THE EISENSTEIN IDEAL WITH SQUAREFREE LEVEL

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ABSTRACT. We use pseudodeformation theory to study the analogue of Mazur’s Eisenstein ideal with certain squarefree levels. Given a prime number $p > 3$ and a squarefree number $N$ satisfying certain conditions, we study the Eisenstein part of the $p$-adic Hecke algebra for $\Gamma_0(N)$, and show that it is a local complete intersection and isomorphic to a pseudodeformation ring. We also show that in certain cases, the Eisenstein ideal is not principal and that the cuspidal quotient of the Hecke algebra is not Gorenstein. As a corollary, we prove that “multiplicity one” fails for the modular Jacobian in these cases. In a particular case, this proves a conjecture of Ribet.

Contents

1. Introduction 1
2. Modular forms 11
3. The pseudodeformation ring 15
4. Toward $R = \mathbb{T}$ 24
5. The case $\epsilon = (-1, 1, 1, \ldots, 1)$ 27
6. The case $\epsilon = (-1, -1)$ 29
7. Generators of the Eisenstein ideal 31
Appendix A. Comparison with the Hecke algebra containing $U_{12}$ 38
Appendix B. Computation of some cup products 41
Appendix C. Algebra 42
References 43

1. Introduction

1.1. Mazur’s results and their squarefree analogues. Let $\ell$ be a prime, and let $T_\ell$ be the $p$-adic Eisenstein completion of the Hecke algebra acting on modular forms of weight 2 and level $\ell$, and let $T_\ell \to T_0^\ell$ be the cuspidal quotient. Let $I_0^\ell \subset T_0^\ell$ be the Eisenstein ideal, and let $m_0^\ell = (p, I_0^\ell)$ be the maximal ideal. Mazur proved the following results:

1. $T_0^\ell/I_0^\ell \cong \mathbb{Z}_p/(\ell-1)\mathbb{Z}_p$,
2. $I_0^\ell$ is principal,
3. $T_0^\ell$ is Gorenstein,
4. $\dim_{\mathbb{F}_p}(J_0(N)\mid m_0^\ell)) = 2$, and
5. if $q \neq \ell$ is a prime such that $q \not\equiv 1 \pmod{p}$ and such that $q$ is not a $p$-th power modulo $\ell$, then $T_q - (q + 1)$ generates $I_0^\ell$.

Date: April 17, 2018.
A prime $q$ as in (5) a good prime for $(\ell, p)$. We note that, of course, (5) implies (2) implies (3). We also note that (2) implies that $\mathbb{T}_\ell$ is Gorenstein also.

The analogue of (1) has been proven for squarefree levels by Ohta [Oht14]. However, as has been noted by many authors, notably Ribet and Yoo [Rib15, Yoo15], the statements (2)-(5) are not true in the squarefree setting. Still, in this paper, we prove, in certain cases, analogues of (2)-(5). Namely, we count the minimal number of generators of the Eisenstein ideal, count the dimension of the Eisenstein kernel of the Jacobian, and give sufficient (and sometimes also necessary) conditions for a list of elements $T_q - (q+1)$ to generate the Eisenstein ideal.

1.2. Pseudomodularity. Our main technical result is an $R = \mathbb{T}$ theorem, where $R$ is a deformation ring for Galois pseudorepresentations and $\mathbb{T}$ is the Eisenstein part of the Hecke algebra. The strategy is similar to that of our previous works [WWE17c, WWE17a], where we gave new proofs and refinements of Mazur’s results. However, there are several points of interest that are new in this setting.

(a) In the case of prime level $\ell$, Calegari and Emerton [CE05] have already applied deformation theory to study Mazur’s Eisenstein ideal. Their method is to rigidify the deformation theory of Galois representations using auxiliary data coming from the prime level $\ell$. In the case of composite level, it is clear that a similar strategy will not work: the deformation problem at prime level is already rigid, and cannot be further rigidified to account for the additional primes.

(b) In the case of squarefree level, there are multiple Eisenstein series, and one has to account for the possibility of congruences among them.

(c) At squarefree level, unlike prime level, the Tate module of the Jacobian may not be free over the Hecke algebra. Since this Tate module is the natural way to construct Galois representations, it is really necessary to work with pseudorepresentations.

(d) We prove $R = \mathbb{T}$ even in some cases where the Galois cohomology groups controlling the tangent space of $R$ are all non-cyclic (see Remark 1.4.8). In this case, the universal pseudodeformation cannot arise from a representation. To address issue (a), we have to develop a theory of Cayley-Hamilton representations and pseudorepresentations with squarefree level, which has the required flexibility; for this, we drew inspiration from our previous joint works [WWE15, WWE17a] and the work of Calegari-Specter [CS16]. The ideas are discussed later in this introduction in §1.8. To address issue (b), we make extensive use of an idea of Ohta [Oht14]: we use the Atkin-Lenher involutions at $\ell | N$ to define $\mathbb{T}$, rather than the usual Hecke operators $U_\ell$.

1.3. Setup. We introduce notation in order to state our main results precisely.

1.3.1. Eisenstein series and Hecke algebras. The Eisenstein series of weight two and level $N$ have a basis $\{E^\epsilon_{2,N}\}$, labeled by elements $\epsilon = (\epsilon_0, \ldots, \epsilon_r)$ in the set $\mathcal{E} = \{\pm 1\}^{r+1} \backslash \{(1, 1, \ldots, 1)\}$. The $E^\epsilon_{2,N}$ are characterized in terms of Hecke eigenvalues by the properties that

1. $T_n E^\epsilon_{2,N} = \left( \sum_{t \mid (n, \ell)} t \right) E^\epsilon_{2,N}$ for all $n$ with $\gcd(n, N) = 1$, and

2. $w_{\ell_1} E^\epsilon_{2,N} = \epsilon_{1} E^\epsilon_{2,N}$ for the Atkin-Lehner involutions $w_{\ell_0}, \ldots, w_{\ell_r}$.
together with the normalization \( a_1(E_{2,N}^\ell) = 1 \). The constant coefficients satisfy

\[
a_0(E_{2,N}^\ell) = -\frac{1}{24} \prod_{i=0}^r (\epsilon_i \ell_i + 1).
\]

Based on the philosophy that congruences between Eisenstein series and cusp forms should happen when the constant term is divisible by \( p \), we expect the most interesting congruences to occur when \( \ell_i \equiv -\epsilon_i \pmod{p} \) for many \( i \).

Consider the Hecke algebra of weight 2 and level \( N \) generated by all \( T_n \) with gcd\((n,N) = 1 \) and by all Atkin-Lehner involutions \( w_{\ell_0}, \ldots, w_{\ell_r} \). Let \( T_{N,\infty} \) denote the completion of this algebra at the maximal ideal generated by \( p \) together with the annihilator of \( E_{2,N}^\ell \).

Let \( I' \) denote the annihilator of \( E_{2,N}^\ell \) in \( T_N \), so \( T_N / I' = \mathbb{Z}_p \), and let \( m^\ell = (I',p) \) be the maximal ideal of \( T_N \). For a Hecke module \( M \), let \( M_{m^\ell} \) denote the tensor product of \( M \) with \( T_N \) over the Hecke algebra. Let \( T_{N,0}^\ell \) denote the cuspidal quotient of \( T_N \), and let \( I'^0 \) be the image of \( I^\ell \) in \( T_{N,0}^\ell \).

1.3.2. Another Hecke algebra. In contrast with our approach, one often studies a different Hecke algebra \( T_{N,U} \), containing the operators \( U_i \) instead of \( w_i \), and with Eisenstein ideal \( I_{N,U}^\ell \) generated by \( T_q - (q + 1) \) for \( q \not\mid N \) and \( U_{\ell_i} - \ell_i^{\epsilon_{i+1}/2} \) for \( i = 0, \ldots, r \). We prove that \( T_{N,U} = T_N \) in some of the cases that we consider — see Appendix A. Our main results together with Appendix A can be used to prove results about \( T_{N,U} \) that are closely related to the results of authors including Ribet [Rib10, Rib15], Yoo ([Yoo15, Yoo17b, Yoo17a] and others) and Hsu [Hsu18]. In general, when \( T_N \neq T_{N,U} \), we believe that \( T_N \) is more natural and better behaved, so we mostly consider \( T_N \).

1.3.3. The field extensions \( K_i \). Let \( \ell_0 \) be a prime such that \( \ell \equiv \pm 1 \pmod{p} \). Then there is a unique degree \( p \) Galois extension \( K_\ell / \mathbb{Q}(\zeta_p) \) such that

1. \( \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \) acts on \( \text{Gal}(K_\ell/\mathbb{Q}(\zeta_p)) \) via the character \( \omega^{-1} \),
2. the prime \( 1 - \zeta_p \) of \( \mathbb{Q}(\zeta_p) \) splits completely in \( K_\ell \), and
3. only the primes above \( \ell \) ramify in \( K_\ell/\mathbb{Q}(\zeta_p) \).

For each \( i \) such that \( \ell_i \equiv \pm 1 \pmod{p} \), let \( K_i = K_{\ell_i} \) (see also Definition 3.10.4).

1.4. Structure of the Hecke algebra. Our main results concern the structure of the Hecke algebra \( T_N \).

**Theorem 1.4.1.** Assume that \( \epsilon = (-1,1,\ldots,1) \). Let

\[
S = \{ i \in \{1, \ldots, r\} \mid \ell_i \equiv -1 \pmod{p}\}
\]

and let \( s = \#S \). Then

1. \( T_N \) is a complete intersection ring.
2. \( T_{N,0}^\ell \) is Gorenstein if and only if \( I^\ell \) is principal.
3. There is a short exact sequence

\[
\begin{align*}
0 & \to \bigoplus_{i=1}^r \mathbb{Z}_p / (\ell_i + 1) \mathbb{Z}_p \to I^\ell / \mathbb{Z}_p^2 \to \mathbb{Z}_p / (\ell_0 + 1) \mathbb{Z}_p \to 0.
\end{align*}
\]

4. The minimal number of generators of \( I^\ell \) is \( s + \delta \) where

\[
\delta = \begin{cases} 
1 & \text{if } \ell_0 \text{ splits completely in } K_i \text{ for all } i \in S, \text{ or } \\
0 & \text{otherwise}.
\end{cases}
\]
Proof. Parts (1) and (3) are proved in §5 (see especially Theorem 5.2.6). It is known to experts that Part (2) follows from (1) (see Lemma 2.4.2). Part (4) is Theorem 7.1.1. □

Remark 1.4.3. In fact, we show that, unless \( s = r \), there are no newforms in \( M_2(N)_{m^*} \), so we can easily reduce to the case \( s = r \) (i.e. the case that \( \ell_i \equiv -1 \pmod{p} \) for all \( i > 0 \)). When \( s = r \), one could use this theorem to prove that there are newforms in \( M_2(N)_{m^*} \), but this is known (see [Rib15, Yoo17b, Thm. 1.3(3)]).

Remark 1.4.4. The criterion of Part (4) determines whether or not the extension class defined by the sequence (1.4.2) is \( p \)-cotorsion. In fact, one can describe this extension class exactly in terms of algebraic number theory, but we content ourselves with the simpler statement (4).

Theorem 1.4.5. Assume \( r = 1 \) and \( \epsilon = (-1, -1) \) and that \( \ell_0 \equiv 1 \pmod{p} \) but \( \ell_1 \not\equiv 1 \pmod{p} \). If \( \ell_1 \) is not a \( p \)-th power modulo \( \ell_0 \), then there are no newforms in \( M_2(N)_{m^*} \). In particular, \( \mathfrak{I}^* \) is principal, and generated by \( T_q - (q + 1) \) where \( q \) is a good prime (of Mazur) for \( (\ell_0, p) \).

Proof. This is Theorem 6.3.1. □

Remark 1.4.6. In the case \( \ell_1 \not\equiv p \), this is a theorem of Ribet [Rib10] and Yoo [Yoo17b, Thm. 2.3]. Yoo has informed us that the method should work for the case \( \ell_1 = p \) as well. In any case, our method is completely different.

Theorem 1.4.7. Assume \( r = 1 \) and \( \epsilon = (-1, -1) \) and that \( \ell_0 \equiv \ell_1 \equiv 1 \pmod{p} \). Assume further that

\[
\ell_i \text{ is not a } p\text{-th power modulo } \ell_j \text{ for } (i, j) \in \{(0, 1), (1, 0)\}.
\]

Then

(1) there are newforms in \( M_2(N)_{m^*} \).
(2) \( T_N^{r^*} \) is a complete intersection ring.
(3) \( T_N^{r^*} \) is not a Gorenstein ring.
(4) \( T_{\epsilon,N}/(\ell_0 - 1)\mathbb{Z}_p \oplus (\ell_1 - 1)\mathbb{Z}_p \).

Proof. Parts (2) and (4) are proven in Theorem 6.4.1. Part (1) follows from part (2) by Theorem 6.4.3. Part (3) follows from (2) and (4) by Lemma 2.4.2. □

Remark 1.4.8. The proof of this theorem may be of particular interest for experts in the deformation theory of Galois representations. The proof is the first (as far as we are aware) example of an \( R = \mathbb{T} \) theorem, where \( R \) is a pseudodeformation ring, and where we do not rely on certain Galois cohomology groups being cyclic. In fact, with the assumptions of the theorem, the relevant cohomology groups are not cyclic. (The cyclicity of these cohomology groups ensures that the pseudorepresentations come from true representations.)

Remark 1.4.9. Outside of the cases considered in these theorems, we cannot expect that \( T_N^{r^*} \) is a complete intersection ring, as the examples in §1.9 below illustrate. Our method, which applies Wiles’s numerical criterion [Wil95], proves that \( T_N^{r^*} \) is a complete intersection ring as a byproduct. A new idea is needed to proceed beyond these cases.
1.5. Applications to multiplicity one. For an application of the main result, we let \( J_0(N) \) be the Jacobian of the modular curve \( X_0(N) \).

**Corollary 1.5.1.** In the following cases, we can compute \( \dim_{\mathbb{F}_p} J_0(N)(\overline{\mathbb{Q}}_p)[m'] \):

1. With the assumptions of Theorem 1.4.1, we have
   \[ \dim_{\mathbb{F}_p} J_0(N)(\overline{\mathbb{Q}}_p)[m'] = 1 + s + \delta, \]
   where \( s \) and \( \delta \) are as in Theorem 1.4.1.

2. With the assumptions of Theorem 1.4.5, we have \( \dim_{\mathbb{F}_p} J_0(N)(\overline{\mathbb{Q}}_p)[m'] = 2 \).

3. With the assumptions of Theorem 1.4.7, we have \( \dim_{\mathbb{F}_p} J_0(N)(\overline{\mathbb{Q}}_p)[m'] = 3 \).

**Proof.** This follows from the named theorems together with Lemma 2.4.3 (which is known to experts).

One says that “multiplicity one holds” if \( \dim_{\mathbb{F}_p} J_0(N)(\overline{\mathbb{Q}}_p)[m'] = 2 \). This corollary implies that multiplicity one holds in case (1) if and only if \( s + \delta = 1 \), always holds in case (2), and always fails in case (3).

1.5.1. Ribet’s Conjecture. Previous works on multiplicity one have used a different Hecke algebra \( T_{N,U} \), defined in §1.3.2 (see, for example, [Yoo15]). Let \( m_U = (I_U, p) \subseteq T_{N,U} \) be its maximal ideal. The previous corollary together with Proposition A.2.3 give the following

**Corollary 1.5.2** (Generalized Ribet’s Conjecture). With the assumptions of Theorem 1.4.1, assume in addition that \( \ell_i \neq 1 \) (mod \( p \)) for \( i > 0 \). Then
   \[ \dim_{\mathbb{F}_p} J_0(N)(\overline{\mathbb{Q}}_p)[m_U] = 1 + s + \delta, \]
   where \( s \) and \( \delta \) are as in Theorem 1.4.1.

The case \( r = s = 1 \) of this corollary was conjectured by Ribet [Rib15] (see also [Yoo17a, pg. 4]).

**Remark 1.5.3.** After we told Yoo about the results of this paper, he found an alternate proof of this corollary in the case \( s = r = 1 \), under the assumption that \( I_U \) is principal if and only if \( T_{N,U}^{0,\epsilon} \) is Gorenstein (this assumption follows from Theorem 1.4.1 and Proposition A.2.3). Yoo’s proof involves a delicate study of the geometry of \( J_0(N) \) and, unlike our proof, does not make use of the fact that \( T_N \) is Gorenstein. The fact that our proof is simpler demonstrates the power of the Gorenstein property and is a reason for our interest in using \( T_N \) rather than \( T_{N,U}^{0,\epsilon} \).

