AN ESTIMATE FOR THE HECKE EIGENVALUES OF MAASS FORMS

DANIEL BUMP, W. DUKE, JEFFREY HOFFSTEIN,
AND HENRYK IWANIEC

Let $\phi(z)$ be an even Maass cusp form for $SL(2, \mathbb{Z})$ which is an eigenfunction of the Hecke operators T_p for all primes p, with eigenvalues $a(p)$. Let $\alpha(p)$ and $\beta(p)$ be the roots of the Hecke polynomial:

$$1 - a(p)X + X^2 = (1 - \alpha(p)X)(1 - \beta(p)X).$$

We will prove the following theorem.

THEOREM. *For all primes p we have $|\alpha(p)|, |\beta(p)| \leqslant p^{5/28}$.*

Previously the best known exponent was 1/5. The estimate $|\alpha|, |\beta| \leqslant p^{1/5}$ first appeared in an unpublished letter of Serre to Deshouillers, as an application of Landau's lemma combined with work of Jacquet, Piatetski-Shapiro, and Shalika ([5], [6], and [7], then unpublished); published versions of this proof may be found in Moreno and Shahidi [8] and M. R. Murty [9]. This proof does not generalize, however, to give a corresponding estimate over a number field. A proof valid over a number field was obtained as part of the far-reaching work of Shahidi [12], [13] on automorphic L-functions. Moreover, Shahidi obtained the strict inequality $|\alpha|, |\beta| < p^{1/5}$. We refer to [11] for further historical remarks.

It is doubtless possible to adapt our method to give the 5/28 estimate for an arbitrary automorphic representation of $GL(2)$ over $\mathbb{Q}$. Over a number field, however, Shahidi still has the best estimate. Both the original proof of the 1/5 estimate and the present proof of the 5/28 bound depend on the gamma functions in the functional equation and give weaker bounds over a number field. It is remarkable that Shahidi's proof, which is based on entirely different principles, does not have this defect.

There are two principal new ingredients in obtaining the exponent 5/28. The first is that the role classically played by Landau's lemma is taken over by a new method of estimating the coefficients of a Dirichlet series, introduced in Duke and Iwaniec [3], which replaces the customary assumption of positivity for the coefficients with

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information about the twists of the series by Dirichlet characters. The second new ingredient is a theorem of Bump and Ginzburg [1] showing that the symmetric square L-function of a cusp form on $GL(r)$ cannot have a pole unless the central character is quadratic. The possibility of such a result, at least when $r = 3$, became evident when Patterson and Piatetski-Shapiro [10] found a Rankin-Selberg convolution for the symmetric square L-function on $GL(3)$. We will apply the symmetric square construction to twists of the Gelbart-Jacquet lift of ϕ to $GL(3)$ and apply the new estimation technique. A significant point is that the crucial inequality (1.3) is a consequence of Deligne's estimate for hyper-Kloosterman sums [2], which emerge on summing the Gauss sums which occur in the functional equations of the twisted Dirichlet series.

Theorem 1 in [3] requires functional equations for a Dirichlet series twisted by nearly all characters modulo a fixed conductor q; what is available from [1] are twists by those characters which are squares. We thus must either modify [1] to obtain further twists, or else we must modify the Theorem of [3] to work with fewer twists. We choose the latter course.

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1. Estimation of coefficients in a Dirichlet series. In this section we prove a slightly modified version of Theorem 1 of Duke and Iwaniec [3]. We will write down an exact statement, but as the proof is only slightly altered, we will assume familiarity with the argument in [3], simply indicating how changes must be made in the original proof to obtain the new theorem.

Let $\mathscr{A} = (a_n)$ be a sequence of complex numbers. Suppose that the series

$$\mathscr{A}(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

converges absolutely in $\operatorname{Re} s > 1$. Set

$$\mathscr{A}(s, \chi) = \sum_{n=1}^{\infty} a_n \chi(n) n^{-s}$$

where χ is a Dirichlet character modulo a prime q. Let $\mathscr{H}_q$ be an exceptional set of Dirichlet characters mod q containing the trivial character, with $|\mathscr{H}_q| \leqslant H$; i.e., the number of exceptions is bounded and independent of q. Assume that there is a fixed integer r such that for $\chi^r \notin \mathscr{H}_q$ the series $\mathscr{A}(s, \chi^r)$ may be analytically continued to an entire function satisfying a functional equation

