (1.16). To develop the required decomposition formula, we begin by proving the following Lemma.

Lemma 6.1 *The linear combinations of depth-two ζ-values,*

$$\begin{aligned}\mathcal{S}(M,N) &= \zeta(2M-1,2N+1)+\sum_{\ell=0}^{N-1}\frac{\varphi_\ell(M,N)\,\Gamma(2M+2\ell)}{(2\ell+2)!\,\Gamma(2M-1)}\zeta(2M+2\ell,2N-2\ell)\\ \mathcal{T}(M,N) &= \zeta(2N+1,2M-1)+\sum_{\ell=0}^{N-1}\frac{\varphi_\ell(M,N)\,\Gamma(2M+2\ell)}{(2\ell+2)!\,\Gamma(2M-1)}\zeta(2N-2\ell,2M+2\ell)\end{aligned} \tag{6.1}$$

obey the following relations.

(a) For $N=0$, the function $\mathcal{S}(M,0)$ is given by,

$$\mathcal{S}(M,0)=\frac{1}{2}(2M-1)\zeta(2M)-\frac{1}{2}\sum_{j=1}^{2M-3}\zeta(j+1)\zeta(2M-1-j) \tag{6.2}$$

(b) For $N\geq 1$, the functions $\mathcal{S}(M,N)$ and $\mathcal{T}(M,N)$ are related as follows,

$$\begin{aligned}\mathcal{S}(M,N) &= -\mathcal{T}(M,N)+\zeta(2M-1)\zeta(2N+1)-\zeta(2M+2N) \\ &\quad+\sum_{\ell=0}^{N-1}\frac{\varphi_\ell(M,N)\,\Gamma(2M+2\ell)}{(2\ell+2)!\,\Gamma(2M-1)}\Big(\zeta(2M+2\ell)\zeta(2N-2\ell)-\zeta(2M+2N)\Big)\end{aligned} \tag{6.3}$$

(c) The functions $\mathcal{S}(M,N)$ and $\mathcal{T}(M,N)$ reduce to a linear combination of products of two odd ζ-values, plus a term valued in $\pi^{2w-2}\mathbb{Q}$, provided the coefficients $\varphi_\ell(M,N)$ are given in terms of the Euler polynomial $E_{2\ell+1}$ by,

$$\varphi_\ell(M,N)=-(2\ell+2)E_{2\ell+1}(0) \tag{6.4}$$

The coefficients φ_ℓ are integer-valued.

(d) For $N\geq 1$, the function $\mathcal{T}(M,N)$ then evaluates as follows,

$$\begin{aligned}\mathcal{T}(M,N) &= \sum_{\alpha=2}^{2M+2N-2}(-)^{\alpha+1}\zeta(\alpha)\zeta(2M+2N-\alpha)\\ &\qquad\times\sum_{n=0}^{2N-1}\frac{1}{2}E_n(0)\binom{\alpha-1}{2N-n}\binom{2M+n-2}{n}\end{aligned} \tag{6.5}$$

While there have been many investigations into the decomposition of linear combinations of double ζ-values onto products of single ζ-values, such as for example in [23, 24, 25], the authors have not been able to find the precise statements of Lemma 6.1 in the literature.[3]

[3]We thank the referee for pointing out that the fact that the sum is totally reducible to a polynomial in the odd weight zeta values shows that the coefficient is a single-valued multiple zeta value.

6.1 Proof of Lemma 6.1

The proof of part *(a)* follows from a well-known formula of Euler, valid for integer $s \geq 2$,

$$\zeta(s,1) = \frac{s}{2}\zeta(s+1) - \frac{1}{2}\sum_{j=1}^{s-2}\zeta(j+1)\zeta(s-j) \tag{6.6}$$

The proof of part *(b)* is obtained by adding the expressions for $\mathcal{S}(M,N)$ and $\mathcal{T}(M,N)$ of (6.1) and using Euler's reflection formula, valid for $s,t \in \mathbb{C}$,

$$\zeta(s,t) + \zeta(t,s) \;\; = \;\; \zeta(s)\zeta(t) - \zeta(s+t) \tag{6.7}$$

The proof of part *(c)* is more involved. Since the sum $\mathcal{S}(M,N)+\mathcal{T}(M,N)$ reduces to the product of two odd ζ-values plus a term valued in $\pi^{2w-2}\mathbb{Q}$ by part *(b)*, it will suffice to prove reducibility for $\mathcal{T}(M,N)$. To do so we express the depth-two ζ-functions in the definition of $\mathcal{T}(M,N)$ in (6.1) in terms of a double sum using (1.25),

