

Types for supercuspidal representations of $\mathrm{GL}(N)$

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Thursday, June 4, 2009

We recall here the basic definitions needed to construct simple types, with no proofs given of the many claims that we make. For a much more detailed account, see [1] and the many other sources cited therein. Note that most of the statements made here could be proven without much difficulty for the reader who has the time and inclination. Any statements requiring a much more elaborate proof are given as propositions.

Let F be a nonarchimedean local field, with ring of integers \mathfrak{o}_F , prime ideal \mathfrak{p}_F (or simply \mathfrak{p} when there is no possibility of confusion), and residue field $\mathbf{k}_F = \mathfrak{o}_F/\mathfrak{p}_F$ of cardinality q .

1 Simple Strata

1.1 Strata

Definition. For a finite-dimensional F -vector space V , a *lattice in V* is a compact open subgroup of V , and an \mathfrak{o}_F -*lattice in V* is a lattice in V that is also an \mathfrak{o}_F -submodule of V . For a finite-dimensional F -algebra A , an \mathfrak{o}_F -*order in A* is an \mathfrak{o}_F -lattice in A that is also a subring of A (with the same 1). Finally, an \mathfrak{o}_F -order \mathfrak{A} in A is called (*left*) *hereditary* if every (left) \mathfrak{A} -lattice is \mathfrak{A} -projective.

We fix once and for all an integer $N > 0$ and an F -vector space V of dimension N , and we let $A = \mathrm{End}_F(V)$ and $G = A^\times = \mathrm{Aut}_F(V) \cong \mathrm{GL}_N(F)$. If \mathfrak{A} is a hereditary \mathfrak{o}_F -order in A and \mathfrak{P} is its Jacobson radical, then it is possible to choose a basis of V and a corresponding partition $N_1 + \cdots + N_e$ of N with respect to which

$$\mathfrak{A} = \begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F & \cdots & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{o}_F & \cdots & \mathfrak{o}_F \\ \vdots & & \ddots & \vdots \\ \mathfrak{p}_F & \mathfrak{p}_F & \cdots & \mathfrak{o}_F \end{pmatrix}$$

and

$$\mathfrak{P} = \begin{pmatrix} \mathfrak{p}_F & \mathfrak{o}_F & \cdots & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \cdots & \mathfrak{o}_F \\ \vdots & & \ddots & \vdots \\ \mathfrak{p}_F & \mathfrak{p}_F & \cdots & \mathfrak{p}_F \end{pmatrix}.$$

(The i, j term in each of these matrices is an $N_i \times N_j$ block. Thus, up to conjugation in G , the choice of such an \mathfrak{A} is equivalent to just choosing an unordered partition of N .) Let $e(\mathfrak{A})$ denote the number of blocks in the block matrix form above (or equivalently the period of the lattice chain associated to \mathfrak{A} , see [1]). We say that \mathfrak{A} is *principal* if $N_i = \frac{N}{e(\mathfrak{A})}$ for all i , *i.e.*, all of the blocks have the same size. (In this case, then, $e(\mathfrak{A})$ divides N , and up to conjugation in G , the choice of a principal hereditary \mathfrak{o}_F -order \mathfrak{A} is equivalent to simply choosing a divisor of N .) Note that \mathfrak{P} is a two-sided ideal of \mathfrak{A} , and is an invertible fractional ideal of \mathfrak{A} in A , with inverse

$$\mathfrak{P}^{-1} = \begin{pmatrix} \mathfrak{o}_F & \mathfrak{p}_F^{-1} & \cdots & \mathfrak{p}_F^{-1} \\ \mathfrak{o}_F & \mathfrak{o}_F & \cdots & \mathfrak{p}_F^{-1} \\ \vdots & & \ddots & \vdots \\ \mathfrak{o}_F & \mathfrak{o}_F & \cdots & \mathfrak{o}_F \end{pmatrix}.$$

Given a hereditary \mathfrak{o}_F -order \mathfrak{A} in A , we can define a discrete valuation $\nu_{\mathfrak{A}}$ on A by

$$\nu_{\mathfrak{A}}(x) = \max\{n \in \mathbb{Z} \mid x \in \mathfrak{P}^n\}.$$

Definition. Let \mathfrak{A} be a hereditary \mathfrak{o}_F -order in A and let $\mathfrak{P} = \text{rad}(\mathfrak{A})$. Define

$$\begin{aligned} U^0(\mathfrak{A}) &= U(\mathfrak{A}) = \mathfrak{A}^\times, \\ U^k(\mathfrak{A}) &= 1 + \mathfrak{P}^k \quad \text{for } k > 0, \end{aligned}$$

and define the normalizer of \mathfrak{A} by

$$\mathfrak{K}(\mathfrak{A}) = \{x \in G \mid x^{-1}\mathfrak{A}x = \mathfrak{A}\}.$$

Then, using the block matrix form above, $U(\mathfrak{A})$ is the standard parahoric subgroup of G (a compact open subgroup) given by the partition $N_1 + \cdots + N_e$ of N associated with \mathfrak{A} . Likewise, the subgroups $U^k(\mathfrak{A})$ for $k > 0$ give the standard filtration of $U(\mathfrak{A})$ by open normal subgroups. Furthermore, $\mathfrak{K}(\mathfrak{A})$ is an open, compact-mod-center subgroup of G , which is the normalizer in G of $U^k(\mathfrak{A})$ for each $k \geq 0$, and within which $U(\mathfrak{A})$ is the unique maximal compact subgroup. Note that

$$U(\mathfrak{A})/U^1(\mathfrak{A}) \cong \prod_{i=1}^e \text{GL}_{N_i}(\mathbf{k}_F).$$

We now fix once and for all an additive character ψ of F of level 1 (*i.e.*, such that ψ is trivial on \mathfrak{p}_F but nontrivial on \mathfrak{o}_F), and we define an additive character ψ_A of A by $\psi_A = \psi \circ \text{Tr}_{A/F}$.

Definition. Let \mathfrak{A} be a hereditary \mathfrak{o}_F -order in A and let $\mathfrak{P} = \text{rad}(\mathfrak{A})$ as usual. For nonnegative integers n and r with $\lfloor \frac{n}{2} \rfloor \leq r < n$ and $\beta \in \mathfrak{P}^{-n}$, define

$$\psi_\beta(x) = \psi_A(\beta(x-1)) \quad \text{for } x \in U^{r+1}(\mathfrak{A}).$$

Then ψ_β is a character of $U^{r+1}(\mathfrak{A})$ that is trivial on $U^{n+1}(\mathfrak{A})$. Furthermore, for $\beta' \in \mathfrak{P}^{-n}$,

$$\psi_\beta = \psi_{\beta'} \quad \text{iff} \quad \beta \equiv \beta' \pmod{\mathfrak{P}^{-r}}.$$

Thus the map $\beta + \mathfrak{P}^{-r} \mapsto \psi_\beta$ is an isomorphism between the (additive) group $\mathfrak{P}^{-n}/\mathfrak{P}^{-r}$ and the (multiplicative) group of characters of $U^{r+1}(\mathfrak{A})/U^{n+1}(\mathfrak{A})$.