$$\mathscr{A}(1 - s, \chi^r) = \varepsilon_{\chi^r} \Phi(s) \mathscr{A}(s, \chi^{-r}) \tag{1.1}$$

where $|\varepsilon_{\chi^r}| = 1$ and $\Phi(s)$ is holomorphic in $\operatorname{Re} s > 1$. Now $\Phi(s)$ depends on q and may depend on the parity of χ^r, but not on χ^r otherwise. Moreover, we assume there exists a constant $c \geqslant 1$ such that

$$\Phi(s) \ll (q|s|)^{(2\sigma-1)c} \tag{1.2}$$

on $\operatorname{Re} s = \sigma > 1$, the implied constant depending on σ. Finally, we assume that for any a the $\varepsilon_{\chi^r}\chi^{-r}(a)$ are randomly distributed on the unit circle, i.e., that

$$K_q(a) = \sum_{\chi^r \notin \mathscr{H}_q} \varepsilon_{\chi^r}\chi^{-r}(a) \ll q^{1/2}. \tag{1.3}$$

Then we have the following proposition.

PROPOSITION 1. *If the above conditions hold for a set of primes q of positive density, then for any $n \geqslant 1$ we have*

$$a_n \ll n^{(2c-1)/(2c+1)+\varepsilon}$$

with any $\varepsilon > 0$, the implied constant depending on ε.

Proof. The original proof of [3] goes through identically, with the following modifications. A new definition is given for $\mathscr{A}_f(q; l)$, namely,

$$\mathscr{A}_f(q; l) = \sum_{n^r \equiv l^r (\operatorname{mod} q)} a_n f(n),$$

and thus if

$$\mathscr{A}_f(\chi) = \sum_n a_n \chi(n) f(n),$$

we have

$$\mathscr{A}_f(q; l) = \frac{1}{q-1} \sum_{\chi (\operatorname{mod} q)} \chi^{-r}(l) \mathscr{A}_f(\chi^r).$$

Also, we set

$$\mathscr{M}_f(q; l) = \frac{1}{q-1} \sum_{\chi \in \mathscr{H}_q} \chi^{-r}(l) \mathscr{A}_f(\chi^r).$$

The only other place where a modification is necessary is in the proof of Theorem 1 of [3], where in place of the first line we have

$$\sum_{n^r \equiv l^r (\operatorname{mod} q)} a_n \omega(nl^{-1}) \ll q^{-1} \sum_{l/2 < n < 2l} |a_n| + q^{c-1/2+\varepsilon}$$

(in the original version $r = 1$). Summing over q as in [3], the term $a_l\omega(1)$ will occur with high multiplicity since every q divides $l^r - l^r$, and if $n \neq l$, then since n and l are positive integers, we have $n^r \neq l^r$; so there are, as in [3], at most $O(\log l)$ prime divisors of $n^r - l^r$ (the constant now depending on r). The remainder of the proof then goes through unchanged. ■

2. The analytic continuation and functional equation of the fourth symmetric power. Let π be the automorphic representation of $GL(2, A)$ corresponding to the Maass cusp form ϕ, A being the adele ring of $\mathbb{Q}$.

Let χ be a primitive Dirichlet character modulo q, where q is a prime. We associate with χ a character of the idele class group $A^\times/\mathbb{Q}^\times$, which by abuse of notation we will also denote as χ, in the usual way: If a is any idele, there exists $\alpha \in \mathbb{Q}^\times$ such that $(\alpha a)_\infty > 0$ and $(\alpha a)_q$ is a unit congruent to 1 modulo q in $\mathbb{Q}_q$; then the idele αa determines an ideal $\mathfrak{a}$ in $\mathbb{Z}$ which is relatively prime to q, and we define $\chi(a) = \chi(\mathfrak{a})$.

