LINKING NUMBERS AND MODULAR COCYCLES

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Abstract. It is known that the 3-manifold \( \text{SL}(2, \mathbb{Z}) \backslash \text{SL}(2, \mathbb{R}) \) is diffeomorphic to the complement of the trefoil knot in \( S^3 \). As is shown by E. Ghys the linking number of the trefoil with a modular knot associated to a hyperbolic conjugacy is related to the classical Dedekind symbol. These symbols arose historically in the transformation property of the logarithm of Dedekind’s eta function. In this paper we study the linking numbers between modular knots associated to two hyperbolic conjugacy classes. To this end we give a generalization of the Dedekind symbol. These new symbols appear in the transformation property of analogs of Dedekind’s eta function. They are also related to the special values of a certain Dirichlet series associated to weight 2 modular integrals.

1. Introduction

Let \( G = \text{SL}(2, \mathbb{R}) \) and \( \Gamma = \text{SL}(2, \mathbb{Z}) \). The homogeneous space \( \Gamma \backslash G \) is diffeomorphic to \( \mathcal{M} = S^3 - \text{Trefoil} \), the complement of a trefoil knot in the 3-sphere \( S^3 \). In [21] Milnor gives a proof (that he attributes to Quillen) of this remarkable fact. The diagonal geodesic flow on \( \Gamma \backslash G \) has arithmetically interesting periodic orbits. Suppose that \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma \) is a primitive hyperbolic element with an eigenvalue \( \epsilon > 1 \). Fix a \( g \in G \) so that \( g^{-1} \gamma g = \left( \begin{smallmatrix} \epsilon & 0 \\ 0 & 1/\epsilon \end{smallmatrix} \right) \). Then

\[
\Gamma g \mapsto \Gamma g \left( \begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix} \right)
\]

where \( t \in [0, \log \epsilon] \) gives a primitive oriented closed orbit in \( \Gamma \backslash G \) which depends only on the conjugacy class of \( \gamma \). The image of this orbit in \( \mathcal{M} \) is a modular knot. Ghys [11] gave the beautiful result that the linking number of this knot with the trefoil (with some orientation) is given by the Rademacher symbol

\[
\Psi(\gamma) = \Phi(\gamma) - 3 \text{sgn}(c(a + d)).
\]

Here \( \Phi(\gamma) \) is the Dedekind symbol defined for all \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma \) by

\[
\Phi(\gamma) = \begin{cases} 
\frac{b}{d} & \text{if } c = 0 \\
\frac{a+d}{c} - 12 \text{sgn } c \cdot s(a,c) & \text{if } c \neq 0,
\end{cases}
\]

where \( s(a,c) \) is the Dedekind sum defined for \( \gcd(a,c) = 1, c > 0 \) by

\[
s(a,c) = \sum_{n=1}^{\lfloor c \rfloor - 1} \left( \frac{n}{c} \right) \left( \frac{na}{c} \right).
\]

As usual, \( \left( \frac{x}{y} \right) = 0 \) if \( x \in \mathbb{Z} \) and otherwise \( \left( \frac{x}{y} \right) = x - \lfloor x \rfloor - 1/2 \).

The Rademacher symbol defined for all \( \gamma \in \Gamma \) by (1.1) is a conjugacy class invariant [25] and, for \( \gamma \) hyperbolic, it is the homogenization of the Dedekind symbol \( \Phi(\gamma) \) [4]. More
\[ \Psi(\gamma) = \lim_{n \to \infty} \frac{\Phi(\gamma^n)}{n} \]

In addition to its role here, the Dedekind sum \( s(a,c) \) occurs in surprisingly diverse contexts (see e.g. [2], [25], [15]). Among its many properties we note here only the famous reciprocity formula for \( a, c > 0 \)
\[ s(a,c) - s(-c,a) = \frac{1}{12} \left( \frac{a^2 + c^2 + 1}{ac} \right) - \frac{1}{4}. \]

The Dedekind symbol arose in Dedekind’s [6] evaluation of the transformation law for the logarithm of
\[ \Delta(z) = q \prod_{m \geq 1} (1 - q^m)^{24} \]
where as usual \( q = e(z) = e^{2\pi i z} \) for \( z \in \mathbb{H} \). Thus for any \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma \) we have
\[ \log \Delta(\gamma z) - \log \Delta(z) = 6 \log(-(cz+d)^2) + 2\pi i \Phi(\gamma), \]
where \( \Phi(\gamma) \) is given by the formula (1.2) and where we choose \( \arg(-(cz+d)^2) \in (-\pi, \pi) \).

Sarnak [26] applied the modular forms connection to study the distribution of modular knots with a given linking number by means of the trace formula. See also [22]. At the end of his paper Ghys mentions the problem of interpreting the linking number between two modular knots. In this paper we will approach this question by giving an appropriate generalization of the Dedekind symbol. To define this, let \( \mathcal{P} \) be the space of holomorphic functions \( f \) on \( \mathbb{H} \) such that \( f(z) \ll y^{\alpha} + y^{-\alpha} \) for some \( \alpha \) depending on \( f \). For any integer \( k \in 2\mathbb{Z}, \gamma \in \Gamma \) acts on \( \mathcal{P} \) by the usual slash action defined via \( f|_k \gamma = (cz+d)^{-k} f(\gamma z) \). A 1-cocycle of weight \( k \) for \( \Gamma \) with coefficients in \( \mathcal{P} \) is a map \( \Gamma \to \mathcal{P} \) given by \( \gamma \mapsto r(\gamma,z) \) with
\[ r(\sigma \gamma, z) = r(\sigma, z)|_k \gamma + r(\gamma, z) \]
for all \( \gamma, \sigma \in \Gamma \). Now given a 1-cocycle \( r(\gamma, z) \) of weight 2 for \( \Gamma \) there will be a unique 1-cocycle \( R(\gamma, z) \) of weight 0 for \( \Gamma \) such that
\[ \frac{d}{dz} R(\gamma, z) = r(\gamma, z), \]
the uniqueness following from the fact that \( H^1(\Gamma, \mathbb{Z}) = \{0\} \). We call \( R(\gamma, z) \) the primitive of \( r(\gamma, z) \).

The weight 2 cocycle relevant to the Dedekind sum is given for \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \) by
\[ r(\gamma, z) = \frac{12c}{cz+d}. \]

It follows from (1.6) that the primitive for this cocycle is
\[ R(\gamma, z) = 6 \log(-(cz+d)^2) + 2\pi i \Phi(\gamma), \]
provided \( c \neq 0 \), from which we have the limit formula for \( \Phi(\gamma) \) in (1.2):
\[ \Phi(\gamma) = \frac{1}{2\pi} \lim_{y \to \infty} \text{Im} R(\gamma, iy). \]

As an attempt to generalize the linking number formula of Ghys to two closed orbits, we will associate to any conjugacy class \( \mathcal{C} \) of hyperbolic \( \sigma \in \Gamma \) with \( \text{tr} \sigma > 2 \) the weight two 1-cocycle defined for \( c \neq 0 \) and \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma \) by
\[ r_c(\gamma, z) := \varepsilon_c \sum \frac{1}{z-w} - \frac{1}{z-w'}, \]
where \( \varepsilon_c \) is chosen so that
\[ r_c(\gamma, z) \ll \varepsilon_c \sum \frac{1}{z-w} - \frac{1}{z-w'} \].
where the sum is over the fixed points $w', w$ of $\sigma \in \mathcal{C}$, satisfying $w' < -d/c < w$ and

$$\varepsilon_{\mathcal{C}} = \begin{cases} 1 & \text{if } \sigma \not\sim \sigma^{-1} \\ 2 & \text{if } \sigma \sim \sigma^{-1} \end{cases}.$$  

If $c = 0$ we let $r_{\mathcal{C}}(\gamma, z) = 0$. We then have

**Theorem 1.** Let $r_{\mathcal{C}}(\gamma, z)$ be defined as in (1.9). Then $r_{\mathcal{C}}(\gamma, z)$ is a weight 2 cocycle for $\Gamma$.