1.5.2. Gorensteinness, and multiplicity one for the generalized Jacobian. The following observations are not used (nor proven) in this paper (although they are familiar to experts), but we include them to illustrate the arithmetic significance of the Gorenstein property for \( T_N \) proved in Theorems 1.4.1, 1.4.5, and 1.4.7.

We learned this point of view from papers of Ohta, especially [Oht05].

As is well-known, and as we explain in §2.4, multiplicity one holds if and only if \( T_N^{0,\epsilon} \) is Gorenstein. The nomenclature “multiplicity one” comes from representation theory. It is related to the question of whether \( H^1_{\text{et}}(X_0(N)\overline{\mathbb{Q}},\mathbb{Z}_p(1))_{m'} \) is a free \( T_N^{0,\epsilon} \)-lattice in the free \( T_N^{0,\epsilon}[1/p] \)-module \( H^1_{\text{et}}(X_0(N)\overline{\mathbb{Q}},\mathbb{Q}_p(1))_{m'} \).

There is another natural lattice to consider, namely \( H^1_{\text{et}}(Y_0(N)\overline{\mathbb{Q}},\mathbb{Z}_p(1))_{m',\text{DM}} \), the image of \( H^1_{\text{et}}(Y_0(N)\overline{\mathbb{Q}},\mathbb{Z}_p(1))_{m'} \) under the Drinfeld-Manin splitting
   \[ H^1_{\text{et}}(Y_0(N)\overline{\mathbb{Q}},\mathbb{Q}_p(1))_{m'} \rightarrow H^1_{\text{et}}(X_0(N)\overline{\mathbb{Q}},\mathbb{Q}_p(1))_{m'}. \]
In a similar manner to the proof of Lemma 2.4.1, one can show that $T_N'$ is Gorenstein if and only if $H^1_{\text{ét}}(Y_0(N)_{\mathcal{O}_p}, \mathbb{Z}_p(1))_m$ is a free $T_N'^e$-module, if and only if
\[ \dim_{\mathbb{F}_p} GJ_0(N)(\mathcal{O}_p)[m'] = 2, \]
where $GJ_0(N)$ is the generalized Jacobian of $J_0(N)$ relative to the cusps (see e.g. [Oht99 §3] for a discussion of generalized Jacobians). Hence our result that $T_N'$ is Gorenstein can be thought of as a multiplicity one result for $GJ_0(N)$.

Finally, we note that these ideas illustrate why the failure of multiplicity one in Corollary 1.5.1 is related to the failure of $I^e$ to be principal: if $T_N'$ is Gorenstein,
\[ H^1_{\text{ét}}(X_0(N)_{\mathcal{O}_p}, \mathbb{Z}_p(1))_m \hookrightarrow H^1_{\text{ét}}(Y_0(N)_{\mathcal{O}_p}, \mathbb{Z}_p(1))_m, \]
has the form, as $T_N'^e$-modules,
\[ T_N'^e \oplus I^e \hookrightarrow T_N'^e \oplus T_N', \]
and hence $H^1_{\text{ét}}(X_0(N)_{\mathcal{O}_p}, \mathbb{Z}_p(1))_m$ is free if and only if $I^e$ is principal.

1.6. **Good primes.** We also prove analogues of Mazur’s good prime criterion (statement (5) of §1.1). In the situation of Theorem 1.4.1, the list of conditions is cumbersome to write down, so we are not precise here. We refer the reader to §1.9.1 for some specific examples and §7.2 for the complete criterion.

**Theorem 1.6.1.** With the assumptions of Theorem 1.4.1, we can specify sufficient conditions on a set of primes $q_1, \ldots, q_{s+\delta}$ not dividing $N$ such that the elements $T_{q_1} - (q_1 + 1), \ldots, T_{q_s+\delta} - (q_{s+\delta} + 1)$ together generate $I^e$.

**Remark 1.6.2.** We can also write down a necessary and sufficient condition, but cannot compute with it, so we doubt its practical use.

In the situation of Theorem 1.4.7, the sufficient condition is very simple to state, and also necessary. To state it, we let
\[ \log_\ell : (\mathbb{Z}/\ell\mathbb{Z})^\times \to \mathbb{F}_p \]
denote an arbitrary surjective homomorphism, for any prime $\ell$ that is congruent to 1 modulo $p$ (the statement below will not depend on the choice).

**Theorem 1.6.3.** With the assumptions of Theorem 1.4.7 fix primes $q_0, q_1$ not dividing $N$ (but possibly dividing $p$). Then the elements $T_{q_0} - (q_0 + 1)$ and $T_{q_1} - (q_1 + 1)$ together generate $I^e$ if and only if
\[ (q_0 - 1)(q_1 - 1) \det \begin{pmatrix} \log_\ell(q_0) & \log_\ell(q_1) \\ \log_\ell(q_0) & \log_\ell(q_1) \end{pmatrix} \in \mathbb{F}_p^\times. \]

**Remark 1.6.4.** For a single prime $\ell$, Mazur’s criterion for $q$ to be a good prime can be written as $(q - 1) \log_\ell(q) \in \mathbb{F}_p$, so this is a natural generalization.

1.7. **Relation to Hida Hecke algebras.** The reader will note that we have allowed for the possibility that $p \mid N$. When $p \mid N$, in Appendix A we also consider a related Hecke algebra $T'_{N,H}$ that contains $U_p$ instead of $w(p)$ (but still has all other $w_{\ell}$ for $\ell \nmid N$) and show that, in many cases we consider, $T'_{N,H} = T_N$.

This is related to Hida theory because (as is well-known for the Hecke algebra $T'_{N,U}$) there is a Hida-theoretic Hecke algebra $T'_{N}$ that is a free module of finite rank over $A \simeq \mathbb{Z}_p[T]$ that satisfies a control theorem with respect to $T'_{N,H}$: there
is an element $\omega_2 \in \Lambda$ such that $T_{N,H}^\Lambda = T_\Lambda^\Lambda/\omega_2 T_\Lambda^\Lambda$. (A proof of this control theorem will appear in forthcoming work of the first-named author with Rob Pollack.)

Then our results about $T_N^\Lambda$ (including its Gorensteinness and the number of generators of its Eisenstein ideal) translate directly to $T_\Lambda^\Lambda$. Subsequently, these results can be specialized to higher weights, as is usual in Hida theory.

1.8. **Method of pseudodeformation theory.** Like our previous work [WWE17c], the method of proof of the theorems in §1.4 is to construct a pseudodeformation ring $R$ and prove that $R = \mathbb{T}$ using the numerical criterion. The ring $R$ is the deformation ring of the residual pseudorepresentation $\tilde{D} = \psi(\omega \oplus 1)$ associated to $E_{2,N}$ that is universal subject to certain conditions (here $\psi$ is the functor associating a pseudorepresentation to a representation, and $\omega$ is the mod $p$ cyclotomic character). These conditions include the conditions considered in our previous works [WWE15, WWE17c] (having cyclotomic determinant, being flat at $p$, being ordinary at $p$), but they also include new conditions at $\ell$ dividing $N$ that are of a different flavor, as we now explain.

1.8.1. The Steinberg at $\ell$ condition. Fix $\ell = \ell_i \mid N$, assume $\ell \neq p$, and let $G_\ell \subset G_\mathbb{Q}$ be a decomposition group at $\ell$. Let $f$ be a normalized cuspidal eigenform of weight 2 and level $\Gamma_0(N)$ and $\rho_f : G_\mathbb{Q} \to \text{GL}_2(O_f)$ is the associated Galois representation, where $O_f$ is a finite extension of $\mathbb{Z}_p$.

If $f$ is old at $\ell$, then $\rho_f|_{G_\ell}$ is unramified. If $f$ is new at $\ell$, we have

\[
\rho_f|_{G_\ell} \sim \begin{pmatrix} \lambda(a_\ell(f))\kappa_{\text{cyc}} & * \\ 0 & \lambda(a_\ell(f)) \end{pmatrix}
\]

where $\lambda(x)$ is the unramified character of $G_\ell$ sending a Frobenius element $\sigma_\ell$ to $x$, and $a_\ell(f)$ is the coefficient of $q^i$ in the $q$-expansion of $f$ (see Lemma 2.3.1). Note that since $\det(\rho_f) = \kappa_{\text{cyc}}$, we have $\lambda(a_\ell(f))^2 = 1$. In fact, $a_\ell(f)$ is the negative of the $w_\ell$-eigenvalue of $f$. We call such representations [1.8.1] “±1-Steinberg at $\ell$”, where $\pm 1 = \mp a_\ell(f)$ is the $w_\ell$-eigenvalue of $f$.

Now assume in addition that $f \in S_2(N)_{\mathbb{A}}$, so that the semi-simplification of the residual representation of $\rho_f$ is $\omega \oplus 1$ and $w_\ell f = \epsilon f$, where $\epsilon = \epsilon_i$. We want to impose a condition on pseudorepresentations that encapsulates the condition that $\rho_f|_{G_\ell}$ is either unramified or $\epsilon$-Steinberg. The main observation is the following, and is inspired by the work of Calegari-Specter [CS16].

**Observation 1.8.2.** Suppose that $\rho : G_\ell \to \text{GL}_2(O)$ is either unramified or $\epsilon$-Steinberg. Then

\[
(\rho(\sigma) - \lambda(-\epsilon)\kappa_{\text{cyc}}(\sigma))(\rho(\tau) - \lambda(-\epsilon)(\tau)) = 0
\]

for all $\sigma, \tau \in G_\ell$ with at least one of $\sigma$ or $\tau$ in the inertia group $I_\ell$.

This is clear if $\rho$ is unramified: the factor involving the one of $\sigma$ or $\tau$ that is in $I_\ell$ will be zero. If $\rho$ is $\epsilon$-Steinberg, then the given product [1.8.3] will have the form

\[
\begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} \begin{pmatrix} * & * \\ * & 0 \end{pmatrix}
\]

and any such product is zero (note that the order is important!).

To impose the unramified-or-$\epsilon$-Steinberg condition on the pseudodeformation ring $R$, we impose the condition [1.8.3] on the universal Cayley-Hamilton algebra, using the theory of [WWE17a] (see [13].
The ordinary at $p$ condition. When $p \mid N$ and $\epsilon_p = -1$, for a newform $f$ of weight 2 and level $N$ the representation $\rho_f|_{G_p}$ is ordinary. In this paper, we define “ordinary pseudorepresentation” exactly as we define the unramified-or-$\epsilon$-Steinberg, following ideas of Calegari-Specter. In our previous paper [WWE15], we gave a different definition of ordinary, and we prove in this paper that the two definitions coincide (see Lemma 3.7.4). This gives an answer to a question of Calegari-Specter [CS16, pg. 2].

1.9. Examples. We conclude this introduction with examples that illustrate the theorems and show that the hypotheses are necessary. For examples where we show that $T^\epsilon_N$ is not Gorenstein, it is helpful to note that $T^\epsilon_N$ is Gorenstein if and only if $\text{Soc}(T^\epsilon_N/pT^\epsilon_N)$ is 1-dimensional, where $\text{Soc}(T^\epsilon_N/pT^\epsilon_N)$ is the annihilator of the maximal ideal (see §C.1).

All computations are using algorithms we have written for the Sage computer algebra software [S^18].

1.9.1. Examples illustrating Theorem 1.4.1

**Example 1.9.1.** Let $p = 5$, $\ell_0 = 41$, $\ell_1 = 19$, so $N = 19 \cdot 41$, and let $\epsilon = (-1, 1)$. In this case, we compute that $K_{19}$ is the field cut out by

$$x^{20} - x^{19} - 7x^{18} + 21x^{17} + 22x^{16} + 223x^{15} - 226x^{14} - 1587x^{13} + 4621x^{12} + 5202x^{11} - 3142x^9 - 439x^8 - 2143x^7 - 2156x^6 - 58x^5 + 1237x^4 + 414x^3 + 148x^2 + 56x + 16$$

and that 41 splits completely in $K_{19}$. The theorem says that $I^\epsilon$ has 2 generators. Moreover, Theorem 1.6.1 says, in this case, that $I^\epsilon$ is generated by $T_{q_0} - (q_0 + 1)$ and $T_{q_1} - (q_1 + 1)$ where $q_0$ is a good prime for $(41, 5)$ and where $q_1$ satisfies

(a) $q_1$ is a prime such that $q_1 \equiv 1 \pmod{5}$,
(b) 41 is not a 5-th power modulo $q_1$, and
(c) $q_1$ does not split completely in $K_{19}$.

A quick search yields that $q_0 = 2$ and $q_1 = 11$ satisfy these criteria. And indeed, we compute that there is an isomorphism

$$\mathbb{F}_p[x, y] / (y^2 - 2x^2, xy) \cong T^\epsilon_{N/pT^\epsilon_N}, \quad (x, y) \mapsto (T_2 - 3, T_{11} - 12).$$

**Example 1.9.2.** Let $p = 5$, $\ell_0 = 11$, $\ell_1 = 19$, $\ell_2 = 29$, so $N = 11 \cdot 19 \cdot 29$, and let $\epsilon = (-1, -1, 1)$. In this case, 11 does not split completely in either of the fields $K_{19}, K_{29}$, and the theorem says that $I^\epsilon$ has 2 generators. Moreover, Theorem 1.6.1 says, in this case, that $I^\epsilon$ is generated by $T_{q_0} - (q_0 + 1)$ and $T_{q_1} - (q_1 + 1)$ where $q_0$ is a good prime for $(11, 5)$ (for example $q_0 = 2$) and where the prime $q_1$ satisfies:

(a) $q_1 \equiv 1 \pmod{5}$,
(b) 11 is not a 5-th power modulo $q_1$,
(c) $q_1$ does not split completely in $K_{19}$, and
(d) $q_1$ does split completely in $K_{29}$.
In this case, $K_{19}$ is the field computed in the previous example and $K_{29}$ is the field cut out by

$$x^{20} - x^{19} - 11x^{18} + 9x^{17} + 124x^{16} - 223x^{15} - 1244x^{14} + 2111x^{13} + 14291x^{12} - 19804x^{11} + 7169x^{10} + 7938x^9 - 10937x^8 + 15603x^7 - 9472x^6 - 2582x^5 + 8233x^4 - 3732x^3 + 1808x^2 - 832x + 256.$$  

A quick search finds that $q_1 = 181$ satisfies the conditions (a)-(d). And indeed, we compute that there is an isomorphism

$$\mathbb{F}_p[x, y] / (x^3 + 2x^2, y^2) \sim \mathbb{T}/\mathbb{F}_p, \quad (x, y) \mapsto (T_2 - 3, T_{181} - 182).$$

Note that these conditions are far from necessary. For example $T_2 - 3$ and $T_7 - 8$ also generate the Eisenstein ideal.

1.9.2. Examples related to Theorem 1.4.2. We give examples illustrating that the assumption is necessary. In fact, it seems that the assumption is necessary even for the Gorensteinness of $\mathbb{T}/\mathbb{F}_p$.

**Example 1.9.3.** Let $p = 5$, $\ell_0 = 11$, $\ell_1 = 23$, so $N = 11 \cdot 23$, and let $\epsilon = (-1, -1)$. Then $\ell_1 \equiv 1 \pmod{11}$ is a 5-th power so the theorem does not apply. We can compute that

$$\mathbb{F}_p[x, y] / (x^2, x^2, y^2) \sim \mathbb{T}/\mathbb{F}_p, \quad (x, y) \mapsto (T_2 - 3, T_3 - 4)$$

has dimension 3. Since $\mathbb{T}/\mathbb{F}_p$ is not Gorenstein.

**Example 1.9.4.** Let $p = 5$, $\ell_0 = 31$, $\ell_1 = 5$, so $N = 5 \cdot 23$, and let $\epsilon = (-1, -1)$. Then note that $\ell_1 = 5 \equiv 7^3 \pmod{31}$, so the theorem does not apply. We can compute that

$$\mathbb{F}_p[x, y] / (x^3, x^2, y^2) \sim \mathbb{T}/\mathbb{F}_p, \quad (x, y) \mapsto (T_2 - 3, 2T_2 + T_3)$$

has dimension 4. Since $\text{rank}_{\mathbb{Z}_p}(\mathbb{T}/\mathbb{F}_p) = 2$, we see that the space of oldforms has dimension 3, and there must be a newform at level $N$. Moreover, $\text{Soc}(\mathbb{T}/\mathbb{F}_p) = x^2y + y^2p$, so $\mathbb{T}/\mathbb{F}_p$ is not Gorenstein.