Let S be the set $\{\infty, q\}$ of places of $\mathbb{Q}$. The partial symmetric fourth power L-function twisted by χ^2 is by definition

$$\begin{aligned} &L_S(s, \pi, \vee^4 \otimes \chi^2) \\ &\quad = \prod_{p \notin S} (1 - \chi^2(p)\alpha(p)^4 p^{-s})^{-1}(1 - \chi^2(p)\alpha(p)^3\beta(p)p^{-s})^{-1}(1 - \chi^2(p)\alpha(p)^2\beta(p)^2p^{-s})^{-1} \\ &\qquad\qquad (1 - \chi^2(p)\alpha(p)\beta(p)^3p^{-s})^{-1}(1 - \chi^2(p)\beta(p)^4p^{-s})^{-1}. \end{aligned}$$

We will also denote by

$$L_S(s, \chi^2) = \prod_{p \notin S} (1 - \chi^2(p)p^{-s})^{-1} = \sum_{n=1}^{\infty} \chi^2(n)n^{-s}$$

the usual Dirichlet L-function associated with χ^2. Let v be (either) purely imaginary number such that $\frac{1}{4} - v^2$ is the eigenvalue of the noneuclidean Laplacian on ϕ. Also, let

$$\begin{aligned} &L(s, \pi, \vee^4 \otimes \chi^2) \\ &\quad = \pi^{-5s/2}\Gamma\left(\frac{s+4v}{2}\right)\Gamma\left(\frac{s+2v}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s-2v}{2}\right)\Gamma\left(\frac{s-4v}{2}\right)L_S(s, \pi, \vee^4 \otimes \chi^2), \end{aligned}$$

$$L(s, \chi^2) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)L_S(s, \chi^2),$$

$$\tau(\chi^2) = \sum_{a \bmod q} \chi^2(a)e^{2\pi i a/q}.$$

We will say that π is *monomial* if $\pi \cong \pi \otimes \eta$ for some quadratic Grössencharacter

η. In this case it is explained in Section 3.7 of [4] that results of Labesse and Langlands imply the Ramanujan conjecture for π. Hence, we may exclude this case.

PROPOSITION 2. *Suppose that π is not monomial and that $\chi^6 \neq 1$. Then the partial L-function $L_S(s, \chi^2)L_S(s, \pi, \bigvee^4 \otimes \chi^2)$ is an entire function of s. We have the functional equation*

$$L(s, \chi^2)L(s, \pi, \textstyle\bigvee^4 \otimes \chi^2) = \tau(\chi^2)^6 q^{-6s} L(1-s, \chi^{-2})L(1-s, \pi, \bigvee^4 \otimes \chi^{-2}).$$

The precise functional equation may be established either by the Rankin-Selberg method (as in the following proof) or by Shahidi's method (see particularly Corollary 6.7 of [12] on the symmetric square L-function, and the discussion on p. 418 of [11] of the standard L-functions for $GL(n) \times GL(m)$.)

Proof. If v is any place of $\mathbb{Q}$, then π_v is a nonramified principal series representation $\pi_v = \pi(\rho_v, \rho_v^{-1})$ of $GL(2, \mathbb{Q}_v)$, where ρ_v is an unramified character of $\mathbb{Q}_v^\times$. Let π'_v be the nonramified principal series representation $\pi(\rho_v^2, 1, \rho_v^{-2})$ of $GL(3, \mathbb{Q}_v)$; it is the local lifting of π_v in the sense of Gelbart and Jacquet [4], Definition 3.1.3. Let $\pi' = \otimes_v \pi'_v$. By Theorem 9.3 of Gelbart and Jacquet [4], π' is automorphic, and since we are assuming that π is not monomial, π' is cuspidal.

We will denote by $\pi'\chi$ the tensor product of π' with the one-dimensional representation $\chi \circ \det$ of $GL(3, A)$. Then

$$L_S(s, \chi^2)L_S(s, \pi, \textstyle\bigvee^4 \otimes \chi^2) = L_S(s, \pi'\chi, \bigvee^2).$$

Since the central character χ^3 of $\pi'\chi$ is not quadratic, it follows from Theorem 7.5 of Bump and Ginzburg [1] that this function is entire.

