INFINITE PRODUCTS OF CYCLOTOMIC POLYNOMIALS

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ABSTRACT. We study analytic properties of certain infinite products of cyclotomic polynomials that generalize some introduced by Mahler. We characterize those that have the unit circle as a natural boundary and use associated Dirichlet series to obtain their asymptotic behavior near roots of unity.

1. INTRODUCTION

In this paper we will investigate analytic properties of certain infinite products of cyclotomic polynomials. The power series expansions of these products have interesting integer coefficients. We will determine those that have the unit circle as a natural boundary and then, at least in some cases, find their asymptotic behavior near roots of unity. This behavior is subtle and is controlled by the size of certain cyclotomic integers and by the residues and special values of associated Dirichlet series.

First we will recall the definition of the cyclotomic polynomials. For $\ell \in \mathbb{Z}^+$ let $\varphi(\ell)$ be Euler's function giving the number of positive integers $\leq \ell$ that are relatively prime to ℓ. Let $\Phi_\ell(x) \in \mathbb{Z}[x]$ be the integral polynomial of degree $\varphi(\ell)$ with $\Phi_\ell(0) = 1$ whose zeros are the primitive ℓ^{th} roots of unity. The first few are

$$\Phi_1(x) = 1 - x, \quad \Phi_2(x) = 1 + x, \quad \Phi_3(x) = 1 + x + x^2, \quad \Phi_4(x) = 1 + x^2, \ldots.$$

Generally, for $\ell \geq 2$ we have

$$\Phi_\ell(x) = \prod_{\substack{a (\mathrm{mod}\ \ell) \\ \gcd(a,\ell)=1}} (x - e(a/\ell))$$

where we set $e(z) = e^{2\pi i z}$. Thus $\Phi_\ell(x)$ is the ℓ^{th} cyclotomic polynomial if $\ell \geq 2$, and is minus the usual cyclotomic polynomial if $\ell = 1$.[1]

For a fixed prime p consider the infinite product

$$F(z) = F_{p,\ell}(z) = \prod_{k\geq 0} \Phi_\ell(z^{p^k}). \tag{1}$$

This product defines an analytic function in the unit disc $\mathbb{D} = \{z; |z| < 1\}$. It is given by a power series with integer coefficients $a(n) \in \mathbb{Z}$:

$$F(z) = \sum_{n\geq 0} a(n) z^n.$$

To see that these coefficients can be interesting let us consider some examples. When $p = 2$ and $\ell = 1$ we have

$$F(z) = \prod_{m\geq 0} (1 - z^{2^m}) = \sum_{n\geq 0} (-1)^{t_n} z^n = 1 - z - z^2 + z^3 - z^4 + z^5 + z^6 - z^7 - z^8 + \cdots. \tag{2}$$

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[1]This non-standard convention serves to make some identities more uniform.

Here t_n is the Thue-Morse sequence defined by $t_n = 0$ or $t_n = 1$ according to whether the sum of the binary digits of n is even or odd. See [1] for a survey about this important sequence. Another interesting example occurs when $p = 2$ and $\ell = 3$:

$$F(z) = \prod_{k\geq 0}(1 + z^{2^k} + z^{2^{k+1}}) = \sum_{n\geq 0} b(n)z^n = 1 + z + 2z^2 + z^3 + 3z^4 + 2z^5 + \cdots . \tag{3}$$

The coefficients $b(n)$ define the Stern diatomic sequence [21]:

$$1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, 8, \ldots .$$

Here $b(n)$ is the number of partitions of the integer n into powers of 2, in which no power of 2 is used more than twice. See [22] for an account of some of the amazing properties of the sequence $\{b(n)\}$. Among them are that $b(n + 1)$ and $b(n)$ are relatively prime and that the sequence of quotients defined by $r(n) = \frac{b(n+1)}{b(n)}$ enumerates all the positive rational numbers:

$$1, 2, \tfrac{1}{2}, 3, \tfrac{2}{3}, \tfrac{3}{2}, \tfrac{1}{3}, 4, \tfrac{3}{4}, \tfrac{5}{3}, \tfrac{2}{5}, \tfrac{5}{2}, \ldots .$$