In this last example, the reader may think that $\ell_0 = 31$ is special because the rank of $\mathbb{T}/\mathbb{F}_p$ is 2. However, we can take $p \ell_1 = 5$ and $\ell_0 = 191$ (note that $\mathbb{T}/\mathbb{F}_p$ is not Gorenstein in this case).

1.9.3. Examples related to Theorem 1.4.7. First, we give examples illustrating that the assumption is necessary. Again, it seems that the assumption is necessary even for the Gorenstein property of $\mathbb{T}/\mathbb{F}_p$.

**Example 1.9.5.** Let $p = 5$, $\ell_0 = 11$, $\ell_1 = 61$, so $N = 11 \cdot 61$, and let $\epsilon = (-1, -1)$. Then note that $11 \equiv 8^5 \pmod{61}$ so the theorem does not apply (but note that 61 is not a 5-th power modulo 11). We can compute that

$$\mathbb{F}_p[x, y] / (x^2, x^2, y^2) \sim \mathbb{T}/\mathbb{F}_p, \quad (x, y) \mapsto (T_3 - T_2 - 1, T_2 - 3).$$
We see that \( \text{Soc}(\mathcal{T}_N/p\mathcal{T}_N) = x\mathbb{F}_p \oplus y^2\mathbb{F}_p \), so \( \mathcal{T}_N \) is not Gorenstein.

**Example 1.9.6.** Let \( p = 5, \ell_0 = 31, \ell_1 = 191 \), so \( N = 31 \cdot 191 \), and let \( \epsilon = (-1, -1) \). We have \( 191 \equiv 7^5 \pmod{31} \) and \( 31 \equiv 61^5 \pmod{191} \), so the assumption of the theorem fails most spectacularly. We can compute that

\[
\mathbb{F}_p[x, y] / ((x, y)^4, 2x^3 + xy^2 + 3y^3, x^3 - x^2y + 2y^3) \cong \mathcal{T}_N/p\mathcal{T}_N,
\]

given by \((x, y) \mapsto (T_2 - 3, T_7 - 8)\).

Letting \( \mathfrak{m}^\epsilon \) denote the maximal ideal of \( \mathcal{T}_N/p\mathcal{T}_N \), we see that \((\mathfrak{m}^\epsilon)^4 = 0 \) but that \((\mathfrak{m}^\epsilon)^3 \) is 2-dimensional, so \( \dim(\text{Soc}(\mathcal{T}_N/p\mathcal{T}_N)) > 1 \) and \( \mathcal{T}_N \) is not Gorenstein.

Finally, we give an example illustrating Theorem 1.6.3.

**Example 1.9.7.** Let \( p = 5, \ell_0 = 11, \ell_1 = 41 \), so \( N = 11 \cdot 41 \), and let \( \epsilon = (-1, -1) \). We see that neither of 11 or 41 is a 5-th power modulo the other, so Theorem 1.6.3 applies. We consider the primes 2, 3, 7 and 13, none of which are congruent to 1 modulo 5.

<table>
<thead>
<tr>
<th>( q )</th>
<th>Is 5-th power modulo 11?</th>
<th>Is 5-th power modulo 41?</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>3</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>7</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>13</td>
<td>No</td>
<td>No</td>
</tr>
</tbody>
</table>

From this we see that

\[
\det \begin{pmatrix}
\log_{11}(3) & \log_{11}(q) \\
\log_{41}(3) & \log_{41}(q)
\end{pmatrix} = \log_{11}(3) \cdot \log_{41}(q) \neq 0,
\]

for any \( q \in \{2, 7, 13\} \). By Theorem 1.6.3, \( \{T_3 - 4, T_q - (q + 1)\} \) generates \( I^\epsilon \) for any \( q \in \{2, 7, 13\} \), and we can see by direct computation that this is true.

More subtly, we can compute that

\[
\det \begin{pmatrix}
\log_{11}(2) & \log_{11}(7) \\
\log_{41}(2) & \log_{41}(7)
\end{pmatrix} \neq 0, \quad \det \begin{pmatrix}
\log_{11}(2) & \log_{11}(13) \\
\log_{41}(2) & \log_{41}(13)
\end{pmatrix} = 0.
\]

By Theorem 1.6.3, this implies that \( \{T_2 - 3, T_7 - 8\} \) generates \( I^\epsilon \), but that \( \{T_2 - 3, T_{13} - 14\} \) does not, and we again verify this by direct computation.

1.10. **Acknowledgements.** We thank Akshay Venkatesh for interesting questions that stimulated this work. We thank Ken Ribet for his inspiring talk [Rib15]. We thank Frank Calegari for helpful comments, and for clarifying the provenance of Ribet’s Conjecture. We thank Hwajong Yoo for helpful comments and for bringing our attention to his results on the subject. We thank Shekhar Khare for helpful discussions about the Steinberg condition and Matt Emerton for asking us about implications for newforms. We thank Rob Pollack for asking us about the case \( p \mid N \) and for his comments on an early draft.

P.W. was supported by the National Science Foundation under the Mathematical Sciences Postdoctoral Research Fellowship No. 1606255. C.W.-E. was supported by Engineering and Physical Sciences Research Council grant EP/L025485/1.
1.11. Notation and Conventions. We let $\partial_{ij}$ denote the Kronecker symbol, which is 1 if $i = j$ and 0 otherwise.

For each prime $\ell \mid Np$, we fix $G_\ell \subset G_0$, a decomposition group at $\ell$, and let $I_\ell \subset G_\ell$ denote the inertia subgroup. We fix elements $\sigma_\ell \in G_\ell$ whose image in $G_\ell/I_\ell \cong \operatorname{Gal}(\overline{\mathbb{F}}_\ell/\mathbb{F}_\ell)$ is the Frobenius. For $\ell \neq p$, we fix elements $\gamma_\ell \in I_\ell$ such that the image in the maximal pro-$p$-quotient $I_\ell^{\text{pro-}p}$ (which is well-known to be pro-cyclic) is a topological generator. Let $\gamma_p \in I_p$ be an element such that the image of $\gamma_p$ in $\operatorname{Gal}(\mathbb{Q}_p^{\text{nr}}(\sqrt{p})/\mathbb{Q}_p^{\text{nr}})$ is non-trivial and $\omega(\gamma_p) = 1$. When $\ell = \ell_i$ for $i \in \{0, \ldots, r\}$ (i.e. $\ell \mid N$), we also write $\sigma_i := \sigma_{\ell_i}$ and $\gamma_i := \gamma_{\ell_i}$ for these elements. We write $G_{\mathbb{Q}, S}$ for the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside of the set places $S$ of $\mathbb{Q}$ supporting $Np\infty$, and use the induced maps $G_\ell \to G_{\mathbb{Q}, S}$. For primes $q \nmid Np$, we write $\operatorname{Fr}_q \in G_{\mathbb{Q}, S}$ for a Frobenius element at $q$.

As in the theory of representations, Cayley-Hamilton representations, actions on modules, pseudorepresentations, and cochains/cocycles/cohomology of profinite groups $G$ discussed in [WWE17a], these objects and categories are implicitly meant to be continuous without further comment. Here all of the targets are finitely generated $A$-modules for some Noetherian local (continuous) $\mathbb{Z}_p$-algebra $A$ with ideal of definition $I$, and the $I$-adic topology is used on the target. Profinite groups used in the sequel satisfy the $\Phi_p$-finiteness condition (i.e. the maximal pro-$p$ quotient of every finite-index subgroup is topologically finitely generated), which allows the theory of [WWE17a] to be applied.

We write

$$H^i(\mathbb{Z}[1/Np], M) = H^i(C^\bullet(\mathbb{Z}[1/Np], M)) = \frac{Z^i(\mathbb{Z}[1/Np], M)}{B^i(\mathbb{Z}[1/Np], M)}$$

for (continuous) cohomology of a $G_{\mathbb{Q}, S}$-module $M$, together with this notation for cochains, cocycles, and coboundaries. We write $x_1 \sim x_2 \in C^\bullet(\mathbb{Z}[1/Np], M_1 \otimes M_2)$ for the cup product of $x_1 \in C^\bullet(\mathbb{Z}[1/Np], M_1)$, and $a_1 \cup a_2 \in H^\bullet(\mathbb{Z}[1/Np], M_1 \otimes M_2)$ for cup product of cohomology classes $a_i \in H^\bullet(\mathbb{Z}[1/Np], M_1)$.

2. Modular forms

In this section, we recall some results about modular curves and modular forms. Our reference is the paper of Ohta [Oht14].

2.1. Modular curves, modular forms, and Hecke algebras. The statements given here are all well-known. We review them here to fix notation.

2.1.1. Modular curves. Let $Y_0(N)/\mathbb{Z}_p$ be the $\mathbb{Z}_p$-scheme representing the functor taking a $\mathbb{Z}_p$-scheme $S$ to the set of pairs $(E, C)$, where $E$ is an elliptic curve over $S$ and $C \subset E[N]$ is a finite-flat subgroup scheme of rank $N$ and cyclic (in the sense of Katz-Mazur [KM83]). Let $X_0(N)/\mathbb{Z}_p$ be the usual compactification of $Y_0(N)/\mathbb{Z}_p$, and let $\{\text{cusps}\}$ denote the complement of $Y_0(N)/\mathbb{Z}_p$ in $X_0(N)/\mathbb{Z}_p$, considered as an effective Cartier divisor on $X_0(N)/\mathbb{Z}_p$. Finally, let $X_0(N) = X_0(N)/\mathbb{Z}_p \otimes \mathbb{Q}_p$.

2.1.2. Modular forms and Hecke algebras. The map $X_0(N)/\mathbb{Z}_p \to \operatorname{Spec}(\mathbb{Z}_p)$ is known to be LCI, and we let $\Omega$ be the sheaf of regular differentials. Let

$$S_2(N; \mathbb{Z}_p) = H^0(X_0(N)/\mathbb{Z}_p, \Omega), \quad M_2(N; \mathbb{Z}_p) = H^0(X_0(N)/\mathbb{Z}_p, \Omega(\{\text{cusps}\}))$$

Let $T_N$ and $T_N^{\text{reg}}$ be the subalgebras of

$$\operatorname{End}_{\mathbb{Z}_p}(M_2(N; \mathbb{Z}_p)), \quad \operatorname{End}_{\mathbb{Z}_p}(S_2(N; \mathbb{Z}_p)),$$
respectively, generated by the standard Hecke operators $T_n$ with $(N, n) = 1$, and all Atkin-Lehner operators $w_{\ell}$ for $\ell \mid N$ (we do not include any $U_{\ell}$ for $\ell \mid N$). These are semi-simple commutative $\mathbb{Z}_p$-algebras (see, e.g. [AL70]).

2.1.3. Eisenstein series and Eisenstein parts. For each $\epsilon \in \{\pm 1\}^{r+1}\setminus\{(1, 1, \ldots, 1)\}$, there is a element $E_{2,N}^{\epsilon} \in M_2(N; \mathbb{Z}_p)$ that is an eigenform for all $T_n$ with $(N, n) = 1$, and has $q$-expansion

$$E_{2,N}^{\epsilon} = -\frac{1}{24} \prod_{i=0}^{r} (\epsilon_i \ell_i + 1) + \sum_{n=1}^{\infty} a_n q^n$$

where $a_n = \sum_{0 < d \mid n} t$ when gcd$(n, N) = 1$ (in particular, $a_1 = 1$), and $w_{\ell_i} E_{2,N}^{\epsilon} = \epsilon_i E_{2,N}^{\epsilon}$ (see [Oht14] Lem. 2.3.4).

Let $I^\epsilon = \text{Ann}_{T_N}^\epsilon(E_{2,N}^{\epsilon})$, and let $T_N^\epsilon$ be the completion of $T_N$ at the maximal ideal $(I^\epsilon, p)$, and let $T_N^0 = T_N^\epsilon \otimes_{\mathbb{T}_N^\epsilon} \mathbb{Z}_p$. Let $I^\epsilon = I^0 \mathbb{T}_N^\epsilon$ and let $I^0 \mathbb{T}_N^\epsilon$ be the image of $I^\epsilon$ in $T_N^0$. For a $T_N^\epsilon$-module $M$, let $M_{\text{Eis}} = M \otimes_{T_N^\epsilon} \mathbb{Z}_p$. The map $T_N^\epsilon \rightarrow \mathbb{Z}_p$, induced by $E_{2,N}^{\epsilon}$ is a surjective ring homomorphism with kernel $I^\epsilon$. We refer to this as the augmentation map for $T_N^\epsilon$.

Note that we have $w_{\ell_i} = \epsilon_i$ as elements of $T_N^\epsilon$. Indeed, this follows from $w_{\ell_i}^{2} = 1$, $w_{\ell_i} - \epsilon_i \in I^\epsilon$, and $p \neq 2$: consider $(w_{\ell_i} - \epsilon_i)(w_{\ell_i} + \epsilon_i) = 0$ and observe that $w_{\ell_i} + \epsilon_i \in (T_N^\epsilon)^\times$. Consequently, $T_N^\epsilon$ is generated as a $\mathbb{Z}_p$-algebra by $T_{\ell}$ for $q \nmid N$.

If $p \nmid N$, let $U_p \in T_N^\epsilon$ denote the unit root of the polynomial

$$X^2 - T_pX + p = 0,$$

which exists and is unique by Hensel’s lemma. Since $T_p - (p+1) \in I^\epsilon$, we see that $U_p - 1 \in I^\epsilon$. Moreover, we see that $T_p = U_p + pU_p^{-1}$.

2.1.4. Duality. As in [Oht14] Thm. 2.4.6], there are perfect pairings of free $\mathbb{Z}_p$-modules

$$M_2(N; \mathbb{Z}_p)_{\text{Eis}}^\epsilon \times T_N^\epsilon \rightarrow \mathbb{Z}_p, \quad S_2(N; \mathbb{Z}_p)_{\text{Eis}}^\epsilon \times T_N^0 \mathbb{T}_N^\epsilon \rightarrow \mathbb{Z}_p$$

given by $(f, t) \mapsto a_1(t \cdot f)$, where $a_1(-)$ refers to the coefficient of $q$ in the $q$-expansion. In particular, $M_2(N; \mathbb{Z}_p)_{\text{Eis}}^\epsilon$ (resp. $S_2(N; \mathbb{Z}_p)_{\text{Eis}}^\epsilon$) is a dualizing (and hence faithful) $T_N^\epsilon$-module (resp. $T_N^0 \mathbb{T}_N^\epsilon$-module).

2.1.5. Oldforms and stabilizations. If $\ell \mid N$ is a prime and $f \in S_2(N/\ell; \mathbb{Z}_p)$ is an eigenform for all $T_n$ with $(n, N/\ell) = 1$, then the subspace

$$\{g \in S_2(N; \mathbb{Z}_p) : a_n(g) = a_n(f) \text{ for all } (n, N/\ell) = 1\}$$

has rank two, with basis $f(z), f(\ell z)$. If we let $f_{\pm}(z) = f(z) \pm \ell f(\ell z)$, then $w_f f_{\pm}(z) = \pm f_{\pm}(z)$. Note that, since $p \neq 2$, we have $f_{\pm} \neq f_{-} \pmod{p}$. In particular, if $\epsilon \in \{\pm 1\}^r$ is the tuple obtained from $\epsilon$ by deleting the entry corresponding to $\ell$, then there are injective homomorphisms given by $f \mapsto f_{\epsilon \ell}$.

$$M_2(N/\ell; \mathbb{Z}_p)_{\text{Eis}}^\epsilon \rightarrow M_2(N; \mathbb{Z}_p)_{\text{Eis}}^\epsilon \quad \text{and} \quad S_2(N/\ell; \mathbb{Z}_p)_{\text{Eis}}^\epsilon \rightarrow S_2(N; \mathbb{Z}_p)_{\text{Eis}}^\epsilon.$$
2.2. Congruence number. We recall this theorem of Ohta, and related results.

**Theorem 2.2.1** (Ohta). There is an isomorphism $\mathbb{T}_N^e / I_N^e \cong \mathbb{Z}_p / a_0(E_{2,N}^e)\mathbb{Z}_p$.

This is [Oht13 Thm. 3.1.3]. His method of proof actually can be used to give the following stronger result, exactly as in [WWE17c Lem. 3.2.2]. See Lemma 2.3.1 for a discussion of fiber products of rings.