Since

$$L_S(s, \pi'\chi, \textstyle\bigvee^2) = \dfrac{L_S(s, (\pi'\chi) \times (\pi'\chi))}{L_S(s, \pi' \times \chi^2)},$$

we may use the global results of Jacquet and Shalika [6] on the Rankin-Selberg method, together with the local results of Jacquet, Piatetski-Shapiro, and Shalika [5] Theorem 3.1, and of Jacquet and Shalika [7] Theorem 5.1 to compute the local functional equations. By means of these results, the local functional equation is reduced to a product of six $GL(1)$ computations, where the local factors are computed in Tate's thesis [14], [15]. To define these, for each place v of $\mathbb{Q}$, we use the "obvious" additive character ψ_v and additive measure on $\mathbb{Q}_v$; these are specified in Tate [14], Section 2.2. We have

$$L_S(s, \pi'\chi, \textstyle\bigvee^2) = \left\{\prod_{v \in S} \gamma_v(s, \pi'_v\chi_v, \bigvee^2, \psi_v)\right\} L_S(1-s, \pi'\chi^{-1}, \bigvee^2)$$

where

$$\gamma_v(s, \pi'_v\chi_v, \vee^2, \psi_v)$$
$$= \gamma_v(s, \rho_v^4\chi_v^2, \psi_v)\gamma_v(s, \rho_v^2\chi_v^2, \psi_v)\gamma_v(s, \chi_v^2, \psi_v)^2\gamma_v(s, \rho_v^{-2}\chi_v^2, \psi_v)\gamma_v(s, \rho_v^{-4}\chi_v^2, \psi_v)$$

and where, for a character σ_v of $\mathbb{Q}_v^\times$ (or, equivalently, of the corresponding Weil group, which has $\mathbb{Q}_v^\times$ as its abelianization)

$$\gamma_v(s, \sigma_v, \psi_v) = \frac{\varepsilon_v(s, \sigma_v, \psi_v)L_v(1 - s, \sigma_v^{-1})}{L_v(s, \sigma_v)}$$

in terms of the local ε and L-functions defined in Tate [15], 3.5.1 and 3.5.2. These factors are made explicit in Tate [15], 3.1 and 3.2. We have

$$\gamma_\infty(s, \rho_\infty^k\chi_\infty^2, \psi_\infty) = \pi^{s-1/2}\frac{\Gamma\left(\frac{1 - s - kv}{2}\right)}{\Gamma\left(\frac{s + kv}{2}\right)},$$

$$\gamma_q(s, \rho_q^k\chi_q^2, \psi_q) = \alpha^k q^{-s}\tau(\chi^2).$$

Hence, we obtain the required functional equation. ■

3. Hecke eigenvalues of Maass forms. In Proposition 1, set $r = 2$ and take for $\mathscr{A}(s)$ the L-series $\zeta(s)L_S(s, \pi, \vee^4)$, and for $\mathscr{A}(s, \chi^2)$ the twisted series $L_S(s, \chi^2)L_S(s, \pi, \vee^4 \otimes \chi^2)$. Take for $\mathscr{H}_q$ the characters such that $\chi^6 = 1$. By Proposition 2 above, (1.1) holds with $\Phi(s) = \Theta(s)/\Theta(1 - s)$, where

$$\Theta(s) = \pi^{-3s}q^{3s}\Gamma\left(\frac{s + 4v}{2}\right)\Gamma\left(\frac{s + 2v}{2}\right)\Gamma\left(\frac{s}{2}\right)^2\Gamma\left(\frac{s - 2v}{2}\right)\Gamma\left(\frac{s - 4v}{2}\right)$$

and $\varepsilon_{\chi^2} = (\tau(\chi^2)/\sqrt{q})^6$, where $\tau(\chi^2)$ is the Gauss sum defined in Section 2. It is then clear that (1.2) holds with $c = 3$. To verify (1.3), note that

$$\sum_{\chi(\bmod q)} \varepsilon_{\chi^2}\chi^{-2}(a) = (q - 1)q^{-3} \sum_{(x_1 \dots x_6)^2 = a^2} e^{2\pi i(x_1 + \dots + x_6)/q},$$

and (1.3) follows as this is simply a sum of two hyper-Kloosterman sums, which are bounded by the Deligne estimate [2]. It then follows from Propositions 1 and 2 that, if A_n denotes the nth coefficient of $\zeta(s)L_S(s, \pi, \vee^4)$, then

(3.1) $$A_n \ll n^{5/7+\varepsilon}$$

for any $\varepsilon > 0$, the implied constant depending on ε. If $|\alpha(p)| > 1$ for some prime p, then $A_{p^k} \sim \alpha(p)^{4k}$ as $k \to \infty$. The "5/28" estimate follows immediately from this and (3.1).

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Bump: Department of Mathematics, Stanford University, Stanford, California 94305

Duke: Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903

Hoffstein: Department of Mathematics, Brown University, Providence, Rhode Island 02912

Iwaniec: Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903