Definition. A *stratum* in A is a 4-tuple $[\mathfrak{A}, n, r, \beta]$ where \mathfrak{A} is a hereditary \mathfrak{o}_F -order in A , n and r are integers with $r < n$, and $\beta \in \mathfrak{P}^{-n}$.

While we are not making all the restrictions here on n and r that we made above, we will see that in all applications of strata to the representation theory of G , these restrictions will play a part. But if we assume temporarily that we are given a stratum $[\mathfrak{A}, n, r, \beta]$ satisfying

$$0 \leq \left\lfloor \frac{n}{2} \right\rfloor \leq r < n \tag{1.1}$$

then this stratum specifies a character ψ_β of the group $U^{r+1}(\mathfrak{A})$ that is trivial on $U^{n+1}(\mathfrak{A})$. Furthermore, note that ψ_β is nontrivial on $U^n(\mathfrak{A})$ iff $\beta \notin \mathfrak{P}^{-n+1}$, *i.e.*, iff

$$\nu_{\mathfrak{A}}(\beta) = -n. \tag{1.2}$$

Thus, under this additional assumption, the parameter n in the stratum specifies the *level* of the character, so to speak. (In general, without assuming (1.2), n specifies a *bound* on the level.) Since different choices of β can give us the same character, it is natural to make the following definition.

Definition. Two strata $[\mathfrak{A}_1, n_1, r_1, \beta_1]$ and $[\mathfrak{A}_2, n_2, r_2, \beta_2]$ are *equivalent*, written

$$[\mathfrak{A}_1, n_1, r_1, \beta_1] \sim [\mathfrak{A}_2, n_2, r_2, \beta_2]$$

if

$$\beta_1 + \mathfrak{P}_1^{-r_1} = \beta_2 + \mathfrak{P}_2^{-r_2},$$

where $\mathfrak{P}_i = \text{rad}(\mathfrak{A}_i)$ for each i .

It is clear that this is an equivalence relation on the set of all strata in A , and it is easy to show that if $[\mathfrak{A}_1, n_1, r_1, \beta_1] \sim [\mathfrak{A}_2, n_2, r_2, \beta_2]$, then $\mathfrak{A}_1 = \mathfrak{A}_2$ and $r_1 = r_2$. (If furthermore both strata satisfy (1.2), then clearly $n_1 = n_2$.) Thus we could rewrite this definition as

$$[\mathfrak{A}, n_1, r, \beta_1] \sim [\mathfrak{A}, n_2, r, \beta_2] \quad \text{iff} \quad \beta_1 \equiv \beta_2 \pmod{\mathfrak{P}^{-r}}.$$

If we assume in addition that both strata satisfy (1.1), then this condition is equivalent to $\psi_{\beta_1} = \psi_{\beta_2}$.

In summary, an equivalence class of strata satisfying (1.1) is nothing more nor less than a choice of a compact open subgroup $U^{r+1}(\mathfrak{A})$ of G and a character ψ_β of this subgroup. The four terms in the tuple $[\mathfrak{A}, n, r, \beta]$ can be thought of as follows:

1. \mathfrak{A} determines a parahoric subgroup $U(\mathfrak{A})$ of G ;
2. r determines a compact open subgroup $U^{r+1}(\mathfrak{A})$ of G , from the standard filtration of $U(\mathfrak{A})$;
3. β determines a character ψ_β of $U^{r+1}(\mathfrak{A})$;
4. n determines a bound on the level of the character ψ_β .

1.2 Pure and simple strata

Definition. A stratum $[\mathfrak{A}, n, r, \beta]$ is called *pure* if

1. $E = F[\beta]$ is a field (*i.e.*, the minimal polynomial of β is irreducible),
2. $E^\times \subset \mathfrak{K}(\mathfrak{A})$ (*i.e.*, E normalizes \mathfrak{A}),
3. $n = -\nu_{\mathfrak{A}}(\beta)$.

Given a pure stratum $[\mathfrak{A}, n, r, \beta]$ with $E = F[\beta]$, we can regard V as an E -vector space, and define

$$B = B_\beta = \text{End}_E(V) = \{x \in A \mid xa = ax \quad \forall a \in E\}.$$

(Note then that B is the centralizer of E in A , or equivalently the centralizer of β in A .) We then define

$$\begin{aligned} \mathfrak{B} &= \mathfrak{B}_\beta = \mathfrak{A} \cap B = \text{the centralizer of } \beta \text{ in } \mathfrak{A}, \\ \mathfrak{Q} &= \mathfrak{Q}_\beta = \mathfrak{P} \cap B = \text{the centralizer of } \beta \text{ in } \mathfrak{P}. \end{aligned}$$

Then \mathfrak{B} is a hereditary \mathfrak{o}_E -order in B , and \mathfrak{Q} is its Jacobson radical. For $k \in \mathbb{Z}$, we define

$$\mathfrak{N}_k(\beta, \mathfrak{A}) = \{x \in \mathfrak{A} \mid \beta x - x\beta \in \mathfrak{P}^k\}.$$

For sufficiently small k (specifically, for all $k \leq \nu_{\mathfrak{A}}(\beta)$), $\mathfrak{N}_k(\beta, \mathfrak{A}) = \mathfrak{A}$, and for sufficiently large k , $\mathfrak{N}_k(\beta, \mathfrak{A}) \subset \mathfrak{B} + \mathfrak{P}$. (Note that if β is scalar, whence $E = F$, then $\mathfrak{A} = \mathfrak{N}_k(\beta, \mathfrak{A}) = \mathfrak{B} + \mathfrak{P}$ for all k .) Hence we define

$$k_0(\beta, \mathfrak{A}) = \begin{cases} \max\{k \in \mathbb{Z} \mid \mathfrak{N}_k(\beta, \mathfrak{A}) \not\subset \mathfrak{B} + \mathfrak{P}\} & \text{if } F[\beta] \neq F \\ -\infty & \text{if } F[\beta] = F. \end{cases}$$

Note that if $F[\beta] \neq F$, then $k_0(\beta, \mathfrak{A}) \geq \nu_{\mathfrak{A}}(\beta)$.

