Lattice Points in Cones and Dirichlet Series

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Abstract

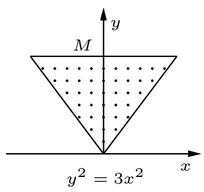
Hecke proved the meromorphic continuation of a Dirichlet series associated to the lattice points in a triangle with a real quadratic slope and found the possible poles in terms of the fundamental unit. An analogous result is proven for certain elliptical cones where now the poles are determined by the spectrum of the Laplacian on an arithmetic Riemann surface.

1 Some results of Hardy-Littlewood and Hecke

In the early 1920's Hardy-Littlewood and Hecke studied the analytic properties of a Dirichlet series whose summatory function counts lattice points in a triangle. More precisely, for fixed $q \in \mathbb{R}^+$, consider the triangle in the x, y-plane whose sides are given by $qx^2 = y^2$ and $y = M$. Letting

$$a(m) = a_q(m) = \#\{x \in \mathbb{Z} \mid qx^2 < m^2\}$$

we clearly have that $\sum_{m<M} a(m)$ is the number of lattice points in this triangle.



The asymptotics of $\sum_{m<M} a(m)$ as $M \to \infty$ is intimately connected to the analytic properties of the associated Dirichlet series

$$\phi(s) = \phi_q(s) = \sum_{m\geq 1} a(m)m^{-s}.$$

These properties are in turn determined by the diophantine properties of $\sqrt{q}$ as reflected in it simple continued fraction:

$$\sqrt{q} = a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\cdots . \tag{1}$$

If $\sqrt{q}$ is rational the problem is trivial and otherwise let $\lambda \in [1, \infty]$ be defined by

$$\lambda = \limsup \frac{\log q_{n+1}}{\log q_n},$$

where q_n is the denominator of the n-th convergent in (1):

$$\frac{p_n}{q_n} = a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\cdots\frac{1}{a_n}.$$

Hardy and Littlewood (see [2] and [3]) showed that $\phi(s)$ is analytic for $\mathrm{Re}(s) > 1 - \lambda^{-1}$ except for a simple pole at $s = 2$ and that, if $\lambda > 1$, the line $\mathrm{Re}(s) = 1 - \lambda^{-1}$ is a natural boundary. However, $\lambda = 1$ for almost all $\sqrt{q}$, in particular when $\sqrt{q}$ is algebraic (see [12] and [16]). Hardy and Littlewood conjectured that when $\lambda = 1$ the imaginary axis is a natural boundary for $\phi(s)$ *unless* $\sqrt{q}$ *is a quadratic irrational*. This remains an important open problem.

Already in 1921 Hecke [4] showed that the situation when $\sqrt{q}$ is a quadratic irrational is truly different. Suppose, for instance, that $q > 0$ is a square free integer with $q \equiv 2$ *or* $3 \pmod 4$. The continued fraction expansion (1) of $\sqrt{q}$ is periodic starting at a_1. If k is the smallest *even* period then let

$$\eta = p_{k-1} + q_{k-1}\sqrt{q}. \tag{2}$$

Hecke showed the following.

Theorem 1 *(Hecke) The function $\phi(s)$ has a meromorphic continuation to the whole s-plane with at most simple poles at $s = 2$ and $s = -2\ell \pm \frac{2\pi n}{\log \eta} i$ with ℓ and n nonnegative integers.*

Let us recall Hecke's method, which uses analysis in the real quadratic field $\mathbb{Q}(\sqrt{q})$. The integers $\mathcal{O}$ in $\mathbb{Q}(\sqrt{q})$ are all elements of the form $\alpha = x_0 + x_1\sqrt{q}$ where $a, b \in \mathbb{Z}$. The conjugate of α is $\alpha' = x_0 - x_1\sqrt{q}$ and α is called totally positive if $\alpha, \alpha' > 0$. Denote by $\mathcal{O}^+$ the set of totally positive elements in $\mathcal{O}$. The units in $\mathcal{O}^+$ form an infinite cyclic group generated by η from (2), which is called the fundamental totally positive unit for $\mathbb{Q}(\sqrt{q})$ (see e.g. p.64. of [21]).