**Lemma 2.2.2.** The composition of the augmentation map $\mathbb{T}_N^e \rightarrow \mathbb{Z}_p$ with the quotient map $\mathbb{Z}_p \rightarrow \mathbb{Z}_p / a_0(E_{2,N}^e)\mathbb{Z}_p$ factors through $\mathbb{T}_N^e$ and induces an isomorphism $\mathbb{T}_N^e \cong \mathbb{T}_N^{0,e} \times \mathbb{Z}_p / a_0(E_{2,N}^e)\mathbb{Z}_p$.

In particular, $\ker(\mathbb{T}_N^e \rightarrow \mathbb{T}_N^{0,e}) = \text{Ann}_{\mathbb{T}_N^e}(I^e)$.

2.3. Eigenforms and associated Galois representations. Let $\nu : \mathbb{T}_N^{0,e} \hookrightarrow \mathbb{T}_N^e$ denote the normalization of $\mathbb{T}_N^{0,e}$.

**Lemma 2.3.1.** We record facts about $\mathbb{T}_N^{0,e}$ and associated Galois representations.

1. Letting $q$ vary over primes $q \nmid Np$, there is an isomorphism

$$h : \mathbb{T}_N^{0,e} \cong \bigoplus_{f \in \Sigma} \mathfrak{O}_f,$$

where $\Sigma \subset S_2(N;\mathbb{Q}_p)^{\text{Eis}}$ is the set of normalized eigenforms, and $\mathfrak{O}_f$ is the valuation ring of the finite extension $\mathbb{Q}_p(a_f)_{q \nmid Np}/\mathbb{Q}_p$.

2. For each $f \in \Sigma$, there is an absolutely irreducible representation $\rho_f : G_{\text{Q},\Sigma} \rightarrow \text{GL}_2(\mathfrak{O}_f[1/p])$ such that the characteristic polynomial of $\rho_f(F_q)$ is $X^2 - a_f(f)X + q$ for any $q \nmid Np$.

3. Assume $\ell_i \neq p$. The representation $\rho_f |_{G_{\ell_i}}$ is unramified if $f$ is old at $\ell_i$.

   Otherwise, $f$ is new at $\ell_i$ and there is an isomorphism

$$\rho_f |_{G_{\ell_i}} \cong \begin{pmatrix} \lambda(a_{\ell_i}(f))^{\kappa_{\text{cyc}}} & * \\ 0 & \lambda(a_{\ell_i}(f)) \end{pmatrix},$$

where $a_{\ell_i}(f) = -\epsilon_i$.

4. There is an isomorphism

$$\rho_f |_{G_p} \cong \begin{pmatrix} \lambda(a_p(f)^{-1})^{\kappa_{\text{cyc}}} & * \\ 0 & \lambda(a_p(f)) \end{pmatrix}.$$

Moreover,

(a) $\rho_f |_{G_p}$ is finite-flat if and only if either

(i) $p \nmid N$, in which case $h : \nu(U_p) \rightarrow (a_p(f))_{f \in \Sigma}$, or

(ii) $p \mid N$ and $f$ is old at $p$.

(b) If $p \mid N$ and $f$ is new at $p$, then $a_p(f) = -\epsilon_p = +1$, i.e. $\epsilon_p = -1$.

**Proof.** For (1)-(3) and (4a) see, for example, [DDT94 Thm. 3.1]. In (4b), the fact that $a_p(f) = -\epsilon_p$ is [AL70 Thm. 3]. To see that $\epsilon_p = -1$, note that the semi-simple residual representation $\delta^{ss}_{\nu^{\text{res}}}$ is $\omega \oplus 1$, but (2.3.3) implies $\delta^{ss}_{\nu^{\text{res}}} |_{G_p} = \lambda(-\epsilon_p)\omega \oplus \lambda(-\epsilon_p)$. Since $\omega |_{G_p}$ is ramified, this implies that $\lambda(-\epsilon_p) = 1$, so $\epsilon_p = -1$.

Combining Lemmas 2.2.2 and 2.3.1, we obtain an injective homomorphism

$$\mathbb{T}_N^e \rightarrow \mathbb{Z}_p \oplus \mathbb{T}_N^{0,e} \rightarrow \mathbb{Z}_p \oplus \bigoplus_{f \in \Sigma} \mathfrak{O}_f$$
determined by sending $T_q$ to $(q + 1, a_q(f)_{f \in \Sigma})$ for $q \nmid Np$ and, if $p \nmid N$, sending $U_p$ to $(1, a_p(f)_{f \in \Sigma})$.

2.4. The kernel of $m^\ell$ on the modular Jacobian and the Gorenstein condition. In this section, we use some results of Ohta (following ideas of Mazur) to relate the structure of the rings $T_N^\ell$ and $T_N^{0,\ell}$ to the geometry of the Néron model $J_0(N)/\mathbb{Z}_p$ of the Jacobian of $X_0(N)$. Let $J_0(N) = J_0(N)/\mathbb{Z}_p \otimes \mathbb{Q}_p$.

For a $\mathbb{Z}_p$-module $M$, let $T_p(M) = \text{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, M)$ be the Tate module of $M$, let $M^* = \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ be the Pontrjagin-dual, and let $M^\vee = \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ be the $\mathbb{Z}_p$-dual. If $M$ is a free $\mathbb{Z}_p$-module, there is an identification $M^* \cong T_p(M)^\vee$.

Let $\mathcal{T} = H_1^b(X_0(N)_{\overline{\mathbb{Q}_p}}, \mathbb{Z}_p(1)) \cong T_p(J_0(N)(\overline{\mathbb{Q}_p}))$.

**Lemma 2.4.1.** There is an exact sequence of $T_N^{0,\ell}[I_p]$-modules

$$0 \to T_N^{0,\ell}(1) \to \mathcal{T}, \to (T_N^{0,\ell})^\vee \to 0.$$  

The sequence splits as $T_N^{0,\ell}$-modules. In particular, we have

$$\dim_{\mathbb{F}_p} J_0(N)[m^\ell]/(\mathbb{F}_p) = \dim_{\mathbb{F}_p}(\mathcal{T}/m^\ell \mathcal{T}) = 2 + \delta(T_N^{0,\ell})$$

where $\delta(T_N^{0,\ell})$ is the Gorenstein defect of $T_N^{0,\ell}$. (See [C.1] for a discussion of Gorenstein defect.)

**Proof.** Ohta has shown in [Oht14, Prop. 3.5.9] that

$$\dim_{\mathbb{F}_p} J_0(N)[m^\ell]/(\overline{\mathbb{F}_p}) = 1.$$  

This implies the result, following [Maz77] §§II.7-II.8 (see also [Maz97]). □

**Lemma 2.4.2.** Suppose that $T_N^\ell$ is Gorenstein. Then there is an isomorphism of $T_N^\ell$-modules

$$I^\ell \xrightarrow{\sim} (T_N^{0,\ell})^\vee.$$  

In particular, the minimal number of generators of $I^\ell$ is $\delta(T_N^{0,\ell}) + 1$, and $I^\ell$ is principal if and only if $T_N^{0,\ell}$ is Gorenstein.

**Proof.** Like the proof of [Oht14, Lem. 3.2.5], there is an exact sequence of $T_N^\ell$-modules

$$0 \to S_2(N; \mathbb{Z}_p)_{\text{Eis}} \to M_2(N; \mathbb{Z}_p)_{\text{Eis}} \xrightarrow{\text{Res}} \mathbb{Z}_p \to 0$$

where $T_N^\ell$ acts on $\mathbb{Z}_p$ via the augmentation map $T_N^\ell \to T_N^\ell/I^\ell = \mathbb{Z}_p$. Since we assume that $T_N^\ell$ is Gorenstein, we see by the duality (2.1.2) that $M_2(N; \mathbb{Z}_p)_{\text{Eis}}$ is a free $T_N^\ell$-module of rank 1. We may choose a generator $f$ of $M_2(N; \mathbb{Z}_p)_{\text{Eis}}$ such that $\text{Res}(f) = 1$. Then we obtain a surjective $T_N^\ell$-module homomorphism

$$T_N^\ell \to \mathbb{Z}_p, \quad T \mapsto \text{Res}(Tf)$$

whose kernel is isomorphic to $S_2(N; \mathbb{Z}_p)_{\text{Eis}}$. Because this map sends 1 to 1, it is a ring homomorphism, and it must be the augmentation map $T_N^\ell \to \mathbb{Z}_p$. Thus $I^\ell \cong S_2(N; \mathbb{Z}_p)_{\text{Eis}}$, so duality (2.1.2) yields the isomorphism of the lemma. The remaining parts follow from [C.1] □

Combining the preceding two lemmas, we obtain the following

**Lemma 2.4.3.** Assume that $T_N^\ell$ is Gorenstein. Then

$$\dim_{\mathbb{F}_p} J_0(N)[m^\ell]/(\overline{\mathbb{F}_p}) = 1 + \dim_{\mathbb{F}_p}(I^\ell/m^\ell I^\ell).$$
3. The pseudodeformation ring

In this section, we set up the deformation theory of Galois pseudorepresentations modeling those that arise from Hecke eigenforms of weight 2 and level $N$ that are congruent to the Eisenstein series $E_{2,N}$. These are the Galois representations of Lemma [2.3.1] See [18] for further introduction.

3.1. Theory of Cayley-Hamilton representations. This section is a summary of [WWE17a]. Just for this section, we work with a general profinite group $G$. All pseudorepresentations are assumed to have dimension 2, for simplicity.

3.1.1. Pseudorepresentations. A pseudorepresentation $D : E \to A$ is the data of an associative $A$-algebra $E$ along with a homogeneous multiplicative polynomial law $D$ from $E$ to $A$. This definition is due to Chenevier [Che14]; see [WWE17a] and the references therein. Despite the notation, the pseudorepresentation $D$ includes the data of a multiplicative function $\psi : A \to A$, and sometimes referred to more briefly as “$\psi$.”

A pseudorepresentation $D : E \to A$ is said to be Cayley-Hamilton if, for every commutative $A$-algebra $B$, every element $x \in E \otimes_A B$ satisfies its characteristic polynomial. We also denote by $D : G \to A$ a pseudorepresentation $D : A[G] \to A$.

3.1.2. Cayley-Hamilton representations. An object of the category of Cayley-Hamilton representations of a profinite group $G$ is a triple

$$\rho : G \to E^\times, E, D : E \to A,$$

and sometimes referred to more briefly as “$\rho$.” Here $\rho$ is a homomorphism (continuous, as always), $E$ is an associative $A$-algebra, $(A, m_A)$ is a Noetherian local $\mathbb{Z}_p$-algebra, and $D$ is a Cayley-Hamilton pseudorepresentation. We call $A$ the scalar ring of $E$. The induced pseudorepresentation of $\rho$ is $D \circ \rho : G \to A$, also denoted $\psi(\rho)$. The functor $\psi$ is essentially surjective. The Cayley-Hamilton representation $\rho$ is said to be over $\psi(\rho) \otimes_A A/\mathfrak{m}_A$, and $\psi(\rho)$ is said to be a pseudodeformation of $\psi(\rho) \otimes_A A/\mathfrak{m}_A$.

Given a pseudorepresentation $\bar{D} : G \to F$ for a field $F$, there is a universal object in the category of Cayley-Hamilton representations over $\bar{D}$. This is denoted by

$$\rho^\rho_{\bar{D}} : G \to E^\times_{\bar{D}}, E_{\bar{D}}, D^\rho_{\bar{D}} : E^\times_{\bar{D}} \to R^\rho_{\bar{D}}).$$

3.1.3. Generalized matrix algebras (GMA). An important example of a Cayley-Hamilton algebra is a generalized matrix algebra (GMA). An $A$-GMA $E$ is given by the data $(B, C, m)$ where $B$ and $C$ are finitely-generated $R$-modules, $m : B \otimes_R C \to R$ is an $R$-module homomorphism satisfying certain conditions, and $E = (B \otimes_R C)$ (see [WWE17a] Example 3.1.7]). There is a Cayley-Hamilton pseudorepresentation $D : E \to A$ given by the usual formula for characteristic polynomial. We write a homomorphism $\rho : G \to E^\times$ as $\rho = \left( \rho^1, \rho^2 \right)$.

If $\bar{D}$ is multiplicity-free (see [WWE17a] Defn. 3.2.1), then $E^\rho_{\bar{D}}$ has a GMA structure such that $D^\rho_{\bar{D}}$ is the induced pseudorepresentation [WWE17a] Thm. 3.2.2].
3.1.4. Reducibility. We will refer to the condition that a Cayley-Hamilton representation or a pseudorepresentation is reducible. We also refer to the reducibility ideal in rings receiving a pseudorepresentations. For these definitions, see [WWE17a §4.2] or [WWE15 §5.7]. The important case for this paper is that, if \((\rho, E, D : \rho \to A)\) is a Cayley-Hamilton representation where \(E\) is the GMA associated to \((B, C, m)\), then the reducibility ideal is the image of \(m\). There are also universal objects, denoted \(\rho^\text{red}\), etc.

3.1.5. Conditions on Cayley-Hamilton representations. We consider two flavors of conditions \(P\) imposed on Cayley-Hamilton representations of \(G\):

1. \(P\) is a condition that certain elements vanish, e.g. Definition 3.4.1
2. \(P\) is a property applying to finite-length \(Z_p[G]\)-modules and satisfying a basic stability condition, e.g. 3.5

In case (1), one produces a universal Cayley-Hamilton \(\rho_\mathcal{P}^\text{univ}\) representation of \(G\) satisfying \(P\) by taking the quotient by the two-sided ideal of \(E_D\) generated by the relevant elements, and then taking a further quotient so that a pseudorepresentation exists. This final quotient is known as the Cayley-Hamilton quotient of \(\rho^\mathcal{P}\) for \(P\). See [WWE17a Defn. 2.4.7] for details; cf. also [WWE15, Defn. 5.9.5].

In case (2), we consider \(E_D\) as a \(G\)-module using its left action on itself by multiplication, and find in [WWE17a §2.4] that the maximal left quotient module satisfying \(P\) can be defined and is an algebra quotient. The subsequent Cayley-Hamilton quotient is then shown to satisfy the desired properties of \(\rho_\mathcal{P}^\text{univ}\).

3.1.6. Conditions on pseudorepresentations. As discussed in [WWE17a §2.5], one says that a pseudorepresentation \(D\) of \(G\) satisfies \(P\) if there exists a Cayley-Hamilton representation \(\rho\) of \(G\) such that \(\psi(\rho) = D\) and \(\rho\) satisfies \(P\). Then the universal pseudodeformation with property \(P\) turns out to be \(\psi(\rho_\mathcal{P})\).

3.2. Universal Cayley-Hamilton representations of Galois groups. Let \(\ell \mid Np\) be a prime. Recall from §1.11 the decomposition group \(G_\ell \to G_{Q,S}\). Let \(\bar{D} : G_{Q,S} \to \mathbb{F}_p\) denote the pseudorepresentation \(\psi(\mathbb{F}_p(1) \oplus \mathbb{F}_p)\).

We denote by

\[
(\rho_\mathcal{P}^\text{univ} : G_{Q,S} \to E^\mathcal{P}_D, E_D^\mathcal{P}, D_D^\mathcal{P} : E_D^\mathcal{P} \to R_D^\mathcal{P})
\]

the universal Cayley-Hamilton representation of \(G_{Q,S}\) over \(\bar{D}\). The scalar ring \(R_D\) is the universal pseudodeformation ring of \(D\), with universal pseudorepresentation \(\psi(\rho_\mathcal{P}^\text{univ})\). Similarly, we let the triple

\[
(\rho_\ell^\text{univ} : G_\ell \to E_\ell, E_\ell, D_\ell^\text{univ} : E_\ell \to R_\ell)
\]

denote the universal Cayley-Hamilton representation of \(G_\ell\) over \(\bar{D}|_{G_\ell}\), and universal pseudodeformation \(\psi(\rho_\ell^\text{univ})\).

**Definition 3.2.1.** Note that \(\bar{D}\) is multiplicity-free, and that, if \(\ell \neq 1 \pmod{p}\), then \(\bar{D}|_{G_\ell}\) is multiplicity-free. In this case, \(E_\ell\) and \(E_D\) have the structure of a GMA. In this paper, whenever we fix such a structure, we assume that \((\rho_\ell^\text{univ})_{1,1} \otimes_{R_D} \mathbb{F}_p \cong \omega|_{G_\ell}\) (resp. \((\rho_\mathcal{P}^\text{univ})_{1,1} \otimes_{R_D} \mathbb{F}_p \cong \omega)\).

