UCLA Mathematics 115A: Homework #1 Solutions

Summer Session C, 2010

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Problems from the supplementary section:

Problem 2.3:

The sentence is equivalent to the statement:

$$(\forall a \in \mathbb{R})P(a,x) \Rightarrow Q(x). \tag{1}$$

However if $a = 0$ then $ax = 0$ for all $x \in \mathbb{R}$; therefore the statement $Q(x)$ may be false when $P(a,x)$ is true!

We can chance the statement (1) to:

$$(\exists a \in \mathbb{R})P(a,x) \Rightarrow Q(x). \tag{2}$$

This is true since whenever $a \neq 0$, $ax = 0$ implies $x = 0$, since in this case a has a multiplicative inverse in $\mathbb{R}$.

Problem 2.4:

I've translated all the statements in the problem into mathematical notation:

a) $(\forall x \in A)(\exists b \in B)(b > x)$.

b) $(\exists x \in A)(\forall b \in B)(b > x)$.

c) $(\forall x, y \in \mathbb{R})(f(x) = f(y)) \Rightarrow (x = y)$. This is equivalent to the statement:
$(\forall x, y \in \mathbb{R})[\neg(f(x) = f(y))] \vee (x = y)$, since $(A \Rightarrow B) \Leftrightarrow (\neg A \vee B)$.

d) $(\forall b \in \mathbb{R})(\exists x \in \mathbb{R})(f(x) = b)$.

e) $(\forall x, y \in \mathbb{R})(\forall \epsilon \in P)(\exists \delta \in P)(|x - y| < \delta) \Rightarrow (|f(x) - f(y)| < \epsilon)$.
This is equivalent to $(\forall x, y \in \mathbb{R})(\forall \epsilon \in P)(\exists \delta \in P)[\neg(|x - y| < \delta)] \vee (|f(x) - f(y)| < \epsilon)$ for the same reason as in part c).

f) $(\forall \epsilon \in P)(\exists \delta \in P)(\forall x, y \in \mathbb{R})(|x - y| < \delta) \Rightarrow (|f(x) - f(y)| < \epsilon)$.
Again, this is equivalent to $(\forall \epsilon \in P)(\exists \delta \in P)(\forall x, y \in \mathbb{R})[\neg(|x - y| < \delta)] \vee (|f(x) - f(y)| < \epsilon)$.

Now you can simply negate the statements and translate into English:

a) $\neg[(\forall x \in A)(\exists b \in B)(b > x)] \Leftrightarrow [(\exists x \in A)(\forall b \in B)(b \leq x)]$. In English:
"There exists an x in A so that for all b in B, b is less than or equal to x."

b) Similarly: "For all x in A there exists b in B such that b is less than or equal to x. "

c) " There exists x, y in $\mathbb{R}$ such that $f(x) = f(y)$ and $x \neq y$."

d) "There exists a b in $\mathbb{R}$ such that for all x in $\mathbb{R}$ $f(x) \neq b$."

e) "There exist $x, y \in \mathbb{R}$ such that there exists an $\epsilon \in P$ so that for all $\delta \in P$, $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$"

f) "There exists an $\epsilon \in P$ such that for all $\delta \in P$ there exist $x, y \in \mathbb{R}$ such that $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$"

Problem 2.9:

Let S denote the collection of students in the school. We want to negate the sentence: "No slow learners attend this school." This is equivalent to the statement: "x is a slow learner $\Rightarrow$ x does not attend this school." This is equivalent to the sentence: "x is not a slow learner or x does not attend this school." We can negate this to get: "x is a slow learner and x attends this school." This is simply saying that some slow learner attends this school. Therefore the answer must be c)!

Problem 2.21:

We want to get rid of all the instances of the "words of negation" in the sentence: "It is false that for every integer $n > 0$ there is some real number $x > 0$ such that $x < 1/n$." This is just the sentence "There is some integer $n > 0$ such that for every real number $x > 0$, $x \geq 1/n$." This statement is clearly false, since there are real numbers which are arbitrarily close to zero; for any n in the above statement let x=1/(n+1) to make the statement false.

Problem 2.22:

We want to get rid of all the instances of the "words of negation" in the sentence: "f is not an increasing function." This is just: "There exist $x, y \in \mathbb{R}$ such that $x < y$ and $f(x) \geq f(y)$."