Hecke's idea is to observe that the function

$$\phi(s, x) = 2^s \sum_{\alpha \in \mathcal{O}^+} (\alpha e^x + \alpha' e^{-x})^{-s}, \tag{3}$$

specializes at $x = 0$ to $\phi(s)$. Due to the action of the totally positive units on $\mathcal{O}^+$, the function $\phi(s, x)$ is periodic in x. It is also even, hence it has a Fourier cosine expansion:

$$\phi(s, x) = c_0(s) + 2\sum_{n>0} c_n(s)\cos(2\pi n x/\log \eta). \tag{4}$$

Here the Fourier coefficient $c_n(s)$ is given by the integral

$$c_n(s) = \frac{1}{\log \eta} \int_0^{\log \eta} \cos(2\pi n x/\log \eta)\phi(s,x)dx.$$

Substituting (3) in this integral, replacing α by $\eta^\ell \alpha$ and α' by $\eta^{-\ell}\alpha'$ and executing the summation over $\ell \in \mathbb{Z}$ gives

$$c_n(s) = \sum_{\substack{(\alpha) \\ \alpha \in \mathcal{O}^+}} \frac{1}{\log \eta} \int_{-\infty}^{\infty} \cos(2\pi n x/\log \eta)(\alpha e^x + \alpha' e^{-x})^{-s} dx,$$

where the sum is now over non-associated totally positive integers α. This integral may be evaluated leading to

$$c_n(s) = \frac{2^{s-1}}{\log \eta} \frac{\Gamma(\frac{s}{2} + \frac{\pi i n}{\log \eta})\Gamma(\frac{s}{2} - \frac{\pi i n}{\log \eta})}{\Gamma(s)} \zeta^+(\tfrac{s}{2}, u_n)$$

where

$$\zeta^+(s, u_n) = \sum_{\substack{(\alpha) \\ \alpha \in \mathcal{O}^+}} u_n(\alpha) N(\alpha)^{-s},$$

with $u_n(\alpha) = \cos\left(\pi n \frac{\log|\alpha/\alpha'|}{\log \eta}\right)$ and $N(\alpha) = \alpha\alpha'$. Write $\zeta^+(s) = \zeta^+(s, u_0)$.

It follows from Hecke's fundamental results about Hecke ζ-functions with Grössencharacters that $\zeta^+(s, u_n)$ is entire of order 1 unless $n = 0$, in which case $(s-1)\zeta^+(s)$ is. Furthermore, by standard analysis using the functional equations for the Hecke ζ-functions, we may deduce that on any compact subset in s that does not contain any of the poles of any $c_n(s)$ the series

$$\phi(s) = \frac{\sqrt{\pi}\Gamma(\frac{s}{2})\zeta^+(\frac{s}{2})}{\log \eta\, \Gamma(\frac{s+1}{2})} + \frac{2^s}{\log \eta} \sum_{n \geq 1} \frac{\Gamma(\frac{s}{2} + \frac{\pi i n}{\log \eta})\Gamma(\frac{s}{2} - \frac{\pi i n}{\log \eta})}{\Gamma(s)} \zeta^+(\tfrac{s}{2}, u_n) \tag{5}$$

converges uniformly, giving the theorem.

It is also clear that one may read off the residues from this expansion in terms of special values of $\zeta^+(s, u_n)$. At $s = 2$ the residue is $4/\sqrt{q}$. Using the Kronecker limit formula it is possible to derive the following remarkable formula for the residue of $\phi(s)$ at $s = 0$:

$$\frac{1}{6 \log \eta}(a_1 - a_2 + a_3 - \cdots + a_k), \tag{6}$$

where a_j are the partial quotients from (1) (see [19, p.183] and [9]).

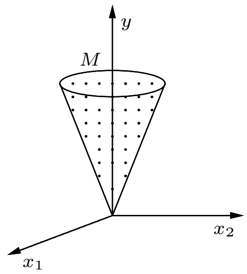
2 An analogue for cones.