Let $R_{\mathcal{C}}(\gamma, z)$ be the unique primitive of $r_{\mathcal{C}}(\gamma, z)$. Next we define the Dedekind symbol for $\mathcal{C}$ and any $\gamma \in \Gamma$ by

$$\Phi_{\mathcal{C}}(\gamma) = \frac{2}{\pi} \lim_{y \to \infty} \Im R_{\mathcal{C}}(\gamma, iy)$$

Then we have

**Theorem 2.** $\Phi_{\mathcal{C}}(\gamma)$ exists and is an integer.

In order to define the linking number of two cycles in a manifold we must assume that they are each homologous to 0 and that they don’t intersect. For two orbits as above one can either fill in the trefoil appropriately to get $S^3$, as is done in [12], or restrict attention to orbits that are null-homologous. We follow the second course and show that the link determined by a primitive hyperbolic element and its inverse is null-homologous in $\mathcal{M}$ and that for two such distinct links, denoted also by $\mathcal{C}_\gamma$ and $\mathcal{C}_\sigma$, their linking number $Lk(\mathcal{C}_\sigma, \mathcal{C}_\gamma)$ is given by the homogenization of $\Phi_{\mathcal{C}_\sigma}$. More precisely;

**Theorem 3.** Let $\mathcal{C}_\sigma$ and $\mathcal{C}_\gamma$ denote also the links associated to two different primitive conjugacy classes and let

$$\Psi_{\mathcal{C}_\sigma}(\gamma) = \lim_{n \to \infty} \frac{\Phi_{\mathcal{C}_\sigma}(\gamma^n)}{n}.$$  

Then

$$Lk(\mathcal{C}_\sigma, \mathcal{C}_\gamma) = \Psi_{\mathcal{C}_\sigma}(\gamma).$$

Of course it is desirable to have a simple closed form expression for $\Phi_{\mathcal{C}}(\gamma)$ like that for $\Phi(\gamma)$ in (1.2). While it seems unlikely that such a simple sum can be given in general, we are able to express $\Phi_{\mathcal{C}}(\gamma)$ in terms of a special value of a certain Dirichlet series that has some properties analogous to the Dedekind sum $s(a, c)$ from (2.5), including the reciprocity formula (1.5). That something like this might be possible is indicated by the fact that for the Dirichlet series

$$L(s, a/c) = \sum_{n \geq 1} \sigma(n)e\left(\frac{a}{c}n\right)n^{-s},$$

where $\sigma(n)$ is the usual divisor sum, we have the limit formula (proven below in Corollary 2.2)

$$s(a, c) = \frac{1}{2\pi i} \lim_{s \to 1} \left[L(s, \frac{a}{c}) + \frac{1}{2s-2}\right],$$

assuming $c > 0$. Hence for each $m \geq 0$ let $j_m$ be the unique modular function holomorphic on $\mathbb{H}$ whose Fourier expansion begins

$$j_m(z) = q^{-m} + O(q)$$

and define for $\alpha \in \mathbb{Q}$ the Dirichlet series

$$L_{\mathcal{C}}(s, \alpha) = \sum_{n \geq 1} a_{\mathcal{C}}(n)e(n\alpha)n^{-s},$$
where the coefficient \( a_c(n) \) is given by the cycle integral

\[
(1.14) \quad a_c(m) = \sqrt{D'} \int_{\gamma_0}^{\gamma_0} j_m(z) \frac{dz}{Q_{\sigma}(z)},
\]

Here \( \sigma = \left( \begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix} \right) \in C \) is primitive and we set \( Q_{\sigma}(z) = (d' - a')z - b' \) and \( D' = (a' + d')^2 - 4 \). The path of integration can be taken as any path from \( z_0 \) to \( \sigma z_0 \). Note that the integral is independent of the choice \( \sigma \in C \) and \( z_0 \). In particular, if \( \lambda \) is the eigenvalue > 1 of \( \sigma^2 \) then

\[
a_c(0) = \log \lambda,
\]

assuming that \( \text{tr} \sigma > 2 \).

Our next theorem gives the connection of this Dirichlet series to \( \Phi_C(\gamma) \).

**Theorem 4.** Let \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma \) with \( c \neq 0 \) and and \( L_C(s, a/c) \) be the Dirichlet series as in (1.13). Then \( L_C(s, a/c) \) converges for \( \Re(s) > 9/4 \), has a meromorphic continuation to \( s > 0 \) and is holomorphic at \( s = 1 \). Moreover

\[
(1.15) \quad \Phi_C(\gamma) = -\frac{1}{\pi^2} \Re L_C(1, a/c).
\]

It is interesting that \( \Phi_C(\gamma) \) depends only on \( a/c \) mod 1. Furthermore, we have the following reciprocity formula, which will be proved in Theorem 4.3:

For \( z_i \in \mathbb{C} \cup \{\infty\} \), let \( [z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)} \) denote the cross ratio. We assume that \( (a, c) = 1 \) and \( ac \neq 0 \). Then

\[
(1.16) \quad \frac{1}{i\pi} [L_C(1, a/c) - L_C(1, -c/a)] = -2 \left( \frac{a^2 + c^2 + 1}{ac} \right) \log \lambda + e_c \sum_{w' < 0 < w} \log \left[ \frac{a}{z'}, w, w', -\frac{c}{a} \right]
\]

Here we interpret the imaginary part of the logarithm of a negative real number to be \( \pi \).

Note that (1.16) is in some sense analogous to (1.5) and allows for a fast calculation of \( L_C(1, a/c) \) and hence also of \( \Phi_C(\gamma) \).

The rest of the paper is organized as follows. In section 2 we define the Dirichlet series associated to a general modular integral, prove its analytic properties. In section 3 we prove Theorem 1 that the function \( r_C(\gamma, z) \) defined by (1.9) is a weight 2 parabolic cocycle for \( \Gamma \). In the following section we restrict ourselves to the rational period function \( r_C(\gamma, z) \) and its associated modular integral \( F_C(z) \). We also give a formula for the unique weight zero cocycle \( R_C(\gamma, z) \) in terms of special values of the Dirichlet series \( L_C(s, a/c) \) associated to \( F_C(z) \). In this section we also give two applications of the cocycle relation for \( R_C(\gamma, z) \).

The first one gives the reciprocity formula for \( L_C(1, a/c) \) where as the second one provides a geometric interpretation for \( L_C(1, a/c) + L_C(1, -d/c) \). In section 5 we turn our attention to the analog of the Dedekind symbol, \( \Phi_C(\gamma) \) and establish that \( \Phi_C(\gamma) \) is an intersection number, hence an integer. In the final section we use a result of Birkhoff together with the geometry of the modular surface to prove that \( \Psi_C(\gamma) \), the homogenization of \( \Phi_C(\gamma) \), is a linking number.

## 2. Dirichlet series associated to weight 2 cocycles

Recall that a (strongly) parabolic cocycle of weight \( k \) for \( \Gamma \) with coefficients in \( P \) is a map \( \Gamma \to P \) given by \( \gamma \mapsto r(\gamma, z) \) with

\[
r(\sigma \gamma, z) = r(\sigma, z)|_k + r(\gamma, z)
\]

for all \( \gamma, \sigma \in \Gamma \) which also satisfies \( r(T, z) \equiv 0 \).
Hence
\[ F|_\gamma(z) = F(z) + r(\gamma, z). \]

Let \( r(\gamma, z) \) be a cocycle of weight 2. We associate to \( r(\gamma, z) \) and its modular integral a Dirichlet series
\[ L(F, s, a/c) = \sum_{n \geq 1} a_n e\left(\frac{an}{c}\right)n^{-s}. \]

In this section we will first prove a general theorem giving the relation of the special value of \( L(F, 1, a/c) \) to the unique weight 0 cocycle \( R(\gamma, z) \) which satisfies \( R'(\gamma, z) = r(\gamma, z) \). More precisely we have the following theorem.

**Theorem 2.1.** Let \( r(\gamma, z) \in \mathcal{P} \) be a cocycle of weight 2 and \( F(z) = \sum_{n \geq 0} a_n q^n \) be its modular integral. Assume that \( a_n \ll n^{\alpha} \) for some \( \alpha > 0 \). Let
\[
\Lambda(s, a/c) = \Lambda(F, s, a/c) = \left(\frac{2\pi}{c}\right)^{-s} \Gamma(s) \sum_{n \geq 1} a_n e\left(\frac{an}{c}\right)n^{-s}
\]
and
\[
H(s, a/c) = \Lambda(s, a/c) + \int_{1}^{\infty} r(-d/c + it/c) t^{1-s} dt + \frac{a_0}{s} - \frac{a_0}{2 - s}.
\]

Then \( H(s, a/c) \) is entire and satisfies the functional equation \( H(s, a/c) = H(2-s, -d/c) \). Moreover if
\[
R(\gamma, z) = iH(1, a/c) + \int_{-\frac{d}{c} + \frac{i}{c}}^{z} r(\gamma, w) dw + a_0 \left(\frac{a+d}{c}\right)
\]

Then \( R(\gamma, z) \) is the weight zero cocycle such that \( R'(\gamma, z) = r(\gamma, z) \).