Definition. A stratum $[\mathfrak{A}, n, r, \beta]$ is called *simple* if it is pure and also satisfies

$$r < -k_0(\beta, \mathfrak{A}).$$

It can be proven (see [1, (2.1.4)]) that if $[\mathfrak{A}, n, r, \beta_1]$ and $[\mathfrak{A}, n, r, \beta_2]$ are simple strata that are equivalent, then

$$\begin{aligned} k_0(\beta_1, \mathfrak{A}) &= k_0(\beta_2, \mathfrak{A}), \\ e(F[\beta_1] \mid F) &= e(F[\beta_2] \mid F), \\ f(F[\beta_1] \mid F) &= f(F[\beta_2] \mid F). \end{aligned} \tag{1.3}$$

This fact will be useful below.

The first examples of simple strata, which will turn out to be the foundation of the whole theory, are given by *minimal elements*:

Definition. Let $E = F[\beta]$ be a field, let ν_E be the normalized valuation on E , and let ϖ_F be a prime element of F . We say that β is *minimal* over F if

1. $\gcd(\nu_E(\beta), e(E|F)) = 1$, and
2. $\varpi_F^{-\nu_E(\beta)} \beta^{e(E|F)} + \mathfrak{p}_E \in \mathfrak{k}_E$ generates the field extension $\mathfrak{k}_E/\mathfrak{k}_F$.

(Note that this is independent of the choice of prime element ϖ_F .)

Equivalently, if $E = F$ (i.e., $\beta \in F$), then β is always minimal over F , and if $E \neq F$, then β is minimal over F iff $\nu_{\mathfrak{A}}(\beta) = k_0(\beta, \mathfrak{A})$ (for any \mathfrak{A} satisfying $E^\times \subset \mathfrak{K}(\mathfrak{A})$). Thus, if $E = F[\beta]$ is a field with β minimal over F , then we can choose a hereditary order \mathfrak{A} in A with $E^\times \subset \mathfrak{K}(\mathfrak{A})$ (such an order will always exist in this situation) and let $n = -\nu_{\mathfrak{A}}(\beta)$. Then $[\mathfrak{A}, n, r, \beta]$ will be a simple stratum for any $r < n$. Strata of this form are referred to in [2] as *alfalfa strata*. Note that by (1.3), any simple stratum equivalent to an alfalfa stratum is also alfalfa.

1.3 Defining sequences for simple strata

Definition. Let $\beta \in A$ such that $E = F[\beta]$ is a field, and let $B = B_\beta$. A *tame corestriction on A relative to E/F* is a linear map $s : A \rightarrow B$ satisfying

1. $s(b_1 a b_2) = b_1 s(a) b_2$ for all $a \in A$ and $b_1, b_2 \in B$ (i.e., s is a (B, B) -bimodule homomorphism), and
2. $s(\mathfrak{A}) = \mathfrak{A} \cap B$ for any hereditary \mathfrak{o}_F -order \mathfrak{A} with $E^\times \subset \mathfrak{K}(\mathfrak{A})$.

For any field extension E of F contained in A , a tame corestriction s exists, and such a map is clearly unique up to multiplication by an element of \mathfrak{o}_E^\times . Furthermore, if \mathfrak{A} is any hereditary \mathfrak{o}_F -order in A with $E^\times \subset \mathfrak{K}(\mathfrak{A})$, and we let $\mathfrak{P} = \text{rad}(\mathfrak{A})$ as usual, then $s(\mathfrak{P}^n) = \mathfrak{P}^n \cap B$ for all $n \in \mathbb{Z}$. Thus in particular if $[\mathfrak{A}, n, r, \beta]$ is a pure stratum and $b \in \mathfrak{P}^{-r}$, then $[\mathfrak{B}_\beta, r, r-1, s(b)]$ is a stratum in B . Such a stratum is called a *derived stratum*, and it is clear by the preceding remarks that the equivalence class of this derived stratum is independent of the choice of s .

The following proposition is the content of [1, (2.4.2)], although its proof and all of the related material consumes much of chapter 2 of [1].

Proposition 1. *Let $[\mathfrak{A}, n, r, \beta]$ be a simple stratum. Then there exists a finite sequence of strata $[\mathfrak{A}, n, r_i, \beta_i]$, $0 \leq i \leq s$, that satisfies the following properties:*

1. $\beta = \beta_0$ and $r = r_0 < r_1 < \dots < r_s < n$,
2. For each i , $F[\beta_i]$ is a field, $F[\beta_i]^\times \subset \mathfrak{K}(\mathfrak{A})$, and $\nu_{\mathfrak{A}}(\beta_i) = -n$,

3. $[\mathfrak{A}, n, r_i, \beta_{i-1}] \sim [\mathfrak{A}, n, r_i, \beta_i]$ for $1 \leq i \leq s$,
4. $r_i = -k_0(\beta_{i-1}, \mathfrak{A})$ for $1 \leq i \leq s$,
5. $k_0(\beta_s, \mathfrak{A}) = -n$ or $-\infty$ (i.e., β_s is minimal over F),
6. Let s_i be a tame corestriction on A relative to $F[\beta_i]/F$. The stratum $[\mathfrak{B}_{\beta_i}, r_i, r_i - 1, s_i(\beta_{i-1} - \beta_i)]$ is equivalent to a simple stratum in B_{β_i} for $1 \leq i \leq s$.

Note that 2 is equivalent to saying that $[\mathfrak{A}, n, r_i, \beta_i]$ is a pure stratum for each i , and that 1 and 2 combined imply that $[\mathfrak{A}, n, r_i, \beta_j]$ is a pure stratum for any i, j . Also, by 3, we have $\beta_{i-1} - \beta_i \in \mathfrak{P}^{-r_i}$, and thus the tuple given in 6 is a derived stratum in B_{β_i} .

Definition. A sequence of strata $[\mathfrak{A}, n, r_i, \beta_i]$, $0 \leq i \leq s$, satisfying the requirements of Proposition 1 will be called a *defining sequence* for the simple stratum $[\mathfrak{A}, n, r, \beta]$.

To interpret Proposition 1 more clearly, note that a defining sequence for $[\mathfrak{A}, n, r, \beta]$ gives us the following strata, all of which are pure:

$$\begin{array}{c}
[\mathfrak{A}, n, r, \beta] = [\mathfrak{A}, n, r_0, \beta_0] \\
\swarrow \\
[\mathfrak{A}, n, r_1, \beta_0] \sim [\mathfrak{A}, n, r_1, \beta_1] \\
\swarrow \\
[\mathfrak{A}, n, r_2, \beta_1] \sim [\mathfrak{A}, n, r_2, \beta_2] \\
\vdots \\
[\mathfrak{A}, n, r_s, \beta_{s-1}] \sim [\mathfrak{A}, n, r_s, \beta_s]
\end{array}$$

(The arrows here do not represent maps, but rather are there to indicate the intended “flow” of the sequence.) In this diagram, the first term on the left is our original simple stratum, and (by parts 1 and 5 of Proposition 1) the final term on the right is an alfalfa stratum (i.e., a simple stratum given by an element that is minimal over F). In between these first and last terms, all of the terms on the right are simple strata (by parts 1 and 4), and the terms on the left are pure strata that are *not* simple, but just barely so (by part 4).