In this paper we shall generalize Theorem 1 to certain right elliptical cones in $\mathbb{R}^3$. Fix $q, r \in \mathbb{R}^+$ and consider the cone in (x_1, x_2, y) with $y > 0$ given by

$qx_1^2 + rx_2^2 = y^2$ and $y = M$. Letting now

$$a(m) = a_{q,r}(m) = \#\{(x_1, x_2) \in \mathbb{Z}^2 \mid qx_1^2 + rx_2^2 < m^2\} \tag{7}$$

we have that $\sum_{m<M} a(m)$ is the number of lattice points in this cone.



$y^2 = 3x_1^2 + 5x_2^2$

We are interested in the analytic properties of the Dirichlet series

$$\phi(s) = \phi_{q,r}(s) = \sum_{m=1}^{\infty} a(m)m^{-s}. \tag{8}$$

Suppose that q and r are positive co-prime square-free integers. We may associate to them a co-finite subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$ consisting of transformations represented by those matrices

$$\begin{pmatrix} x_0 + x_1\sqrt{q} & \sqrt{r}(x_2 + x_3\sqrt{q}) \\ \sqrt{r}(x_2 - x_3\sqrt{q}) & x_0 - x_1\sqrt{q} \end{pmatrix} \tag{9}$$

of determinant 1 having each x_i integral. Let $\mathcal{H}$ be the upper half plane. It is well known that $\Gamma\backslash\mathcal{H}$ is a compact Riemann surface if and only if Γ contains no elliptic or parabolic transformations. This is equivalent to

$$|qx_1^2 + rx_2^2 - qrx_3^2| < 2 \tag{10}$$

having no non-trivial integral solutions. For simplicity, we shall assume that this holds. The Gauss-Bonnet formula then gives

$$\mathrm{v} = \mathrm{vol}(\Gamma\backslash\mathcal{H}) = \int_{\Gamma\backslash\mathcal{H}} \frac{dx\,dy}{y^2} = 4\pi(\mathrm{g} - 1), \tag{11}$$

where g is the genus of $\Gamma\backslash\mathcal{H}$. The volume, hence g, is computed explicitly in terms of q and r in [1].

The Laplacian

$$\triangle = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

on the compact Riemann surface $\Gamma\backslash\mathcal{H}$ has purely discrete spectrum

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \ldots \tag{12}$$

distributed according to Weyl's law:

$$\#\{n \mid \lambda_n \leq x\} = (\mathrm{g}-1)x + O(\sqrt{x}). \tag{13}$$

It is conjectured that $\lambda_1 \geq \frac{1}{4}$, which would follow from the Selberg eigenvalue conjecture via the Jacquet-Langlands correspondence ([11], see also [8] and [20]). The best known result in this direction is due to Kim and Sarnak [14]:

$$\lambda_1 \geq \tfrac{975}{4096} = .238\ldots.$$

We write as usual $\lambda_n = \frac{1}{4} + r_n^2$. The following is the main result of this paper.

Theorem 2 *The function $\phi(s)$ has meromorphic continuation to the whole s-plane with at most simple poles at $s = 3$, $s = 1$ and $s = -2\ell + \frac{1}{2} \pm ir_n$ when $r_n \neq 0$, and at most double poles at $-2\ell + \frac{1}{2}$. Here n is any positive integer and ℓ any nonnegative integer.*

Infinitely many examples satisfying the condition that (10) has no non-trivial integral solutions occur by taking r a prime $\equiv 1$ modulo 4 and q a quadratic non-residue modulo r. For these examples [1] yields, after some calculation, a simple formula for the genus.[1] In case q is odd we have

$$g = \frac{1}{4}(r+1)\prod_{p|q}\left(p + \left(\frac{q}{p}\right)\right) + 1,$$

while a similar formula holds when q is even. For instance, $\Gamma\backslash\mathcal{H}$ has genus 7 when $q = 3$ and $r = 5$.