Problems from Section 1.2

Problem 1.2.7:

First I show that $f = g$ on the set $S = \{0, 1\}$:

1. **t=0:** $f(0) = 2 \cdot 0 + 1 = 1 = 1 + 4 \cdot 0 - s \cdot 0^2 = g(0)$
2. **t=1:** $f(1) = 2 \cdot 1 + 1 = 3 = 1 + 4 \cdot 1 - s \cdot 1^2 = g(1)$

Again, all we have to do is check that the function h equals $f + g$ on the set S.

1. **t=0:** $f(0) + g(0) = (2 \cdot 0 + 1) + (1 + 4 \cdot 0 - s \cdot 0^2) = 2 = 5^0 + 1 = h(0)$
2. **t=1:** $f(1) + g(1) = (2 \cdot 1 + 1) + (1 + 4 \cdot 1 - s \cdot 1^2) = 6 = 5^1 + 1 = h(1)$

We've verified both statements, so we're done.

Problem 1.2.8:

Claim: In any vector space V over the scalar field $\mathbb{F}$ the equation $(a+b)(\mathbf{x}+\mathbf{y}) = a\mathbf{x} + a\mathbf{y} + b\mathbf{x} + b\mathbf{y}$ is true for all $\mathbf{x}, \mathbf{y} \in V$ and $a, b \in \mathbb{F}$.

Proof: We will have to use the appropriate vector space axioms ((VS 1),..., (VS 8)) along with closure under both '+' and '·'.

$$\begin{aligned} (a+b)(\mathbf{x}+\mathbf{y}) &= a(\mathbf{x}+\mathbf{y}) + b(\mathbf{x}+\mathbf{y}) && \text{by VS 8} \\ &= a\mathbf{x} + a\mathbf{y} + b\mathbf{x} + b\mathbf{y} && \text{by VS 7} \end{aligned}$$

□

Problem 1.2.8:

Corollary 1: The zero vector is unique.

Proof: Suppose that $\mathbf{0}'$ is a vector for which $\mathbf{v} + \mathbf{0}' = \mathbf{v}$ for all $\mathbf{v} \in V$. Then consider the equation

$$\begin{aligned} \mathbf{0}' + \mathbf{0} =& \mathbf{0} \\ =& \mathbf{0}' \end{aligned}$$

Thus $\mathbf{0} = \mathbf{0}'$ so the zero vector is unique.
□

Corollary 2: The vector $\mathbf{y}$ satisfying (VS 4) is unique.

Proof: Suppose we have that $\mathbf{x} + \mathbf{y} = \mathbf{0}$, and also that $\mathbf{x} + \mathbf{y}' = \mathbf{0}$. Then $\mathbf{x} + \mathbf{y} = \mathbf{x} + \mathbf{y}'$. Via the *Cancellation Law for Addition* we have $\mathbf{y} = \mathbf{y}'$.
□

Problem 1.2.11:

Claim: The set $V = \{\mathbf{0}\}$ with addition and multiplication as defined in the problem is a vector space.

Proof: We need to check closure under addition and scalar multiplication as well as (VS 1) through (VS 8). The closure is automatic by the hypothesis of the problem. Since there's only one element, any equation must reduce to the form $\mathbf{0} = \mathbf{0}$. Thus the vector space axioms will automatically be satisfied.
□

Problem 1.2.14:

Even more is true than the problem would have you know: If V is any vector space over the complex numbers, $\mathbb{C}$, then it is also a vector space over $\mathbb{R}$. Further more, if V is a vector space over any field $\mathbb{F}$ and $\mathbb{K}$ is a subfield of $\mathbb{F}$ then V is also a vector space over $\mathbb{K}$. First since $\mathbb{K}$ is a subfield, closure under addition and scalar multiplication is trivially satisfied. Furthermore, the vector space axioms are automatically satisfied since $\mathbb{K}$ is a subfield.

Problem 1.2.15:

Unlike the previous problem, the n-tuples of real numbers (which form a $\mathbb{R}$-vector space) are not a $\mathbb{C}$-vector space since for $(a_1, \ldots, a_n) \in \mathbb{R}^n$ we have that $\sqrt{-1} \cdot (a_1, \ldots, a_n) = (\sqrt{-1}a_1, \ldots, \sqrt{-1}a_n)$ which isn't in $\mathbb{R}^n$.

Problem 1.2.21:

Just use the definition to check (VS 1) through (VS 8).

