A RESTRICTED DIVISOR PROBLEM

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ABSTRACT. A new kind of restricted divisor function is introduced. The associated divisor problem involves aspects of the usual Dirichlet divisor problem as well as the Hardy-Littlewood problem of counting lattice points in an expanding right triangle. Both problems depend on the Diophantine nature of a defining parameter: the amount of restriction in the divisor problem and a slope in the triangle problem. A method Hecke invented to analyze a Dirichlet series that arises in the Hardy-Littlewood problem is adapted to the new problem to give a "vertical" Voronoi-type formula for the restricted divisor function.

1. INTRODUCTION

Dirichlet proved that for d(n) the usual divisor function,

(1.1)
$$\Delta(x) = \sum_{n \le x} d(n) - x \log x - (2\gamma - 1)x = O(x^{\frac{1}{2}}),$$

where γ is Euler's constant. Voronoi found an explicit formula for $\Delta(x)$ that can be expressed in the form

(1.2)
$$\sum_{n \le x} \left(d(n) - \log n - 2\gamma \right) = \frac{x^{\frac{1}{4}}}{\pi\sqrt{2}} \sum_{n \ge 1} \frac{d(n)}{n^{\frac{3}{4}}} \cos(4\pi\sqrt{nx} - \frac{1}{4}\pi) + O(\log x).$$

This formula reflects the functional equation of the Dirichlet series

$$\zeta^{2}(s) = \left(\sum_{n \ge 1} n^{-s}\right)^{2} = \sum_{n \ge 1} d(n)n^{-s}$$

After truncating the sum on the RHS of (1.2) at $n = x^{\frac{1}{3}}$, a nontrivial estimate of the remainder implies that

(1.3)
$$\Delta(x) = O(x^{\frac{1}{3}+\epsilon}).$$

This kind of argument has been further developed using difficult estimates of exponential sums. Currently, the best known exponent (rounded) is .315. It is conjectured that

$$\Delta(x) = O(x^{\frac{1}{4}+\epsilon}),$$

which would be essentially best possible. See [29] for a survey and references on the Dirichlet divisor problem.

In this paper I will consider a divisor function that counts divisors that are restricted in a certain way. For a fixed $\alpha > 1$ define

(1.4)
$$d(\alpha, n) = \#\{d|n; \ \alpha^{-1}n \le d^2 \le \alpha n\}.$$

Thus $d(\alpha, n)$ counts divisors of n that are quite close together; it can be written as

(1.5)
$$d(\alpha, n) = \#\{n = d_1 d_2; \ \alpha^{-1} d_1 \le d_2 \le \alpha d_1\}.$$

Date: March 17, 2024.



FIGURE 1. Restricted divisor function

On average, the number of divisors so restricted is much smaller than the total number. Let

(1.6)
$$\Delta(\alpha, x) = \sum_{n \le x} d(n, \alpha) - x \log \alpha.$$

A simple count of lattice points using the area of the hyperbolic sector determined by (1.5) for the main term (see Figure 1) yields the estimate

(1.7)
$$\Delta(\alpha, x) = O(x^{\frac{1}{2}}).$$

In this paper α is always fixed and implied constants may depend on it, without that being explicitly stated.

The growth of $\Delta(\alpha, x)$ depends on the arithmetic (Diophantine) nature of α . The restricted divisor problem may be thought of as a degree two analogue of the classical problem of Hardy-Littlewood [19],[20] on counting lattice points in the right triangle in the first quadrant bounded by the line $\alpha x + y = n$. That problem comes down to understanding the behavior of the sum

(1.8)
$$S(\alpha, x) = \sum_{1 \le n \le x} (\{n\alpha\} - \frac{1}{2}),$$

where $\{a\} = a - \lfloor a \rfloor$. It is to be expected that the restricted divisor problem that we are considering should be more difficult than the degree one triangle problem. In fact, they are related in a way that allows us to obtain some information about $\Delta(\alpha, x)$ from known results about $S(\alpha, x)$. Define

(1.9)
$$q = q(\alpha, x) = \lfloor \left(\frac{x}{\alpha}\right)^{\frac{1}{2}} \rfloor.$$

A standard application of Vinogradov's lattice point counting method, which originally yielded a different proof of (1.3), gives the next result.

Theorem 1. Let $\Delta(\alpha, x)$ be given in (1.6). Then for a fixed $\alpha \ge 1$ and any $\epsilon > 0$ we have that

$$\Delta(\alpha, x) = -2S(\alpha, q) + O(x^{\frac{1}{3}+\epsilon})$$

The implied constant depends on α and ϵ .