**Proof.** Let \( z_t = -d/c + it/c \) so that \( \gamma z_t = a/c + it/c \) and \( cz_t + d = i/t \). Then
\[
\Lambda(s, a/c) = \int_{0}^{\infty} (F(\gamma z_t) - a_0) t^{s-1} dt
\]
\[= \int_{0}^{1} (F(\gamma z_t) - a_0) t^{s-1} dt + \int_{1}^{\infty} (F(\gamma z_t) - a_0) t^{s-1} dt
\]
\[= -\frac{a_0}{s} - \int_{1}^{\infty} F(\gamma z_{1/t})(it)^{-2} t^{1-s} dt + \int_{1}^{\infty} (F(\gamma z_t) - a_0) t^{s-1} dt
\]
\[= -\frac{a_0}{s} - \int_{1}^{\infty} [F(z_{1/t}) + r(\gamma, z_{1/t})] t^{1-s} dt + \int_{1}^{\infty} (F(\gamma z_t) - a_0) t^{s-1} dt
\]
\[= -\frac{a_0}{s} + \frac{a_0}{2 - s} - \int_{1}^{\infty} r(\gamma, z_{1/t}) t^{1-s} dt - \int_{1}^{\infty} (F(z_{1/t}) - a_0) t^{1-s} dt + \int_{1}^{\infty} (F(\gamma z_t) - a_0) t^{s-1} dt
\]

Hence
\[
H(s, a/c) = \Lambda(s, a/c) + \int_{1}^{\infty} r(\gamma, -d/c + it/c) t^{1-s} dt + \frac{a_0}{s} - \frac{a_0}{2 - s}
\]
\[= -\int_{1}^{\infty} (F(z_{1/t}) - a_0) t^{1-s} dt + \int_{1}^{\infty} (F(\gamma z_t) - a_0) t^{s-1} dt
\]

Both integrals in (2.4) converge for all \( s \in \mathbb{C} \) due to the exponential decay of the integrands proving the analytic continuation of \( H(s, a/c) \) to the whole complex plane. The functional equation \( H(s, a/c) = H(2-s, -d/c) \) also follows easily from (2.4) since \( z_{1/t} = -d/c + it/c \) and \( \gamma z_t = a/c + it/c \).
We next take the limit \( s \to 1 \) to get

\[
H(1, \frac{a}{c}) = -\frac{c}{i} \int_{z_1}^{i\infty} (F(z) - a_0) \, dz + \frac{c}{i} \int_{\gamma z_1}^{i\infty} (F(z) - a_0) \, dz
\]

\[
= -\frac{c}{i} \left( G(\gamma z_1) - G(z_1) - a_0 \left( \frac{a + d}{c} \right) \right)
\]

where \( G(z) = a_0 z + \sum_{n \geq 1} \frac{a_n}{2\pi i n} q^n \). Since \( G'(z) = F(z) \),

\[
G(\gamma z) - G(z) = \int_{z_1}^{z} r(\gamma, w) \, dw + \Phi(\gamma)
\]

with \( \Phi(\gamma) = (G(\gamma z_1) - G(z_1)) \).

Hence

\[
R_\gamma(z) = \int_{z_1}^{z} r(\gamma, w) \, dw + (G(\gamma z_1) - G(z_1)) = G(\gamma z) - G(z)
\]

is a cocycle being the boundary of a function \( G \). This finishes the proof of the theorem since clearly \( R'(\gamma, z) = r(\gamma, z) \).

As an immediate corollary of Theorem 2.1 we prove the limit formula (1.12) for the classical Dedekind sums.

**Corollary 2.2.** Let \( s(a, c) \) be the Dedekind sum defined for \( \gcd(a, c) = 1, c \neq 0 \) by

\[
s(a, c) = \sum_{n=1}^{\left| c \right|-1} \left( \left( \frac{n}{c} \right) \left( \frac{na}{c} \right) \right)
\]

and

\[
L(s, a/c) = \sum_{n \geq 1} \sigma(n) e \left( \frac{a}{c} n \right) n^{-s},
\]

Then

\[
s(a, c) = \frac{1}{2\pi i} \lim_{s \to 1} \left[ L(s, \frac{a}{c}) + \frac{1}{2s-2} \right].
\]

**Proof.** We apply Theorem 2.1 in the case of Eisenstein series \( E_2 = 1 - 24 \sum \sigma(n) q^n \) and its cocycle \( r(\gamma, z) = \frac{6}{\pi i} e \left( \frac{cz}{c} \right) \). For simplicity assume \( c > 0 \). As a primitive of \( r(\gamma, z) \) we choose \( \frac{6}{\pi i} \log \left( \frac{cz}{i} + d \right) \). Using (2.2) and (2.3) we have

\[
R(\gamma, z) = \lim_{s \to 1} \left[ -\frac{24}{2\pi i} L(s, a/c) - \frac{6}{\pi i} \frac{1}{s-1} \right] + \frac{6}{\pi i} \int_{-d/c+i/c}^{\infty} \frac{c}{cw + d} \, dw + \left( \frac{a + d}{c} \right)
\]

\[
= \frac{12}{2\pi i} \log \frac{cz}{i} + d + \Phi(\gamma)
\]

where

\[
\Phi(\gamma) = \lim_{s \to 1} \left[ -\frac{24}{2\pi i} L(s, a/c) - \frac{6}{\pi i} \frac{1}{s-1} \right] + \left( \frac{a + d}{c} \right)
\]

The limit formula (1.12) now follows from Dedekind’s formula (1.2) for \( \Phi(\gamma) \).
3. Weight 2 rational cocycles for the modular group

In this section we restrict ourselves to cocycles of weight 2 which are rational functions. The simplest example is \( r(\gamma, z) = 12c/(cz + d) \) whose poles are in \( \mathbb{Q} \). In the case \( r(\gamma, z) \) is a rational cocyle whose poles are not rational it is known that \( r(S, z) \) can be written as a finite linear combination of functions of the form

\[
\sqrt{D} \sum_{AC < 0} \frac{\text{sign } A}{Az^2 + Bz + C}
\]

where \( Q(X, Y) = AX^2 + BXY + CY^2 \) runs over quadratic forms in the class \( \mathcal{C} \) (see [1, 5, 23]). Rational period functions were introduced by Knopp in the 1970s [17, 18] who showed using results from [16] that they have modular integrals. His construction arises from a meromorphic Poincaré series formed out of cocycles and is very difficult to compute (see also [10]). On the other hand in [8] and [9] certain explicit modular integrals were constructed whose Fourier coefficients are given by cycle integrals of weakly holomorphic forms. These functions are parametrized by classes of indefinite quadratic forms \( \mathcal{C} \) and are given by the Fourier expansion

\[
F_{\mathcal{C}}(z) = \sum_{m \geq 0} a_{\mathcal{C}}(m)e(mz).
\]

with

\[
a_{\mathcal{C}}(m) = \sqrt{D} \int_{z_0}^{\sigma z_0} j_m(z) \frac{dz}{Q(z)}.
\]

Here \( j_m \) is the unique modular function whose Fourier expansion has the form \( q^{-m} + O(q) \), \( Q \) is any quadratic form in the class \( \mathcal{C} \), \( \sigma = \sigma_Q \) is a distinguished generator of the group of automorphs of \( Q \). The value of the integral is independent of the path and the point \( z_0 \in \mathbb{H} \).

The association \( Q \mapsto \sigma_Q \) sets up a bijection between elements of the class \( \mathcal{C} \) of the quadratic form \( Q \) and the conjugacy class of \( \sigma_Q \), which by abuse of notation will also be denoted by \( \mathcal{C} \). Since it is more convenient for us to express our results in terms of the hyperbolic conjugacy class, we briefly recall this correspondence. If \( Q(X, Y) = AX^2 + BXY + CY^2 \) has discriminant \( D = B^2 - 4AC \), and \( t, u \) are the smallest positive solutions of Pell’s equation \( t^2 - Du^2 = 4 \) then

\[
\sigma_Q = \left( \frac{t+Bu}{2}, \frac{Cu}{2-Au} \right).
\]

Conversely if \( \sigma = (\alpha, \beta) \in \mathcal{C} \) is a primitive hyperbolic element and we set \( Q_{\sigma}(z) = (\alpha'X^2 + (d' - \alpha')XY - b'Y^2) \), and \( Q = \frac{1}{u}Q_{\sigma} \) with \( u = \gcd(\alpha', d'-\alpha', b') \) then \( \sigma_Q = \sigma \). It follows that with \( D' = (a' + d')^2 - 4 \) we also have

\[
a_{\mathcal{C}}(m) = \varepsilon_{\sigma} \sqrt{D'} \int_{z_0}^{\sigma z_0} j_m(z) \frac{dz}{Q_{\sigma}(z)}
\]

as in (1.14).

As in [8] one can show that \( a_{\mathcal{C}}(m) \ll m^{5/4+\epsilon} \) for any \( \epsilon > 0 \) and \( F_{\mathcal{C}} \) satisfies the transformation property

\[
z^{-2}F_{\mathcal{C}}(-1/z) = F_{\mathcal{C}}(z) + \varepsilon_{\mathcal{C}} \sum_{w_1' \leq w < w_2'} \frac{1}{z-w} - \frac{1}{z-w'}.
\]

Here for \( Q \in \mathcal{C} \), \( w_1' < w_1 \) are the two roots of \( Q(t, 1) = 0 \). If \( \sigma = \sigma_Q \) then these are also the fixed points \( w_{1,2}' < w_{1,2} \) of \( \sigma \), and \( \varepsilon_{\mathcal{C}} \) is defined as in (1.10).
If $\mathcal{C}$ denotes the class of $Q$ or the class of the hyperbolic element $\sigma_Q$ we let
\begin{equation}
\mathcal{W}_\mathcal{C} = \{(w'_Q, w_Q) : Q \in \mathcal{C}\} = \{(w'_\sigma, w_\sigma) : \sigma \in \mathcal{C}\}
\end{equation}
the ordered pairs of roots of $Q \in \mathcal{C}$ or equivalently the fixed point of $\sigma$.