Another well-known class of examples comes when ℓ is prime and $p > \ell$:

$$F(z) = \prod_{k\geq 0}(1 + z^{p^k} + z^{2p^k} + \cdots + z^{(\ell-1)p^k}) = \sum_{n\geq 0} a(n)z^n. \tag{4}$$

Now $a(n) = 1$ if the base p expansion of n has only digits less than ℓ and $a(n) = 0$ otherwise. When $\ell = 2$ the sequence $a(n)$ was studied by Lehmer, Mahler and van der Poorten [10]. When $p = \ell$ we have the trivial example

$$F(z) = \sum_{n\geq 0} z^n = \frac{1}{1-z}, \tag{5}$$

as follows by the uniqueness of the base p expansion of an integer.

A famous result of Carlson and Pólya (see [20]) says that a power series with integral coefficients that converges in $\mathbb{D}$ is either a rational function or has the unit circle as a natural boundary. As (5) illustrates, for our $F(z)$ this dichotomy is settled by whether or not $p|\ell$.

Theorem 1. *The function $F(z)$ is rational if and only if $p|\ell$, in which case*

$$F(z) = \frac{1}{\Phi_m\left(z^{p^{r-1}}\right)}, \tag{6}$$

where $\ell = p^r m$ with $r \geq 1$ and $p \nmid m$. Otherwise $F(z)$ has the unit circle as a natural boundary.

We will prove this result, parts of which are already known, in the next section. The main originator of this line of research was Mahler. The fact that (2) and (4) have the unit circle as a natural boundary follows from Mahler's early papers [11] and [12], respectively. For (4) one can apply the well-known result that a power series with coefficients from a finite set is rational if and only if the coefficients are eventually periodic (see [19, p.138, #158]). That the generating function of the Stern sequence (3) has the unit circle as a natural boundary is also known (see [5]). More generally, it follows from [8] that $F(z)$ has the unit circle as a natural boundary when $p \nmid \ell$ and ℓ is square-free. The reader may consult [4] and [18] and their references for background on the relation of such results to transcendence theory.

Our proof that $F(z)$ with $p \nmid \ell$ has the unit circle as a natural boundary is short and uses an approach given by Mahler in one of his last papers [14]. Actually, Mahler's result includes the case $\ell = 1$ but not $\ell > 1$. By suitably modifying his method, we will see that $F(z) \to 0$ as z approaches a primitive $(p^n\ell)^{th}$ root of unity along a radius of the unit circle, for any nonnegative integer n. This implies the result since the set of such roots of unity is dense in the unit circle. In fact, this approach can be developed much further with interesting applications. In the paper [8], Dumas and Flajolet give a very precise asymptotic formula for the m^{th} *coefficient* of the *reciprocal* function

$$F(z)^{-1} = \prod_{k\geq 0} \Phi_\ell(z^{p^k})^{-1},$$

when $p \nmid \ell$ and ℓ is square-free.[2] They apply Cauchy's formula in the manner of the circle method and utilize asymptotic expansions of $\log F(z)$ near $(p^n\ell)^{th}$ roots of unity. Their method is a refinement of that of de Bruijn [3], who considered the case when $\ell = 1$ and $p = 2$, which had been studied earlier by Mahler [13].