Naturally, a defining sequence for a simple stratum need not be unique. But by repeated application of (1.3), it is easy to see that the sequence of integers r_i (and the length s) of any defining sequence for $[\mathfrak{A}, n, r, \beta]$ is uniquely determined (in fact by the equivalence class of $[\mathfrak{A}, n, r, \beta]$), and furthermore that the equivalence classes of all the simple strata $[\mathfrak{A}, n, r_i, \beta_i]$ in the defining sequence are uniquely determined.

2 Simple Characters

The next major step toward the definition of simple types is to define *simple characters*, which are abelian characters of a very specific kind defined on certain compact open subgroups of G . We must first define these subgroups, which occur within natural decreasing filtrations of subgroups of G , denoted $H^k(\beta, \mathfrak{A})$, $k \geq 0$. Although we will not need them until later, we will also define another closely related family of decreasing filtrations of subgroups, denoted $J^k(\beta, \mathfrak{A})$, $k \geq 0$. As the notation suggests, the requirements for β and \mathfrak{A} are familiar ones: $\beta \in A$, \mathfrak{A} a hereditary \mathfrak{o}_F -order in A , $E = F[\beta]$ a field with $E^\times \subset \mathfrak{K}(\mathfrak{A})$, and $\nu_{\mathfrak{A}}(\beta)$ and $k_0(\beta, \mathfrak{A})$ both negative. In other words, we require that $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum (where $n = -\nu_{\mathfrak{A}}(\beta)$ of course). Unfortunately, the definitions of the desired groups can't be given uniformly in terms of just β and \mathfrak{A} , except in the case where β is minimal over F . In the general case, we must specify these rings in terms of a defining sequence for this simple stratum.

It may be useful to keep in mind throughout this section the following motivation: simple characters are based on the character ψ_β of $U^{\lfloor \frac{n}{2} \rfloor + 1}(\mathfrak{A})$ and certain properties of it. Note that this is the character naturally associated to the (pure but not necessarily simple) stratum $[\mathfrak{A}, n, \lfloor \frac{n}{2} \rfloor, \beta]$. It will turn out that for all $k \geq 0$,

$$U^k(\mathfrak{B}_\beta) \subseteq H^k(\beta, \mathfrak{A}) \subseteq J^k(\beta, \mathfrak{A}) \subseteq U^k(\mathfrak{A}),$$

and that for all $k \geq \lfloor \frac{n}{2} \rfloor + 1$, the last two of these containments are equalities. Naturally then, for such values of k , the only simple character of $H^k(\beta, \mathfrak{A})$ will be ψ_β . For smaller values of k , we will obtain simple characters by extending ψ_β from $H^{\lfloor \frac{n}{2} \rfloor + 1}(\beta, \mathfrak{A})$ to the larger group $H^k(\beta, \mathfrak{A})$.

2.1 The groups $H^k(\beta, \mathfrak{A})$ and $J^k(\beta, \mathfrak{A})$

At this point, our exposition deviates slightly from that of [1]. There, the groups $H^k(\beta, \mathfrak{A})$ and $J^k(\beta, \mathfrak{A})$ are defined in terms of a pair of rings $\mathfrak{H}(\beta, \mathfrak{A})$ and $\mathfrak{J}(\beta, \mathfrak{A})$ and a filtration of ideals of each. For the sake of brevity, we have chosen here to define these groups directly, as we will have no need for the aforementioned rings and ideals. What follows is adapted from [1, (3.1.7) - (3.1.15)].

Definition. Let β and \mathfrak{A} be such that $[\mathfrak{A}, n, 0, \beta]$ is a simple stratum (with $n = -\nu_{\mathfrak{A}}(\beta)$). If β is minimal over F , define

$$H^k(\beta, \mathfrak{A}) = \begin{cases} U^k(\mathfrak{B}_\beta)U^{\lfloor \frac{n}{2} \rfloor + 1}(\mathfrak{A}) & \text{if } 0 \leq k < \lfloor \frac{n}{2} \rfloor + 1, \\ U^k(\mathfrak{A}) & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \leq k; \end{cases}$$

$$J^k(\beta, \mathfrak{A}) = \begin{cases} U^k(\mathfrak{B}_\beta)U^{\lfloor \frac{n+1}{2} \rfloor}(\mathfrak{A}) & \text{if } 0 \leq k < \lfloor \frac{n+1}{2} \rfloor, \\ U^k(\mathfrak{A}) & \text{if } \lfloor \frac{n+1}{2} \rfloor \leq k. \end{cases}$$

If β is not minimal over F , let $r = -k_0(\beta, \mathfrak{A})$ and choose a simple stratum $[\mathfrak{A}, n, r, \beta']$ that is equivalent to the (pure but not simple) stratum $[\mathfrak{A}, n, r, \beta]$,

as in the construction of a defining sequence. Assuming that $H^k(\beta', \mathfrak{A})$ and $J^k(\beta', \mathfrak{A})$ have already been defined for all $k \geq 0$, we can define

$$H^k(\beta, \mathfrak{A}) = \begin{cases} U^k(\mathfrak{B}_\beta)H^{\lfloor \frac{r}{2} \rfloor + 1}(\beta', \mathfrak{A}) & \text{if } 0 \leq k < \lfloor \frac{r}{2} \rfloor + 1, \\ H^k(\beta', \mathfrak{A}) & \text{if } \lfloor \frac{r}{2} \rfloor + 1 \leq k; \end{cases}$$

$$J^k(\beta, \mathfrak{A}) = \begin{cases} U^k(\mathfrak{B}_\beta)J^{\lfloor \frac{r+1}{2} \rfloor}(\beta', \mathfrak{A}) & \text{if } 0 \leq k < \lfloor \frac{r+1}{2} \rfloor, \\ J^k(\beta', \mathfrak{A}) & \text{if } \lfloor \frac{r+1}{2} \rfloor \leq k. \end{cases}$$

The existence of a defining sequence for a given simple stratum (*i.e.*, Proposition 1) guarantees that the second part of this definition can be iterated, thereby defining the sets $H^k(\beta, \mathfrak{A})$ and $J^k(\beta, \mathfrak{A})$ for all β and \mathfrak{A} as above. That these sets are actually groups follows from the fact that $U^k(\mathfrak{A})$ is normalized by $U^m(\mathfrak{B}_\beta)$ for all $m \geq 0$, and proceeding by induction. However, it is obviously not clear that these groups are well-defined, due to the choice made in the second part of this definition (*i.e.*, the fact that a defining sequence for a simple stratum may not be unique). But postponing that matter for a moment, we note the following, which for small values of k (and always at least for $k = 0$ or 1) may be taken as an alternative to the above definition.