3.3. Case \(\ell \mid Np\): unramified. For \(\ell \mid Np\), we want Galois representations to be unramified at \(\ell\). We impose this by considering representations of \(G_{Q,S}\), as opposed to \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\).
3.4. Case $\ell \neq p$ and $\ell | N$: the unramified-or-Steinberg condition. In this subsection, we write $\ell$ for one of the factors of $N$ referred to elsewhere in this manuscript as $\ell_i$. Likewise, we write $\epsilon_\ell$ for $\epsilon_i$.

**Definition 3.4.1.** Let $(\rho : G_\ell \to E, E, D : E \to A)$ be a Cayley-Hamilton representation of $G_\ell$ over $\bar{D}|G_\ell$. We call $\rho$ unramified-or-$\epsilon_\ell$-Steinberg (or US$_\ell^\epsilon$) if

\[ V^{\epsilon_\ell}_\rho(\sigma, \tau) := (\rho(\sigma) - \lambda(-\epsilon_\ell)(\sigma)\kappa_{cyc}(\sigma))(\rho(\tau) - \lambda(-\epsilon_\ell)(\tau)) \in E \]

is equal to 0 for all $(\sigma, \tau)$ ranging over the set

\[ I_\ell \times G_\ell \cup G_\ell \times I_\ell \subset G_\ell \times G_\ell. \]

Write $V^{\epsilon_\ell}_\rho$ for the set of all elements $V^{\epsilon_\ell}_\rho(\sigma, \tau)$ over this range.

A pseudodeformation $D : G_\ell \to A$ of $\bar{D}|G_\ell$ is called US$_\ell^\epsilon$ if there exists a US$_\ell^\epsilon$ Cayley-Hamilton representation $\rho$ such that $\psi(\rho) = D$. 

**Definition 3.4.3.** Let $(E^{\epsilon_\ell}_\ell, D^{\epsilon_\ell}_\ell, E^{\epsilon_\ell}_\ell \to R^{\epsilon_\ell}_\ell)$ be the Cayley-Hamilton quotient of $(E_\ell, D_\ell)$ by $V^{\epsilon_\ell}_\rho$. Let

\[ (\rho^{\epsilon_\ell}_\ell : G_\ell \to (E^{\epsilon_\ell}_\ell)^\times, E^{\epsilon_\ell}_\ell, D^{\epsilon_\ell}_\ell : E^{\epsilon_\ell}_\ell \to R^{\epsilon_\ell}_\ell), \]

be the corresponding Cayley-Hamilton representation.

As discussed in 3.1.5, $\rho^{\epsilon_\ell}_\ell$ is the universal Cayley-Hamilton representation satisfying US$_\ell^\epsilon$ over $\bar{D}$, and $\psi(\rho^{\epsilon_\ell}_\ell)$ is the universal US$_\ell^\epsilon$ pseudodeformation over $\bar{D}$.

**Lemma 3.4.4.** If $\ell \neq p$, then, for any $\epsilon_\ell$, we have $D^{\epsilon_\ell}_\ell(\tau) = 1$ and $\text{Tr}_{D^{\epsilon_\ell}_\ell}(\tau) = 2$ for all $\tau \in I_\ell$. That is, $(D^{\epsilon_\ell}_\ell)|_{I_\ell} = \psi(1 + 1)$.

**Proof.** Let $\tau \in I_\ell$. We see in (3.4.2) that $V^{\epsilon_\ell}_\rho(\tau, \tau) = (\rho^{\epsilon_\ell}_\ell(\tau) - 1)^2 = 0$. Thus by [Che14] Lem. 2.1.5(iii), we see $\text{Tr}_{D^{\epsilon_\ell}_\ell}(\tau - 1) = D^{\epsilon_\ell}_\ell(\tau - 1) = 0$. As traces are additive, we have $\text{Tr}_{D^{\epsilon_\ell}_\ell}(\tau) = \text{Tr}_{D^{\epsilon_\ell}_\ell}(1) = 2$. Applying (3.1.1) with $x = \tau$ and $t = 1$, we find that $D^{\epsilon_\ell}_\ell(\tau) = 1$. \hfill $\square$

**Lemma 3.4.5.** Suppose that $\epsilon_\ell = +1$ and $\ell \equiv -1, 0 \pmod{p}$. Then $\rho^{\epsilon_\ell}_\ell$ is unramified (i.e. $\rho^{\epsilon_\ell}_\ell|_{I_\ell} = 1$).

**Proof.** Let $\sigma \in G_\ell$ be the element $\sigma_\ell$ defined in 8.1.1. By definition of $E^{\epsilon_\ell}_\ell$,

\[ V^{\epsilon_\ell}_\rho(\tau, \sigma) = (\rho^{\epsilon_\ell}_\ell(\tau) - 1)(\rho^{\epsilon_\ell}_\ell(\sigma) + 1) = 0, \]

for any $\tau \in I_\ell$. To prove the lemma, it suffices to show that $(\rho^{\epsilon_\ell}_\ell(\sigma) + 1) \in (E^{\epsilon_\ell}_\ell)^\times$. By the Cayley-Hamilton property, we know that any element $x \in E^{\epsilon_\ell}_\ell$ satisfies $x^2 - \text{Tr}_{D^{\epsilon_\ell}_\ell}(x)x + D^{\epsilon_\ell}_\ell(x) = 0$. In particular, we see that $x \in (E^{\epsilon_\ell}_\ell)^\times$ if and only if $D^{\epsilon_\ell}_\ell(x) \in (R^{\epsilon_\ell}_\ell)^\times$. Hence it will suffice to show that $D^{\epsilon_\ell}_\ell(\sigma + 1) \in (R^{\epsilon_\ell}_\ell)^\times$.

Writing $m \subset R^\times_\ell$ for the maximal ideal, we know that $D^{\epsilon_\ell}_\ell = \bar{D} \pmod{m}$, so it will suffice to show that $\bar{D}(\sigma + 1) \in R^\times_\ell$. Because $\ell \neq p$ and $\bar{D} = \psi(\omega^{\epsilon_\ell} + 1)$, we apply (3.1.1) with $x = \sigma$ and $t = -1$, calculating that $\bar{D}(\sigma + 1) = 2(\ell + 1) \in \mathbb{F}_p$. This is a unit because $p$ is odd and $\ell \equiv -1 \pmod{p}$. \hfill $\square$
3.5. The finite-flat case: \( \ell = p \) and \( p \nmid N \). A finite-length \( \mathbb{Z}_p[G_p] \)-module \( V \) is said to be finite-flat when it arises as \( \mathcal{G}(\overline{G}_p) \), where \( \mathcal{G} \) is a finite flat group scheme over \( \mathbb{Z}_p \). In [WWE17a, §5.2] we check that the theory of [3.1.5.2] can be applied to the finite-flat condition. This theory gives us

\[
(\rho_{\text{flat}} : G_p \to (E_p^\text{flat})^\times, E_p^\text{flat}, D_p^\text{flat} : E_p^\text{flat} \to R_p^\text{flat}),
\]

the universal finite-flat Cayley-Hamilton representation of \( G_p \) over \( \overline{D}|G_p| \). The pseudodeformation \( \psi(\rho_{\text{flat}}) = D_p^\text{flat} \circ \rho_{\text{flat}} : G_p \to R_p^\text{flat} \) is the universal finite-flat pseudodeformation of \( \overline{D}|G_p| \).

Consider a GMA structure on \( E_p \) as in Definition 3.2.1, which we write as

\[
\rho_{\text{flat}} = \left( \rho_{1,1}^\text{flat}, \rho_{1,2}^\text{flat}, \rho_{2,1}^\text{flat}, \rho_{2,2}^\text{flat} \right) : G_p \to \left( R_p^\text{flat}, B_p^\text{flat}, C_p^\text{flat}, R_p^\text{flat} \right)^\times.
\]

**Lemma 3.5.1.** For any such GMA structure on \( E_p \), \( C_p^\text{flat} = 0 \).

**Proof.** The proof is implicit in [WWE17c] but not stated in this form there. One simply combines the following facts. See [WWE17c] §B.4 for the notation.

- As the maximal ideal of \( R_p^\text{flat} \) contains the reducibility ideal, we have \( \text{Hom}_{R_p^\text{flat}}(C_p^\text{flat}, E_p) = \text{Ext}_{\text{fgs}}^1(\mu_p, \mathbb{Z}/p\mathbb{Z}) \), where \( \text{fgs}/\mathbb{Z}_p \) is the category of finite flat group schemes over \( \mathbb{Z}_p \), by [WWE17a Thm. 4.3.5].

- We see in [WWE17c Lem. 6.2.1(1)] that \( \text{Ext}_p^1(\mu_p, \mathbb{Z}/p\mathbb{Z}) = 0 \).

As \( C_p^\text{flat} \) is a finitely-generated \( R_p^\text{flat} \)-module, this implies that \( C_p^\text{flat} = 0 \).

Now that we know that \( C_p^\text{flat} = 0 \), \( \rho_{i,i}^\text{flat} \) are \( R_p^\text{flat} \)-valued characters of \( G_p \), for \( i = 1, 2 \). Similarly to [WWE17c §5.1], using the fact that \( \omega|G_p| \neq 1 \), we see the following

**Lemma 3.5.2.** A pseudodeformation \( D \) of \( \overline{D}|G_p| \) is finite-flat if and only if \( D = \psi(\kappa_{\text{cy}}}^1 \chi_1 \oplus \chi_2) \) where \( \chi_1, \chi_2 \) are unramified deformations of the trivial character.

3.6. The finite-flat case: \( \ell = p \), \( p \mid N \), and \( \epsilon_p = +1 \). By Lemma 3.5.1(4), we see that, if \( \epsilon_p = +1 \), then the residually Eisenstein cusp forms are old at \( p \) with associated \( G_{\mathcal{O},S} \)-representation being finite-flat at \( p \). We impose this condition exactly as in [3.5.1.4]. Namely, we say that a Cayley-Hamilton representation of \( G_p \) is unramified-or-(+1)-Steinberg (or US\(_p^{+1} \)) if it is finite-flat.

3.7. The ordinary case: \( \ell = p \), \( p \mid N \), and \( \epsilon_p = -1 \). Based on the form of Galois representations arising from \( p \)-ordinary eigenforms given in Lemma 2.3.1(4), we proceed exactly as in the case \( \ell \neq p \) given in 3.4.

**Definition 3.7.1.** We say that a Cayley-Hamilton representation or a pseudodeformation over \( \overline{D}|G_p| \) is ordinary (or US\(_p^{-1} \)) when it satisfies Definition 3.4.1 simply letting \( \ell = p \).

We let \( (E_p^\text{ord}, D_p^\text{ord}) \) be the Cayley-Hamilton quotient of \( (E_p, D_p) \) by \( V_p^{-1} \), and let \( (\rho^\text{ord}_p, E_p^\text{ord}, D_p^\text{ord}, R_p^\text{ord}) \) be the corresponding Cayley-Hamilton representation. As discussed in [3.1.5.1], \( \rho^\text{ord}_p \) is the universal ordinary Cayley-Hamilton representation over \( \overline{D} \) and \( \psi(\rho^\text{ord}_p) \) is the universal ordinary pseudodeformation over \( \overline{D} \).
Remark 3.7.2. If one applies $V^{+1} = 0$ in the case $\epsilon_p = +1$, one does not get the desired finite-flat condition of \([3.6]\) that agrees with Lemma \([2.3.1]\)(4b). Instead, one finds that $E^{+1}_p = 0$ (i.e. no deformations of $\tilde{D}$ satisfy this condition).

We set up the following notation, which includes all cases: $\epsilon_p = \pm 1$ or $p \nmid N$.

Definition 3.7.3. For any $N$ and $\epsilon$, we define
\[
(r^\text{ord}_p, E^\text{ord}_p, D^\text{ord}_p, R^\text{ord}_p) := \begin{cases} (r^\text{ord}_p, E^\text{ord}_p, D^\text{ord}_p, R^\text{ord}_p) & \text{if } p \mid N \text{ and } \epsilon = -1, \\
(r^\text{flat}_p, E^\text{flat}_p, D^\text{flat}_p, R^\text{flat}_p) & \text{otherwise.}
\end{cases}
\]

In \([WWE15, \S 5]\), we developed an alternative definition of ordinary Cayley-Hamilton algebra. (This definition applies to general weight, which we specialize to weight 2 here.) Choose a GMA structure on $E_p$, as in Definition \([3.2.1]\). Let $J^p_{\text{ord}} \subset E_p$ be the two-sided ideal generated by the subset
\[
\rho_{2,1}(G_p) \bigcup (\rho_{1,1} - \kappa_{\text{cyc}})(I_p) \bigcup (\rho_{2,2} - 1)(I_p).
\]
As in \([WWE15, \text{Lem. 5.9.3}]\), $J^\text{ord}_p$ is independent of the choice of GMA-structure.

Lemma 3.7.4. The Cayley-Hamilton quotient of $E_p$ by $J^\text{ord}_p$ is equal to $E^\text{ord}_p$.

Proof. Let $(V^\text{ord}_p)$ denote the kernel of $E_p \rightarrow E^{\text{ord}}_p$, which contains (but may not be generated by) $V^\text{ord}_p$ (see \([3.1.5]\)). It will suffice to show that $(V^\text{ord}_p) = J^\text{ord}_p$.

The inclusion $(V^\text{ord}_p) \subset J^\text{ord}_p$ is straightforward: see the calculations in \([WWE15, \S 5.9]\), from which it is evident that the Cayley-Hamilton quotient of $\rho^\text{ord}_p$ by $J^\text{ord}_p$ is a Cayley-Hamilton representation that is ordinary (in the sense of Definition \([3.4.1]\)). It remains to show that $J^{\text{ord}}_p \subset (V^{\text{ord}}_p)$.

First we will show that $D^\text{ord}_p|_{I_p} = \psi(\kappa_{\text{cyc}} \oplus 1)|_{I_p} \otimes \mathbb{Q}_p R^\text{ord}_p$. For any $\tau \in I_p$, $\rho^\text{ord}_p(\tau)$ satisfies both polynomials
\[
T^2 - \text{Tr}_{D^\text{ord}_p}(\tau)T - D^\text{ord}_p(\tau) \quad \text{and} \quad (T - \kappa_{\text{cyc}}(\tau))(T - 1),
\]
the first by the Cayley-Hamilton condition and the second by Definition \([3.7.1]\). If $\omega(\tau) \neq 1$, Hensel's lemma implies that these two polynomials are identical. For such $\tau$, we have $D^\text{ord}_p(\tau) = \kappa_{\text{cyc}}(\tau)$ and $\text{Tr}_{D^\text{ord}_p}(\tau) = \kappa_{\text{cyc}}(\tau) + 1$. Now choose an arbitrary element of $I_p$ and write it as $\sigma \tau$ with $\omega(\sigma), \omega(\tau) \neq 1$. We immediately see that $D^\text{ord}_p(\sigma \tau) = \kappa_{\text{cyc}}(\sigma \tau)$, since both sides are multiplicative. Let $r_\sigma = \rho^\text{ord}_p(\sigma)$ and $r_\tau = \rho^\text{ord}_p(\tau)$. Since $E^\text{ord}_p$ is Cayley-Hamilton, we have
\[
(t_\sigma r_\sigma + t_\tau r_\tau)^2 - \text{Tr}_{D^\text{ord}_p}(t_\sigma r_\sigma + t_\tau r_\tau)(t_\sigma r_\sigma + t_\tau r_\tau) + D^\text{ord}_p(t_\sigma r_\sigma + t_\tau r_\tau) = 0
\]
in the polynomial ring $E^\text{ord}_p[t_\sigma, t_\tau]$. We can expand $D^\text{ord}_p(t_\sigma r_\sigma + t_\tau r_\tau)$ using \([\text{Che14 Example 1.8}]\). Taking the coefficient of $t_\sigma t_\tau$ and writing $\text{Tr} = \text{Tr}_{D^\text{ord}_p}$ for brevity, we see
\[
(r_\sigma r_\tau + r_\sigma r_\tau - \text{Tr}(\sigma)r_\tau - \text{Tr}(\tau)r_\sigma - \text{Tr}(\sigma \tau) + \text{Tr}(\sigma \tau)\text{Tr}(\tau) = 0.
\]
Substituting for $r_\sigma r_\tau$ using $V^\text{ord}_p(\sigma, \tau) = 0$ and for $r_\sigma r_\tau$ using $V^\text{ord}_p(\tau, \sigma) = 0$, one obtains the desired conclusion $\text{Tr}(\sigma \tau) = \kappa_{\text{cyc}}(\sigma \tau) + 1$.