Our main object is to determine the asymptotic behavior of $F(z)$ as z approaches a q^{th} root of unity $e(a/q)$ along a radius of the unit circle when $p \nmid q$ and $q \nmid \ell$. We will see that this behavior depends on the value of the cyclotomic integer

$$S = \prod_{k=1}^{\mathrm{ord}_q(p)} \Phi_\ell(e(\tfrac{p^k a}{q})). \tag{7}$$

Here $\mathrm{ord}_q(b)$ for any integer b with $\gcd(b,q) = 1$ is the usual multiplicative order of b modulo q. Clearly S is a non-zero real number. Set

$$\alpha_q = (\mathrm{ord}_q(p) \log p)^{-1}.$$

Theorem 2. *Suppose that $p \nmid q$ and $q \nmid \ell$. If $|S| < 1$ then for some constant c depending on a/q we have that*

$$F(e(\tfrac{a}{q} + iy)) = c + o(1)$$

as $y \to 0^+$. If $|S| \geq 1$ there exists a continuous 1-periodic function $g(x)$ depending on a/q so that as $y \to 0^+$

$$F(e(\tfrac{a}{q} + iy)) \sim g(\alpha_q \log y) y^{-\alpha_q \log |S|}.$$

Now S is an element of $\mathbb{Z}[e(1/r)]$ where $r = \varphi(\ell)/\mathrm{ord}_q(p)$. In fact, it follows from a well-know result about resultants of cyclotomic polynomials (see [2]) that

$$\prod_{\substack{a (\mathrm{mod\ q}) \\ \gcd(a,q)=1}} \Phi_\ell(e(\tfrac{a}{q})) = \begin{cases} p_1^{\varphi(\ell)} & \text{if } q/\ell = p_1^m \text{ for some prime } p_1, \\ 1 & \text{otherwise.} \end{cases}$$

In particular, if q/ℓ is not a power of a prime then S is a cyclotomic unit. In the special case when p is also a primitive root modulo q we see that $S = 1$. Some other cases where S can be evaluated follow from the results of [9].

For example, when $p = 2$ is a primitive root modulo q with q an odd prime (e.g. $q = 3, 5, 11, 13, \ldots$) we have for the Thue-Morse function (2)

$$F(e(\tfrac{1}{q} + iy)) \sim g(\tfrac{\log y}{(q-1)\log 2}) y^{-\frac{\log q}{(q-1)\log 2}} \tag{8}$$

while if $q > 3$ we have for the Stern function (3)

$$F(e(\tfrac{1}{q} + iy)) \sim g(\tfrac{\log y}{(q-1)\log 2}). \tag{9}$$

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[2]Actually they allow p to be any integer ≥ 2 that is prime to ℓ.

Example (8) should be compared with results about the behavior of partial sums of the Thue-Morse sequence in progressions first obtained by Newman [16] (see also [6] and [7] and the references cited there.)

We remark that Theorem 2 holds as well when $p \mid \ell$. Of course, after Theorem 1 it is not very interesting in this case.

We will see that the value of c and the Fourier expansion of g are determined by special values and residues of Dirichlet series formed from the coefficients of $F(z)$ twisted by exponentials. Dirichlet series associated to more general infinite products are studied in Section 3. The main result of that section, Theorem 3, is then applied to prove Theorem 2 in Section 4. We end the paper in Section 5 with some concluding remarks.

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2. Proof of Theorem 1.

To prove that (6) holds when $p|\ell$ we need some basic results about cyclotomic polynomials.

Lemma 1. *If m is odd and $r \geq 1$ then*

$$\Phi_{p^r m}(z) = \frac{\Phi_m(z^{p^r})}{\Phi_m(z^{p^{r-1}})}.$$

Proof. We have the well-known identities (see [15, p.160]) for $n \in \mathbb{Z}^+$ and p a prime:

$$\Phi_{pn}(z) = \Phi_n(z^p) \tag{10}$$

when $p \mid n$ and

$$\Phi_{pn}(z) = \frac{\Phi_n(z^p)}{\Phi_n(z)} \tag{11}$$

when $p \nmid n$. Taking $n = p^{r-1}m$ and applying equation (10) $r-1$ times and then applying (11) once we get the result. □