3 The spectral expansion

The proof of Theorem 2 uses analysis in the quaternion algebra consisting of all matrices in (9) with rational x_i, clearly a division algebra under our assumptions. Let $\mathcal{O}$ be the order consisting of those matrices with integral x_i and $\mathcal{O}^+$ be the positive definite matrices in $\mathcal{O}$:

$$\mathcal{O}^+ = \left\{\alpha = \begin{pmatrix} x_0 + x_1\sqrt{q} & x_2\sqrt{r} \\ x_2\sqrt{r} & x_0 - x_1\sqrt{q}\end{pmatrix};\ x_0^2 - qx_1^2 - rx_2^2 > 0,\ \ x_0 > 0\right\}. \tag{14}$$

The group $\mathcal{O}^*$ of proper units of $\mathcal{O}$ consists of those matrices in $\mathcal{O}$ with determinant 1. Clearly $\gamma \in \mathcal{O}^*$ acts on $\alpha \in \mathcal{O}^+$ by $\alpha \mapsto \alpha[\gamma] \equiv {}^t\gamma\alpha\gamma$, where ${}^t\gamma$ is the transpose of γ. We shall make use of the well known fact that the set of $\alpha \in \mathcal{O}^+$ with a given determinant splits into finitely many classes modulo this action. Also, we have that $\Gamma = \mathcal{O}^*/\pm 1$, where Γ is as in (9).

[1] This calculation is complicated by the fact that the relevant order in the associated quaternion algebra is not an Eichler order.

In order to define the analogue of (3), we use the realization of the upper half plane $\mathcal{H}$ as the space $\mathcal{SP}_2$ of all 2 by 2 positive definite matrices of determinant one, where $z \in \mathcal{H}$ is identified with

$$z = \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix} \left[\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \in \mathcal{SP}_2.$$

It can be checked that $g \in \mathrm{PSL}(2, \mathbb{R})$ acting on $z \in \mathcal{SP}_2$ by $z \mapsto z[g]$ corresponds to the usual linear fractional action of g^{-1} on $x + iy \in \mathcal{H}$.

Define for $z \in \mathcal{SP}_2$ and $\mathrm{Re}(s)$ sufficiently large

$$\phi(s, z) = 2^s \sum_{\alpha \in \mathcal{O}^+} (\operatorname{tr} \alpha^* z)^{-s}, \tag{15}$$

where $\alpha^* = (\det \alpha)\alpha^{-1}$. Our interest in $\phi(s, z)$ stems from the easily seen fact that $\phi(s, I) = \phi(s)$ from (8). For fixed s with $\mathrm{Re}(s)$ large enough the series (15) and all of its partial derivatives in x and y up to the second order converge absolutely and uniformly for z in a fundamental domain for Γ. This follows from the next elementary lemma.

Lemma 1 *For a fixed $T > 0$ and positive definite*

$$\begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix},$$

the region in $(x_0, x_1, x_2) \in \mathbb{R}^3$ with $x_0 > 0$ defined by the conditions

$$x_0^2 - q x_1^2 - r x_2^2 \geq 0 \quad \textit{and} \quad x_0(y_1 + y_3) + x_1 \sqrt{q}(y_3 - y_1) - 2 x_2 \sqrt{r} y_2 \leq T$$

is bounded.

Proof: First make the change of variables

$$x_0 \mapsto x_0, \quad x_1 \mapsto q^{-1/2} x_1 \quad \text{and} \quad x_2 \mapsto r^{-1/2} x_2,$$

which does not change the boundedness condition. It leads to the two conditions

$$x_0^2 - x_1^2 - x_2^2 \geq 0 \quad \text{and} \quad x_0(y_1 + y_3) + x_1(y_3 - y_1) - 2 x_2 y_2 \leq T$$

for which the boundedness is insured if the plane from the second inequality intersects the circular cone from the first inequality in an ellipse. This is equivalent to the condition $\cos\theta > \frac{1}{\sqrt{2}}$ where θ is the angle between the normal to the plane and the x_0-axis. But

$$\cos\theta = \frac{y_1 + y_3}{\sqrt{2y_1^2 + 4y_2^2 + 2y_3^2}} > \frac{1}{\sqrt{2}}$$