For any $\epsilon > 0$ we deduce from [19] (see also [30]) that irrational α exists so that

$$S(\alpha, x) = O(x^{1-\epsilon})$$

does not hold. Thus by Theorem 1 and (1.9), for any $\epsilon > 0$ there exist irrational α for which (1.10) $\Delta(\alpha, x) = O(x^{\frac{1}{2}-\epsilon})$ does not hold. That is, for general irrational α we cannot essentially improve the trivial bound (1.7). Note that if $\alpha \in \mathbb{Q}$ we get another main term in the count coming from points on the edges.

If the simple continued fraction expansion of irrational α has bounded partial quotients, which includes quadratic irrational α , then we have the essentially optimal results

(1.11)
$$S(\alpha, n) = O(\log n) \text{ and } S(\alpha, n) = \Omega_{\pm}(\log n),$$

which were proven independently by Hardy-Littlewood [19], [20] and Ostrowski [33]. Thus for such α

(1.12)
$$\Delta(x,\alpha) = O(x^{\frac{1}{3}+\epsilon}).$$

Numerical evidence suggests that for these α the much stronger bound

(1.13)
$$\Delta(x,\alpha) = O(x^{\epsilon}) \quad \text{or even} \quad \Delta(x,\alpha) = O(\log x)$$

should hold, unlike the bound for $\Delta(x)$ in the usual divisor problem. The methods used to prove (1.11) rely heavily on continued fractions. To use similar methods to bound $\Delta(\alpha, x)$ seems to be an interesting and non-trivial problem. Here I will take a different approach to the problem for certain special α .

Just like analytic properties of $\zeta^2(s) + \zeta'(s) - 2\gamma\zeta(s)$ affect the behavior of $\Delta(x)$, analytic properties of the Dirichlet series

(1.14)
$$\psi(s,\alpha) = \sum_{n \ge 1} (\{n\alpha\} - \frac{1}{2})n^{-s}$$

affect the behavior of $S(\alpha, x)$. For many α , the function $\psi(s, \alpha)$ has a natural boundary. For some real quadratic α , Hecke applied his Zeta functions with Grössencharakteren to give the meromorphic continuation of this Dirichlet series $\psi(s, \alpha)$ to \mathbb{C} , along with the explicit determination of its infinitely many poles and their residues. He made a clever and intricate application of (a slight extension of) the Schnee-Landau theorem to get the bound

(1.15)
$$S(\alpha, x) \ll x^{\epsilon}$$
 for all $\epsilon > 0$.

Although this is weaker than (1.11), the analytic properties of $\psi(s, \alpha)$ used in the proof can also be applied to obtain results about averages of $S(\alpha, x)$ that are otherwise inaccessible (see [26, p. 330]).

In this article I will apply a variation of Hecke's method to obtain a Voronoi-type formula for certain averages of $\Delta(\alpha, n)$, at least for some special real quadratic α , namely

(1.16)
$$\alpha = \frac{1}{2}(a + \sqrt{a^2 - 4}),$$

where $D = a^2 - 4 > 5$ is the discriminant of a real quadratic field. The number α , which is totally positive, is the fundamental unit in $\mathbb{F} = \mathbb{Q}(\sqrt{D})$. The first few eligible *a* are

$$a = 4, 5, 8, 9, 12, 13, 15, 17, 19, 21, 24, 28, 31, 32, \dots$$

The simple continued fraction of α is

(1.17)
$$\alpha = a - 1 + \frac{1}{1+} \frac{1}{(a-2)+} \frac{1}{1+} \frac{1}{(a-2)+} \cdots,$$

and the ring of integers of $\mathbb F$ is given by

$$\mathcal{O} = \mathbb{Z} + \alpha \mathbb{Z}.$$

Let $\nu_0(\beta) = 1$ and $\nu_1(\beta) = \operatorname{sgn}(\beta\beta')$ be Hecke's sign characters and set

$$\lambda(\beta) = e^{\frac{\pi i \log |\beta/\beta'|}{\log \alpha}}$$

For $n \in \mathbb{Z}$ the Grössencharaktere $\nu_j(\beta)\lambda^n(\beta)$ satisfies $\nu_j(\alpha)\lambda^n(\alpha) = 1$ and so is well-defined on (narrow) principal ideals $(\beta) \subset \mathcal{O}$. The associated zeta function is given for Re s > 1 by

(1.18)
$$\zeta(s,\nu_j\lambda^n) = \sum_{(\beta)} \nu_j(\beta)\lambda^n(\beta)|N(\beta)|^{-s},$$

where the sum is over distinct nonzero (narrow) principal ideals of \mathcal{O} . The function $\zeta(s, \nu_1 \lambda^n)$ has analytic continuation to an entire function in s. Our main result is the following "vertical" analogue of Voronoi's formula (1.2).