If $\ell = p^r m$ with $r \geq 1$ and $p \nmid m$ then by Lemma 1 we have the telescoping product

$$\prod_{0 \leq k \leq n} \Phi_{p^r m}(z^{p^k}) = \frac{\Phi_m(z^{p^r})}{\Phi_m(z^{p^{r-1}})} \cdots \frac{\Phi_m(z^{p^{r+n}})}{\Phi_m(z^{p^{r+n-1}})} = \frac{\Phi_m(z^{p^{r+n}})}{\Phi_m(z^{p^{r-1}})}.$$

Thus by (1) we get (6) since

$$F_{p,p^r m}(z) = \lim_{n \to \infty} \prod_{0 \leq k \leq n} \Phi_{p^r m}(z^{p^k}) = \frac{1}{\Phi_m(z^{p^{r-1}})}.$$

Turning to the case when $p \nmid \ell$, we have the following estimate.

Lemma 2. *Suppose that $p \nmid \ell$. There is an absolute constant $C > 0$ so that for $\gcd(a, \ell) = 1$ and $0 < y < 1$*

$$|F\big(e(\tfrac{a}{\ell} + iy)\big)| \ll_{p,\ell} e^{-C(\log y)^2}.$$

Proof. Recall that

$$|\Phi_\ell(z)| = \prod_\epsilon |z-\epsilon|,$$

where the product runs over all primitive ℓ^{th} roots of unity ϵ. Thus there is a constant A depending only on ℓ so that for any such ϵ we have the estimate

$$|\Phi_\ell(\epsilon e^{-2\pi y})| \le A(1-e^{-2\pi y}).$$

Since $p \nmid \ell$ is odd we know that $e(\frac{p^k a}{\ell})$ is a primitive ℓ^{th} root of unity for all k so we have

$$\begin{aligned}|F(e(\tfrac{a}{\ell}+iy))| &\le \prod_{0\le k\le \log(y^{-1})} A\big(1-e^{-2\pi(p^k y)}\big) \prod_{k>\log(y^{-1})} |\Phi_\ell\big(e(p^k(\tfrac{a}{\ell}+iy))\big)| \\ &\ll_\ell \prod_{0\le k\le \log(y^{-1})} (2\pi A)p^k y \ll e^{-C(\log y)^2}.\end{aligned}$$

□

Now it follows from (1) that for any $n \ge 1$

$$F(z) = \prod_{k=0}^{n-1} \Phi_\ell(z^{p^k})F(z^{p^n})$$

so by Lemma 2 we have that $F(z) \to 0$ as z approaches any primitive $(p^n\ell)^{th}$ root of unity along a radius of the unit circle. Since the set of all these points is dense in the unit circle F must have the unit circle as a natural boundary.

3. Associated Dirichlet series

Our proof of Theorem 2 makes use of analytic properties of a Dirichlet series associated to $F(z)$. In this section we will proceed a bit more generally. Let

$$P(\tau) = \sum_{m=0}^{d} c_m e(m\tau),$$

be a trigonometric polynomial of degree $d \ge 1$ with $c_m \in \mathbb{C}$ and $c_0 = 1$. Let $b \ge 2$ be an integer and consider the Fourier series defined for $\tau \in \mathcal{H}$, the upper half-plane, by

$$f(\tau) = \prod_{k\ge 0} P(b^k\tau). \tag{12}$$

This is easily seen to define an analytic function in $\mathcal{H}$ by comparison with $\sum_{n\ge 0} e(b^n\tau)$. Thus f has a Fourier expansion

$$f(\tau) = 1 + \sum_{n\ge 1} a(n)e(n\tau), \tag{13}$$

which converges uniformly on compact subsets of $\mathcal{H}$. Note that $a(n) = a_\ell(n)$ from the Introduction in case $P(\tau) = \Phi_\ell(e(\tau))$ and $b = 2$.