$$\mathcal{T}(M,N) = \frac{1}{2}\sum_{m,n=1}^{\infty} F(m,n) \tag{6.8}$$

The function $F(m,n)$ is chosen to be symmetric in m,n, and is given by,

$$\begin{aligned} F(m,n) \;\; = \;\; & \frac{1}{(m+n)^{2N+1}\, m^{2M-1}} + \frac{1}{(m+n)^{2N+1}\, n^{2M-1}} \\ & + \sum_{\ell=0}^{N-1} \frac{\varphi_\ell(M,N)\,\Gamma(2M+2\ell)}{(2\ell+2)!\,\Gamma(2M-1)} \left(\frac{1}{(m+n)^{2N-2\ell}\, m^{2M+2\ell}} + \frac{1}{(m+n)^{2N-2\ell}\, n^{2M+2\ell}} \right) \end{aligned} \tag{6.9}$$

Next, we obtain the conditions on the coefficients $\varphi_\ell(M,N)$ required for reducibility of $\mathcal{T}(M,N)$, and show that $\varphi_\ell(M,N)$ is independent of M,N. To do so, we proceed by partial fraction decomposition of F in the variable m (or equivalently in n). Homogeneity of F in the variables m,n of degree $2M+2N$ restricts the decomposition to the following form,

$$F(m,n) = \sum_{\alpha=0}^{2N} \frac{\tilde{f}_\alpha(M,N)}{(m+n)^{2N+1-\alpha}\, n^{2M+\alpha-1}} + \sum_{\alpha=1}^{2M+2N-2} \frac{f_\alpha(M,N)}{m^\alpha\, n^{2M+2N-\alpha}} \tag{6.10}$$

We determine the coefficients $\varphi_\ell(M,N)$ by requiring $\tilde{f}_\alpha(M,N) = 0$ for all $0 \leq \alpha \leq 2N$, namely the absence of double ζ-values in the sum. Although the system appears overdetermined with $2N+1$ conditions on N variables φ_ℓ with $0 \leq \ell \leq N-1$, the symmetry in m,n guarantees that it is uniquely solvable. To set $\tilde{f}_\alpha(M,N) = 0$, we require that

$(m+n)^{2N+1}F(m,n)$ and its $2N$ derivatives in m vanish at $m=-n$. The vanishing of $\tilde f_0(M,N)$ is automatic. For $1\le\alpha\le 2N$, the condition $\tilde f_\alpha=0$ is given by,

$$\sum_{\ell=0}^{N-1}\frac{\varphi_\ell(M,N)}{2\ell+2}\left(\delta_{2\ell+1,\alpha}+\binom{\alpha}{2\ell+1}\right)=1 \tag{6.11}$$

To solve these conditions, we proceed as follows. The sum over ℓ receives no contributions from $2\ell+1>\alpha$. Therefore, the upper range of the sums may be replaced by $[\alpha/2]$, and the conditions on $\varphi_\ell(M,N)$ are now independent of M and N, and so are their solutions. Separating the equations for even and odd α, we set respectively $\alpha=2p+1$ and $\alpha=2p+2$. The relations determining $\varphi_\ell=\varphi_\ell(M,N)$ are thus given as follows,

$$\sum_{\ell=0}^{p}\frac{\varphi_\ell}{2\ell+2}\left(\delta_{\ell,p}+\binom{2p+1}{2\ell+1}\right)=\sum_{\ell=0}^{p}\frac{\varphi_\ell}{2\ell+2}\binom{2p+2}{2\ell+1}=1 \tag{6.12}$$

These equations are equivalent to one another. We solve the second equation by multiplying by $x^{2p+1}/\Gamma(2p+2)$ and summing over all $p\ge 0$. The sum over p may be carried out explicitly in terms of hyperbolic functions. Introducing the generating function,

$$\varphi(x)=\sum_{\ell=0}^{\infty}\frac{\varphi_\ell\, x^{2\ell}}{\Gamma(2\ell+3)} \tag{6.13}$$

it is found that $\varphi(x)$ must satisfy $\varphi(0)=\frac{1}{2}$ and the following linear differential equation,

$$1=x\varphi'(x)+\left(1+\frac{x}{\mathrm{th}(x)}\right)\varphi(x) \tag{6.14}$$

Its integral is given by $\varphi(x)=(e^x-1)x^{-1}(e^x+1)^{-1}$. In terms of the Euler polynomials $E_n(t)$ which are defined by,

$$\frac{2\,e^{xt}}{e^x+1}=\sum_{n=0}^{\infty}E_n(t)\frac{x^n}{n!} \tag{6.15}$$

we find the explicit expression $\varphi_\ell=-(2\ell+2)E_{2\ell+1}(0)$. This completes the proof of part *(c)*. While $E_{2\ell+1}(0)$ are generally rational numbers, the coefficients φ_ℓ are integers.