For a fixed $\gamma \in SL_2(\mathbb{Z})$, we let as in (1.9),
\begin{equation}
r_C(\gamma, z) := \varepsilon_C \sum \frac{1}{z - w} - \frac{1}{z - w'}
\end{equation}
where the sum is over $(w', w) \in \mathcal{W}_C$, satisfying $w' < -d/c < w$ if $c \neq 0$ and $r_C(\gamma, z) \equiv 0$ otherwise.

For later use we give another description of $r_C(\gamma, z)$. For $\sigma \in \mathcal{C}$ a fixed hyperbolic element, let $w_\sigma, w'_\sigma$ be its two fixed points, $\Gamma_\sigma = \{g \in \Gamma : g^{-1}\sigma g = \sigma\}$, and $S_\sigma$ be the semicircle whose endpoints are $w_\sigma$ and $w'_\sigma$. Let $\partial \mathbb{H} = \mathbb{R} \cup i\infty$ and $\overline{\mathbb{H}} = \mathbb{H} \cup \partial \mathbb{H}$. For $z_1, z_2 \in \mathbb{H}$ we denote the geodesic segment joining $z_1$ and $z_2$ by $\ell_{z_1, z_2}$ and let
\begin{equation}
X_C(z_1, z_2) = \{\alpha \in \Gamma_\sigma \setminus \Gamma : \alpha S_\sigma \text{ intersects } \ell_{z_1, z_2}\}.
\end{equation}
Then we also have
\begin{equation}
r_C(\gamma, z) = \sum_{\alpha \in X_C(-d/c, \infty)} \text{sign}(\alpha w_\sigma - \alpha w'_\sigma) \left( \frac{1}{z - \alpha w_\sigma} - \frac{1}{z - \alpha w'_\sigma} \right)
\end{equation}

**Theorem 3.1.** For any $\gamma, \sigma \in \Gamma$, with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
\begin{equation}
r_C(\sigma \gamma, z) = r_C(\sigma, \gamma z)(cz + d)^{-2} + r_C(\gamma, z)
\end{equation}

**Proof.** To ease the notation the dependence on $\mathcal{C}$, which is fixed, is suppressed. As usual let $T$ and $S$ denote the two generators of $\Gamma$ corresponding to the translation $z \to z + 1$ and the inversion $z \to -1/z$ respectively. First note that $r(T\gamma, z) = r(\gamma, z)$. Hence if we prove
\begin{equation}
r(S\gamma, z) = r(S, \gamma z)(cz + d)^{-2} + r(\gamma, z)
\end{equation}
the proposition follows by induction on the word length expressing $\sigma$ in terms of the generators $S$ and $T$.

Since $S\gamma = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$ to prove (3.8) we have to prove that
\begin{equation}
\sum_{w' < -b/a < w \atop w' < -d/c < w} \frac{1}{z - w} - \frac{1}{z - w'} - \sum_{w' < -c < w \atop w' < -d/c < w} \frac{1}{z - w} - \frac{1}{z - w'} = \sum_{w' < 0 < w} \frac{1}{z - \gamma^{-1}w} - \frac{1}{z - \gamma^{-1}w'}
\end{equation}
all sums over pairs $(w', w) \in \mathcal{W}$.

Assume first that $ac > 0$ so that $-d/c < -b/a$. On the left hand side of (3.9) we have
\begin{equation}
\sum_{-d/c < w < -b/a < w} \frac{1}{z - w} - \frac{1}{z - w'} - \sum_{w' < -d/c < w < -b/a} \frac{1}{z - w} - \frac{1}{z - w'}
\end{equation}
On the other hand we can write for the right hand side of (3.9)
\begin{equation}
\sum_{w' < 0 < w} \frac{1}{z - \gamma^{-1}w} - \frac{1}{z - \gamma^{-1}w'} =
\end{equation}
\begin{equation}
\sum_{w' < 0 < w < a/c} \frac{1}{z - \gamma^{-1}w} - \frac{1}{z - \gamma^{-1}w'} + \sum_{w' < 0 < a/c < w} \frac{1}{z - \gamma^{-1}w} - \frac{1}{z - \gamma^{-1}w'}
\end{equation}
Now note that
\[ \gamma^{-1} z = -\frac{d}{c} - \frac{1}{c^2(z-a/c)} \]
and the function \( x \to -\frac{d}{c} - \frac{1}{c^2(x-a/c)} \) is monotonic for \( x \in (-\infty, a/c) \) and also for \( x \in (a/c, \infty) \).

It follows that for \( w' < 0 < w < a/c \),
\[ -d/c < \gamma^{-1} w' < -b/a < \gamma^{-1} w \tag{3.12} \]
and similarly that for \( w' < 0 < a/c < w \)
\[ \gamma^{-1} w < -d/c < \gamma^{-1} w' < -b/a. \tag{3.13} \]

Using (3.12) and (3.13) in (3.11) we get that
\[ \sum_{w' < 0 < w < a/c} \left( \frac{1}{z - \gamma^{-1} w} - \frac{1}{z - \gamma^{-1} w'} \right) = \sum_{w' < 0 < a/c < w} \left( \frac{1}{z - \gamma^{-1} w} - \frac{1}{z - \gamma^{-1} w'} \right) + \sum_{-d/c < w' < -b/a < w} \left( \frac{1}{z - w} - \frac{1}{z - w'} \right) - \sum_{-d/c < w < -b/a} \left( \frac{1}{z - w} - \frac{1}{z - w'} \right) \]
This proves (3.9) when \( ac > 0 \). The case \( ac < 0 \) follows in the same manner. The case \( ac = 0 \) can be checked easily since \( c = 0 \) corresponds to \( \gamma = T^m \) whereas \( a = 0 \) rises from \( \gamma = ST^m \).

This proves Theorem 3.1 and hence also Theorem 1 from the introduction. \( \square \)

Extending our earlier work we show that

**Theorem 3.2.** For any hyperbolic conjugacy class \( C \) the function \( F_C(z) \) is holomorphic on \( \mathbb{H} \) and satisfies
\[ (cz + d)^{-2} F_C(\gamma z) = F_C(z) + r_C(\gamma, z) \tag{3.15} \]

**Proof.** The claim is trivial for \( T \) and has been established for the generator \( S = \left( \begin{smallarray} 0 & -1 \\ 1 & 0 \end{smallarray} \right) \) in [9]. It is possible to give a proof of the general case along the lines of the proof of (3.3) given in [9]. However the algebraic proof above already established that the rational function \( r_C(\gamma, z) \) defined in (1.9) is a weight 2 cocycle. Since it agrees with the cocycle associated to \( F_C(z) \) for the generators \( \gamma = S = \left( \begin{smallarray} 0 & -1 \\ 1 & 0 \end{smallarray} \right) \) and \( T = \left( \begin{smallarray} 1 & 1 \\ 0 & 1 \end{smallarray} \right) \) the difference is a 1-cocycle that vanishes on both \( S \) and \( T \) and so must vanish identically. \( \square \)

**4. The Dirichlet Series associated to** \( F_C(z) \)**

Guided by the example of the Eisenstein series \( E_2(z) \) and its primitive \( \log \Delta(z) \), it is natural to study a primitive of a general modular integral, and the associated weight zero cocycle that appears in its transformation property.

We look at this problem in the case of the function \( F_C(z) \) and determine the unique weight 0 primitive \( R_C(\gamma, z) \) of the cocycles \( r_C(\gamma, z) \) in terms of the special values of the Dirichlet series \( L(F_C, s, a/c) \). The next theorem and its corollary proves Theorem 4 from the introduction.
Theorem 4.1. Let $F_C(z)$ be the modular integral in (3.1) and $L_C(s,a/c) := L(F_C, s, a/c)$ be its associated Dirichlet series. Then $L_C(s,a/c)$ converges for $\Re(s) > 9/4$, has a meromorphic continuation to $s > 0$ and is holomorphic at $s = 1$. Moreover if $R_C(\gamma, z)$ is the unique weight 0 cocycle such that $R_C'(\gamma, z) = r_C(\gamma, z)$ then

\begin{equation}
R_C(\gamma, z) = \varepsilon_C \sum_{w < -d/c < w'} \log(z - w) - \log(z - w') + \frac{1}{2\pi i} L_C(1, a/c) + a_C(0) \left( \frac{a + d}{c} \right)
\end{equation}

Proof. The convergence of $L_C(s,a/c)$ for $\Re(s) > 9/4$ follows from the bound $a_C(m) \ll m^{5/4+\epsilon}$ which was proved in Proposition 6 of [8].