We are concerned with the associated Dirichlet series

$$\psi(s) = \sum_{n\ge 1} a(n)n^{-s}$$

and, more generally, its twist defined for $a \in \mathbb{Z}$ and $q \in \mathbb{Z}^+$ by

$$\psi(s, a/q) = \sum_{n\ge 1} a(n)e(na/q)n^{-s}.$$

We will show that these series converge absolutely for $\mathrm{Re}(s)$ sufficiently large. Also, $\psi(s,a/q)$ may be meromorphically continued to the entire s-plane, provided $\gcd(q,b)=1$, and its possible poles lie on or to the left of

$$\mathrm{Re}(s)=\frac{\log|S(a/q)|}{\mathrm{ord}_q(b)\log b},$$

where

$$S(a/q)=\prod_{k=1}^{\mathrm{ord}_q(b)}\sum_{m=0}^{d}c_m e(m\tfrac{ab^k}{q}) \tag{14}$$

and $\mathrm{ord}_q(b)$ is as before the order of b modulo q.

In particular, if $S(a/q)=0$ then $\psi(s,a/q)$ is entire. In case $q=1$ this last fact was proven by Mahler [14].

Lemma 3. *Let $M=\max_m|c_m|$ and $A=\frac{\log M(d+1)}{\log b}$. Then*

(a) $a(n)=O(n^A)$ and

(b) $f(\tau)=O((\mathrm{Im}\,\tau)^{-A-1})$ for $\mathrm{Im}(\tau)\le 1$.

Proof. For the first statement, note that the number of factors in (12) that contribute a term greater than 1 to $a(n)e(n\tau)$ in (13) is less than or equal to

$$N=\frac{\log n}{\log b}+1=\frac{\log nb}{\log b}.$$

The number of possible products from these factors is $(d+1)^N$ and the coefficient of each product is bounded in absolute value by M^N so

$$|a(n)|\le((d+1)M)^N=(d+1)Mn^A.$$

For the second statement we use the first to obtain

$$\sum_{n\ge1}|a(n)||e(n\tau)|\le\sum_{n\ge1}n^Ae^{-2\pi ny}=O(y^{-A-1})$$

for $y\le 1$. □

Remark: If $P(0)=0$, Mahler showed in [14] that for all $c>0$ we have $f(\tau)=O((\mathrm{Im}\,\tau)^c)$ from which the fact that $\psi(s)$ is entire follows easily from the Mellin transform.

By Lemma 3(a) we see that the Dirichlet series $\psi(s,a/q)$ converges absolutely for $\mathrm{Re}(s)>A+1$ and that we have the Mellin representation

$$\Psi(s,a/q)\overset{(\mathrm{def})}{=}(2\pi)^{-s}\Gamma(s)\psi(s,a/q)=\int_0^\infty(f(\tfrac{a}{q}+iy)-1)y^s\frac{dy}{y}, \tag{15}$$

also for $\mathrm{Re}(s)>A+1$.

It follows immediately from the definition (12) that $f(\tau)$ satisfies the functional equation

$$f(\tau)=f(b^n\tau)f_n(\tau) \tag{16}$$

for all positive integers n, where

$$f_n(\tau)=\prod_{k=0}^{n-1}P(b^k\tau).$$

Note that $f_1(\tau) = P(\tau)$ and that we may write

$$f_n(\tau) = \sum_{m=0}^{d_n} c_{n,m} e(m\tau), \tag{17}$$

where

$$d_n = d(b^n - 1)(b-1)^{-1}.$$

It is also convenient to define the Dirichlet polynomial

$$\psi_n(s, a/q) = \sum_{m=1}^{d_n} c_{n,m} e(m\tfrac{a}{q}) m^{-s}. \tag{18}$$

Observe that $c_{n,m} = a(m)$ for $n > \frac{\log m}{\log b}$.