Theorem 2. Fix α as in (1.16) and r > 3. Then for $d(\alpha, n)$ defined in (1.4),

$$\sum_{n < x} (1 - \frac{n}{x})^r \left(d(n, \alpha) - \log \alpha \right) = \frac{3 - a}{12 \log \alpha} \log x + c_0 + \frac{\Gamma(r+1)}{\log \alpha} \sum_{n \neq 0} (-1)^n \frac{\zeta(in\kappa, \nu_1 \lambda^n)}{(in\kappa)_{r+1}} e^{\frac{\pi i n \log x}{\log \alpha}} + o(1).$$

Here c_0 is a constant that depends on r and a, and $\kappa = \frac{\pi}{\log \alpha}$. The sum is absolutely convergent. Also, $(\cdot)_{r+1}$ is the Pochhammer symbol.

By trivially estimating the infinite sum on the RHS of the formula we get for r > 3

(1.19)
$$\sum_{n \le x} (1 - \frac{n}{x})^r \left(d(n, \alpha) - \log \alpha \right) = \frac{3 - a}{12 \log \alpha} \log x + O(1).$$

It follows, in particular, that $|\Delta(\alpha, x)|$ is unbounded. More precisely, there is a C > 0 so that

$$\Delta(\alpha, x) < -C\log x$$

for arbitrarily large x. See e.g. [23, p. 22].

The proof of Theorem 2 requires us to modify Hecke's method in some rather non-obvious ways. In place of $\psi(s, \alpha)$ we study

(1.20)
$$\varphi(s,\alpha) = \sum_{n} \left(d(\alpha,n) - \log \alpha \right) n^{-s}.$$

In addition to its meromorphic continuation and the explicit determination of its infinitely many poles and their residues, we need to obtain growth estimates for $\varphi(s, \alpha)$ in certain vertical strips. This was done by Hecke for $\psi(s, \alpha)$, but in our case the problem is more difficult for two reasons: 1) our Dirichlet series is of degree two, and 2) we must estimate a certain hypergeometric function uniformly when the parameters contain two independent variables that can get large, which is well-known to be problematic.

Remarks. i) A modified version of Theorem 2 holds for a = 3 so that (1.19) is still valid. It is also possible to treat the restricted sum of divisors function

$$\sigma_w(n,\alpha) = \sum_{\{d \mid n; \alpha^{-1}n \leq d^2 \leq \alpha n\}} d^w,$$

in a similar manner to that done here when w = 0.

ii) Other kinds of restricted divisor functions have been introduced and studied. For some examples see [39], [9] (also [36, p.207]), [35], [16], [17], [3], listed chronologically. Various generalizations and applications of Hecke's original method are given in the papers [4], [5], [11], [2].

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2. Counting lattice points

In this section I will prove Theorem 1. Theorem 2 follows from Propositions 1 and 2, which are stated and proven in §3 and §4.

The proof of Theorem 1 uses the lattice point interpretation of $d(\alpha, n)$. We employ a classical method Vinogradov used to count lattice points under a nice curve and, in particular, to give the Voronoi exponent. Variations on the proof I give are clearly possible, and I make no effort to improve the $\frac{1}{3}$ or keep track of dependence of the estimates on α .



FIGURE 2. Region R_x

Proof of Theorem 1. For x > 0 let $L(\alpha, x)$ denote the number of lattice points $(r, s) \in (\mathbb{Z}^+)^2$ in the region R_x determined by

$$r \le \frac{x}{s}$$
 and $s \le \sqrt{\frac{x}{\alpha}}$.

These are the lattice points in the union of the two shaded regions of Figure 2. Then

$$L(\alpha, x) = \sum_{1 \le n \le \sqrt{\frac{x}{\alpha}}} \lfloor \frac{x}{n} \rfloor.$$

The following standard asymptotic formula is from e.g. [37, p.22].

Lemma 1.

$$\sum_{1 \le n \le \sqrt{\frac{x}{\alpha}}} \frac{x}{n} = x \log\left(\frac{x}{\alpha}\right)^{\frac{1}{2}} + \left(\frac{1}{2} - \left\{\left(\frac{x}{\alpha}\right)^{\frac{1}{2}}\right\}\right) \sqrt{\alpha x} + x\gamma + O(1).$$

Thus we have

(2.1)
$$L(\alpha, x) = -L_1(\alpha, x) + x \log\left(\frac{x}{\alpha}\right)^{\frac{1}{2}} + \left(\frac{1}{2} - \left\{\left(\frac{x}{\alpha}\right)^{\frac{1}{2}}\right\}\right) \sqrt{\alpha x} + x\gamma + O(1),$$

where we set

$$L_1(\alpha, x) = \sum_{1 \le n \le \sqrt{\frac{x}{\alpha}}} \{ \frac{x}{n} \}.$$

Choose integral t_0 such that $q 2^{-t_0} \ge 2x^{\frac{1}{3}} \ge q 2^{-t_0-1}$, where $q = \lfloor (\frac{x}{\alpha})^{\frac{1}{2}} \rfloor$ was defined in (1.9). Therefore

(2.2)
$$L_1(\alpha, x) = \sum_{t=0}^{t_0} \sum_{2^{-t-1}q \le n \le 2^{-t}q} \{ \frac{x}{n} \} + O(x^{\frac{1}{3}}).$$

The next lemma is due to Vinogradov, given in the form presented in e.g. [28, Thm 11.3].