Let $\sigma \in I_p$, and let $\tau \in I_p$ be such that $\omega(\tau) \neq 1$. Using the fact that $\rho^\text{ord}_p|_{I_p}$ is reducible, we see that the $(1, 1)$-coordinate of $V^\text{ord}_p(\sigma, \tau)$ is
\[
(\rho^\text{ord}_p(1, 1) - \kappa_{\text{cyc}}(\sigma))(\rho^\text{ord}_p(1, 1) - 1) = 0
\]
Since \( \rho_{1,1}^{ord} \) is a deformation of \( \omega \), we have \( \rho_{1,1}^{ord}(\tau) - 1 \in (R_p^{ord})^\times \), so this implies \( \rho_{1,1}^{ord}(\sigma) - \kappa_{cyc}(\sigma) = 0 \). This shows that \( (\rho_{1,1} - \kappa_{cyc})(I_p) \subset (V_p^{ord}) \), and a similar argument gives \( (\rho_{2,2} - 1)(I_p) \subset (V_p^{ord}) \).

It remains to show that \( \rho_{2,1}^{ord}(G_p) = 0 \). Let \( m \subset R_p^{ord} \) be the maximal ideal. In fact, we will show that \( \bar{\rho}_{2,1} \) generates the finitely generated \( R_p^{ord} \)-module \( C_p^{ord} \). We work with \( \bar{\rho}_{ord} := \rho_{ord} \pmod{m} \). Since \( \rho_{ord}^{ord} \) is reducible, we can consider \( \bar{\rho}_{2,1}^{ord} \in Z^1(G_p, \bar{C}_p^{ord}/mC_p^{ord} \otimes_{\mathbb{F}_p} \mathbb{F}_p(-1)) \), and [BC09, Thm. 1.5.5] implies that there is an injection
\[
\text{Hom}_p(C_p^{ord}/mC_p^{ord}, \mathbb{F}_p) \hookrightarrow H^1(G_p, \mathbb{F}_p(-1))
\]
sending \( \phi \) to the class of the cocycle \( \phi \circ \bar{\rho}_{2,1}^{ord} \). So to show that \( C_p^{ord}/mC_p^{ord} \) is zero, it is enough to show that \( \bar{\rho}_{2,1}^{ord} \) is a coboundary, or, equivalently, that \( \bar{\rho}_{2,1}^{ord}(\sigma) = 0 \) for all \( \sigma \in \ker(\omega) \subset G_p \). However, we compute that the \((2,1)\)-entry of \( V_p^{ord}(\sigma, \tau) \) is
\[
\rho_{2,1}(\sigma)(\rho_{1,1}(\tau) - 1) + (\rho_{2,2}(\sigma) - \kappa_{cyc}(\sigma))\rho_{2,1}(\tau).
\]
Taking \( \sigma \in \ker(\omega) \) and \( \tau \in I_p \) such that \( \omega(\tau) \neq 1 \), we see that \( \rho_{1,1}(\tau) - 1 = \omega(\tau) - 1 \not\equiv 0 \pmod{m} \) and \( \rho_{2,2}(\sigma) - \kappa_{cyc}(\sigma) \in m \), so this implies \( \bar{\rho}_{2,1}^{ord}(\sigma) = 0 \). \( \square \)

We have the following consequence, following [WWW15, §5.9].

**Proposition 3.7.5.** A Cayley-Hamilton representation \( \rho : G_p \to E_p^\times, E, D : E \to A \) over \( \bar{D}|_{G_p} \) is ordinary if and only if it admits a GMA structure such that

1. it is upper triangular, i.e. \( \rho_{1,1} = 0 \), and
2. the diagonal character \( \rho_{1,1} \) (resp. \( \rho_{2,2} \)) is the product of \( \kappa_{cyc} \otimes_{\mathbb{Z}_p} A \) (resp. the constant character \( A \)) and an unramified \( A \)-valued character.

**Corollary 3.7.6.** Any finite-flat Cayley-Hamilton representation of \( G_p \) over \( \bar{D}|_{G_p} \) is ordinary. The resulting morphism of universal Cayley-Hamilton representations of \( G_p \), \( (\rho_p^{ord}, \mathbb{C}_p^{ord}, D_p^{ord}) \to (\rho_p^{flat}, \mathbb{C}_p^{flat}, D_p^{flat}) \), induces an isomorphism on universal pseudodeformation rings \( R_p^{ord} \cong R_p^{flat} \). The universal pseudorepresentations have the form \( \psi(\kappa_{cyc}(\chi_1 + \chi_2)) \), where \( \chi_1, \chi_2 \) are unramified deformations of the trivial character.

**Proof.** The Cayley-Hamilton representation \( \rho_p^{flat} \) satisfies conditions (1) and (2) of Proposition 3.7.5 by Lemmas 3.5.1 and 3.5.2 respectively. The isomorphism of universal pseudorepresentations becomes evident by comparing Lemma 3.5.2 and Proposition 3.7.5. \( \square \)

### 3.8. Global formulation.

We now combine the local constructions to define what it means for a global Cayley-Hamilton representation or pseudorepresentation to be unramified-or-Steinberg of level \( N \) and type \( \epsilon \).

**Definition 3.8.1.** Let \( \rho : G_{0,N} \to E_p^\times \) be a Cayley-Hamilton representation with local scalar ring \( A \), induced pseudorepresentation \( D : G \to A \), and residual pseudorepresentation \( \bar{D} \). We say that \( \rho \) is unramified-or-Steinberg of level \( N \) and type \( \epsilon \) (or \( \text{US}_{N}^\epsilon \)) when \( \rho|_{G_\ell} \) is \( \text{US}_{N}^\epsilon \) for all primes \( \ell \mid N \), and, if \( p \nmid N \), \( \rho|_{G_p} \) is finite-flat.

Let \( D : G \to A \) be a pseudodeformation of \( \bar{D} \). We say that \( D \) is unramified-or-Steinberg of level \( N \) and type \( \epsilon \) (or \( \text{US}_{N}^\epsilon \)) when there exists a Cayley-Hamilton representation \( \rho : G \to E_p^\times, E, D_E : E \to A \) such that \( D = \psi(\rho) \) and \( \rho \) is \( \text{US}_{N}^\epsilon \).
Recall the Cayley-Hamilton representation $\rho_D^\ell$ set up in \[3.2\]. There are maps of Cayley-Hamilton algebras $\iota_t : (E_t, R_t) \rightarrow (E_D, R_D)$ arising from the fact that the restriction of $\rho_D^\ell$ to $G_t$ is a Cayley-Hamilton representation of $G_t$ over $D_{|G_t}$. For any $\ell \mid Np$, write $J_t^\ell$ for the kernel of $E_t \rightarrow E_t^\ell$ (refer to Definition \[3.7.3\] for $E_p^\ell$).

**Definition 3.8.2.** Let $(E_N^\ell, R_N^\ell)$ denote the Cayley-Hamilton algebra quotient of $E_D$ by the two-sided ideal generated by the union of $\iota_t(J_t^\ell)$ over all $\ell \mid Np$. We denote the quotient Cayley-Hamilton representation of $G_{\mathbb{Q},S}$ by

$$\rho_N^\ell : G \rightarrow (E_N^\ell)^\times, E_N^\ell : E_N \rightarrow R_N^\ell$$

Using \[3.1.5\], we see that $\rho_N^\ell$ (resp. $D_N^\ell$) is the universal US$_N^\ell$ Cayley-Hamilton representation (resp. pseudodeformation) over $D$. In particular, a homomorphism $R_D \rightarrow A$ factors through $R_N^\ell$ if and only if the corresponding to a pseudorepresentation $D : G_{\mathbb{Q},S} \rightarrow A$ satisfies US$_N^\ell$.

**Proposition 3.8.3.** Let $D : G_{\mathbb{Q},S} \rightarrow A$ be a pseudodeformation of $\tilde{D}$ satisfying US$_N^\ell$. Then $D(\tau) = \kappa_{\text{cyc}}(\tau)$ for all $\tau \in G_{\mathbb{Q},S}$.

*Proof.* It suffices to show that $D(\tau) = \kappa_{\text{cyc}}(\tau)$ for all $\tau \in I_\ell$ and all $\ell \mid Np$, since this will show that $D\kappa_{\text{cyc}}^{-1}$ is a character of $G_{\mathbb{Q},S}$ that is unramified everywhere and hence trivial. For $\ell \neq p$, this follows from Lemma \[3.4.4\] and for $\ell = p$ this follows from Corollary \[3.7.6\]. $\square$

### 3.9. Information about $B_p^\ell$ and $C_p^\ell$.**

Recall that we fixed a GMA structure on $E_p$ in \[3.7\]. This defines a GMA structure on $E_p^\ell$ and $E_N^\ell$ via the Cayley-Hamilton algebra morphisms $E_p \rightarrow E_p^\ell$ and $E_p^\ell \rightarrow E_N^\ell$. We write this GMA structure as

$$E_N^\ell = \begin{pmatrix} R_N^\ell & B_N^\ell \\ C_N^\ell & R_N^\ell \end{pmatrix}, \quad \rho_N^\ell(\tau) = \begin{pmatrix} a_\tau & b_\tau \\ c_\tau & d_\tau \end{pmatrix}.$$

\[3.9.1\]

**3.9.1. Computation of $B_{\text{flat}}^\ell$ and $C_{\text{flat}}^\ell$.** First we work in the case that either $p \mid N$ or $\ell = 1$, so $E_p^\ell = E_p^\flat$, with a GMA structure chosen. Let $(E_{\text{flat}}, R_{\text{flat}})$ represent the Cayley-Hamilton quotient of $E_D$ by $\psi(J_p^\ell)$, with a GMA structure coming from $E_{\text{flat}} \rightarrow E_{\text{flat}}$. Write this GMA structure as

$$E_{\text{flat}} \cong \begin{pmatrix} R_{\text{flat}} & B_{\text{flat}} \\ C_{\text{flat}} & R_{\text{flat}} \end{pmatrix}, \quad \rho_{\text{flat}}(\tau) = \begin{pmatrix} a_{\text{flat},\tau} & b_{\text{flat},\tau} \\ c_{\text{flat},\tau} & d_{\text{flat},\tau} \end{pmatrix}.$$

\[3.9.2\]

Let $J_{\text{flat}}^\ell = \ker(R_{\text{flat}} \rightarrow \mathbb{Z}_p)$, where $R_{\text{flat}} \rightarrow \mathbb{Z}_p$ corresponds to $\psi(\mathbb{Z}_p(1) + \mathbb{Z}_p)$, which is obviously finite-flat. Let

$$B_{\text{flat}}^\ell = B_{\text{flat}} / J_{\text{flat}}^\ell B_{\text{flat}}, \quad C_{\text{flat}}^\ell = C_{\text{flat}} / J_{\text{flat}}^\ell C_{\text{flat}}.$$

By [WWE17], Prop. 2.5.1, we have, for any finitely-generated $\mathbb{Z}_p$-module $M$, isomorphisms

$$\text{Hom}_{\mathbb{Z}_p}(B_{\text{flat}}^\ell, M) \cong H^1_{\text{flat}}(\mathbb{Z}[1/Np], M(1))$$

$$\text{Hom}_{\mathbb{Z}_p}(C_{\text{flat}}^\ell, M) \cong H^1_{(p)}(\mathbb{Z}[1/Np], M(1)),$$

\[3.9.3\]

where

$$H^1_{\text{flat}}(\mathbb{Z}[1/Np], M(1)) = \ker \left( \left( H^1(\mathbb{Z}[1/Np], M(1)) \rightarrow \frac{H^1(\mathbb{Q}_p, M(1))}{\text{Ext}^1_{\mathbb{Z}_p}(M, \mathbb{Z}_p(1))} \right) \right)$$

and

$$H^1_{(p)}(\mathbb{Z}[1/Np], M(1)) = \ker(\text{Ext}^1_{\mathbb{Z}_p}(\mathbb{Z}_p(1), M(1)) \rightarrow H^1(\mathbb{Q}_p, M(-1))).$$
The Galois cohomology computations of [WWE17c, §6.3] allow us to compute these.

**Lemma 3.9.4.** There are isomorphisms

\[ Z_p^\otimes r + 1 \cong B^\text{min}_\text{ord}, \quad \bigoplus_{i=0}^{r} Z_p/((\ell^2_i - 1)Z_p) \cong C^\text{min}_\text{ord} \]

given by \( e_i \mapsto b_{\text{ord}, \gamma_i} \) and \( e_i \mapsto c_{\text{ord}, \gamma_i} \), where \( e_i \in Z_p^\otimes r + 1 \) is the \( i \)-th standard basis vector.

3.9.2. Computation of \( B^\text{min}_\text{ord} \) and \( C^\text{min}_\text{ord} \). Next we compute in the case \( p \mid N \) and \( \epsilon_p = -1 \), so \( E_p^\epsilon = E_p^\text{ord} \). Let \( (E_\text{ord}, R_\text{ord}) \) be the Cayley-Hamilton quotient of \( E_D \) by \( \iota_p(J_p^\epsilon) \), receiving a GMA structure via \( E_\text{ord} \to E_\text{ord} \). Write this GMA structure as

(3.9.5) \[ E_\text{ord} \cong \left( \begin{array}{ccc} R_\text{ord} & B_\text{ord} & \rho_\text{ord}(\tau) \\ C_\text{ord} & R_\text{ord} & \end{array} \right), \quad \rho_\text{ord}(\tau) = \left( \begin{array}{cc} a_{\text{ord}, \tau} & b_{\text{ord}, \tau} \\ c_{\text{ord}, \tau} & d_{\text{ord}, \tau} \end{array} \right). \]

Let \( J_\text{ord}^\text{min} = \ker(R_\text{ord} \to Z_p) \), where \( R_\text{ord} \to Z_p \) corresponds to \( \psi(Z_p(1) \oplus Z_p) \), which is obviously ordinary. Let

\[ B^\text{min}_\text{ord} = B_\text{ord}/J_\text{ord}^\text{min} B_\text{flat}, \quad C^\text{min}_\text{ord} = C_\text{ord}/J_\text{ord}^\text{min} C_\text{ord}. \]

Just as in [WWE17c, Lem. 4.15], we have, for any finitely-generated \( Z_p \)-module \( M \), isomorphisms

(3.9.6) \[
\begin{aligned}
\text{Hom}_{Z_p}(B^\text{min}_\text{ord}, M) &\cong H^1(Z[1/N_p], M(1)), \\
\text{Hom}_{Z_p}(C^\text{min}_\text{ord}, M) &\cong H^1(Z[1/N_p], M(-1)).
\end{aligned}
\]

The Galois cohomology computations of [WWE17c, §6.3] allow us to compute these. Recall that \( \gamma_i \) is defined in [1.11] even when \( \ell_i = p \).

**Lemma 3.9.7.** There are isomorphisms

\[ Z_p^\otimes r + 1 \cong B^\text{min}_\text{ord}, \quad \bigoplus_{i=0}^{r} Z_p/((\ell^2_i - 1)Z_p) \cong C^\text{min}_\text{ord} \]

given by \( e_i \mapsto b_{\text{ord}, \gamma_i} \) and \( e_i \mapsto c_{\text{ord}, \gamma_i} \), where \( e_i \in Z_p^\otimes r + 1 \) is the \( i \)-th standard basis vector.

3.9.3. Information about \( B_N^\epsilon\text{min} \) and \( C_N^\epsilon\text{min} \). Let \( J_N^\text{min} := \ker(R_N^\epsilon \to Z_p) \), where this homomorphism is induced by the US \( \kappa_N \) pseudodeformation \( \psi(Z_p(1) \oplus Z_p) \) of \( \bar{D} \).

**Lemma 3.9.8.** We consider \( B_N^\epsilon\text{min} = B_N^\epsilon/J_N^\text{min} B_N^\epsilon \) and \( C_N^\epsilon\text{min} = C_N^\epsilon/J_N^\text{min} C_N^\epsilon \).

1. If \( \epsilon_i = 1 \) and \( \ell_i \neq p \), then the image of \( b_{\gamma_i} \) in \( B_N^\epsilon\text{min} \) is 0.
2. If \( \epsilon_i = -1 \) and \( \ell_i \equiv -1 \pmod{p} \), then the image of \( c_{\gamma_i} \) in \( C_N^\epsilon\text{min} \) is 0.