To prove part *(d)*, we determine the coefficients $f_\alpha(M,N)$ in (6.10). Using the expression we have obtained for φ_ℓ in (6.4), we readily find $f_1(M,N)=0$. Furthermore, symmetry of F in m,n implies the reflection property, $f_{2M+2N-\alpha}(M,N)=f_\alpha(M,N)$. Combining (6.8) and (6.10) with $\tilde f_\alpha(M,N)=0$, we obtain $\mathcal{T}(M,N)$ by summing over m,n,

$$\mathcal{T}(M,N)=\sum_{\alpha=2}^{2M+2N-2}\frac{1}{2}f_\alpha(M,N)\,\zeta(\alpha)\,\zeta(2M+2N-\alpha) \tag{6.16}$$

To compute the expressions for the remaining coefficients $f_\alpha(M,N)$, we match the poles in n at $n=0$ between the expressions (6.9) and (6.10) with $\tilde f_\alpha(M,N)=0$, and we obtain,

$$f_\alpha(M,N) = (-)^{\alpha+1} \sum_{n=0}^{2N-1} E_n(0) \binom{\alpha-1}{2N-n} \binom{2M+n-2}{n} \tag{6.17}$$

Note that $E_0(0)=1$, and $E_{2\ell}(0)=0$ for $\ell \in \mathbb{N}$, which simplifies the above sum to ranging only over $n=0$ and the odd positive integers. Substituting the expression for $f_\alpha(M,N)$ into (6.16) gives (6.5) and thereby proves part (d).

6.2 Derivation of the decomposition formula

The starting point for the decomposition formula is the expression for c_{2-w} in terms of double ζ-values, given in (1.22), (1.23) and (1.24) of Theorem 1.1. We change summation variable in the definition of $Z(a_1,a_2,a_3)$ in (1.24) from ℓ to $m=k+\ell-1$, so that we have,

$$\begin{aligned} Z(a_1,a_2,a_3) &= \sum_{m=1}^{2a_1-1} \sum_{k=k_-}^{k_+} \binom{a_1+a_2-k-1}{a_2-1} \binom{a_1+a_2-m+k-2}{a_2-1} \\ &\qquad \times \binom{m-1}{k-1} \binom{2w-m-3}{w-k-1} \zeta(2w-m-2,m) \end{aligned} \tag{6.18}$$

where $k_\pm$ are defined by $k_+ = \min(a_1,m)$ and $k_- = \max(1, m-a_1+1)$. Next, we eliminate the contributions of the odd-odd double ζ-values, which arise when m is odd, in favor of the function $\mathcal{S}$ plus even-even ζ-values. The most interesting formula is obtained upon symmetrizing in a_2 and a_3, and we find,

$$\begin{aligned} Z(a_1,a_2,a_3) + Z(a_1,a_3,a_2) &= \sum_{n=0}^{a_1-1} \Big(Z_n(a_1,a_2,a_3) + Z_n(a_1,a_3,a_2) \Big) \mathcal{S}(w-1-n,n) \\ &\qquad + \sum_{n=1}^{a_1-1} X_n(a_1,a_2,a_3) \zeta(2w-2n-2,2n) \end{aligned} \tag{6.19}$$

where $Z_\alpha(a_1,a_2,a_3)$ is given in (1.29), and the function $X_n(a_1,a_2,a_3)$ is defined by,

$$\begin{aligned} X_n(a_1,a_2,a_3) &= \sum_{\ell=n}^{a_1-1} E_{2\ell-2n+1}(0) Z_\ell(a_1,a_2,a_3) \binom{2w-2n-3}{2w-2\ell-4} \\ &\quad + \sum_{k=k'_-}^{k'_+} \binom{a_1+a_2-k-1}{a_2-1} \binom{a_1+a_2-2n+k-2}{a_2-1} \\ &\qquad \times \binom{2n-1}{k-1} \binom{2w-2n-3}{w-k-1} + (a_2 \leftrightarrow a_3) \end{aligned} \tag{6.20}$$