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Lemma 2. For $k \ge 1$ and $f \in C^2[M, M + M']$ with $\frac{1}{C} \le |f''(y)| \le \frac{k}{C}$

we have

$$\sum_{n=M}^{M+M'-1} \{f(n)\} = \frac{M'}{2} + O(k^2 M' \log C + kC) C^{-\frac{1}{3}}).$$

In Lemma 2 choose

$$f(y) = \frac{x}{y}, \quad M = M' = q2^{-t-1}, \quad C = q^3 x^{-1} 2^{-(3t+1)}$$

to conclude from (2.2) that

$$L_1(\alpha, x) = \frac{1}{2}q + O(x^{\frac{1}{3}}\log^2 x)$$

Thus from (2.1) we get



FIGURE 3

(2.3)
$$L(\alpha, x) = \frac{1}{2}x\log x + \frac{x}{2}\log\alpha + (\frac{1}{2} - \{(\frac{x}{\alpha})^{\frac{1}{2}}\})(\alpha x)^{\frac{1}{2}} + x\gamma - \frac{q}{2} + O(x^{\frac{1}{3}}\log^2 x)$$

Let $M(\alpha, x)$ denote the number of lattice points (r, s) in the region R_x that also satisfy $s < \frac{r}{\alpha}$, i.e the lattice points in the union of the two shaded regions of Figure 3 not on the line $s = \frac{r}{\alpha}$, the hypotenuse depicted there. Then, for $S(\alpha, q)$ from (1.8),

$$\begin{split} L(\alpha, x) - M(\alpha, x) &= \sum_{1 \le n \le q} \lfloor n\alpha \rfloor \\ &= \alpha \sum_{1 \le n \le q} n - \frac{q}{2} - S(\alpha, q) \\ &= \frac{\alpha}{2}q(q+1) - \frac{q}{2} - S(\alpha, q) \\ &= \frac{\alpha}{2}\big(\big(\frac{x}{\alpha}\big)^{\frac{1}{2}} - \big\{\big(\frac{x}{\alpha}\big)^{\frac{1}{2}}\big\}\big)^2 + \frac{\alpha q}{2} - \frac{q}{2} - S(\alpha, q) \\ &= \frac{x}{2} - \alpha\big(\frac{x}{\alpha}\big)^{\frac{1}{2}}\big\{\big(\frac{x}{\alpha}\big)^{\frac{1}{2}}\big\} + \frac{\alpha q}{2} - \frac{q}{2} - S(\alpha, q) + O(1) \\ &= \frac{x}{2} - \alpha\big(\frac{x}{\alpha}\big)^{\frac{1}{2}}\big\{\big(\frac{x}{\alpha}\big)^{\frac{1}{2}}\big\} + \frac{\alpha}{2}\big(\frac{x}{\alpha}\big)^{\frac{1}{2}} - S(\alpha, q) + O(1). \end{split}$$

Therefore by (2.3),

$$M(\alpha, x) = \frac{1}{2}x\log x + \frac{x}{2}\log\alpha + x(\gamma - \frac{1}{2}) + S(\alpha, q) + O(x^{\frac{1}{3}}\log^2 x).$$

Using the bound (1.3) for $\Delta(x)$ from (1.1), we get by subtraction that

$$\Delta(\alpha, x) = -2S(\alpha, q) + O(x^{\frac{1}{3}+\epsilon}).$$

Thus Theorem 1 follows.

3. Associated Dirichlet series

Next we turn to the proof of Theorem 2. First recall some basic properties of the Hecke zeta functions from (1.18). Let

(3.1)
$$A = \frac{1}{\pi}\sqrt{D}$$
 and $\kappa = \frac{\pi}{\log \alpha}$.

Lemma 3. For $j \in \{0, 1\}$ and $n \in \mathbb{Z}$, the completed zeta function given by

$$\xi(s,\nu_j\lambda^n) = A^s \Gamma(\frac{s+j}{2} + \frac{in\kappa}{2}) \Gamma(\frac{s+j}{2} - \frac{in\kappa}{2}) \zeta(s,\nu_j\lambda^n)$$

is entire and satisfies the functional equation

$$\xi(1-s,\nu_j\lambda^n) = \xi(s,\nu_j\lambda^{-n}),$$

except that, when k = j = 0, it has simple poles at s = 1, 0 with residues $2 \log \alpha$, $-2 \log \alpha$, respectively. Away from these poles, the function $\xi(s, \nu_j \lambda^n)$ is bounded in vertical strips. We have the evaluations

(3.2)
$$\operatorname{res}_{s=1}\zeta(s,\nu_0) = \frac{2\log\alpha}{\sqrt{D}}, \quad \zeta(0,\nu_0) = 0 \quad \text{and} \quad \zeta(0,\nu_1) = \frac{a-3}{6}.$$

Proof. The first two statements follow from [24] (see also [26, \S 3]). The last is due to Hecke as well [25].