Moreover, there are surjections

\[ \bigoplus_{i=0}^{r} Z_p/((\epsilon_i + 1)Z_p) \to B_N^\epsilon\text{min}, \quad \bigoplus_{i=0}^{r} Z_p/((\ell_i + \epsilon_i)Z_p) \to C_N^\epsilon\text{min}. \]

given by \( e_i \mapsto b_{\gamma_i} \) and \( e_i \mapsto c_{\gamma_i} \), respectively.

**Proof.** Note that for \( \rho_N^\epsilon\text{min} = \rho_N^\epsilon \otimes R_N^\epsilon \), in the GMA structure, we have

\[ \rho_N^\epsilon\text{min} = \left( \begin{array}{cc} \kappa_{\text{cyc}} & b \\ c & 1 \end{array} \right). \]
Note that we have
\[V^\epsilon_i \rho_N^r (\gamma_i, \sigma_i) = (\rho_N^r (\gamma_i) - 1)(\rho_N^r (\sigma_i) + \epsilon_i) = 0.\]

In GMA notation, this is
\[0 = \begin{pmatrix} 0 & b_{\gamma_i} \\ c_{\gamma_i} & 0 \end{pmatrix} \begin{pmatrix} \ell_i + \epsilon_i & b_{\sigma_i} \\ c_{\sigma_i} & 1 + \epsilon_i \end{pmatrix} = \begin{pmatrix} 0 & (1 + \epsilon_i) b_{\gamma_i} \\ (\ell_i + \epsilon_i) c_{\gamma_i} & 0 \end{pmatrix}.\]

In case (1), \((1 + \epsilon_i)\) is invertible, so \(b_{\gamma_i} = 0.\) In case (2), \((\ell_i + \epsilon_i)\) is invertible, so \(c_{\gamma_i} = 0.\)

The final statement follows from (1) and (2) and Lemma 3.9.7 if \(p \mid N\) and \(\epsilon_p = 1\) or Lemma 3.9.4 otherwise.

### 3.10. Labeling some cohomology classes

Later, in \[\text{loc. cit.}\], it will be convenient to have notation for the extension classes, taken as Galois cohomology classes, arising from homomorphisms \(B_N^{N,\text{min}} \to F_p\) and \(C_N^{N,\text{min}} \to F_p.\)

**Definition 3.10.1.** We call a cohomology class \(x \in H^i(Z[1/Np], M)\) ramified at a prime \(\ell\) when its image in \(H^i(I_\ell, M)\) is non-zero. For certain \(i, 0 \leq i \leq r,\) we designate \(b_i\) and \(c_i\) as follows.

- For \(i = 0, \ldots, r,\) let \(\tilde{b}_i\) denote the \(F_p^\times\)-scaling the Kummer cocycle of \(\ell_i\) such that \(\tilde{b}_i(\gamma_i) = 1,\) and let \(b_i \in \text{H}^1(Z[1/Np], F_p(1))\) be the class of \(\tilde{b}_i.\)
- Let \(T = \{0 \leq j \leq r : \ell_j \equiv \pm 1 (mod p)\}.\) For \(i \in T,\) let \(c_i \in \text{H}^1_{(p)}(Z[1/Np], F_p(-1))\) be an element that is ramified exactly at \(\ell_i\) and such that \(c_i(\gamma_i) = 1\) for any cocycle \(c_i\) representing \(c_i.\)

**Lemma 3.10.2.** The sets \(\{b_i\}_{i=0}^r\) and \(\{c_i\}_{i \in T}\) are well-defined and satisfy the following properties:

1. \(b_i\) is characterized up to \(F_p^\times\)-scaling by being ramified at \(\ell_i\) and unramified outside \(\{\ell_i, p\}\).
2. If \(p \not\mid N,\) the set \(\{b_i\}_{i=0}^r\) is a basis of \(\text{H}^1(Z[1/Np], F_p(1)).\)
3. The subset \(\{b_i : \ell_i \not= p\}\) is a basis of \(\text{H}^1_{\text{flat}}(Z[1/Np], F_p(1)).\)
4. The set \(\{c_i\}_{i \in T}\) is a basis of \(\text{H}^1_{(p)}(Z[1/Np], F_p(-1)).\)

**Proof.** See [WWE17c, Prop. 6.3.3], combined with Tate duality of Thm. B.3.2 of loc. cit., for the existence of \(c_i \in \text{H}^1_{(p)}(F_p(-1))\) characterized up to \(F_p^\times\)-scaling by being ramified exactly at \(\ell_i.\) These statements also imply Part (iv). Because \(\omega|_{\ell_i} = 1, c_i|_{\ell_i} : I_{\ell_i} \to F_p\) is a homomorphism not dependent on the choice of \(c_i.\)

The value of \(\tilde{b}_i(\gamma_i)\) is well-defined for the same reason when \(\ell_i \not= p,\) and \(b_{\gamma_p(\gamma_p)}\) is well-defined by the choice of \(\gamma_p\) (in §1.11). Parts (i), (ii), and (iii) follow from Kummer theory.

The stated bases are almost dual bases, with the exception arising from the possibility that \(b_i\) is ramified at \(p\) even when \(\ell_i \not= p.\)

**Lemma 3.10.3.** Under the perfect pairings

1. \(B_{\text{flat}} \otimes_{R_{\text{flat}}} F_p \times H^1_{\text{flat}}(Z[1/Np], F_p(1)) \to F_p,\)
2. \(C_{\text{ord}} \otimes_{R_{\text{ord}}} F_p \times H^1_{(p)}(Z[1/Np], F_p(-1)) \to F_p,\)
3. \(C_{\text{flat}} \otimes_{R_{\text{flat}}} F_p \times H^1_{(p)}(Z[1/Np], F_p(-1)) \to F_p,

defined by (3.9.3) and (3.9.6), the following are respective dual basis pairs

1. \(\{b_{\text{flat}, \gamma_i} : i = 0, \ldots, r \text{ if } \ell_i \not= p\} \text{ and } \{b_i : i = 0, \ldots, r \text{ if } \ell_i \not= p\} \)
(2) \( \{ c_{\text{ord}, \gamma_i} : i \in T \} \) and \( \{ c_i : i \in T \} \)
(3) \( \{ c_{\text{rat}, \gamma_i} : i \in T \} \) and \( \{ c_i : i \in T \} \)

Also, for \( 0 \leq i, j \leq r \) such that \( \ell_i = p \) or \( \ell_j \neq p \), we have \( b_i(b_{\text{ord}, \gamma_j}) = \partial_{ij} \).

**Proof.** We give the proof for (1), the other parts being similar. The pairing \((3.9.3)\) sends a class \( x \) by the choice of GMA structure on \( b \), we see that \( b \) is a particular cocycle representing \( x \) (the choice is determined by the choice of GMA structure on \( E_{\text{flat}} \)). However, if \( \omega(\tau) = 1 \), the value of \( \tilde{x}(\tau) \) is independent of the choice of cocycle, and we may write this value as \( x(\tau) \). Hence we see that \( b_i(b_{\text{flat}, \gamma_j}) = b_i(\gamma_j) = \partial_{ij} \). \( \square \)

**Definition 3.10.4.** For each \( i \in T \), let \( K_i \) be the fixed field of \( \ker(\tilde{c}_i|_{G_{\mathbb{Q}(\zeta_p), S}}) \), where \( \tilde{c}_i \) is any cocycle \( \tilde{c}_i : G_{\mathbb{Q}, S} \to \mathbb{F}_p(-1) \) representing \( c_i \).

One readily verifies that \( K_i \) is the unique extension of \( \mathbb{Q}(\zeta_p) \) satisfying the properties of \( \mathbb{Q}(\zeta_p) \).

4. TOWARD \( R = \mathbb{T} \)

4.1. The map \( R_N^\epsilon \to \mathbb{T}_N^\epsilon \). We prove the following proposition, following the construction technique of Calegari-Emerton [CE05, Prop. 3.12].

**Proposition 4.1.1.** There is a surjective homomorphism \( R_N^\epsilon \to \mathbb{T}_N^\epsilon \) of augmented \( \mathbb{Z}_p \)-algebras. Moreover, \( \mathbb{T}_N^\epsilon \) is generated as a \( \mathbb{Z}_p \)-algebra by \( T_q \) for \( q \nmid Np \).

**Proof.** For this proof, it is important to note that the elements \( \text{Tr}(\rho_N^\epsilon(\mathbb{F}_q)) \) for primes \( q \nmid Np \) generate \( R_N^\epsilon \) as a \( \mathbb{Z}_p \)-algebra. This follows the fact that \( R_N^\epsilon \) is a quotient of the (unrestricted) universal pseudodeformation ring \( R_{\mathbb{D}} \), the fact that traces of elements of \( G_{\mathbb{Q}, S} \) under the universal pseudodeformation generate \( R_{\mathbb{D}} \) (because the residue characteristic is not 2, see [Che14, Prop. 1.29]), and Chebotarev density.

In the rest of the proof, we use the notation \( \Sigma, \rho_f \) and \( \mathcal{O}_f \) established in Lemma 2.3.1.

We proceed in three steps:

**Step 1.** Construct a homomorphism \( R_N^\epsilon \to \mathcal{O}_f \) for each \( f \in \Sigma \) that sends \( \text{Tr}(\rho_N^\epsilon(\mathbb{F}_q)) \) to \( a_q(f) \) for each prime \( q \nmid Np \).

**Step 2.** Show that the resulting map \( R_N^\epsilon \to \mathbb{Z}_p \oplus \bigoplus_{f} \mathcal{O}_f \) sends \( \text{Tr}(\rho_N^\epsilon(\mathbb{F}_q)) \) to the image of \( T_q \) under the map \( \mathbb{T}_N^\epsilon \to \mathbb{Z}_p \oplus \bigoplus_{f} \mathcal{O}_f \) of \( (2.3.4) \), for each \( q \nmid Np \). This gives a homomorphism \( R_N^\epsilon \to \mathbb{T}_N^\epsilon \) whose image is the \( \mathbb{Z}_p \)-span of the \( T_q \) for all \( q \nmid Np \). This completes the proof if \( p \nmid N \).

**Step 3.** In the case that \( p \nmid N \), show that the image of \( R_N^\epsilon \to \mathbb{T}_N^\epsilon \) contains \( U_p \) and \( U_p^{-1} \). This shows both that \( R_N^\epsilon \to \mathbb{T}_N^\epsilon \) is surjective and that \( \mathbb{T}_N^\epsilon \) is generated as a \( \mathbb{Z}_p \)-algebra by \( T_q \) for \( q \nmid Np \).

**Proof of Step 1.** Let \( f \in \Sigma \). Then \( \psi(\rho_f) = \bar{D} \), so \( \psi(\rho_f) \) induces a map \( R_{\bar{D}} \to \mathcal{O}_f \). For each prime \( q \nmid Np \), we have \( \text{Tr}(\rho_f)(\mathbb{F}_q) = a_q(f) \), so \( R_{\bar{D}} \to \mathcal{O}_f \) sends \( \text{Tr}(\rho_{\bar{D}}^\epsilon(\mathbb{F}_q)) \) to \( a_q(f) \).

In order to show that \( R_{\bar{D}} \to \mathcal{O}_f \) factors through \( R_{\epsilon}^\epsilon \), we prove that \( \psi(\rho_f) \) and \( \rho_f \) are USN as per Definition 3.8.1.

- For \( \ell \nmid N \) with \( \ell \neq p \), \( \rho_f|_{G_\ell} \) is USN by Lemma 2.3.1(3).
- If \( p \nmid N \), or if \( p \mid N \) and \( f \) is odd at \( p \), then \( \rho_f|_{G_p} \) is finite-flat by Lemma 2.3.1(4a). Also, when \( p \mid N \), this implies that \( \rho_f|_{G_p} \) is USN by definition if \( \epsilon_p = +1 \) and by Corollary 3.7.6 if \( \epsilon_p = -1 \).
• If \( f \) is new at \( p \), then \( \epsilon_p = -1 \) and \( \rho_f|_{G_p} \) is \( \text{US}_p^{-1} \) by Lemma 2.3.14(b).

Proof of Step 2. By construction, the map \( R_N^* \to \mathbb{Z}_p \oplus \bigoplus_f \mathbb{O}_f \) sends \( \text{Tr}(\rho_N^*\pi_{q}) \) to \((1 + q, \bigoplus_j a_q(f))\), which, by 2.3.4, is the image of \( T_q \).

Proof of Step 3. Let \( r \in I_p \) be an element such that \( \omega(r) \neq 1 \). Let \( x = \kappa_{\text{cyc}}(r) \in \mathbb{Z}_p \), so that \( 1 - x \in \mathbb{Z}_p^\times \). Let \( \sigma_p \in G_p \) be the element defined in (1.11) and let \( z = \kappa_{\text{cyc}}(\sigma_p) \). By Lemma 2.3.14, we see that \( \text{Tr}(\rho_f)(\sigma_p) = x a_p(f) - 1 + a_p(f) \) and \( \text{Tr}(\rho_f)(\tau \sigma_p) = x z a_p(f) - 1 + a_p(f) \). Hence we have

\[
\begin{align*}
a_p(f) &= \frac{1}{x-1} \left( x \text{Tr}(\rho_f)(\sigma_p) - \text{Tr}(\rho_f)(\tau \sigma_p) \right) \quad \text{and} \\
a_p(f)^{-1} &= \frac{1}{x-z} \left( \text{Tr}(\rho_f)(\sigma_p) - \text{Tr}(\rho_f)(\tau \sigma_p) \right).
\end{align*}
\]

We see that \( U_p \) is the image of \( \frac{1}{x-1} \left( x \text{Tr}(\rho_N^*)(\sigma_p) - \text{Tr}(\rho_N^*)(\tau \sigma_p) \right) \) and \( U_p^{-1} \) is the image of \( \frac{1}{x-z} \left( \text{Tr}(\rho_N^*)(\sigma_p) - \text{Tr}(\rho_N^*)(\tau \sigma_p) \right) \). Since \( T_N^* \) is generated by \( T_q \) for \( q \nmid Np \) and \( T_p = U_p + pU_p^{-1} \), we see that \( R_N^* \to T_N^* \) is surjective.

4.2. Computation of \( (R_N^*)^{\text{red}} \). In this section, we will frequently make use of the elements \( \sigma_i \) and \( \gamma_i \) defined in (1.11). We denote by \( M^{p\text{-part}} \) the maximal \( p \)-primary quotient of a finite abelian group \( M \).

Consider the group \( \text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})^{p\text{-part}} \). We have isomorphisms

\[
\text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})^{p\text{-part}} \cong \bigoplus_{i=0}^r \text{Gal}(\mathbb{Q}(\zeta_{\ell_i})/\mathbb{Q})^{p\text{-part}} \cong \prod_{i=0}^r \mathbb{Z}_p/(\ell_i - 1)\mathbb{Z}_p.
\]

Since \( \mathbb{Q}(\zeta_{\ell_i})/\mathbb{Q} \) is totally ramified at \( \ell_i \), we can and do choose the second isomorphism so that the image of \( \gamma_i \) is \((0, \ldots, 0, 1, 0, \ldots, 0) \) (with 1 in the \( i \)-th factor). We define \( \alpha_i^j \) to be the \( j \)-th factor of the image of \( \sigma_i \), so that \( \sigma_i \mapsto (\alpha_0^j, \alpha_1^j, \ldots, \alpha_r^j) \) (we can and do assume that \( \alpha_i^0 = 0 \)).

Remark 4.2.1. Note that if \( \ell_j \equiv 1 \pmod{p} \), we may choose a surjective homomorphism \( \log_{\ell_j} : \mathbb{Z}/\ell_j\mathbb{Z}^\times \to \mathbb{F}_p^\times \) such that \( \log_{\ell_j}(\ell_i) \equiv \alpha_i^j \pmod{p} \). By abuse of notation, we denote by \( \log_j = \log_{\ell_j} \) a \( \mathbb{F}_p^\times \)-valued character of \( G_{Q,S} \) produced by composition with the canonical surjection \( G_{Q,S} \to \text{Gal}(\mathbb{Q}(\zeta_{\ell_j})/\mathbb{Q}) \cong (\mathbb{Z}/\ell_j\mathbb{Z})^\times \).

