1. Let V be a vector space over a field F.

   (a) Let $a \in F$ and $v, w \in V$. Prove that if $a \neq 0$ and $av = aw$ then $v = w$.

   Assume $a \neq 0$ and $av = aw$. Since $a \in F$ and $a \neq 0$, there exists a multiplicative inverse a^{-1} for a in F. Now $a^{-1}(av) = a^{-1}(aw)$, so $(a^{-1}a)v = (a^{-1}a)w$, so $1v = 1w$, so $v = w$.

   (b) Let $a, b \in F$ and $v \in V$. Prove that if $v \neq 0$ and $av = bv$ then $a = b$.

   Assume $v \neq 0$ and $av = bv$. Then $av - bv = 0$, so $(a-b)v = 0$. Since $(a-b)0 = 0$ also, we have $(a-b)v = (a-b)0$. If $a-b \neq 0$, then by part (a), we would have $v = 0$. By contrapositive, if $v \neq 0$ then $a-b = 0$. Therefore, since we are assuming $v \neq 0$, we have $a - b = 0$. So $a = b$.

2. Define $T : \mathbb{R}^3 \to P_2(\mathbb{R})$ by

$$T(a, b, c) = (a + c)X^2 + (a + b)X + (a + 3b - 2c).$$

(a) Find a basis for $R(T)$.

$T(1,0,0) = X^2 + X + 1$

$T(0,1,0) = X + 3$

$T(0,0,1) = X^2 - 2$

So $\{X^2 + X + 1, X + 3, X^2 - 2\}$ is a generating set for $R(T)$.

But since $(X^2 + X + 1) - (X + 3) - (X^2 - 2) = 0$, this set is linearly dependent.

$\{X^2 + X + 1\}$ is lin. indep.

$\{X^2 + X + 1, X + 3\}$ is lin. indep. because $X + 3$ is not a scalar multiple of $X^2 + X + 1$.

So $\{X^2 + X + 1, X + 3\}$ is a basis of $R(T)$.

(b) What is rank(T)? Is T onto?

rank$(T) = 2$ since the basis found in part (a) has 2 elements.

Since $\dim(P_2(\mathbb{R})) = 3$ and rank$(T) < 3$, T is $\underline{not}$ onto.

(c) What is nullity(T)? Is T one-to-one?

By the Rank-Nullity Theorem, rank(T) + nullity(T) = $\dim(\mathbb{R}^3) = 3$,

so nullity$(T) = 3 - 2 = 1$.

Since nullity$(T) \neq 0$, T is $\underline{not}$ one-to-one.

3. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called *odd* if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$, and is called *even* if $f(-x) = f(x)$ for all $x \in \mathbb{R}$.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Define functions f_1 and f_2 by

$$f_1(x) = \frac{f(x) - f(-x)}{2} \qquad \text{and} \qquad f_2(x) = \frac{f(x) + f(-x)}{2}.$$

Prove that f_1 is odd and f_2 is even.

For any $x \in \mathbb{R}$, $f_1(-x) = \frac{f(-x) - f(-(-x))}{2} = -\frac{f(x) - f(-x)}{2} = -f_1(x)$

so f_1 is odd.

For any $x \in \mathbb{R}$, $f_2(-x) = \frac{f(-x) + f(-(-x))}{2} = \frac{f(x) + f(-x)}{2} = f_2(x)$

so f_2 is even.

(b) Let V be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$, and let W_1 be the subspace of all odd functions and W_2 be the subspace of all even functions. Prove that $V = W_1 \oplus W_2$. *(Hint: Use part (a)!)*

Let $f \in W_1 \cap W_2$. Then $f : \mathbb{R} \to \mathbb{R}$ is both even and odd.

So for any $x \in \mathbb{R}$, $f(x) = f(-x) = -f(x)$, so $2f(x) = 0$,

so $f(x) = 0$.

Thus $f = 0$.

Therefore $W_1 \cap W_2 \subseteq \{0\}$. Obviously $W_1 \cap W_2 \supseteq \{0\}$. So $W_1 \cap W_2 = \{0\}$.

Now let $f \in V$ be any function. Define f_1 and f_2 as in part (a). Then, as shown in part (a), f_1 is odd and f_2 is even, so $f_1 \in W_1$ and $f_2 \in W_2$.

Since $(f_1 + f_2)(x) = \frac{f(x) - f(-x)}{2} + \frac{f(x) + f(-x)}{2} = f(x)$ for all $x \in \mathbb{R}$, $f_1 + f_2 = f$.

Thus $f \in W_1 + W_2$.

Therefore $V \subseteq W_1 + W_2$. Obviously $V \supseteq W_1 + W_2$. So $V = W_1 + W_2$.

Since $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$, $V = W_1 \oplus W_2$.

4. Let $V = P_2(\mathbb{R})$ and $W = P_3(\mathbb{R})$, and define a linear transformation $T : V \to W$ by

$$T(f) = \int_1^x f(t)dt.$$

Let $\beta = \{6x^2 - 2x, 4x + 1, 2x - 3\}$ and $\gamma = \{1, x + 1, x^2 + 1, x^3 + 1\}$ (so β and γ are ordered bases of V and W, respectively.) Compute $[T]_\beta^\gamma$.

$$T(6X^2 - 2X) = \int_1^X (6t^2 - 2t)\,dt = 2t^3 - t^2 \Big|_1^X = 2X^3 - X^2 - 1$$

$$= \underline{(-2)}(1) + \underline{0}(X+1) + \underline{(-1)}(X^2+1) + \underline{2}(X^3+1)$$

So $\left[T(6X^2 - 2X)\right]_\gamma = \begin{pmatrix} -2 \\ 0 \\ -1 \\ 2 \end{pmatrix}$

$$T(4X+1) = \int_1^X (4t+1)dt = 2t^2 + t \Big|_1^X = 2X^2 + X - 3$$

$$= \underline{-6}(1) + \underline{1}(X+1) + \underline{2}(X^2+1) + \underline{0}(X^3+1)$$

So $\left[T(4X+1)\right]_\gamma = \begin{pmatrix} -6 \\ 1 \\ 2 \\ 0 \end{pmatrix}$

$$T(2X-3) = \int_1^X (2t-3)dt = t^2 - 3t \Big|_1^X = X^2 - 3X + 2$$

$$= \underline{4}(1) + \underline{-3}(X+1) + \underline{1}(X^2+1) + \underline{0}(X^3+1)$$

So $\left[T(2X-3)\right]_\gamma = \begin{pmatrix} 4 \\ -3 \\ 1 \\ 0 \end{pmatrix}$

Therefore $[T]_\beta^\gamma = \begin{pmatrix} -2 & -6 & 4 \\ 0 & 1 & -3 \\ -1 & 2 & 1 \\ 2 & 0 & 0 \end{pmatrix}$