Theorem 3. *Suppose that* $\gcd(b, q) = 1$. *If* $S(a/q) = 0$ *then* $\psi(s, a/q)$ *has an analytic continuation to an entire function. Otherwise,* $\psi(s, a/q)$ *has a meromorphic continuation to the entire* s*-plane with at most simple poles at*

$$s_{j,k} = \frac{1}{\operatorname{ord}_q(b) \log b}\big(\log|S(a/q)| + i \arg S(a/q) \pm 2\pi i j - k\big)$$

for $j, k = 0, 1, 2, \ldots$. *Moreover, there is a constant* $C > 0$ *so that for*

$$\operatorname{Re}(s) \geq Re(s_{0,0}) - \tfrac{1}{2}$$

we have

$$\big(b^{s \operatorname{ord}_q(b)} - S(a/q)\big)\psi(s, a/q) = O\big((|\operatorname{Im} s| + 1)^C\big). \tag{19}$$

Proof. We establish a kind of recursion formula for $\psi(s, a/q)$ that allows us to meromorphically continue $\psi(s, a/q)$ in vertical strips of width 1, inductively.

Set $n = \mathrm{ord}_q(b)$, which is defined since $\gcd(b, q) = 1$. By the functional equation (16) and (15) we have for $\mathrm{Re}(s) > A + 1$ that

$$\begin{aligned}
\Psi(s, a/q) = &\int_0^\infty f_n(\tfrac{a}{q} + iy)[f(\tfrac{a}{q} + iyb^n) - 1]y^s \frac{dy}{y} + \int_0^\infty (f_n(\tfrac{a}{q} + iy) - 1)y^s \frac{dy}{y} \\
= &\sum_{m=0}^{d_n} c_{n,m} e(m\tfrac{a}{q}) \int_0^\infty e(imy)[f(\tfrac{a}{q} + iyb^n) - 1]y^s \frac{dy}{y} \\
+ &\sum_{m=1}^{d_n} c_{n,m} e(m\tfrac{a}{q}) \int_0^\infty e(imy) y^s \frac{dy}{y} \\
= &b^{-ns} \sum_{m=0}^{d_n} c_{n,m} e(m\tfrac{a}{q}) \int_0^\infty e(imb^{-n}y)[f(\tfrac{a}{q} + iy) - 1]y^s \frac{dy}{y} \\
+ &(2\pi)^{-s}\Gamma(s)\psi_n(s, a/q) \quad \text{by } (18) \\
= &b^{-ns} \sum_{m=0}^{d_n} c_{n,m} e(m\tfrac{a}{q}) \int_0^\infty [e(imb^{-n}y) - 1][f(\tfrac{a}{q} + iy) - 1]y^s \frac{dy}{y} \\
+ &b^{-ns} \sum_{m=0}^{d_n} c_{n,m} e(m\tfrac{a}{q}) \int_0^\infty [f(\tfrac{a}{q} + iy) - 1]y^s \frac{dy}{y} + (2\pi)^{-s}\Gamma(s)\psi_n(s, a/q) \\
= &b^{-ns} \sum_{m=1}^{d_n} c_{n,m} e(m\tfrac{a}{q}) \int_0^\infty [e(imb^{-n}y) - 1][f(\tfrac{a}{q} + iy) - 1]y^s \frac{dy}{y} \\
+ &b^{-ns} f_n(a/q)\Psi(s, a/q) + (2\pi)^{-s}\Gamma(s)\psi_n(s, a/q) \quad \text{by } (17).
\end{aligned}$$

Thus for $\mathrm{Re}(s) > A + 1$ we have, upon using $n = \mathrm{ord}_q(b)$ and (14), that

$$\begin{aligned}
\Psi(s, a/q)(b^{ns} - S(a/q)) = &b^{ns}(2\pi)^{-s}\Gamma(s)\psi_n(s, a/q) \\
&+ \sum_{m=1}^{d_n} c_{n,m} e(m\tfrac{a}{q}) \int_0^\infty [e(imb^{-n}y) - 1][f(\tfrac{a}{q} + iy) - 1]y^s \frac{dy}{y}.
\end{aligned}$$