In order to use the Hecke zeta functions to prove Theorem 2, we will express the Dirichlet series

(3.3)
$$\varphi^*(s,\alpha) = \sum_n d(\alpha,n)n^{-s},$$

with $d(\alpha, n)$ defined in (1.4), in terms of them. We use the following lemma, which is easily shown by direct calculation.

Lemma 4. Let $\mathcal{O} = \mathbb{Z} + \alpha \mathbb{Z}$ where $\alpha = \frac{1}{2}(a + \sqrt{a^2 - 4})$, with $D = a^2 - 4$ the discriminant of a real quadratic field. The map

$$\beta \mapsto (d_1, d_2) = \left(\left| \frac{\beta}{\sqrt{D}} \right| + \left| \frac{\beta'}{\sqrt{D}} \right|, \left| \frac{\alpha'\beta}{\sqrt{D}} \right| + \left| \frac{\alpha\beta'}{\sqrt{D}} \right| \right)$$

gives a bijection from

$$\{\beta \in \mathcal{O}; \ \beta > 0 \text{ and } \beta' < 0\}$$
 to $\{(d_1, d_2) \in (\mathbb{Z}^+)^2; \ \alpha^{-1}d_1 < d_2 < \alpha d_1\}.$

This lemma can be adapted to apply to more general units α , but we must restrict to (d_1, d_2) that satisfy a certain congruence (see [10, Lemma 2]).

For $\operatorname{Re} s > 1$ let

$$\Phi_j(s) = \Phi_j(s,\alpha) = \sum_{\beta \in \mathcal{O}}' \nu_j(\beta) (|\beta| + |\beta'|)^{-s} (|\alpha'\beta| + |\alpha\beta'|)^{-s},$$

where as usual the prime in the sum means to leave out $\beta = 0$. Convergence follows easily since the sum is over a two dimensional lattice. The next identity follows straight from Lemma 4.

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Lemma 5. For $\varphi^*(s, \alpha)$ defined in (3.3) we have the identity

$$\varphi^*(s,\alpha) = \frac{D^s}{4} \big(\Phi_0(s) - \Phi_1(s) \big),$$

when $\operatorname{Re} s > 1$.

Thus to study $\varphi^*(s, \alpha)$, hence $\varphi(s, \alpha)$ from (1.20), we are reduced to considering the Dirichlet series $\Phi_j(s)$ for j = 1, 2.

Proposition 1. Fix $j \in \{0,1\}$ and a as above. Suppose that $s = \sigma + it$ is in a compact subset of \mathbb{C} that does not contain any of the points $-k \pm \frac{\pi in}{\log \alpha}$, for $n, k \in \{0, 1, 2, ...\}$. Then we have the uniformly convergent expansion

(3.4)

$$\Phi_{j}(s) = B \sum_{n \in \mathbb{Z}} (-1)^{n} \frac{\Gamma(\frac{s+1-j+in\kappa}{2})\Gamma(\frac{s+1-j-in\kappa}{2})}{(\log \alpha)\Gamma(2s)} {}_{2}F_{1}(s+in\kappa,s-in\kappa;s+\frac{1}{2};\frac{1}{4}-\frac{a}{2})\xi(s,\nu_{j}\lambda^{n}),$$

where $B = B(s) = (2\sqrt{\pi})^{2s-2}D^{-\frac{s}{2}}$.

We will first restrict s so that $\operatorname{Re} s > 1$ and prove the following variant identity.

Lemma 6. For $j \in \{0, 1\}$ and Re s > 1

(3.5)
$$\Phi_j(s) = \sum_{n \in \mathbb{Z}} (-1)^n \frac{\Gamma(s+in\kappa)\Gamma(s-in\kappa)}{(\log \alpha)\Gamma(2s)} (\frac{a}{2})^{-s+in\kappa} {}_2F_1(\frac{s+1}{2} - \frac{in\kappa}{2}, \frac{s}{2} - \frac{in\kappa}{2}; s + \frac{1}{2}; \frac{D}{a^2}) \zeta(s, \nu_j \lambda^n).$$

Proof. Define for $x \in \mathbb{R}$ and $\operatorname{Re} s > 1$ by

$$\Phi_j(s,x) = \sum_{\beta \in \mathcal{O}}' \nu_j(\beta) (|\beta|e^x + |\beta'|e^{-x})^{-s} (|\alpha'\beta|e^x + |\alpha\beta'|e^{-x})^{-s}$$