since $y_1, y_3 > 0$ and $y_2^2 < y_1 y_3$. $\square$

Since clearly $\phi(s,z) = \phi(s, z[\gamma])$ for $\gamma \in \Gamma$, we have that $\phi(s,z)$ defines a C^2 function on the compact Riemann surface $\Gamma\backslash\mathcal{H}$. The Hilbert space $L^2(\Gamma\backslash\mathcal{H})$, with respect to the (normalized) invariant measure $d\mu = d\mu(z) = \mathrm{v}^{-1} dx\, dy/y^2$, has an orthonormal basis consisting of real eigenfunctions u_n for the Laplacian

$$\triangle u_n + \lambda_n u_n = 0,$$

where λ_n are in (12). The following result may now be deduced from [5].

Proposition 1 *For fixed s with $\mathrm{Re}(s)$ sufficiently large we have the identity*

$$\phi(s,z) = \sum_{n\geq 0} c_n(s) u_n(z), \quad \text{where} \quad c_n(s) =< \phi(s,z), u_n(z) >, \tag{16}$$

the convergence being absolute and uniform for $z \in \Gamma\backslash\mathcal{H}$. In particular, if $\mathrm{Re}(s)$ is sufficiently large,

$$\phi(s) = \sum_{n\geq 0} c_n(s) u_n(I).$$

4 The spectral coefficients

To proceed we must compute the spectral coefficients $c_n(s)$ from (16). Suppose $\mathrm{Re}(s)$ is sufficiently large. We have

$$c_n(s) =< \phi(s,z), u_n(z) >= 2^s \int_{\Gamma\backslash\mathcal{H}} \sum_{\alpha\in\mathcal{O}^+} (\mathrm{tr}\, \alpha^* z)^{-s} u_n(z) d\mu$$

$$= 2^s \sum_{\gamma\in\Gamma} \int_{\Gamma\backslash\mathcal{H}} \sum_{(\alpha)} (\mathrm{tr}\, \alpha[\gamma]^* z)^{-s} u_n(z) d\mu,$$

where the sum over (α) runs over a complete set of representatives of $\alpha \in \mathcal{O}^+$ modulo the action $\alpha \mapsto \alpha[\gamma]$ by $\gamma \in \Gamma$, using that Γ contains no elliptic transformations. Since γ^{-1} runs over all of Γ as γ does, using that $u_n(z) = u_n(z[\gamma])$ we get

$$c_n(s) = 2^s \sum_{\gamma\in\Gamma} \int_{\Gamma\backslash\mathcal{H}} \sum_{(\alpha)} (\mathrm{tr}\, \alpha^* z[\gamma])^{-s} u_n(z[\gamma]) d\mu = 2^s \int_{\mathcal{H}} \sum_{(\alpha)} (\mathrm{tr}\, \alpha^* z)^{-s} u_n(z) d\mu,$$

where in the last equality we have made a change of variable and unfolded the integral. Thus we have

$$c_n(s) = 2^s \sum_{(\alpha)} (\det \alpha)^{-\frac{s}{2}} \int_{\mathcal{H}} (\mathrm{tr}\, \alpha_0^{-1} z)^{-s} u_n(z) d\mu \tag{17}$$

where $\alpha_0 = \alpha/\sqrt{\det \alpha} \in \mathcal{SP}_2$. Now the kernel function $k(w,z) = (\frac{1}{2}\mathrm{tr}\, w^{-1} z)^{-s}$ on $\mathcal{SP}_2$ is clearly a point pair invariant: $k(w[g], z[g]) = k(w,z)$ for any $g \in$