To prove (4.1), we let $r(\gamma, z) = r_C(\gamma, z) = \varepsilon_C \sum_{W_C} \frac{1}{z-w} - \frac{1}{z-w'}$. As a primitive of $r_C(\gamma, z)$ we choose

\begin{equation}
\varepsilon_C \sum_{w' < -d/c < w} \log(z - w) - \log(z - w').
\end{equation}

Once again using (2.2) and (2.3) we have

\begin{equation}
R_C(\gamma, z) = -\frac{i}{c} \lim_{s \to 1} \left[ \left( \frac{2\pi}{c} \right)^{-s} \Gamma(s) L_C(s, a/c) + \int_{1}^{\infty} r_C(\gamma, -d/c + it/c) t^{1-s} dt \right]
+ \int_{z_1}^{z} r_C(\gamma, w) dw + a_C(0) \left( \frac{a + d}{c} \right).
\end{equation}

where $z_1 = -d/c + i/c$.

Contrary to the case of $E_2$, the Dirichlet series $L_C(s, a/c)$ has no pole at $s = 1$. This is due to the fact that at $s = 1$ the first integral in (4.2) has the finite value

\begin{equation}
\varepsilon_C \sum_{w' < -d/c < w} \log(z_1 - w) - \log(z_1 - w').
\end{equation}

To finish the proof of Theorem 4.1 we combine the two integrals in (4.2) to get

\begin{equation}
R_C(\gamma, z) = \frac{1}{2\pi i} L_C(1, a/c) + \int_{\infty}^{z} r_C(\gamma, z) dw + a_C(0) \left( \frac{a + d}{c} \right)
= \frac{1}{2\pi i} L_C(1, a/c) + \varepsilon_C \sum_{w' < -d/c < w} \log(z - w) - \log(z - w') + a_C(0) \left( \frac{a + d}{c} \right).
\end{equation}

Since $a_C(0) = \log \lambda$ is real, the following corollary easily follows from (4.1)

Corollary 4.2. Let $\Phi_C(\gamma) = \frac{2}{\pi} \lim_{y \to \infty} \Im R_C(\gamma, iy)$. Then

\begin{equation}
\phi_C(\gamma) = -\frac{1}{\pi} \Re L_C(1, a/c).
\end{equation}

In the rest of the section we will give two applications of Theorem 4.1 and the cocycle relation

$$R_C(\sigma \gamma, z) = R_C(\sigma, \gamma z) + R_C(\gamma, z).$$

The first one is an analog of the Dedekind’s reciprocity formula (1.5) for the Dirichlet series $L_C(1, a/c)$. More precisely we have
Theorem 4.3. For \( z_i \in \mathbb{C} \cup \{\infty\} \), let \( [z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)} \) denote the cross ratio. We assume that \((a, c) = 1\) and \(ac \neq 0\). Then

\[
\frac{1}{i\pi} [L_c(1, a/c) - L_c(1, -c/a)] = -2 \left( \frac{a^2 + c^2 + 1}{ac} \right) \log \lambda + \varepsilon_c \sum_{w' < 0 < w} \log \left[ \frac{a}{c}, w, w', -\frac{c}{a} \right]
\]

Here we interpret the imaginary part of the logarithm of a negative real number to be \( \pi \).

Proof. Let \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and \( S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). From (4.3) we have

\[
R_c(\gamma, z) = R_c(1, a/c) + \int_{i\infty}^{z} r_c(\gamma, w)dw + a_c(0) \left( \frac{a + d}{c} \right)
\]

Since \( R_c(\gamma, z) \) is a cocycle it satisfies

\[
R_c(S\gamma, z) = R_c(S, \gamma z) + R_c(\gamma, z).
\]

Hence

\[
R_c(S\gamma, z) = \frac{1}{2\pi i} L_c(1, a/c) + \int_{i\infty}^{z} r_c(S\gamma, w)dw + a_c(0) \left( \frac{b - c}{a} \right)
\]

\[
= \frac{1}{2\pi i} L_c(1, 0) + \int_{i\infty}^{\gamma z} r_c(S, w)dw + \frac{1}{2\pi i} L_c(1, a/c) + \int_{i\infty}^{z} r_c(\gamma, w)dw + a_c(0) \left( \frac{a + d}{c} \right)
\]

We let \( z \to i\infty \) to get

\[
\frac{1}{2\pi i} [L_c(1, -c/a) - L_c(1, a/c)] = a_c(0) \left( \frac{a^2 + c^2 + 1}{ac} \right)
\]

\[
+ \frac{1}{2\pi i} L_c(1, 0) + \int_{i\infty}^{a/c} r_c(S, w)dw
\]

Hence

\[
\frac{1}{2\pi i} [L_c(1, -c/a) - L_c(1, a/c)] = a_c(0) \left( \frac{a^2 + c^2 + 1}{ac} \right) + \frac{1}{2\pi i} L_c(1, 0) + \varepsilon_c \sum_{w' < 0 < w} \log \left( \frac{a}{c} - w \right) - \log \left( \frac{a}{c} - w' \right)
\]

Now replacing the roles of \(-c\) with \(a\) and \(a\) with \(c\) gives

\[
\frac{1}{2\pi i} [L_c(1, a/c) - L_c(1, -a/c)] = -a_c(0) \left( \frac{a^2 + c^2 + 1}{ac} \right) + \frac{1}{2\pi i} L_c(1, 0) + \varepsilon_c \sum_{w' < 0 < w} \log \left( \frac{-c}{a} - w \right) - \log \left( \frac{-c}{a} - w' \right)
\]

Finally noting that \(a_c(0) = \log \lambda\) and taking the difference of the last two equations prove (4.4). \(\square\)

As a second application we have the following geometric interpretation of the special value of the Dirichlet series \(L_c(s, a/c)\).
Theorem 4.4. Let $L_C(s, a/c)$ be the Dirichlet series associated to $F_C(z)$. Then
\begin{equation}
\frac{1}{2\pi i} [L_C(1, a/c) + L_C(1, -d/c)]
\end{equation}
\begin{equation}
\begin{aligned}
&= -\varepsilon_C \sum_{w' < \frac{-d}{c} < w} \log\left(\frac{-d}{c} - w\right) - \log\left(\frac{-d}{c} - w'\right) \\
&= -\varepsilon_C \left(2 \log \prod_{w' < \frac{-d}{c} < w} \tan\left(\frac{\theta_w}{2}\right) + i\pi \sum_{w' < \frac{-d}{c} < w} 1\right)
\end{aligned}
\end{equation}
where the sum and the product runs over elements $(w', w) \in W_C$ that are separated by $\frac{-d}{c}$. \(\theta_w\) is the angle of intersection of the vertical line $(-d/c, -d/c + i\infty)$ with the semicircle with end points $w'$ and $w$. Here $\theta_w$ is the angle containing the line segment connecting this intersection to $w'$.

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Using the cocycle relation $0 = R_C(\gamma, \gamma^{-1}z) + R_C(\gamma^{-1}, z)$, the formula \(4.1\) and taking the limit as $z \to i\infty$ leads to the first equality \(4.12\). Since $-d/c, w, w'$ all lie on the real axis, the argument of each logarithm term in the sum in \(4.12\) is $\pi$. Here we interpret the imaginary part of the logarithm of a negative real number to be $\pi$. This proves that the imaginary part of \(4.12\) is indeed given by $\pi \sum_{w' < \frac{-d}{c} < w} 1$.

The fact that the real part \(4.12\) is given as in \(4.13\) follows easily using elementary geometry. (See also [3] p.116.)

\[\square\]

Note that since $L_C(1, a/c)$ depends only on $a/c \mod 1$, so does $\Phi_C(\gamma)$ and hence we can write $\Phi_C(a/c) = \Phi_C(\gamma)$. The following is a simple corollary of Theorem 4.4.

Corollary 4.5. Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then
\[\Phi_C(0) = \Phi_C(S) = -\varepsilon_C \sum_{w' < 0 < w} 1\]

Finally we remark that once all the conjugates of $\sigma \in \mathcal{C}$ whose fixed points are separated by $0$ are listed Theorem 4.3, Theorem 4.4 and Corollary 4.5 allow a fast calculation of $L_C(1, a/c)$.

5. Intersection Numbers

In this section we restrict ourselves to the imaginary part of $R_C(\gamma, z)$. Recall that
\[\Phi_C(\gamma) = \frac{1}{\pi} \lim_{y \to \infty} \text{Im} R_C(\gamma, iy) = -\frac{1}{\pi} \text{Re} L_C(1, a/c).
\]

Our first goal is to prove that $\Phi_C(\gamma)$ is an intersection number, hence an integer.

To this end let $\sigma \in \mathcal{C}$ be a hyperbolic element and $w, w'$ its two fixed points, $S_\sigma$ the semicircle whose endpoints are $w$ and $w'$. For $X_C(z_1, z_2)$ as in \(3.5\) we denote the cardinality of $X_C(z_1, z_2)$ by $I_C(z_1, z_2)$; $I_C(z_1, z_2) = |X_C(z_1, z_2)|$.