Corollary. *Let $[\mathfrak{A}, n, r_i, \beta_i]$, $0 \leq i \leq s$, be a defining sequence for the simple stratum $[\mathfrak{A}, n, 0, \beta]$, and let $r = -k_0(\beta, \mathfrak{A})$. Then for $0 \leq k \leq \lfloor \frac{r}{2} \rfloor + 1$,*

$$H^k(\beta, \mathfrak{A}) = U^k(\mathfrak{B}_{\beta_0})U^{\lfloor \frac{r_1}{2} \rfloor + 1}(\mathfrak{B}_{\beta_1}) \cdots U^{\lfloor \frac{r_s}{2} \rfloor + 1}(\mathfrak{B}_{\beta_s})U^{\lfloor \frac{n}{2} \rfloor + 1}(\mathfrak{A}),$$

and for $0 \leq k \leq \lfloor \frac{r+1}{2} \rfloor$,

$$J^k(\beta, \mathfrak{A}) = U^k(\mathfrak{B}_{\beta_0})U^{\lfloor \frac{r_1+1}{2} \rfloor}(\mathfrak{B}_{\beta_1}) \cdots U^{\lfloor \frac{r_s+1}{2} \rfloor}(\mathfrak{B}_{\beta_s})U^{\lfloor \frac{n}{2} \rfloor + 1}(\mathfrak{A}).$$

Proposition 2. *Let β and \mathfrak{A} be such that $[\mathfrak{A}, n, 0, \beta]$ is a simple stratum (with $n = -\nu_{\mathfrak{A}}(\beta)$).*

1. *For all $k \geq 0$, $H^k(\beta, \mathfrak{A})$ and $J^k(\beta, \mathfrak{A})$ are well-defined independently of the choices made in the definition above. Furthermore, they depend only on the equivalence class of the simple stratum $[\mathfrak{A}, n, 0, \beta]$.*
2. *$H^0(\beta, \mathfrak{A})$ and $J^0(\beta, \mathfrak{A})$ are compact open subgroups of G , and the $H^k(\beta, \mathfrak{A})$ for $k > 0$ form a decreasing filtration of $H^0(\beta, \mathfrak{A})$ by open normal subgroups (and likewise for $J^k(\beta, \mathfrak{A})$).*
3. *For all $k \geq 0$,*

$$U^k(\mathfrak{B}_\beta) \subseteq H^k(\beta, \mathfrak{A}) \subseteq J^k(\beta, \mathfrak{A}) \subseteq U^k(\mathfrak{A}).$$

4. *For all $k \geq 0$, $H^k(\beta, \mathfrak{A})$ and $J^k(\beta, \mathfrak{A})$ are normalized by $\mathfrak{K}(\mathfrak{B}_\beta)$.*
5. *For all $k > 0$, $H^k(\beta, \mathfrak{A})$ is a normal subgroup of $J^0(\beta, \mathfrak{A})$.*

6. Let $r = -k_0(\beta, \mathfrak{A})$. Then for $0 \leq m < k \leq \lfloor \frac{r}{2} \rfloor + 1$, we have a natural isomorphism

$$H^m(\beta, \mathfrak{A})/H^k(\beta, \mathfrak{A}) \cong U^m(\mathfrak{B}_\beta)/U^k(\mathfrak{B}_\beta).$$

Similarly for $0 \leq m < k \leq \lfloor \frac{r+1}{2} \rfloor$, we have a natural isomorphism

$$J^m(\beta, \mathfrak{A})/J^k(\beta, \mathfrak{A}) \cong U^m(\mathfrak{B}_\beta)/U^k(\mathfrak{B}_\beta).$$

It will turn out that the three groups

$$H^1(\beta, \mathfrak{A}) \triangleleft J^1(\beta, \mathfrak{A}) \triangleleft J^0(\beta, \mathfrak{A})$$

will be the ones needed in the definition of simple types. In this context, we note the natural isomorphism

$$J^0(\beta, \mathfrak{A})/J^1(\beta, \mathfrak{A}) \cong U(\mathfrak{B}_\beta)/U^1(\mathfrak{B}_\beta)$$

given above, which will play a significant role later.

2.2 Simple characters

We are now ready to define simple characters. Once again, the definition of, as well as the computation of (and many of the properties of), these characters, are relatively straightforward in the case where β is minimal over F , but in the general case must be done in terms of a defining sequence for the associated simple stratum.

To begin with, let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum, and let \det_{B_β} denote the determinant map from B_β to $F[\beta]$. The starting point of the definition of simple characters is the observation of the following two facts about ψ_β :

1. the restriction of ψ_β to $U^{\lfloor \frac{n}{2} \rfloor + 1}(\mathfrak{A}) \cap B_\beta^\times$ factors through \det_{B_β} ;
2. $\psi_\beta(x^{-1}ax) = \psi_\beta(a)$ for all $x \in \mathfrak{K}(\mathfrak{B}_\beta)$ and all $a \in U^{\lfloor \frac{n}{2} \rfloor + 1}(\mathfrak{A})$ (i.e., $\mathfrak{K}(\mathfrak{B}_\beta)$ normalizes ψ_β).

The first of these facts requires some effort to prove, but does not merit stating as a proposition. The second is clear.

Definition. Let β and \mathfrak{A} be such that $[\mathfrak{A}, n, 0, \beta]$ is a simple stratum (with $n = -\nu_{\mathfrak{A}}(\beta)$), and let $0 \leq m < n$. Assume that β is minimal over F . If $\lfloor \frac{n}{2} \rfloor \leq m < n$, define $\mathcal{C}(\mathfrak{A}, m, \beta) = \{\psi_\beta\}$. If $0 \leq m < \lfloor \frac{n}{2} \rfloor$, then define $\mathcal{C}(\mathfrak{A}, m, \beta)$ to be the set of all characters θ of $H^{m+1}(\beta, \mathfrak{A})$ which satisfy

1. the restriction of θ to $H^{\lfloor \frac{n}{2} \rfloor + 1}(\beta, \mathfrak{A})$ is equal to ψ_β ;
2. the restriction of θ to $H^{m+1}(\beta, \mathfrak{A}) \cap B_\beta^\times$ factors through \det_{B_β} .

It follows immediately from this definition and the second observation above that every $\theta \in \mathcal{C}(\mathfrak{A}, m, \beta)$ is normalized by $\mathfrak{K}(\mathfrak{B}_\beta)$. We now proceed with the general case.