Problems from section 1.3:

Problem 1.3.10

For the first part of this problem notice that if $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ both satisfy the equation $x_1 + \ldots + x_n = 0$ then $\mathbf{a} + \mathbf{b}$ does as well. Similarly for any scalar $c \in \mathbb{F}$ we have that $c \cdot \mathbf{a}$ satisfies the equation. Finally, the zero vector $(0, \ldots 0)$ satisfies the equation. These are the things you have to check to show a subset is a subspace via theorem 1.3.

However if we look at solutions to the equation $x_1 + \ldots + x_n = 1$ we see that the zero vector is not a solution. Thus if $\mathbf{a} = (a_1, \ldots, a_n)$ solves this second equation, we have that $0 \cdot \mathbf{a} = \mathbf{0}$ doesn't, so the second subset can't be a subspace by definition.

Problem 1.3.11

The answer is no. The reason is that two polynomials of degree n may add together, canceling the degree n term. If any of the lower degree terms of the sum are not zero, then closure under addition fails.

Problem 1.3.13

The zero function is in our collection of functions. The sum of two functions which are identically zero on S are still identically zero. Any multiple of a function which is zero on S is still zero on S. Thus by theorem 1.3, our collection of functions is a subspace.

Problem 1.3.17

Suppose that W is a subspace of V. Then $\mathbf{0} \in W$ so $W \neq \varnothing$. Furthermore, since W is a subspace it is closed under addition and scalar mult. Thus for all $\mathbf{x}, \mathbf{y} \in W$, $\mathbf{x} + \mathbf{y} \in W$ and also for all $c \in \mathbb{F}$, $c \cdot \mathbf{x} \in W$.

Conversely, Suppose that $W \neq \varnothing$ and that it is closed under addition and scalar multiplication. Since W is nonempty there is some $\mathbf{x} \in W$. Then $0 \cdot \mathbf{x}$ must be the zero vector.

To see this let's form the sum $0 \cdot \mathbf{x} + \mathbf{x} = (0 + 1) \cdot \mathbf{x} = 1 \cdot \mathbf{x} = \mathbf{x}$. Now we look at the sums of the form $0 \cdot \mathbf{x} + \mathbf{y} = 0 \cdot \mathbf{x} + \mathbf{y} + \mathbf{x} + (-1)\mathbf{x}$. Re-arranging terms we can use the first identity to show that this is just $\mathbf{y} + \mathbf{x} + (-1)\mathbf{x} = \mathbf{y}$ Since $\mathbf{y}$ was arbitrary, $0 \cdot \mathbf{x}$ must be the zero vector.

Since W satisfies the hypotheses of theorem 1.3 it must be a subspace.

Problem 1.3.22

First note that the zero function is both even and odd. Next note that any scalar multiple of an even function is even and any scalar multiple of an odd function is still odd. Finally note that the sum of two even functions is even and the sum of two odd functions is odd. Then these two sets satisfy theorem 1.3 respectively and are hence subspaces.

Problem 1.3.23

For part (a) note that sums of the form $x + 0$ where $x \in W_1$ are in $W_1 + W_2$ as are sums of the form $0 + y$ for $y \in W_2$. Thus W_1, W_2 are subspaces of $W_1 + W_2$.

For part (b) if U is a subspace that contains both W_1 and W_2 then by its closure property it contains all sums of the form $x + y$ where $x \in W_1$ and $y \in W_2$. The collection of such elements is just $W_1 + W_2$. Thus $W_1 + W_2 \subset U$.

Problem 1.3.24

Clearly any element of $\mathbb{F}^n$ can be written as a sum of an element in $x \in W_1$ and $y \in W_2$ Now checking any $(a_1, \ldots, a_n) \in W_1 \cap W_2$ we see that $a_n = 0$ and $a_2 = \ldots = a_{n-1} = 0$ Thus $W_1 \cup W_2 = \{0\}$. Hence $\mathbb{F}^n = W_1 \oplus W_2$.

Problem 1.3.25

Clearly any polynomial can be written as a sum of even powers and a sum of odd powers. Thus we have that $\mathbf{P}(\mathbb{F}) = W_1 + W_2$. Similarly any polynomial that has only zero coefficients for the even power terms and zero coefficients for the odd power terms must be the zero polynomial. Thus $W_1 \cap W_2 = \{0\}$. Hence, $\mathbf{P}(\mathbb{F}) = W_1 \oplus W_2$.