Section 2.5, #27:

16th November 2003

Problem Statement:

Show that

$$\frac{d}{dx}\int_0^x f(x,y)dy = f(x,x) + \int_0^x \frac{df}{dx}(x,y)dy$$

First, an explanation:

Note that $\int_0^x f(x,y)dy$ is a function of x (not of x and y) because when you take a definite integral with respect to y, you eliminate all occurrences of the variable y. So on the left hand side of the equation, you are taking the derivative with respect to x of a function of x. Similarly, the integral on the right hand side is also a function of just x, not x and y.

To solve this proof, you will actually need to use three "big" theorems from calculus. Two of them are mentioned in the statement of the problem: the chain rule, and "differentiation under the integral sign." The third big theorem you will need is the Fundamental Theorem of Calculus, specifically the second part of this theorem (sometimes called the Second Fundamental Theorem of Calculus.)

Probably the most mysterious of these right now, since it hasn't been mentioned previously, is the theorem on differentiation under the integral sign. This theorem simply says that, under the right conditions, given a function of two or more variables, if you integrate it with respect to one variable and differentiate it with respect to a different variable, then it doesn't matter in which order you do these two operations. To state this as an equation:

$$\frac{d}{dx}\int_a^b f(x,y)dy = \int_a^b \frac{d}{dx}f(x,y)dy$$

In other words, you can take a differentiation symbol ($\frac{d}{dx}$) and simply move it inside the integral. Two things are worth noting here. First, you can only do this when the limits of integration are *not* dependent on x. The point of this homework problem is to generalize this a little bit: you need to show how to "differentiate under the integral sign" when one of the limits of integration *is* x. Second, note that I said "under the right conditions" above. I said that because this theorem only works when the function $f(x, y)$ is "nice" enough, for example, when it is continuously differentiable with respect to both variables. You may of course assume that the theorem works for this problem.

Hint:

I want to motivate this hint a little bit before I simply give it to you. As I pointed out in the first paragraph above, the integral in the left hand side of the equation is just a function of x, and you are just differentiating it with respect to x. So where are you going to use the multivariable chain rule, which is the topic of this section? To apply that, you need to be differentiating a function of two or more variables. But note also that the variable x appears in two distinct places in the definition of this function: once inside the expression $f(x, y)$, and once in the upper limit of the integral. So what you might want to do is create a function of two variables by replacing one of these occurrences of x with a different variable.

So here's the hint: define a new function, say $h(x, t)$, as follows:

$$h(x, t) = \int_0^t f(x, y)dy$$

Now we can view the desired expression (the left hand side of the equation that you are trying to prove) as simply the derivative, with respect to x, of $h(x, x)$. Apply the chain rule to this, then use the other two theorems mentioned above, to arrive at the right hand side of the equation. I strongly encourage you to try to figure this out on your own. But if you absolutely can't, you can scroll down to the next page for a detailed proof.

Full Solution:

As I said above, start by defining

$$h(x,t) = \int_0^t f(x,y)dy.$$

Now we wish to take the derivative

$$\frac{d}{dx}h(x,x).$$

Note that we have replaced t with x, so effectively the *variable* t has been replaced with a *function* of x, which we could call $t(x)$. In other words, we want to find

$$\frac{d}{dx}h(x,t(x))$$

where $t(x) = x$. Now, by the chain rule (recall problem 3 in this section) we have

$$\frac{d}{dx}h(x,t(x)) = \frac{dh}{dx} + \frac{dh}{dt}\frac{dt}{dx}.$$

Now we need to figure out what all of these partial derivatives are. We can get the first one by differentiating under the integral sign:

$$\frac{dh}{dx} = \frac{d}{dx}\int_0^t f(x,y)dy = \int_0^t \frac{d}{dx}f(x,y)dy.$$

Note that this is okay in this case, because now the limits of the integral don't depend on x. (Since we are taking the partial derivative with respect to x, we treat t like a constant here, so it does not depend on x.) The second partial derivative we need is *precisely* an application of the Second Fundamental Theorem of Calculus:

$$\frac{dh}{dt} = \frac{d}{dt}\int_0^t f(x,y)dy = f(x,t).$$

And the third one is trivial, since $t(x) = x$:

$$\frac{dt}{dx} = 1.$$

Combining all of these, we get

$$\frac{d}{dx}h(x,t(x)) = \int_0^t \frac{d}{dx}f(x,y)dy + f(x,t).$$

Finally, substituting x in place of t everywhere, we get the desired result:

$$\frac{d}{dx}\int_0^x f(x,y)dy = \frac{d}{dx}h(x,t(x)) = \int_0^x \frac{d}{dx}f(x,y)dy + f(x,x).$$