$\mathrm{PSL}(2,\mathbb{R})$. In fact, a calculation shows that $\frac{1}{2}\mathrm{tr}\, w^{-1}z = \cosh\rho(z,w)$, where $\rho(z,w)$ is the hyperbolic distance from z to w. We conclude that

$$\int_{\mathcal{H}} k(w,z)u_n(z)d\mu(z) = \lambda_n(s)u_n(w)$$

where $\lambda_n(s)$ is the Selberg transform of $k(w,z)$ (see [10]). Thus from (17)

$$c_n(s) = \lambda_n(s)\sum_{(\alpha)}(\det\alpha)^{-\frac{s}{2}}u_n(\alpha_0) = \lambda_n(s)\zeta^+(\tfrac{s}{2},u_n), \tag{18}$$

where

$$\zeta^+(s,u_n) = \sum_{(\alpha)} u_n(\alpha_0)(\det\alpha)^{-s}. \tag{19}$$

Before discussing the analytic properties of $\zeta^+(s,u_n)$ let us compute $\lambda_n(s)$. Since $\mathrm{Im}(z)^{1/2+ir_n}$ has the same eigenvalue for $\triangle$ on $\mathcal{H}$ as u_n we have

$$\lambda_n(s)\,\mathrm{Im}(w)^{1/2+ir_n} = \int_{\mathcal{H}}(\mathrm{tr}(w^{-1}z)/2)^{-s}\mathrm{Im}(z)^{1/2+ir_n}d\mu.$$

If $w = u+iv$ make the change of variables $z\mapsto z[g]$ where

$$g = \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ u & 1\end{pmatrix}$$

to get

$$2^{-s}\lambda_n(s) = \int_{\mathcal{H}}(\mathrm{tr}\, z)^{-s}\,\mathrm{Im}(z)^{1/2+ir_n}d\mu = \mathrm{v}^{-1}\int_0^\infty\int_{-\infty}^\infty (x^2+y^2+1)^{-s}y^{s+ir_n-3/2}dxdy$$

which is readily evaluated to give

$$\lambda_n(s) = \frac{2^{s-1}\sqrt{\pi}}{\mathrm{v}}\,\frac{\Gamma(\frac{s-1/2+ir_n}{2})\Gamma(\frac{s-1/2-ir_n}{2})}{\Gamma(s)}.$$

Thus we have proven the following.

Proposition 2 *For fixed s with $\mathrm{Re}(s)$ sufficiently large we have the identity for the spectral coefficients $c_n(s)$ from Proposition 1:*

$$c_n(s) = \frac{2^{s-1}\sqrt{\pi}}{\mathrm{v}}\,\frac{\Gamma(\frac{s-1/2+ir_n}{2})\Gamma(\frac{s-1/2-ir_n}{2})}{\Gamma(s)}\zeta^+(\tfrac{s}{2},u_n) \tag{20}$$

where $\zeta^+(s,u_n)$ is defined in (19) and v in (11).

5 The Maass ζ-function

In order to finish the proof of Theorem 2, we must establish some needed analytic properties of the Maass ζ-function

$$\zeta^+(s, u_n) = \sum_{m \geq 1} h_n(m) m^{-s}, \quad \text{where} \quad h_n(m) = \sum_{\substack{(\alpha) \\ \det \alpha = m}} u_n(\alpha_0). \tag{21}$$

Write $\zeta^+(s) = \zeta^+(s, u_0) = \sum_{m \geq 1} h_0(m) m^{-s}$ and observe that $h_0(m)$ is a non-negative integer. The function $\zeta^+(s)$ was introduced by Siegel [18], who showed that it is entire except for a simple pole at $s = 3/2$ with residue $\mathrm{v}/(2\sqrt{qr})$, and satisfies a functional equation. In general, $\zeta^+(s, u_n)$ was first studied by Maass in [15] where he established, among other things, that it is a meromorphic function. Later Hejhal [6, 7] generalized and strengthened certain of the results of Maass. We shall quote the needed results from [6, 7].

