Name: SOLUTIONS

Instructions:

- No calculators or notes are allowed.
- You must show all relevant work in order to receive credit (except on the first problem).
- The actual Midterm will be of similar length and format.
- The problems on this practice exam are weighted toward the earlier material; some later sections you are responsible for are not represented.

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1. (20 points) Circle the correct answer. If more than one answer is circled you will receive no credit.

1. (5) Where did Euclid teach?
   
   (a) Alexandria
   (b) Cyrene
   (c) Athens
   (d) Samos

2. (5) Where was Pythagoras born?
   
   (a) Alexandria
   (b) Athens
   (c) Cyrene
   (d) Samos

3. (5) Which of the following results is not in Euclid’s Elements?
   
   (a) A proof that an integer has a unique factorization into primes.
   (b) A proof that there are infinitely many primes.
   (c) A proof that no odd perfect number exists.
   (d) A formula giving an even perfect number.

4. (5) Which of the following numbers is perfect?
   
   (a) 1
   (b) 8
   (c) 12
   (d) 28
2. (25 points)

(a) (5) Define what it means for an integer \( a \) to divide an integer \( b \), written \( a \mid b \).

\[
a \mid b \text{ means that there exists } c \in \mathbb{Z}
\]

\[
such \text{ that } \quad b = ac.
\]

(b) (20) Suppose that \( p \) is a prime number and that \( a, b \) are integers. Suppose that \( p \mid (ab) \)
and that \( p \) does not divide \( a \). Prove using a consequence of the Euclidean algorithm that
\( p \mid b \).

Assume \( p \mid ab \) and \( p \mid a \). Thus
\[
\gcd(p, a) = 1.
\]

The Euclidean algorithm implies
\[
1 = mp + na \quad \text{for } m, n \in \mathbb{Z}.
\]

Thus
\[
b = mbp + nab.
\]

Now \( p \mid mbp \) and \( p \mid nab \) so
\[
p \mid mbp + nab \text{ which implies that } p \mid b.
\]
3. (30 points)

(a) (15) Compute the radius of the circle that circumscribes a regular pentagon whose sides have length 1. Write your answer in the form

\[ r = \sqrt{\frac{a + b\sqrt{5}}{c}} \]

where \(a, b, c\) are integers. You may use the formula \(\cos \frac{2\pi}{5} = \frac{\sqrt{5} - 1}{4}\).

Let \( r = \text{radius} \)

\[ \theta = \frac{2\pi}{5} \]

Law of Cosines

\[ 1^2 = r^2 + r^2 - 2r^2 \cos \theta \]

\[ = 2r^2(1 - \cos \frac{2\pi}{5}) \]

\[ = 2r^2 \left(1 - \frac{\sqrt{5} - 1}{4}\right) \]

\[ = r^2 \left(\frac{5 - \sqrt{5}}{2}\right) \]

Thus \( r^2 = \frac{2}{5 - \sqrt{5}} = \frac{5 + \sqrt{5}}{10} \)

So \( r = \sqrt{\frac{5 + \sqrt{5}}{10}} \)

\[ r = \sqrt{\frac{5 + \sqrt{5}}{10}} \]

Or \( a = 5, b = 1, c = 10 \)
Problem 3 part 2

(15) Compute the radius of the circle inscribed in a regular pentagon whose sides have length 1. Write your answer in the same form as in Part 1.

![Diagram of a regular pentagon with a circle inscribed, showing the radius r and the side length s.]

*Hint:* Use part 1.

Find \( R \):

From part 1 we know \( r = \sqrt{\frac{5 + \sqrt{5}}{10}} \)

By the Pythagorean Theorem, \( r^2 = R^2 + \frac{1}{4} \)

So \( R^2 = r^2 - \frac{1}{4} = \frac{5 + \sqrt{5}}{10} - \frac{1}{4} = \frac{10 + 2\sqrt{5} - 5}{20} = \frac{5 + 2\sqrt{5}}{20} \).

Thus \( R = \sqrt{\frac{5 + 2\sqrt{5}}{20}} \).
4. (25 points) Find a formula for the sum of the proper divisors of $2^n \times 9$, where $n$ is a positive integer. Write your final answer in the form $a \cdot 2^{n+1} + b \cdot 2^n + c$ by finding the numbers $a, b, c$. You must justify your answer. If you simply experiment and guess you might not receive any credit.

Unique Factorization with primes:

the proper divisors of $2^n \times 9 = 2^n \cdot 3^2$

are:

$1, 2, 2^2, \ldots, 2^n$  
$3, 3 \cdot 2, 3 \cdot 2^2, \ldots, 3 \cdot 2^{n-1}$  
$3^2, 3^2 \cdot 2, \ldots, 3^2 \cdot 2^{n-1}$

Using $1 + r + r^2 + \ldots + r^n = \frac{r^{n+1} - 1}{r - 1}$

we get that the sum of the proper divisors of $2^n \times 3^2$ is

$1 + 2 + \ldots + 2^n$

$+ 3 \left( 1 + 2 + \ldots + 2^n \right)$

$+ 9 \left( 1 + 2 + \ldots + 2^{n-1} \right)$

$= 2^{n+1} - 1 + 3 \left( 2^{n+1} - 1 \right) + 9 \left( 2^n - 1 \right)$

$= \frac{4 \cdot 2^{n+1} + 9 \cdot 2^n - 13}{9}$

or

$a = 4, b = 9, c = -13$