Note that if we denote the net of $\sigma$ as
\begin{equation}
\mathcal{N}_\sigma := \bigcup_{g \in \Gamma} g S_\sigma = \bigcup_{g \in \Gamma} S_{g^{-1}\sigma g},
\end{equation}
then $I_C(\alpha, \beta)$ counts the number of intersections of the geodesic segment $\ell_{\alpha, \beta}$ with the semicircles in $\mathcal{N}_\sigma$, the net of $\sigma$.

First we note that as a simple corollary of Theorem 4.4 we have
Proposition 5.1. Let $C$ be the conjugacy class of a hyperbolic element $\sigma$, and $\gamma = (a/b,c/d)$ another hyperbolic element in $\Gamma$. Then

$$\Phi_C(\gamma) + \Phi_C(\gamma^{-1}) = -2I_C(-d/c,i\infty)$$

The next result shows that $\Phi_C(\gamma)$ is already an integer.

Theorem 5.2. Let $I_C(-d/c,i\infty)$ as above. Then $\Phi_C(\gamma) = -I_C(-d/c,i\infty)$ and hence $\Phi_C(\gamma) \in \mathbb{Z}$.

Proof. Let

$$H(\gamma_1, \gamma_2) = \Phi_C(\gamma_1 \gamma_2) - \Phi_C(\gamma_1) - \Phi_C(\gamma_2).$$

Note that $I_C(-d_1/c_1,i\infty) = I_C(\gamma_1^{-1}i\infty,i\infty)$. We will show that

$$H(\gamma_1, \gamma_2) = I_C(\gamma_1^{-1}i\infty,i\infty) + I_C(\gamma_2^{-1}i\infty,i\infty) - I_C((\gamma_1 \gamma_2)^{-1}i\infty,i\infty).$$

This will prove the theorem since this gives $\gamma \mapsto \Phi_C(\gamma) + I_C(\gamma^{-1}i\infty,i\infty)$ is a homomorphism of $\Gamma$ into $\mathbb{C}$ and so is identically 0. First note that if either $\gamma_1$ or $\gamma_2$ is $T^n$ for some $n \in \mathbb{Z}$ then $H(\gamma_1, \gamma_2) = 0$ and the identity holds trivially. So we assume that $\gamma_1, \gamma_2$ are not parabolic.

To prove (5.2) note that from definition (1.11) of $\Phi_C(\gamma)$ and the cocycle property we have

$$H(\gamma_1, \gamma_2) = \frac{2\varepsilon_C}{\pi} \lim_{y \to \infty} \text{Im}(R_C(\gamma_1, \gamma_2iy) - R_C(\gamma_1, iy))$$

which by (4.3) equals

$$\frac{2\varepsilon_C}{\pi} \lim_{y \to \infty} \left[ \sum \text{arg} \left( \frac{\gamma_2iy - w}{\gamma_2iy - w'} \right) - \sum \text{arg} \left( \frac{iy - w}{iy - w'} \right) \right]$$

the sums are over $(w', w) \in W_C$, $w' < -d_1/c_1 < w$. The second sum in the limit clearly goes to zero. Since $\gamma_2iy \to a_2/c_2$ when $y \to \infty$

$$H(\gamma_1, \gamma_2) = 2\varepsilon_C n(\gamma_1^{-1}, \gamma_2)$$

where $n(\gamma_1^{-1}, \gamma_2)$ is the number of $(w', w) \in W_C$, for which $w' < -d_1/c_1, a_2/c_2 < w$. By the definition (3.5) we have

$$\varepsilon_C n(\gamma_1^{-1}, \gamma_2) = |X_C(\gamma_1^{-1}i\infty,i\infty) \cap X_C(\gamma_2i\infty,i\infty)|$$

Any geodesic that does not go through the vertices of an ideal hyperbolic triangle intersects exactly two sides of the triangle if it intersects the triangle at all. Applying this fact to the ideal hyperbolic triangle with vertices $i\infty, a_2/c_2 = \gamma_2i\infty$ and $-d_1/c_1 = \gamma_1^{-1}i\infty$ shows that the sets

$$X_C(\gamma_1^{-1}i\infty,i\infty) \cap X_C(\gamma_2i\infty,i\infty),$$

$$X_C(\gamma_1^{-1}i\infty,i\infty) \cap X_C(\gamma_2i\infty, \gamma_1^{-1}i\infty) \text{ and}$$

$$X_C(\gamma_2i\infty, \gamma_1^{-1}i\infty) \cap X_C(\gamma_2i\infty,i\infty)$$

are mutually disjoint. A standard inclusion exclusion argument gives

$$H(\gamma_1, \gamma_2) = I_C(\gamma_1^{-1}i\infty,i\infty) + I_C(\gamma_2i\infty,i\infty) - I_C(\gamma_1^{-1}i\infty, \gamma_2i\infty)$$

Finally we use that $I_C(z_2,z_1) = I_C(z_1,z_2) = I_C(\gamma z_1, \gamma z_2)$ for all $\gamma \in \Gamma$ to establish that

$$H(\gamma_1, \gamma_2) = I_C(\gamma_1^{-1}i\infty,i\infty) + I_C(\gamma_2^{-1}i\infty,i\infty) - I_C(\gamma_1^{-1}i\infty, \gamma_2i\infty).$$

□

It will be important for us to compare $I_{C_\sigma}(\gamma^{-1}z_0, z_0)$ and $I_{C_\sigma}(\gamma^{-1}i\infty,i\infty)$. Before we formulate our result we need a simple lemma about hyperbolic quadrangles. Recall that for $z_1, z_2 \in \mathbb{H}$ the geodesic segment connecting $z_1$ and $z_2$ is denoted by $\ell_{z_1,z_2}$. 
Lemma 5.3. Let $z_1, z_2 \in \mathbb{H}$ and $x_1, x_2 \in \partial \mathbb{H}$. If $\ell$ is a geodesic that intersects neither the geodesic half line $\ell_{z_1, x_1}$ nor the geodesic half line $\ell_{z_2, x_2}$ then $\ell$ intersects either both $\ell_{z_1, x_1}$ and $\ell_{z_2, x_2}$ or it intersects neither of them.

Proof. By applying a hyperbolic isometry if necessary we may assume that $\ell = \ell_{0, i\infty}$. The geodesic arc from $z_1$ to $x_1$ does not intersect $\ell = (0, i\infty)$, so $x_1$ and $\text{Re}(z_1)$ have the same sign. Similarly the geodesic arc from $z_2$ to $x_2$ does not intersect $(0, i\infty)$, so $x_2$ and $\text{Re}(z_2)$ have the same sign. Finally the arc from $z_1$ to $z_2$ intersects $(0, i\infty)$ if and only if their real parts have opposite signs. This proves that $\ell$ either intersects both the arc from $z_1$ to $z_2$ and the geodesic from $x_1$ to $x_2$ or that intersects neither of them. 

Proposition 5.4. Let $\sigma, \gamma$ be hyperbolic elements, and fix a point $z_0 \in S_\gamma$. Then

\begin{equation}
|I_{C_\sigma}(\gamma^{-1}z_0, z_0) - I_{C_\sigma}(\gamma^{-1}i\infty, i\infty)| \leq 2I_{C_\sigma}(z_0, i\infty).
\end{equation}

Note that we do not assume $\gamma$ to be primitive.

Proof. Let $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$. Consider the geodesic circular arc $L_1$ connecting $\gamma^{-1}z_0$ to $\gamma^{-1}i\infty = -d/c$ and the half-line $L_2$ connecting $z_0$ to $i\infty$. Assume that $\alpha S_\sigma$ intersects neither $L_1$ nor $L_2$. Then it follows from Lemma 5.3 that $\alpha S_\sigma$ either intersects both the arc from $z_0$ to $\gamma^{-1}z_0$ and the line from $-d/c$ to $i\infty$ or that intersects neither of them. 

Hence we have shown that the symmetric difference of the sets $X_{C}(-d/c, i\infty)$ and $X_{C}(z_0, \gamma^{-1}z_0)$ is a subset of $X_{C}(z_0, i\infty) \cup X_{C}(-d/c, \gamma^{-1}z_0)$;

$X_{C}(-d/c, i\infty) \Delta X_{C}(z_0, \gamma^{-1}z_0) \subset X_{C}(z_0, i\infty) \cup X_{C}(-d/c, \gamma^{-1}z_0)$

Since

$|I_{C}(z_0, \gamma^{-1}z_0) - I_{C}(-d/c, i\infty)| \leq |X_{C}(-d/c, i\infty) \Delta X_{C}(z_0, \gamma^{-1}z_0)|$

and $X_{C}(z_0, i\infty)$ and $X_{C}(-d/c, \gamma^{-1}z_0)$ have the same cardinality $I_{C_\sigma}(z_0, i\infty)$ this proves the proposition. 