To prove (3.5) we apply the basic principle of Fourier analysis, which was used in brilliant and unexpected ways by Hecke to study algebraic numbers [25, p.338]:

Wenn eine Funktion bei einer Substitution (von unendlich hoher Ordnung) invariant bleibt, so entwickle man die Funktion in eine Fouriersche Reihe nach einer geeignet gewählten Variablen, welche diese Invarianz in Evidenz setzt.¹

Clearly $\Phi_i(s, x)$ is a "nice" function for fixed s with $\operatorname{Re} s > 1$ and

$$\Phi_j(s, x + \log \alpha) = \Phi_j(s, x).$$

We have the convergent Fourier expansion

$$\Phi_j(s,x) = \sum_{n \in \mathbb{Z}} A_j(n,s) e(\frac{nx}{\log \alpha}) \qquad e(z) = e^{2\pi i z}.$$

We apply Hecke's well-known unfolding trick to compute $A_i(n,s)$:

$$A_j(n,s) = \frac{1}{\log \alpha} \int_0^{\log \alpha} e(-\frac{nx}{\log \alpha}) \sum_{\beta \in \mathcal{O}}' \nu_j(\beta) (|\beta|e^x + |\beta'|e^{-x})^{-s} (|\alpha'\beta|e^x + |\alpha\beta'|e^{-x})^{-s}$$
$$= \frac{2}{\log \alpha} K_n(s) \zeta(s; \nu_j \lambda^n),$$

 $^{^{1}}$ If a function remains invariant under a substitution (of infinitely high order), then expand the function in a Fourier series with respect to a suitably chosen variable that makes this invariance evident.

where

$$K_n(s) = \int_{-\infty}^{\infty} \left((e^x + e^{-x})(\alpha' e^x + \alpha e^{-x}) \right)^{-s} e(-\frac{nx}{\log \alpha}) dx$$

Finally, we compute this integral which, for $\operatorname{Re} s > 0$ and $n \in \mathbb{Z}$, can be evaluated using [13, (22) p. 121] and [12, (14) p. 111] to be

(3.6)
$$K_n(s) = (-1)^n \frac{\Gamma(s+in\kappa)\Gamma(s-in\kappa)}{2\Gamma(2s)} (\frac{a}{2})^{-s+in\kappa} {}_2F_1(\frac{s+1}{2} - \frac{in\kappa}{2}, \frac{s}{2} - \frac{in\kappa}{2}; s + \frac{1}{2}; \frac{D}{a^2}).$$

Proof of Proposition 1. To derive (3.4) from Lemma 6, first note that by applying a quadratic transformation to the hypergeometric function in (3.6) (see e.g. [12, (16) p.112]) we get:

(3.7)
$$(\frac{a}{2})^{-s+i\kappa n} {}_2F_1(\frac{s+1}{2}-\frac{in\kappa}{2},\frac{s}{2}-\frac{in\kappa}{2};s+\frac{1}{2};\frac{D}{a^2}) = {}_2F_1(s+in\kappa,s-in\kappa;s+\frac{1}{2};\frac{1}{2}-\frac{a}{4}).$$

In order to analytically continue the sum in (3.4) in s to the left, we need to estimate the hypergeometric function uniformly in the large parameter n, which occurs in both the first and second parameter of the hypergeometric function. That this is possible is due to the fundamental paper [38] of Watson (see also [31, p.237]), from which we get

(3.8)
$${}_{2}F_{1}(s+in\kappa,s-in\kappa;s+\frac{1}{2};\frac{1}{2}-\frac{a}{4}) \ll |n|^{-\sigma},$$

where the implied constant depends on a and s. Here we are applying Watson's formulation to the RHS of (3.7).

After using the duplication formula

$$\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2}) = 2^{1-s}\sqrt{\pi}\Gamma(s),$$

Proposition 1 follows by the properties of the completed Hecke zeta functions given in Lemma 3, together with (3.8) and the standard fact that

(3.9)
$$\lim_{|t| \to \infty} \frac{\Gamma(\sigma + it)}{e^{-\frac{\pi}{2}|t|}|t|^{\sigma - \frac{1}{2}}} = \sqrt{2\pi}.$$

Remarks. i) Note that $\zeta(s, \nu_j \lambda^k)$ need not have an Euler product unless \mathbb{F} has (wide) class number one. It is known (see [7], also [6]) that this holds exactly for

$$D = 12, 21, 77, 437,$$

with corresponding values of a given by 4, 5, 9, 21. In these cases we have

$$\zeta(s,\nu_0) = \zeta(s)L(s,\chi_D)$$
 and $\zeta(s,\nu_1) = L(s,\chi_{D_1})L(s,\chi_{D_2}),$

where $L(s, \chi_D)$ is the Dirichlet *L*-functions with Kronecker symbol χ_D and $D = D_1 D_2$ with 12 = (-3)(-4), 21 = (-3)(-7), 77 = (-7)(-11) and 437 = (-19)(-23).