A straightforward calculation shows that Γ from (9) maps under $\mathcal{W}$, defined in [6, p.413], to a subgroup of finite index in $\mathcal{G}_{rq}$, defined in [6, p.414]. Also, $\zeta^+(s, u_n)$ is the same, up to a simple factor, as $F_a(s; S)$ from [6, p.416]. Theorem 2 of [6] then implies that $\zeta^+(s, u_n)$ converges absolutely for $\mathrm{Re}(s) > 3/2$ and has analytic continuation to the entire s-plane (except at $s = 3/2$ when $n = 0$). It also satisfies a functional equation of the form

$$\begin{aligned}(2\pi)^{2s} \zeta^+(3/2 - s, u_n) = {} & \frac{\Gamma(s - \frac{1/2 + ir_n}{2}) \Gamma(s - \frac{1/2 - ir_n}{2})}{\Gamma(\frac{1}{2} + s) \Gamma(\frac{1}{2} - s)} \psi_1(s, u_n) \\ & + \frac{\Gamma(s - \frac{1/2 + ir_n}{2}) \Gamma(s - \frac{1/2 - ir_n}{2})}{\Gamma(\frac{1/2 + ir_n}{2}) \Gamma(\frac{1/2 - ir_n}{2})} \psi_2(s, u_n),\end{aligned} \tag{22}$$

where $\psi_1(s, u_n)$ and $\psi_2(s, u_n)$ are Dirichlet series of the same type that are absolutely convergent for $\mathrm{Re}(s) > 3/2$. Furthermore, it is shown in [7, (7.1)] that in a fixed vertical strip $\mathcal{S}$

$$\zeta^+(s, u_n) \ll_{n, \mathcal{S}, c} e^{c|\mathrm{Im}(s)|} \tag{23}$$

for any $c > \pi/2$.

Since we must establish convergence of the spectral expansion for s outside of the region of convergence of $\zeta^+(s, u_n)$, we will need to estimate $\zeta^+(s, u_n)$ uniformly in the eigenvalue λ_n. By [17] we have

$$\sup_{z \in \Gamma \backslash \mathcal{H}} |u_n(z)| \ll \lambda_n^{1/4}. \tag{24}$$

It follows from (21), (24) and [6, (4.1)] that

$$\sum_{m \leq x} |h_n(m)| \ll \lambda_n^{1/4} x^{3/2}.$$

Similarly, from [6, (4.1)] the same bound holds for the partial sums of the coefficients of $\psi_1(s, u_n)$ and $\psi_2(s, u_n)$. Thus $\zeta^+(s, u_n), \psi_1(s, u_n)$, and $\psi_2(s, u_n)$

are all $\ll \lambda_n^{1/4}$ for $\mathrm{Re}(s) \geq 2$, say. Now a standard argument using (22), (23), well known properties of the gamma function and the Phragmen-Lindelöf theorem proves the second statement in the following.

Proposition 3 *The Maass ζ-function $\zeta^+(s, u_n)$ in entire except when $n = 0$, where it has only a simple pole at $s = 3/2$ with residue $\mathrm{v}/(2\sqrt{qr})$. Furthermore, for s in any compact subset of the s-plane, we have for $n > 0$ that*

$$\zeta^+(s, u_n) \ll \lambda_n^A,$$

where A is a positive constant.

6 The final formula

We now may complete the proof of Theorem 2. We have from Propositions 1 and 2 that if $\mathrm{Re}(s)$ is sufficiently large,

$$\phi(s) = \frac{2\pi}{\mathrm{v}} \frac{\zeta^+(\frac{s}{2})}{s-1} + \frac{2^{s-1}\sqrt{\pi}}{\mathrm{v}} \sum_{n \geq 1} \frac{\Gamma(\frac{s-1/2+ir_n}{2})\Gamma(\frac{s-1/2-ir_n}{2})}{\Gamma(s)} \zeta^+(\tfrac{s}{2}, u_n) u_n(I).$$

By Proposition 3, (24) and Weyl's law (13) we see that this formula provides the needed meromorphic continuation of $\phi(s)$ since it is absolutely and uniformly convergent on any compact subset in the s-plane not containing any of the poles.

As with Hecke's formula (5), one may read off the residues of the poles; the poles $s = 3$ and $s = 1$ both occur in the first term. The residue at $s = 3$ is $\frac{\pi}{\sqrt{qr}}$, while that at $s = 1$ may be written

$$(2\mathrm{g} - 2)^{-1} \zeta^+(1/2).$$

In view of (6), it might be interesting to determine $\zeta^+(1/2)$ in elementary terms.