Definition. Let β and \mathfrak{A} be such that $[\mathfrak{A}, n, 0, \beta]$ is a simple stratum (with $n = -\nu_{\mathfrak{A}}(\beta)$), and let $0 \leq m < n$. Assume now that β is *not* minimal over F , let $r = -k_0(\beta, \mathfrak{A})$, and choose a simple stratum $[\mathfrak{A}, n, r, \beta']$ that is equivalent to the (pure but not simple) stratum $[\mathfrak{A}, n, r, \beta]$, as in the construction of a defining sequence. We will assume that $\mathcal{C}(\mathfrak{A}, m', \beta')$ has already been defined for all m' . If $r \leq m < n$, define $\mathcal{C}(\mathfrak{A}, m, \beta) = \mathcal{C}(\mathfrak{A}, m, \beta')$. Otherwise, let $\mathcal{C}(\mathfrak{A}, m, \beta)$ be the set of all characters θ of $H^{m+1}(\beta, \mathfrak{A})$ which satisfy

1. (a) if $\lfloor \frac{r}{2} \rfloor \leq m < r$, then $\theta = \theta_0 \cdot \psi_{\beta - \beta'}$ for some $\theta_0 \in \mathcal{C}(\mathfrak{A}, m, \beta')$;
(b) if $0 \leq m < \lfloor \frac{r}{2} \rfloor$, then the restriction of θ to $H^{\lfloor \frac{r}{2} \rfloor + 1}(\beta, \mathfrak{A})$ is equal to $\theta_0 \cdot \psi_{\beta - \beta'}$ for some $\theta_0 \in \mathcal{C}(\mathfrak{A}, \lfloor \frac{r}{2} \rfloor, \beta')$;
2. the restriction of θ to $H^{m+1}(\beta, \mathfrak{A}) \cap B_\beta^\times$ factors through \det_{B_β} ;
3. $\mathfrak{K}(\mathfrak{B}_\beta)$ normalizes θ .

A few remarks are in order. First, recall that for $m' \geq \lfloor \frac{r}{2} \rfloor + 1$, $H^{m'+1}(\beta, \mathfrak{A}) = H^{m'+1}(\beta', \mathfrak{A})$. Thus defining the elements of $\mathcal{C}(\mathfrak{A}, m, \beta)$ in terms of those of $\mathcal{C}(\mathfrak{A}, m', \beta')$ makes sense as long as $m' \geq \lfloor \frac{r}{2} \rfloor$, which is true in all cases of the definition above. Second, note that $\beta - \beta' \in \mathfrak{P}^{-r}$, so in part 1, $\psi_{\beta - \beta'}$ is a character of $U^{\lfloor \frac{r}{2} \rfloor + 1}(\mathfrak{A})$ that is trivial on $U^{r+1}(\mathfrak{A})$. To show that these two definitions are actually much more uniform than they may at first appear, we note the following, which may be taken as an alternative to the pair of definitions above.

Corollary. Let $[\mathfrak{A}, n, r_i, \beta_i]$, $0 \leq i \leq s$, be a defining sequence for the simple stratum $[\mathfrak{A}, n, 0, \beta]$. Then for $0 \leq m < n$ and $0 \leq i \leq s$, $\mathcal{C}(\mathfrak{A}, m, \beta_i)$ is the set of all characters θ of $H^{m+1}(\beta_i, \mathfrak{A})$ satisfying the following criteria:

1. the restriction of θ to $H^{m+1}(\beta, \mathfrak{A}) \cap B_\beta^\times$ factors through \det_{B_β} ;
2. $\mathfrak{K}(\mathfrak{B}_\beta)$ normalizes θ ;
3. (a) if $i = s$, let $m' = \max\{m, \lfloor \frac{n}{2} \rfloor\}$; then the restriction of θ to $H^{m'+1}(\beta_s, \mathfrak{A})$ is equal to ψ_{β_s} ;
(b) if $i < s$, let $m' = \max\{m, \lfloor \frac{r_i+1}{2} \rfloor\}$; then the restriction of θ to $H^{m'+1}(\beta_i, \mathfrak{A})$ is equal to $\theta_0 \cdot \psi_{\beta_i - \beta_{i+1}}$ for some $\theta_0 \in \mathcal{C}(\mathfrak{A}, m', \beta_{i+1})$.

Once again, it is not clear that the sets $\mathcal{C}(\mathfrak{A}, m, \beta)$ are well-defined in general, since defining sequences for simple strata are not unique. In this case, establishing this takes a significant effort. Furthermore, it is not obvious except in certain cases that these sets are nonempty. However, as we hinted at above, the basic idea in the definition of simple characters is to start with ψ_β

on $H^{\lfloor \frac{n}{2} \rfloor + 1}(\beta, \mathfrak{A})$, and extend to larger subgroups. Thus we may hope that, for $m' > m$, restriction of characters from $H^{m'+1}(\beta, \mathfrak{A})$ to $H^{m+1}(\beta, \mathfrak{A})$ will map characters of $\mathcal{C}(\mathfrak{A}, m, \beta)$ to $\mathcal{C}(\mathfrak{A}, m', \beta)$. This is indeed the case, and in fact these maps are always surjective and their fibers can be described explicitly. The following proposition is a summary of the culminating results of sections (3.2) and (3.3) of [1].

Proposition 3. *Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum, and let $0 \leq m < n$. The set $\mathcal{C}(\mathfrak{A}, m, \beta)$ is well-defined and nonempty, and depends only on the equivalence class of the (pure but not necessarily simple) stratum $[\mathfrak{A}, n, m, \beta]$. More specifically, let $[\mathfrak{A}, n, r_i, \beta_i]$, $0 \leq i \leq s$ be a defining sequence for $[\mathfrak{A}, n, 0, \beta]$. We can compute all of the sets $\mathcal{C}(\mathfrak{A}, m, \beta)$ inductively as follows.*

1. (a) If $\lfloor \frac{n}{2} \rfloor \leq m < n$, then

$$\mathcal{C}(\mathfrak{A}, m, \beta_s) = \{\psi_{\beta_s}\}.$$

- (b) If $0 \leq m < \lfloor \frac{n}{2} \rfloor$, then all characters in $\mathcal{C}(\mathfrak{A}, m, \beta_s)$ are extensions of ψ_{β_s} . Furthermore, if $\theta \in \mathcal{C}(\mathfrak{A}, m, \beta_s)$, then

$$\mathcal{C}(\mathfrak{A}, m, \beta_s) = \{\theta \cdot \chi \mid \chi \in X\},$$

where X is the group of characters of $U^{m+1}(\mathfrak{B}_{\beta_s})$ that are trivial on $U^{\lfloor \frac{n}{2} \rfloor + 1}(\mathfrak{B}_{\beta_s})$ and factor through $\det_{B_{\beta_s}}$.