6. Linking Numbers

If $\gamma$ is a primitive hyperbolic element such that $\text{tr} \, \gamma > 2$ there is an associated closed periodic orbit of the geodesic flow whose linking number with the trefoil is given by the Rademacher symbol [11]; [4]

$\Psi(\gamma) = \Phi(\gamma) - 3 \text{ sign } c(a + d) = \lim_{n \to \infty} \frac{\Phi(\gamma^n)}{n},$

Our goal in this section is to provide a similar interpretation for

$\Psi_{C}(\gamma) := \lim_{n \to \infty} \frac{\Phi_{C}(\gamma^n)}{n}$

as a linking number.

To make this explicit note that if $\gamma \in \Gamma$ has $\text{tr} \, \gamma > 2$ and fixed points $w' < w$ then both $\gamma$ and $\gamma^{-1}$ are diagonalized by $M = \frac{1}{\sqrt{w-w'}} \left( \begin{smallmatrix} w & w' \\ 1 & 1 \end{smallmatrix} \right)$. By replacing $\gamma$ with $\gamma^{-1}$ if necessary we may assume that

$\gamma M = M \left( \begin{smallmatrix} \varepsilon & 0 \\ 0 & 1/\varepsilon \end{smallmatrix} \right)$

where $\varepsilon > 1$. When $a + d > 2$ this is equivalent to sign $c > 0$. Both

$\tilde{\gamma}_+(t) = M \phi(t)$ and $\tilde{\gamma}_-(t) = MS \phi(t)$

are periodic orbits of the geodesic flow $g \mapsto g \phi(t)$ on $\Gamma \setminus SL_2(\mathbb{R})$. Here $\phi(t) = \left( \begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix} \right)$. 
For the convenience of the reader we sketch Ghys’ argument for the identification of the Rademacher symbol with the linking number with the trefoil. Define $\tilde{\Delta} : SL_2(\mathbb{R}) \to \mathbb{C}$ by
\[
\tilde{\Delta}(g) = \Delta(gi)j_{12}(g, i)
\]
where for $g = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in SL_2(\mathbb{R})$
\[
j_{12}(g, z) = (cz + d)^{-12}.
\]
Similar lifts of the classical Eisenstein series $E_4$ and $E_6$ give an embedding of $\Gamma \backslash SL_2(\mathbb{R})$ into $\mathbb{C}^2$. A general topological argument then shows that the linking number of the closed periodic orbit $\tilde{\gamma}_+ + \gamma$ with the trefoil is the same as the winding number of $\tilde{\Delta}(\tilde{\gamma}_+(t))$ around 0. This of course can be computed as follows
\[
2\pi i \text{ind}(\tilde{\Delta}(\tilde{\gamma}_+(t)), 0) = \int_{\tilde{\gamma}_+} \frac{d\tilde{\Delta}}{\Delta} = \int_{\tilde{\gamma}_+} \frac{d\Delta}{\Delta} + \int_{\tilde{\gamma}_+} \frac{dj_{12}}{j_{12}}.
\]
The first integral can be evaluated from the transformation formula of $\log \Delta$ from $\tilde{\gamma}_+(0)i = M\tilde{\gamma}_0 + z_0$ to $\tilde{\gamma}_+(\log \varepsilon)i = \gamma z_0$
\[
\log \Delta(\gamma z_0) - \log \Delta(z_0) = 12 \log \left( \frac{cz_0 + d}{i \text{sign } c} \right) + 2\pi i \Phi(\gamma)
\]
with $\Phi(\gamma)$ as in (1.2). (See [25] equation (60) on page 49.)

Similarly the value of the second integral is $12 \log(cz_0 + d)$ and the linking number of the closed orbit of a hyperbolic $\gamma$ is given by
\[
\frac{6}{\pi i} \left( \log \left( \frac{cz_0 + d}{i \text{sign } c} \right) - \log(cz_0 + d) \right) + \Phi(\gamma)
\]
Finally for $\text{Im } z_0 > 0$
\[
\frac{6}{\pi i} \left( \log \left( \frac{cz_0 + d}{i \text{sign } c} \right) - \log(cz_0 + d) \right) = -3 \text{sign } c
\]

We now move on to interpret $\Psi_C$ in terms of linking numbers. Let $[\tilde{\gamma}_+]$ and $[\tilde{\gamma}_-]$ be the homology class of the curves $t \mapsto M\phi(t), t \in [0, \log \varepsilon]$ and $t \mapsto MS\phi(t), t \in [0, \log \varepsilon]$, respectively. Note that $\tilde{\gamma}_+(t)i, t \in [0, \log \varepsilon]$ maps into a geodesic arc in $\mathbb{H}$ connecting $Mi$ to $\gamma Mi$ on the semicircle with endpoints $w$ and $w'$. On the quotient space $\Gamma \backslash \mathbb{H}$ this is a closed geodesic, and $\tilde{\gamma}_-(t)i$ simply travels this closed geodesic backwards.

This suggests immediately that $[\tilde{\gamma}_+] + [\tilde{\gamma}_-]$ is null-homologous in $\Gamma \backslash SL_2(\mathbb{R})$ and indeed we do have

**Lemma 6.1.** $[\tilde{\gamma}_+] + [\tilde{\gamma}_-]$ is null-homologous in $\Gamma \backslash SL_2(\mathbb{R})$.

**Proof.** In fact we even have that $M\phi(t)$ and $MS\phi(-t)$ are homotopic via
\[
H : [0, \log \varepsilon] \times [0, \pi/2] \to G
\]
\[
(t, \theta) \mapsto M\phi(t)k(\theta)
\]
where as usual
\[
k(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.
\]
Note the image of $H$ is an immersed submanifold $X_γ$ in the quotient space $Γ\backslash SL_2(ℝ)$. This follows readily from the fact that $φ(t_1)k(θ_1) = φ(t_2)k(θ_2)$, for $θ ∈ [0, π/2]$ implies $t_1 = t_2, θ_1 = θ_2$ and so the image of $H$ when viewed in $SL_2(ℝ)$ is an embedded submanifold.

Now assume that $C_σ$ and $C_γ$ are two (different) primitive conjugacy classes. The above construction of the null-homologous chains associated to $σ, γ$ have a well-defined linking number [13], [20] which we denote by $Lk(C_σ, C_γ)$. (This is well defined as the chains themselves depend only on the conjugacy class.) A geometric interpretation of this linking number between the trivial homology class $[γ_+] + [γ_-]$ and $[σ_+] + [σ_-]$ is given as the number of signed intersections of $X_γ$ (the surface defined above by the homotopy map $H$) and $σ_+(s)$ and $σ_-(s), s ∈ [0, log λ]$, the closed orbits associated to $σ$. The geodesic flow has the interesting property that all intersections of $X_γ$ and $σ_±$ have the same sign. To show this we fix the sign by fixing an orientation as follows. We think of $X$ and $σ_±$ themselves depend only on the conjugacy class. We think of $X$ as a subspace of the space of real $2 × 2$ matrices. The tangent space at the identity is the set of $2 × 2$ matrices with trace $0$ where we fix the basis (see [14] pg 27)

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and we say the orientation of three tangent vectors tangent to $SL_2(ℝ)$ at $g$ is positive, i.e. three matrices $v_1, v_2, v_3$ are positively oriented if $g^{-1}v_1, g^{-1}v_2, g^{-1}v_3$, are positively oriented at the identity. We then have the following proposition.

**Proposition 6.2.** Let $N = \frac{1}{\sqrt{w_σ - w'_σ}} (\begin{smallmatrix} w_σ & w'_σ \\ 1 & 1 \end{smallmatrix})$, where $w_σ, w'_σ$ are the two fixed points of $σ$. Assume that the trajectory $Nφ(s)$ is disjoint from $[γ_+] + [γ_-]$ and intersects $X_γ$ at a point $g$. Then the sign of the intersection is negative.

**Proof.** Let

$$g = Mφ(t)k(θ) = Nφ(s).$$

To compute the sign of the intersection we have to compute the determinant of the coefficient matrix of the tangent vectors

$$g^{-1}Mφ'(t)k(θ), \ g^{-1}Mφ(t)κ'(θ) \text{ and } g^{-1}Nφ'(s).$$

Since $φ'(t) = φ(t)h$ and $κ'(θ) = κ(θ)(y - x)$ we have

$$g^{-1}Mφ'(t)k(θ) = k(θ)hk(θ), \ g^{-1}Mφ(t)k'(θ) = (y - x),$$

and

$$g^{-1}Nφ'(s) = h.$$

Since $k(θ) (\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}) k(θ) = (\begin{smallmatrix} \cos 2θ & -\sin 2θ \\ \sin 2θ & -\cos 2θ \end{smallmatrix}) = -\sin 2θx - \sin 2θy + \cos 2θh$ the value of the determinant we need to compute is $-2\sin 2θ$, always negative since $θ ∈ (0, π/2)$.