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References

[1] J.H.H. Chalk, Units of indefinite quaternion algebras. Proc. Roy. Soc. Edinburgh Sect. A 87 (1980/81), no. 1-2, 111–126.

[2] G.H. Hardy and J.E. Littlewood, *Some problems of Diophantine approximation: the lattice points of a right-angled triangle*, Hamburg. Math.Abh., 1 (1921) 212–249.

[3] G.H Hardy and J.E.Littlewood, *Some problems of Diophantine approximation: the analytic properties of certain Dirichlet series associated with the distribution of numbers to modulus unity*, Trans. Cambridge Phil. Soc., **XXII**, 1923, 519-534.

[4] E. Hecke, *Über Analytische Funktionen und die verteilung von zahlen mod eins*, Hamburg. Math.Abh., 1(1921) 54–76

[5] D.A. Hejhal, The Selberg trace formula for PSL$(2, R)$. Vol. I. Lecture Notes in Mathematics, Vol. 548. Springer-Verlag, Berlin-New York, 1976. vi+516 pp.

[6] D.A. Hejhal , *Some Dirichlet series with coeffcients related to periods of Automorphic eigenforms I.* Proc. Japan Acad., **58** Ser. A , 1982, 413–417.

[7] D.A. Hejhal , *Some Dirichlet series with coeffcients related to periods of Automorphic eigenforms II.* Proc. Japan Acad., **59** Ser. A , 1983, 335–338.

[8] Hejhal, Dennis A. A classical approach to a well-known spectral correspondence on quaternion groups. Number theory (New York, 1983–84), 127–196, Lecture Notes in Math., 1135, Springer, Berlin, 1985.

[9] D. Hickerson, Continued fractions and density results for Dedekind sums. J. Reine Angew. Math. 290 (1977), 113–116.

[10] H. Iwaniec, Spectral methods of automorphic forms. Second edition. Graduate Studies in Mathematics, 53. American Mathematical Society, Providence, RI; Revista Matemtica Iberoamericana, Madrid, 2002

[11] H. Jacquet and R. P. Langlands, Automorphic Forms on GL(2), Lecture Notes in Math. 114, Springer, Berlin, 1970.

[12] A.Ya Khinchin, A. Ya. Continued fractions. With a preface by B. V. Gnedenko. Translated from the third (1961) Russian edition. Reprint of the 1964 translation. Dover Publications, Inc., Mineola, NY, 1997. xii+95 pp.

[13] H. Kim, "Functoriality of the exterior square of GL_4 and the symmetric fourth power of GL_2", *JAMS*, Vol. 16, 1 (2003), 139-183.

[14] H. Kim and P. Sarnak, Appendix to [13], *JAMS* **16** (2003), no. 1, 139–183.

[15] H. Maass, Über die räumliche Verteilung der Punkte in Gittern mit indefiniter Metrik. Math. Ann. 138 1959 287–315.

[16] K.F. Roth, Rational approximations to algebraic numbers. Mathematika 2 (1955), 1–20; corrigendum, 168.

[17] A. Seeger and C.D. Sogge, Bounds for eigenfunctions of differential operators. Indiana Univ. Math. J. 38 (1989), no. 3, 669–682.

[18] C.L. Siegel, Über die Zetafunktionen indefiniter quadratischer Formen, Math. Zeit. 43, 682–708 (1938).

[19] C.L. Siegel, Lectures on advanced analytic number theory. Notes by S. Raghavan. Tata Institute of Fundamental Research Lectures on Mathematics, No. 23 Tata Institute of Fundamental Research, Bombay 1965 iii+331+iii pp.

[20] Strömbergsson, Andreas. Some remarks on a spectral correspondence for Maass waveforms. Internat. Math. Res. Notices 2001, no. 10, 505–517.

[21] B.A. Venkov, Elementary number theory. Translated from the Russian and edited by Helen Alderson Wolters-Noordhoff Publishing, Groningen 1970 ix+249 pp.