ii) To study $\psi(s, \alpha)$ from (1.14) for certain real quadratic α , Hecke used the simpler "degree one" functions

$$\Psi_j(s) = \sum_{\beta}' \nu_j(\beta) (|\beta| + |\beta'|)^{-s},$$

with the appropriate summation over β . The fact that $\Phi_j(s)$ has degree two accounts for one of the new difficulties in treating the restricted divisor problem using Hecke's method, due to its increased growth in vertical strips. Also, for $\Psi_j(s)$ no hypergeometric function occurs in the corresponding Fourier coefficient.

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4. Growth in vertical strips

In this section we will complete the proof of Theorem 2. In addition to the degree of $\Phi_j(s, \alpha)$, another serious difficulty arises when we want to apply Proposition 1 to obtain asymptotic formulas for sums of the coefficients of $\varphi(s, \alpha)$ by using standard methods, like Perron's formula. Namely, we must now bound the hypergeometric function uniformly in terms of its parameters as they vary with two (or even three) independent variables. Doing this accurately is in general an unsolved problem. However, the next result, in which we restrict Re s, gives enough information to prove Theorem 2. Here we use the hypergeometric function as expressed in the LHS of (3.7).

Lemma 7. For $n \in \mathbb{Z}$, κ from (3.1) and $s = \sigma + it$ with $-1 < \sigma$ we have

$$(1 - \alpha^{-2s}) \frac{\Gamma(\sigma + i(t + n\kappa))\Gamma(\sigma + i(t - n\kappa))}{\Gamma(2\sigma + 2it)} {}_{2}F_{1}(\frac{\sigma + 1}{2} + \frac{i}{2}(t + n\kappa), \frac{\sigma}{2} + \frac{i}{2}(t - n\kappa); \sigma + \frac{1}{2} + it; 1 - \frac{4}{a^{2}})$$

$$\ll |t|^{-\sigma + \frac{1}{2}} |t + n\kappa|^{\frac{\sigma}{2} + \frac{1}{2}} |t - n\kappa|^{\frac{\sigma}{2}} e^{-\frac{\pi}{4}(|t + n\kappa| + |t - n\kappa|)} e^{\frac{\pi}{2}|t|},$$

where the implied constant depends on a and σ .

Proof. The $(1 - \alpha^{-2s})$ is there to kill the poles of the gamma quotient. The bound follows from the integral representation (see e.g. [31, pp. 58, 235])

$${}_{2}F_{1}(a,b;c,z) = 1 + \frac{a\Gamma(c)z}{\Gamma(b)\Gamma(c-b)} \int_{0}^{1} \int_{0}^{1} v^{b}(1-v)^{c-b-1}(1-zuv)^{-a-1}dv \, du,$$

which is valid for $\operatorname{Re}(c-b) > 0$, $\operatorname{Re}(b) > -1$ and $|\arg(1-z)| < \pi$, together with (3.9).

We also need the following application of the Phragmén-Lindelöf theorem, which is proven in [34].

Lemma 8. For $-\delta \leq \sigma \leq 1 + \delta$ and |t| > 1, where $0 < \delta \leq \frac{1}{2}$,

$$\zeta(\sigma + it, \nu_j \lambda^n) \ll \left((1 + |t - n\kappa|)(1 + |t + n\kappa|) \right)^{-\frac{\nu}{2} + \frac{1}{2} + \frac{\nu}{2}}$$

with the implied constant depending only on \mathbb{F} and ϵ .

Proposition 2. The function
$$(1 - \alpha^{-s})\varphi(s, \alpha)$$
 is holomorphic for $\sigma > -1$ and satisfies
 $(1 - \alpha^{-2s})\varphi(s, \alpha) \ll |t|^{3-\sigma+\delta},$

for $-\delta \leq \sigma \leq 1 + \delta$ where $0 < \delta \leq \frac{1}{2}$.