Let $A_γ$ denote the image of the curve $Mφ(t)i$ in $ℍ$. Next we show that the number of intersections of $X_γ$, the image surface of the homotopy $H$ from Lemma 6.1, with $σ_+(s)$ and $σ_-(s)$ is given by the number of intersections of $A_γ$ with the net of $σ, N_σ$ which is defined in (5.1), when counted properly. According to Dehorney [7] this is a special case of a general theorem of Birkhoff. The interpretation of this geometric idea requires both multiplicities arising from self-intersections and some care for $Γ = SL_2(ℤ)$ due to the presence of elliptic elements. We avoid this by going directly to $I_C(z_0, γz_0)$ which counts group elements in $X_C(z_0, γz_0)$. More precisely we have
Theorem 6.3. If we let \( z_0 = Mi \in \mathcal{S}_\gamma \) then
\[
Lk(C_\sigma, C_\gamma) = -I_{C_\sigma}(z_0, \gamma z_0)
\]

The theorem will follow from a series lemmas.

Lemma 6.4. For \( M = \frac{1}{\sqrt{\lambda}} \begin{pmatrix} w_\lambda & w'_\lambda \\ 1 & 1 \end{pmatrix}, \) \( N = \frac{1}{\sqrt{\lambda}} \begin{pmatrix} w_\lambda & w'_\lambda \\ 1 & 1 \end{pmatrix}, \) with \( \{w_\lambda, w'_\lambda\} \) and \( \{w_\sigma, w'_\sigma\} \), the fixed points of \( \gamma \) and \( \sigma \) respectively, let
\[
A = \{(s, t, \theta) \in [0, \log \lambda] \times [0, \log \varepsilon] \times [0, \pi/2] : \exists \alpha \in \Gamma, M\phi(t)k(\theta) = \alpha N\phi(s)\}
\]
and
\[
B = \{(s, t, \theta) \in [0, \log \lambda] \times [0, \log \varepsilon] \times [0, \pi/2] : \exists \alpha \in \Gamma, M\phi(t)k(\theta) = \alpha NS\phi(s)\}.
\]
Then \( A \cap B = \emptyset \).

Proof. First note that each point in \( A \) (resp in \( B \)) corresponds to an intersection of \( X_\gamma \) with the curve \( \tilde{\sigma}_+ \) (resp \( \tilde{\sigma}_- \)).

Assume \( (s, t, \theta) \in A \cap B \) with \( M\phi(t)k(\theta) = \alpha N\phi(s) \) and \( M\phi(t)k(\theta) = \beta NS\phi(s) \) for some \( \alpha, \beta \in \Gamma \). It follows that \( \beta^{-1}\alpha = NS^{-1} \). Recall that \( N = \frac{1}{\sqrt{w_\sigma - w'_\sigma}} \begin{pmatrix} w_\sigma & w'_\sigma \\ 1 & 1 \end{pmatrix} \), where \( w_\sigma, w'_\sigma \) are the two fixed points of \( \sigma \). Now a simple matrix multiplication shows that the matrix \( NS^{-1} \) cannot have integer entries, contradicting \( \beta^{-1}\alpha \in SL(2, \mathbb{Z}) \). Hence \( A \cap B = \emptyset \).

Lemma 6.5. There is a bijection between \( B \) in the Lemma 6.4 and
\[
B' = \{(s, t, \theta) \in [0, \log \lambda] \times [0, \log \varepsilon] \times [\pi/2, \pi] : \exists \alpha \in \Gamma, M\phi(t)k(\theta) = \alpha N\phi(s)\}
\]
given by for \( s \neq 0 \)
\[
(s, t, \theta) \mapsto (\log \lambda - s, t, \theta + \pi/2).
\]
and for \( s = 0 \) by
\[
(0, t, \theta) \mapsto (0, t, \theta + \pi/2)
\]

Proof. Assume \( (s, t, \theta) \in B \). The case \( s = 0 \) is trivial and otherwise \( \exists \alpha \in \Gamma \) such that
\[
M\phi(t)k(\theta) = \alpha NS\phi(s).
\]
Since \( \sigma N = N\phi(\log \lambda) \)
\[
M\phi(t)k(\theta) = \alpha^{-1} N\phi(\log \lambda - s)S.
\]
This gives the claim since \( S^{-1} = -k(\pi/2) \).

Lemma 6.6. There is a bijection between the set \( A \cup B' \) and \( X_{C_\sigma}(z_0, \gamma z_0) \) and so \( |A \cup B'| = I_C(z_0, \gamma z_0) \).

Proof. We define a map
\[
f : A \cup B' \to \Gamma/\Gamma_\sigma
\]
(6.1)
\[
(s, t, \theta) \mapsto \alpha \Gamma_\sigma.
\]
Here \( \alpha \) is the unique element in \( \Gamma \) given by
\[
M\phi(t)k(\theta)\phi(-s)N^{-1} = \alpha.
\]
(6.2)
To see that \( f \) is injective let \( f(s, t, \theta) = f(s', t', \theta') \) with \( M\phi(t)k(\theta)\phi(-s)N^{-1} = \alpha \) and \( M\phi(t')k(\theta')\phi(-s')N^{-1} = \beta \). Then \( \alpha \sigma^k = \beta \) for some \( k \in \mathbb{Z} \). Hence
\[
\phi(t)k(\theta)\phi(-s)N^{-1} \sigma^k N = \phi(t')k(\theta')\phi(-s').
\]
Since \( N^{-1} \sigma^k N = \phi(k \log \lambda) \) we have
\[ \phi(t - t')k(\theta)\phi(k \log \lambda - s + s') = k(\theta'). \]

Now a simple matrix multiplication shows that this equality holds only if \((s, t, \theta) = (s', t', \theta')\), proving the injectivity of \(f\).

To show that \(f(s, t, \theta) \in X_C(z_0, \gamma z_0)\), let \((s, t, \theta), \alpha\) be such that

\[ M\phi(t)k(\theta) = \alpha N\phi(s). \]

Now \(M\phi(t)\) is in \(A_\gamma\), the geodesic arc connecting \(z_0 = Mi\) and \(\gamma z_0\) where as \(N\phi(s)i\) is in \(S_\sigma\) and hence \(\alpha \Gamma_\sigma \in X_C(z_0, \gamma z_0)\).

Finally to see that this map is surjective let \(\alpha\) be such that \(\alpha \Gamma_\sigma \in X_C(z_0, \gamma z_0)\) so that there is \(\tau \in S_\sigma\) for which \(\alpha \tau \in A_\gamma\), and so \(\alpha \tau = M\phi(t)i\) for some \(t \in [0, \log \varepsilon]\), and also \(\tau = \sigma^k N\phi(s)i\) for some \(s \in [0, \log \lambda]\). Since the stabilizer of \(i\) in \(SL_2(R)\) is \(SO(2)\), there exists \(\theta \in [0, 2\pi]\), such that

\[ M\phi(t)k(\theta) = \alpha \sigma^k N\phi(s). \]

Replacing \(\alpha\) by \(-\alpha\) if necessary we may assume that \(\theta \in [0, \pi]\) proving surjectivity.

\[ \square \]

**Proof of the Theorem 6.3.**

First note that each point in the set \(A\) corresponds to an intersection of the surface \(X_\gamma\) with the curve \([\hat{\sigma}^+]\) and similarly points in \(B\) correspond to intersections of \(X_\gamma\) with the curve \([\hat{\sigma}^-]\). Hence for the linking number we have \(\text{Lk}(C_\sigma, C_\gamma) = -|A| - |B|\). Since by Lemma 6.4 \(A \cap B = \emptyset\) we have \(\text{Lk}(C_\sigma, C_\gamma) = -|A \cup B|\). Finally by Lemma 6.5 and Lemma 6.6, \(|A \cup B| = I_{C_\sigma}(z_0, \gamma z_0)\).

This finishes the proof of the Theorem 6.3.

\[ \square \]

We are now ready to prove the main result of this section.

**Theorem 6.7.** Let \(C_\sigma\) and \(C_\gamma\) be different primitive conjugacy classes. Then

\[ \text{Lk}(C_\sigma, C_\gamma) = \Psi_{C_\sigma}(\gamma) \]

**Proof.** By Theorem 6.3 we have

\[ \text{Lk}(C_\sigma, C_\gamma^n) = -I_C(z_0, \gamma^{-n}z_0) \]

and by Theorem 5.2

\[ \Phi_C(\gamma^n) = -I_C(\gamma^{-n}i\infty, i\infty) \]

Hence

\[ |n \text{Lk}(C_\sigma, C_\gamma) - \Phi_{C_\sigma}(\gamma^n)| = |I_C(z_0, \gamma^{-n}z_0) - I_C(\gamma^{-n}i\infty, i\infty)| \]

Now using Proposition 5.4 we have

\[ |\text{Lk}(C_\sigma, C_\gamma) - \frac{\Phi_{C_\sigma}(\gamma^n)}{n}| \leq \frac{2I_C(z_0, i\infty)}{n} \]

Since \(I_C(z_0, i\infty)\) is independent of \(n\) this proves

\[ \text{Lk}(C_\sigma, C_\gamma) = \lim_{n \to \infty} \frac{\Phi_{C_\sigma}(\gamma^n)}{n} = \Psi_{C_\sigma}(\gamma). \]

\[ \square \]
REFERENCES