2. Let $0 \leq i < s$ and abbreviate $r = r_{i+1}$.

- (a) If $r \leq m < n$, then

$$\mathcal{C}(\mathfrak{A}, m, \beta_i) = \mathcal{C}(\mathfrak{A}, m, \beta_{i+1}).$$

- (b) If $\lfloor \frac{r}{2} \rfloor \leq m < r$, then

$$\mathcal{C}(\mathfrak{A}, m, \beta_i) = \{\theta \cdot \psi_{\beta_i - \beta_{i+1}} \mid \theta \in \mathcal{C}(\mathfrak{A}, m, \beta_{i+1})\}.$$

- (c) If $0 \leq m < \lfloor \frac{r}{2} \rfloor$, then all characters in $\mathcal{C}(\mathfrak{A}, m, \beta_i)$ are extensions of characters in $\mathcal{C}(\mathfrak{A}, \lfloor \frac{r}{2} \rfloor, \beta_i)$, i.e., restriction of characters defines a surjective map from $\mathcal{C}(\mathfrak{A}, m, \beta_i)$ to $\mathcal{C}(\mathfrak{A}, \lfloor \frac{r}{2} \rfloor, \beta_i)$. Furthermore, if $\theta \in \mathcal{C}(\mathfrak{A}, m, \beta_i)$ restricts to $\bar{\theta} \in \mathcal{C}(\mathfrak{A}, \lfloor \frac{r}{2} \rfloor, \beta_i)$, then the fiber of $\bar{\theta}$ under this restriction map is

$$\{\theta \cdot \chi \mid \chi \in X\},$$

where X is the group of characters of $U^{m+1}(\mathfrak{B}_{\beta_i})$ that are trivial on $U^{\lfloor \frac{r}{2} \rfloor + 1}(\mathfrak{B}_{\beta_i})$ and factor through $\det_{B_{\beta_i}}$.

3 Simple Types

As mentioned above, the groups that will be used to define simple types are precisely the groups $H^1(\beta, \mathfrak{A}) \subseteq J^1(\beta, \mathfrak{A}) \subseteq J^0(\beta, \mathfrak{A})$. Note that $\mathcal{C}(\mathfrak{A}, 0, \beta)$ is a set of characters defined on the smallest of these three groups. Thus the last remaining task before we can define simple types will be to further extend these characters to the groups $J^1(\beta, \mathfrak{A})$ and $J^0(\beta, \mathfrak{A})$.

3.1 Extending simple characters

Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum in A , and let $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$, *i.e.*, let θ be a simple character of $H^1(\beta, \mathfrak{A})$. It is straightforward to prove that there is a unique irreducible representation η of $J^1(\beta, \mathfrak{A})$ whose restriction to $H^1(\beta, \mathfrak{A})$ contains θ . (It also follows from the proof that the restriction of η to $H^1(\beta, \mathfrak{A})$ is in fact a direct sum of $[J^1(\beta, \mathfrak{A}) : H^1(\beta, \mathfrak{A})]^{\frac{1}{2}}$ copies of θ .)

Unfortunately, the issue of extending η to $J^0(\beta, \mathfrak{A})$ is much more subtle. Many extensions exist, so we must choose only certain ones, and we would still like to be able to say something about how many such extensions exist. We begin by recalling a standard set of definitions.

Definition. Temporarily let G be any group, let H_1 and H_2 be subgroups of G , and let ρ_1 and ρ_2 be irreducible representations of H_1 and H_2 , respectively. Let $g \in G$. Let $H_1^g = g^{-1}H_1g$ and denote by ρ_1^g the irreducible representation of H_1^g given by $h \mapsto \rho_1(hg^{-1})$. We say that g intertwines ρ_1 with ρ_2 if

$$\text{Hom}_{H_1^g \cap H_2}(\rho_1^g, \rho_2) \neq 0.$$

We define the G -intertwining of ρ_1 with ρ_2 by

$$I_G(\rho_1|H_1, \rho_2|H_2) = \{g \in G \mid g \text{ intertwines } \rho_1 \text{ with } \rho_2\}.$$

(When there is no ambiguity, we will often drop the subgroups from this notation and just write $I_G(\rho_1, \rho_2)$.) Finally, we say that ρ_1 and ρ_2 *intertwine in G* if $I_G(\rho_1, \rho_2) \neq \emptyset$.

We note the following basic facts:

1. $I_G(\rho_1|H_1, \rho_2|H_2)$ is always a union of double-cosets H_1gH_2 .
2. The identity element always intertwines ρ with itself.
3. g intertwines ρ_1 with ρ_2 iff g^{-1} intertwines ρ_2 with ρ_1 .
4. By the previous two points, the concept of two representations intertwining in G is reflexive and symmetric; it is not transitive.

If we have a single subgroup H of G and an irreducible representation ρ of H , we simply say g *intertwines ρ* if g intertwines ρ with itself. Similarly, in place of $I_G(\rho|H, \rho|H)$, we simply write $I_G(\rho|H)$ (or just $I_G(\rho)$), and we refer to this set as the G -intertwining of ρ .

There are many ways to motivate the concept of intertwining (as it has many useful applications), but perhaps one of the simplest is this: in the case that G is a compact group and H_1 and H_2 are closed subgroups, $\left\langle \text{ind}_{H_1}^G(\rho_1), \text{ind}_{H_2}^G(\rho_2) \right\rangle_G$ (i.e., $\dim \text{Hom}_G(\text{ind}_{H_1}^G(\rho_1), \text{ind}_{H_2}^G(\rho_2))$) is equal to the number of distinct double-cosets $H_1 g H_2$ in $I_G(\rho_1, \rho_2)$. This follows from an easy application of Mackey's Theorem, and hence this statement has generalizations to many other settings in which some form of Mackey's Theorem applies.

We return to the problem at hand. In particular, we resume using our previous notation, in which $G = A^\times \cong \text{GL}_N(F)$.

Definition. Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum in A , let $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$, and let η be the unique irreducible representation of $J^1(\beta, \mathfrak{A})$ whose restriction to $H^1(\beta, \mathfrak{A})$ contains θ . A β -extension of η is an irreducible representation κ of $J^0(\beta, \mathfrak{A})$ such that

1. The restriction of κ to $J^1(\beta, \mathfrak{A})$ is equal to η , and
2. $B_\beta^\times \subseteq I_G(\kappa)$, i.e., κ is intertwined by every element of B_β^\times .