where $k'_\pm$ are defined by $k'_+ = \min(a_1, 2n)$ and $k'_- = \max(1, 2n - a_1 + 1)$. The first sum on the right arises from the elimination in (1.24) of the odd-odd double ζ-values in favor of $\mathcal{S}$, while the second term arises from the contribution of the even-even ζ-values to (1.24). The symmetrization in a_2, a_3 applies to the entire expression. The formula for X_n may be rendered more explicit by expressing Z_ℓ as the sum given in (1.29), changing variables $n \to a_1 - n$ and $\ell \to a_1 - \ell$, and recognizing that the second sum may be combined with the first corresponding to the Euler polynomial with index zero, $E_0(0) = 1$, and we find,

$$\begin{aligned} X_{a_1-n}(a_1,a_2,a_3) &= \sum_{k=0}^{a_1-1}\sum_{\ell=0}^{a_1-1} \theta(2n-k-\ell-1) E_{2n-k-\ell-1}(0) \binom{a_2-1+k}{k}\binom{a_2-1+\ell}{\ell} \\ & \quad \times \binom{2a_1-k-\ell-2}{a_1-k-1}\binom{2a_+ 2a_3+k+\ell-2}{a_2+a_3+k-1}\binom{2a_2+2a_3+2n-3}{2n-k-\ell-1} \\ & \quad +(a_2 \leftrightarrow a_3) \end{aligned} \tag{6.21}$$

Note that the index of the Euler polynomial is now allowed to run over even and odd integers though, of course, we have $E_{2k}(0) = 1$ for all $k \geq 1$.

Conjecture 6.2 *For $a_1, a_2, a_3 \in \mathbb{N}$ and for $1 \leq n \leq a_1 - 1$ the following identities hold,*

$$X_n(a_1,a_2,a_3) = 0 \tag{6.22}$$

Using MAPLE, the conjecture has been verified by showing that $X_n(a_1, a_2, a_3) = 0$ as functions of a_2 and a_3 for all n in the range $1 \leq n \leq a_1 - 1$ and for $1 \leq a_1 \leq 65$. An analytical proof of the conjecture is, however, outstanding.

Assuming the validity of Conjecture 6.2, we prove the *decomposition formula* of Conjecture 1.4 by expressing c_{2-w} in terms of $\mathcal{S}$. To this end, we combine the formulas (1.22), (1.23), (6.19) and the result of Conjecture 6.2, to obtain,

$$c_{2-w} = c^0_{2-w}\zeta(2w-2) + 2\sum_{\sigma\in\mathfrak{S}_3}\sum_{n=0}^{a_1-1} Z_n(a_1,a_2,a_3)\mathcal{S}(w-1-n,n) \tag{6.23}$$

Evaluating $\mathcal{S}$ with the help of formulas (6.2), (6.3) and (6.5) of Lemma 6.1 allows us to express $\mathcal{S}$ in terms of a sum over pairs of odd ζ-values whose weights add up to $2w-2$, as well as terms taking values in $\pi^{2w-2}\mathbb{Q}$ (arising from the contributions of even ζ-values in (6.2), (6.3) and (6.5), the latter for even α). By Theorem 1.2, the contributions valued in $\pi^{2w-2}\mathbb{Q}$ cancel. The remaining contributions give γ_k of (1.28) as follows. The first term on the right in (1.28) arises from $\mathcal{S}(w-1,0)$ given by (6.2), the second term from the second term on the right in the sum of $\mathcal{S}$ and $\mathcal{T}$ in (6.3), and the remaining terms from the terms for odd values of α in the decomposition of $\mathcal{T}$ in (6.5). This completes the proof of the decomposition formula of Conjecture 1.4) assuming the validity of Conjecture 6.2.

Acknowledgments

It is a pleasure to thank Michael Green and Pierre Vanhove for many discussions and earlier collaborations out of which the present work has grown. We are also happy to thank Justin Kaidi for some assistance in articulating Lemma 1 and for helpful comments on the manuscript. ED is grateful to the Kavli Institute for Theoretical Physics in Santa Barbara for their hospitality while part of this work was being completed. This research is supported in part by the National Science Foundation under research grants PHY-16-19926 and PHY-1125915 for ED and DMS 1701638 for WD. ED is pleased to acknowledge support by a Fellowship from the Simons Foundation.

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