Proof. To prove this we will use the identity (3.5) of Lemma 6 for $\Phi_j(s)$, which after Proposition 1 remains valid for all s away from poles. Apply absolute values inside the sum in (3.5) and then split the sum into those n with $0 \leq |n| \leq \frac{|t|}{\kappa}$ and those with $\frac{|t|}{\kappa} < |n|$. By Lemmas 6, 7 and 8, together with the obvious identity

$$|x - y| + |x + y| = 2\max(|x|, |y|)$$
 for $x, y \in \mathbb{R}$,

we deduce that

$$(1 - \alpha^{-2s})\Phi_j(\sigma + it) \ll |t|^{3 - \sigma + \epsilon} + e^{\frac{\pi}{2}|t|} \sum_{\frac{|t|}{\kappa} < |n|} |n|^{\frac{1}{2}} e^{-\frac{\pi}{2}n\kappa} \ll |t|^{3 - \sigma + \epsilon} + |t|^{\frac{1}{2}}.$$

We must also bound the residues at the poles of $\Phi_j(s)$. Using Lemma 6, we have for any $n \neq 0$ and any $\epsilon > 0$ the bound

(4.1)
$$\operatorname{res}_{s=in\kappa} \Phi_j(s,\alpha) = (-1)^n \frac{1}{\log \alpha} \zeta(in\kappa,\nu_j\lambda^n) \ll |t|^{\frac{1}{2}+\epsilon},$$

where for the last inequality we have applied Lemma 8. Now Proposition 2 follows from Lemma 5. $\hfill \Box$

Given Propositions 1 and 2, the proof of Theorem 2 can be completed using the standard contour shifting method. The following lemma is obtained by an easy modification of the lemma on p. 105. of [8], for example.

Lemma 9. For r, x, c, T > 0

(4.2)
$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\Gamma(r+1)\Gamma(s)}{\Gamma(r+1+s)} x^s \, ds = \chi_{(1,\infty)}(x) \left(1 - \frac{1}{x}\right)^r + O\left(\frac{x^c}{T^r} \min\left(1, \frac{1}{T|\log x|}\right)\right)$$

where $\chi_{(1,\infty)}(x)$ is the usual characteristic function.

Now apply Lemma 9 with the Dirichlet series $\varphi(s, \alpha)$ from (1.20) to give the Cesaro sums of its coefficients. This yield the following particular version of Perron's formula.

Lemma 10. For r, T > 0 and c > 1

(4.3)
$$\sum_{n < x} (1 - \frac{n}{x})^r \left(d(n, \alpha) - \log \alpha \right) = \frac{1}{2\pi i} \int_{c - iT}^{c + iT} \frac{\Gamma(r+1)\Gamma(s)}{\Gamma(r+1+s)} \varphi(s, \alpha) x^s \, ds + O\left(\frac{x^c}{T^{r+1}}\right) + O\left(\frac{x^{c-1}}{T^r}\right).$$

Proof. After summing the formula (4.2) against $d(n, \alpha) - \log \alpha$, use that for any $0 < \epsilon \leq 1$ we have

$$d(n,\alpha) - \log \alpha \ll n^{\epsilon}$$

and for x > 1 the estimate

$$\sum_{\substack{|n-x| \ge 1\\n \ge 1}} \frac{1}{n^{1+\epsilon} |\log \frac{x}{n}|} = O_{\epsilon}(\epsilon^{-1}).$$

Proof of Theorem 2. To prove Theorem 2, we choose T = x and move the contour of the integral in Lemma 10 to $\text{Re } s = -\delta$ for $\delta > 0$ sufficiently small. For r > 3 the estimate of Proposition 2 together with (4.1) and (3.2) allows us to conclude the result from Lemma 10. Here the integrals over the horizontal segments are

$$\ll T^{-r-1} \int_{-\delta}^{c} T^{3+\delta-\sigma} x^{\sigma} d\sigma = o(1).$$

Also, the new integral over the vertical segment is

$$\ll T^{3-r+\delta} = o(1).$$

By Proposition 1, only the residues from poles of $\Phi_1(s, \alpha)$ contribute. Finally, we extend the summation over residues to the infinite series of the formula of Theorem 2, using (4.1).

Remarks. i) A different proof giving the meromorphic continuation of the Dirichlet series $\psi(s, \alpha)$ from (1.14) for certain real quadratic α can be found in [21]. Their proof is based on properties of the double zeta function of Barnes. This Hardy-Littlewood method was developed further in [32] and then in [14] (see also [15]), to cover much more general Dirichlet series. In fact, results of [14] yield another proof that our $\varphi(s, \alpha)$ has a meromorphic continuation to a function of finite order and even can be applied to determine the location of the poles, but without the explicit determination of their residues or the growth estimates needed in our proof of Theorem 2.

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ii) If one could prove sufficiently strong uniform estimates for the hypergeometric function, it should be possible to improve Theorem 2 by allowing smaller values of r. For this it would be natural to apply (a slight extension of) the summability version of the Schnee-Landau theorem given in [1], together with an argument along the lines Hecke used for $\psi(s, \alpha)$. This argument needs good growth estimates on vertical lines (away from poles) that are uniform with respect to the horizontal position of the line. However, the growth of $\varphi(s, \alpha)$ itself makes it unlikely that this analytic approach could yield the first statement of (1.13) for our real quadratic α , which would correspond to Hecke's result (1.15) for $S(\alpha, x)$.

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