Let $E = F[\beta]$, $B = B_\beta$, and $\mathfrak{B} = \mathfrak{B}_\beta$. For any character χ of \mathbf{k}_E^\times , we can view χ as a character of \mathfrak{o}_E^\times that is trivial on $1 + \mathfrak{p}_E$, and then $\chi \circ \det_B$ defines a character of $U(\mathfrak{B})$ that is trivial on $U^1(\mathfrak{B})$. Recalling the canonical isomorphism $U(\mathfrak{B})/U^1(\mathfrak{B}) \cong J^0(\beta, \mathfrak{A})/J^1(\beta, \mathfrak{A})$, we can thus view $\chi \circ \det_B$ as a character of $J^0(\beta, \mathfrak{A})$ that is trivial on $J^1(\beta, \mathfrak{A})$. The following proposition is [1, (5.2.2)].

Proposition 4. *Let all notation be as above.*

1. A β -extension of η exists.
2. If κ is a β -extension of η , then all other β -extensions of η are of the form $\kappa \otimes (\chi \circ \det_B)$ for some character χ of \mathbf{k}_E^\times as above.
3. Distinct characters χ of \mathbf{k}_E^\times yield distinct (nonisomorphic) representations $\kappa \otimes (\chi \circ \det_B)$. Thus the number of distinct β -extensions of η is equal to $q_E - 1$.

3.2 Simple types

We are finally ready to define simple types.

Definition. A *simple type in G* is a pair (J, λ) consisting of a compact open subgroup J of G and an irreducible representation λ of J , of one of the following two forms.

1. $J = J^0(\beta, \mathfrak{A})$ and $\lambda = \kappa \otimes \sigma$, where
 - (a) \mathfrak{A} is a *principal* hereditary \mathfrak{o}_F -order in A and $\beta \in A$ such that $[\mathfrak{A}, n, 0, \beta]$ is a simple stratum (with $n = -\nu_{\mathfrak{A}}(\beta)$ of course);

- (b) for some $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$, κ is a β -extension of the unique irreducible representation η of $J^1(\beta, \mathfrak{A})$ whose restriction to $H^1(\beta, \mathfrak{A})$ contains θ ;
- (c) if we let $E = F[\beta]$, $\mathfrak{B} = \mathfrak{B}_\beta$, $e = e(\mathfrak{B})$, and $N_0 = \frac{N}{[E:F]}$, so that we have isomorphisms

$$J^0(\beta, \mathfrak{A})/J^1(\beta, \mathfrak{A}) \cong U(\mathfrak{B})/U^1(\mathfrak{B}) \cong \prod_{i=1}^e \mathrm{GL}_{N_0}(\mathbf{k}_E),$$

then σ is the inflation to $J^0(\beta, \mathfrak{A})$ of the e -fold tensor product $\sigma_0 \otimes \cdots \otimes \sigma_0$ for some irreducible cuspidal representation σ_0 of $\mathrm{GL}_{N_0}(\mathbf{k}_E)$.

2. $J = U(\mathfrak{A})$, where \mathfrak{A} is a *principal* hereditary \mathfrak{o}_F -order in A , and if we let $e = e(\mathfrak{A})$ so that we have an isomorphism

$$U(\mathfrak{A})/U^1(\mathfrak{A}) \cong \prod_{i=1}^e \mathrm{GL}_{\frac{N}{e}}(\mathbf{k}_F),$$

then λ is the inflation to $U(\mathfrak{A})$ of the e -fold tensor product $\sigma_0 \otimes \cdots \otimes \sigma_0$ for some irreducible cuspidal representation σ_0 of $\mathrm{GL}_{\frac{N}{e}}(\mathbf{k}_F)$.

Note that, with the terminology above, if $\beta \in F$, then $E = F$, $\mathfrak{B} = \mathfrak{A}$, and $J^0(\beta, \mathfrak{A}) = U(\mathfrak{A})$. Thus the second form in this definition is almost a special case of the first form, in which $\beta \in F$, but κ is the trivial character of $J^0(\beta, \mathfrak{A})$. (Note that the trivial character of $H^1(\beta, \mathfrak{A})$ is never a simple character, and thus as this definition is written, κ cannot be trivial. This is why we must treat the second form as a separate case.) Thus, in either case, there is a simple stratum $[\mathfrak{A}, n, 0, \beta]$ associated to the type (J, λ) , such that $J = J^0(\beta, \mathfrak{A})$.

Definition. Let (J, λ) be a simple type, and assume all the notation of the previous definition. We say that (J, λ) is a *maximal simple type* if $e(\mathfrak{B}) = 1$. (Naturally, in the second case of the previous definition, this means $e(\mathfrak{A}) = 1$.)

Note that the condition in this definition is equivalent to requiring that $e(E|F) = e(\mathfrak{A})$, since

$$e(\mathfrak{B}) = \frac{e(\mathfrak{A})}{e(E|F)}.$$

Also, using the block matrix form of hereditary orders, it is equivalent to requiring that $\mathfrak{B} \cong M_{N_0}(\mathfrak{o}_E)$, where $N_0 = \frac{N}{[E:F]}$. The following is a summary of the culminating results of chapter 6 of [1] (part of which is not proved until the end of chapter 8 of *loc. cit.*).

Proposition 5.

1. Let (J, λ) be a maximal simple type. If (π, V) is a smooth representation of G that contains λ (i.e., $\langle \pi, \lambda \rangle_J > 0$), then π is supercuspidal. Furthermore, if (π', V') is any other smooth representation of G that contains λ , then π' is inertially equivalent to π (i.e., there exists an unramified character χ of F^\times such that $\pi' \cong (\chi \circ \det) \otimes \pi$).

2. Conversely, let (π, V) be an irreducible supercuspidal representation of G . Then there exists a maximal simple type (J, λ) such that π contains λ . Furthermore, if (J', λ') is any other simple type, then π contains λ iff (J', λ') is conjugate to (J, λ) (i.e., $J' = J^g$ and $\lambda' \cong \lambda^g$ for some $g \in G$). Thus in particular any simple type contained in π is maximal. Also, any simple type (J, λ) contained in π occurs in π with multiplicity 1 (i.e., $\langle \pi, \lambda \rangle_J = 1$).
3. Let (J, λ) be a maximal simple type, with an associated simple stratum $[\mathfrak{A}, n, 0, \beta]$, and let $E = F[\beta]$ as usual. If Λ is any extension of λ from J to $E^\times J$, then the representation

$$\pi = \text{Ind}_{E^\times J}^G(\Lambda)$$

is irreducible and supercuspidal.

4. Conversely, let (π, V) be an irreducible supercuspidal representation of G that contains a maximal simple type (J, λ) , with an associated simple stratum $[\mathfrak{A}, n, 0, \beta]$, and let $E = F[\beta]$. Then there exists a unique extension Λ of λ from J to $E^\times J$ such that

$$\pi \cong \text{Ind}_{E^\times J}^G(\Lambda).$$

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