Now write for K a non-negative integer

$$e(imb^{-n}y) - 1 = \sum_{k=1}^{K} \frac{(-2\pi mb^{-n}y)^k}{k!} + E_K(my)$$

where $E_K(y) = O(y^K)$ for $y \geq 1$ while for $y \leq 1$

$$E_K(y) = O(y^{K+1}). \tag{20}$$

Thus we have for $\mathrm{Re}(s) > A + 1$

$$\begin{aligned}
\Psi(s, a/q)(b^{ns} - S(a/q)) = &b^{ns}(2\pi)^{-s}\Gamma(s)\psi_n(s, a/q) \\
&+ \sum_{k=1}^{K} \frac{b^{-nk}(-2\pi)^k}{k!} \psi_n(-k, a/q)\Psi(s + k, a/q) + G_K(s),
\end{aligned} \tag{21}$$

where by Lemma 3(b) and (20)

$$G_K(s) = \sum_{m=1}^{d_n} c_{n,m} e(m\tfrac{a}{q}) \int_0^\infty E_K(my)[f(\tfrac{a}{q} + iy) - 1]y^s \frac{dy}{y}$$

is holomorphic for $\mathrm{Re}(s) > A - K$.

Now $b^{ns} - S(a/q) = 0$ if and only if

$$s = \frac{1}{\mathrm{ord}_q(b)\log b}\big(\log|S(a/q)| + i\arg S(a/q) + 2\pi i j\big)$$

for $j \in \mathbb{Z}$.

We use (21) for $K = 0$ to define $\psi(s, a/q)$ for $\mathrm{Re}(s) > A$ with possible simple poles at these points. We then continue this process with $K = 1, 2, \ldots$.

Finally, (19) follows from (21) and the fact that

$$\frac{\Gamma(s+k)}{\Gamma(s)} = O(|s|^k).$$

□

4. Proof of Theorem 2.

In Theorem 3 take $P(\tau) = \Phi_\ell(e(\tau))$ and $b = p$. To prove Theorem 2 we use the inverse Mellin transform from (15) to represent $F(e(a/q + iy))$ in terms of $\Psi(s, a/q)$:

$$\begin{aligned} F(e(a/q+iy)) - 1 =& \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=c} \Psi(s, a/q) y^{-s} ds \\ =& \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=c} (2\pi)^{-s}\Gamma(s)\psi(s, a/q) y^{-s} ds \end{aligned}$$

where c is sufficiently large. Now we push the contour to the line $\mathrm{Re}(s) = \mathrm{Re}(s_{0,0}) - 1/2$ and pick up residues at the (possible) simple poles $s_{j,0}$. Using (19) and the exponential decay of the gamma function on vertical lines we now easily derive the asymptotic formula of Theorem 2. Explicitly, if $|S| < 1$ we get the main contribution from the pole of $\Gamma(s)$ at $s = 0$:

$$F(e(\tfrac{a}{q} + iy)) = c + O(y^{-\alpha_q \log|S|}),$$

where $c = 1 + \psi(0, a/q)$. If $|S| \geq 1$ we get the Fourier expansion

$$g(x) = \sum_{n\in\mathbb{Z}} \rho(n) e(nz)$$

where

$$\rho(n) = \mathrm{res}_{s=s_n}\Psi(s, a/q)$$

and

$$s_n = \alpha_q\big(\log|S| - 2\pi i n\big).$$

The absolute convergence of this Fourier series follows from (19) and the exponential decay of $\Gamma(s)$ on vertical lines.

5. Concluding remarks

As should be clear, many of the results of this paper can be generalized in various ways. In particular, a generalization of Theorem 2 may be given for more general products of the form (12). Our restriction to cyclotomic polynomials and $b = p$ was mainly to give easily stated and perhaps more elegant results that apply to the Thue-Morse and Stern diatomic sequences. Also, there could be some interest in further understanding the nature of the residues and special values of the associated Dirichlet series $\psi(s, a/q)$ in this case. Is it possible to express them in terms of invariants of cyclotomic fields? The Dirichlet series used by de Bruijn, Dumas and Flajolet in [3] and [8] arise from the Mellin transform of $\log F(z)$, and are directly related to classical zeta functions. It might be interesting to try to connect them with $\psi(s, a/q)$.

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