This isomorphism determines an isomorphism of group rings

\[
\mathbb{Z}_p[\text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})^{p\text{-part}}] \cong \mathbb{Z}_p \left[ \prod_{i=0}^r \mathbb{Z}_p/(\ell_i - 1)\mathbb{Z}_p \right] \cong \mathbb{Z}_p[y_0, \ldots, y_r]/(y_i^{v_i} - 1)
\]

where \( v_i = v_p(\ell_i - 1) \), and where the second isomorphism sends \( y_i \) to the group-like element \((0, \ldots, 0, 1, 0, \ldots, 0) \) (with 1 in the \( i \)-th factor). Let

\[
\langle - \rangle : G_{Q,S} \to (\mathbb{Z}_p[y_0, \ldots, y_r]/(y_i^{v_i} - 1))^{\times}
\]

be the character obtained by the quotient \( G_{Q,S} \to \text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})^{p\text{-part}} \) followed by this isomorphism. Note that

\[
\langle \gamma_i \rangle = y_i, \quad \langle \sigma_i \rangle = \prod_{j=0}^r y_j^{\alpha_i^j}.
\]

Let \( R_{\text{flat}}(\kappa_{\text{cyc}}) \) (resp. \( R_{\text{ord}}^{\text{red}}(\kappa_{\text{cyc}}) \)) be the quotient of the finite-flat global deformation ring \( R_{\text{flat}} \) (resp. ordinary global deformation ring \( R_{\text{ord}} \)) defined in 3.9.1 (resp.
The composite of \[ D^\gamma_B (\gamma) - \kappa_{\text{cyc}}(\gamma) : \gamma \in G_{Q,S} \]. That is, we are insisting that the determinant is \( \kappa_{\text{cyc}} \).

**Lemma 4.2.2.** The surjection \( R_{\text{ord}} \to R_{\text{flat}} \) induces an isomorphism \( R_{\text{ord}}^{\kappa_{\text{cyc}}} \cong R_{\text{flat}}^{\kappa_{\text{cyc}}} \). Moreover, they are both isomorphic as rings to

\[
\mathbb{Z}_p[y_0, \ldots, y_r]/(y_i^{\alpha_i} - 1)
\]

and the universal reducible pseudorepresentation pulls back to \( D^{\text{red}} = \psi(\kappa_{\text{cyc}}(-)^{-1} \oplus (-)) \) via these isomorphisms.

**Proof.** The quotient map \( R_{\text{ord}} \to R_{\text{flat}} \) comes from the first part of Corollary 3.7.6 and the two rings differ only in the local condition at \( p \). After imposing the reducibility and determinant conditions, the universal pseudodeformations both have the form \( \psi(\kappa_{\text{cyc}} \chi^{-1} \oplus \chi) \) for a character \( \chi \) that deforms the trivial character. By the latter parts of the corollary, the finite-flat and ordinary conditions on such pseudodeformations are identical. The last statement is proven just as in [WWE17c, Lem. 5.1.1].

Let \( Y_i = 1 + y_i \).

**Lemma 4.2.3.** There is an isomorphism

\[
(R_N^\text{red}) \cong \mathbb{Z}_p[Y_0, \ldots, Y_r]/a
\]

where \( a \) is the ideal generated by the elements

\[
Y_i^2, (\ell_i - 1)Y_i, (\epsilon_i + 1)Y_i, Y_i \left( \prod_{j=0}^{r} (1 - \alpha_i^j Y_j) - 1 \right), Y_i \left( \prod_{j=0}^{r} (1 + \alpha_i^j Y_j) - 1 \right),
\]

for \( i = 0, \ldots, r \), where \( \alpha_i^j \in \mathbb{Z}_p \) is any lift of \( \alpha_i^j \in \mathbb{Z}_p/(\ell_j - i)\mathbb{Z}_p \) (note that \( a \) is independent of the choice of this lift).

**Proof.** We consider \( (E_N^\epsilon)^{\text{red}} = E_N^\epsilon \otimes R_N^{\text{red}} \). We write the base-change of \( \rho_N^\epsilon \) to this algebra as \( \rho^{\text{red}} \), for simplicity. Write \( \langle \cdot \rangle : G_{Q,S} \to ((R_N^\text{red})^\times) \) for the composite of \( \langle \cdot \rangle \) with the quotient \( R_{\text{flat}}^{\kappa_{\text{cyc}}} \to (R_N^\text{red})^\times \), which exists by Proposition 3.8.3 (We use \( R_{\text{flat}}^{\kappa_{\text{cyc}}} \) even in the ordinary case, in light of Lemma 4.2.2)

First we show that the map \( R_{\text{flat}}^{\kappa_{\text{cyc}}} \to (R_N^\text{red})^\times \) factors through \( \mathbb{Z}_p[Y_0, \ldots, Y_r]/a \). We can write \( \rho^{\text{red}} \) in GMA notation as

\[
\rho^{\text{red}} = \begin{pmatrix} \kappa_{\text{cyc}}\langle \cdot \rangle^{-1} & * \\ * & \langle \cdot \rangle \end{pmatrix}.
\]

Since \( V_{\rho^{\text{red}}}^\epsilon(\gamma_i, \gamma_i) = (\rho^{\text{red}}(\gamma_i) - 1)^2 = 0 \in (E_N^\epsilon)^{\text{red}} \), we see that \( Y_i^2 = 0 \) in \( (R_N^\text{red}) \). Since \( (1 + Y_i)^{p^\epsilon_i} - 1 = 0 \), this implies that \( p^\epsilon_i Y_i = 0 \) in \( (R_N^\text{red}) \). Moreover, by Lemma 3.4.5 if \( \epsilon_i = +1 \) and \( v_i > 0 \), then \( \rho^{\text{red}}(\gamma_i) = 1 \); for such \( i \), this implies that \( Y_i = 0 \) in \( (R_N^\text{red}) \). We can rephrase this as \( (\epsilon_i + 1)Y_i = 0 \) for all \( i \).

From now on, consider \( i \) such that \( \epsilon_i = -1 \). Already, we see that

\[
\langle \sigma_i \rangle = \prod_{j=0}^{r} y_j^{\alpha_i^j} = \prod_{j=0}^{r} (1 + \alpha_i^j Y_j)
\]
Since $V_{\rho, \sigma} = (\rho(\gamma_i) - 1)(\rho(\sigma) - 1) = 0$ in $(E_\sigma)^{\text{red}}$, we obtain
\[
\langle \gamma_i \rangle^{-1} - 1(\ell_i \langle \sigma_i \rangle^{-1} - 1) = 0, \quad (\langle \gamma_i \rangle - 1)(\langle \sigma_i \rangle - 1) = 0.
\]
These imply
\[
0 = Y_i \left( \prod_{j=0}^{r} (1 - \alpha_i^j Y_j) - 1 \right) = Y_i \left( \prod_{j=0}^{r} (1 + \alpha_i^j Y_j) - 1 \right).
\]
This shows that $R_{\text{flat}}(\kappa_{\text{cyc}}) \to (R_N)^{\text{red}}$ factors through $\mathbb{Z}_p[Y_0, \ldots, Y_r]/\mathfrak{a}$. It remains to verify that the pseudorepresentation $D : G_{\mathbb{Q}, S} \to \mathbb{Z}_p[Y_0, \ldots, Y_r]/\mathfrak{a}$ defined by $\psi(\kappa_{\text{cyc}}) \equiv -1 + \langle \gamma_i \rangle^{-1} \oplus \langle \sigma_i \rangle^{-1}$ is $US_N^\epsilon$. This is checked easily.

5. The case $\epsilon = (-1, 1, 1, \ldots, 1)$

In this section, we consider the case where $\epsilon_0 = -1$ and $\epsilon_i = 1$ for $0 < i \leq r$.

We can and do, for this section, assume that the primes $\{\ell_i\}_{i=0}^r$ are ordered so that $\ell_i \equiv -1 \pmod{p}$ for $i = 1, \ldots, s$ and $\ell_i \not\equiv -1 \pmod{p}$ for $s < i \leq r$. Here $s$ is an integer between 0 and $r$. The most interesting case is $s = r$, and, in fact, we immediately reduce to this case.

5.1. Reduction to the case $s = r$. Let $N(s) = \prod_{i=0}^{s} \ell_i$ and $(s) \in \{\pm 1\}^{s+1}$ be defined by $\epsilon(s)_0 = -1$ and $\epsilon(s)_i = 1$ for $0 < i \leq s$. There is a natural map $T_N \to T_{N(s)}$ by restricting to the space of forms that are old at $\ell_i$ for $s < i \leq r$.

There is also a natural surjection $R_N^{\epsilon(s)} \to R_N^{\epsilon(s)}$, since $\rho_N^{\epsilon(s)}$ is unramified (resp. finite-flat) at $\ell_i$ when $\ell_i \neq p$ (resp. $\ell_i = p$) and $s < i \leq r$.

Lemma 5.1.1. The natural map $R_N \to R_N^{\epsilon(s)}$ is an isomorphism. Moreover, if the map $R_N^{\epsilon(s)} \to T_N^{\epsilon(s)}$ is an isomorphism, then the maps $R_N^{\epsilon(s)} \to T_N^{\epsilon(s)}$ are isomorphisms.

Proof. The isomorphy of $R_N^{\epsilon(s)} \to R_N^{\epsilon(s)}$ can be rephrased as saying that, for all $s < i \leq r$, $\rho_N$ is unramified (resp. finite-flat) at $\ell_i$ if $\ell_i \neq p$ (resp. if $\ell_i = p$). This follows from Lemma 3.4.5 and [3.6]. For the second statement, consider the commutative diagram of surjective ring homomorphisms

\[
\begin{array}{ccc}
R_N^{\epsilon(s)} & \longrightarrow & T_N^{\epsilon(s)} \\
\downarrow & & \downarrow \\
R_N^{\epsilon(s)} & \longrightarrow & T_N^{\epsilon(s)} \\
\end{array}
\]

5.2. The case $s = r$. Now we assume that $s = r$ (i.e. that $\ell_i \equiv -1 \pmod{p}$ for $i = 1, \ldots, r$). We write $J_{\min} \subseteq R_N^{\epsilon(s)}$ for the augmentation ideal, and $J^{\text{red}} = \ker(R_N^{\epsilon(s)} \to (R_N^{\epsilon(s)})^{\text{red}})$. We have the following consequence of Wiles’s numerical criterion [Wil95 Appendix].

Proposition 5.2.1. The surjection $R_N^{\epsilon(s)} \to T_N^{\epsilon(s)}$ is a surjection of complete intersection rings if and only if
\[
\#J_{\min} / J_{\min}^2 \leq p^{v_p(\ell_i - 1)} \prod_{i=1}^{r} p^{v_p(\ell_i + 1)}.
\]
If this is the case, then equality holds.

Proof. The surjection comes from Proposition 4.1.1. Note that
\[ p^{v_p(t_0-1)} \prod_{i=1}^r p^{v_p(t_i+1)} = \#\mathbb{Z}_p/a_0(E^\epsilon)\mathbb{Z}_p. \]
The proposition follows from Theorem 2.2.1 and the numerical criterion, as in [WWE17c, Thm. 7.1.1]. □

Lemma 5.2.2. There is an isomorphism
\[ J_{\text{min}}/J_{\text{red}} \cong \mathbb{Z}_p/((\ell_0-1)\mathbb{Z}_p) \]
sending \( d_{\gamma_0} - 1 \) to 1, and \( J_{\text{min}}^2 \subset J_{\text{red}} \).

Proof. By Lemma 4.2.3 we have
\[ (R_{N}^e)_{\text{red}} = \mathbb{Z}_p[Y_0]/((\ell_0-1)Y_0, Y_0^2), \]
and we can easily see that \( d_{\gamma_0} - 1 \) maps to \( Y_0 \) and generates the image of \( J_{\text{min}} \). Since \( Y_0^2 = 0 \), we have the second statement. □

Lemma 5.2.3. There is a surjection
\[ \mathbb{Z}_p/(\ell_0-1)\mathbb{Z}_p \oplus \bigoplus_{i=1}^r \mathbb{Z}_p/(\ell_i+1)\mathbb{Z}_p \twoheadrightarrow J_{\text{red}}/J_{\text{min}} J_{\text{red}} \]
given by \( e_i \mapsto b_{\gamma_0} c_{\gamma_i} \).

Proof. By Lemma 3.9.8 we have surjections
\[ \mathbb{Z}_p \twoheadrightarrow B_{N}^{e,\min}, \quad 1 \mapsto b_{\gamma_0} \]
and
\[ \mathbb{Z}_p/(\ell_0-1)\mathbb{Z}_p \oplus \bigoplus_{i=1}^r \mathbb{Z}_p/(\ell_i+1)\mathbb{Z}_p \twoheadrightarrow C_{N}^{e,\min}, \quad e_i \mapsto c_{\gamma_i}. \]
As is true in any GMA (see e.g. [BC09, Prop. 1.5.1]), we have a surjection
\[ B_{N}^{e,\min} \otimes C_{N}^{e,\min} \twoheadrightarrow J_{\text{red}}/J_{\text{min}} J_{\text{red}}, \quad b \otimes c \mapsto bc. \]
Combining these, we have the lemma. □

Lemma 5.2.5. The element \( b_{\gamma_0} c_{\gamma_i} \in R_{N}^e \) is in \( J_{\text{min}}^2 \).

Proof. Since \( V_{\epsilon_0}^e(\gamma_0, \gamma_0) = (\rho_N(\gamma_0)-1)^2 = 0 \), we see that \( (a_{\gamma_0} - 1)^2 + b_{\gamma_0} c_{\gamma_0} = 0 \). Since \( a_{\gamma_0} - 1 \in J_{\text{min}} \), we have the lemma. □

We have arrived at the main theorem.

Theorem 5.2.6. Let \( N = \ell_0 \ell_1 \cdots \ell_r \) and \( \epsilon = (-1, 1, \ldots, 1) \). Then the map \( R_{N}^e \to \mathbb{Z}_p \) is an isomorphism of augmented \( \mathbb{Z}_p \)-algebras, and both rings are complete intersection. The ideal \( J_{\text{min}} \) is generated by the elements \( b_{\gamma_0} c_{\gamma_i} \) for \( i = 1, \ldots, r \) together with \( d_{\gamma_0} - 1 \). There is an exact sequence
\[ 0 \to \bigoplus_{i=1}^r \mathbb{Z}_p/(\ell_i+1)\mathbb{Z}_p \to I'/I' \to \mathbb{Z}_p/(\ell_0-1)\mathbb{Z}_p \to 0. \]
Proof. By Lemma \[5.2.2\] there is an exact sequence
\[0 \to J^\text{red} / J^\text{min}^2 \to J^\text{min} / J^\text{min}^2 \to \mathbb{Z}_p / (\ell_0 - 1)\mathbb{Z}_p \to 0 \tag{5.2.8}\]
Combining Lemmas \[5.2.3\] and \[5.2.5\] we see that there is a surjection given by \(J\) that \(\text{red} / J^\text{min}^2\) is an isomorphism. Using Lemma \[5.2.2\] and Nakayama’s lemma, this shows that this inequality is actually equality. This implies that \(J^\text{min}\) is generated by the stated elements. Since \(J^\text{min}\) maps isomorphically onto \(I^e\), the desired sequence follows from \(5.2.8\). \(\square\)

6. The case \(\epsilon = (-1, -1)\)

In this section, we assume that \(r = 1\) and also that \(\epsilon = (-1, -1)\).

6.1. No interesting primes. If \(\ell_i \neq 1 \pmod{p}\) for \(i = 0, 1\), then there are no cusp forms congruent to the Eisenstein series.

**Theorem 6.1.1.** If \(\ell_i \neq 1 \pmod{p}\) for \(i = 0, 1\), then \(T_N^\epsilon = \mathbb{Z}_p\) and \(T_N^{\epsilon, 0} = 0\).

**Proof.** It is enough to show that \(R_N^\epsilon = \mathbb{Z}_p\). By Lemma \[4.2.3\] we have \(R_N^\epsilon, \text{red} = \mathbb{Z}_p\), and by Lemma \[3.9.8\] we have \(C_N^\epsilon = 0\), so \(J^{\text{red}} = 0\). This implies \(R_N^\epsilon = \mathbb{Z}_p\). \(\square\)

6.2. Generators of \(B_N^\epsilon\). Since nothing interesting happens if there are no interesting primes, we now assume that \(\ell_0 \equiv 1 \pmod{p}\). We emphasize that, in this section, we do not assume that \(\ell_1 \neq p\). Recall the notation \(a_\tau, b_\tau, c_\tau, d_\tau\) for \(\tau \in G_{Q,S}\) from \(3.9.1\) and the elements \(\gamma_i, \sigma_i \in G_{Q,S}\) from \[1.11\].

**Lemma 6.2.1.** Assume that \(\ell_1\) is not a \(p\)-th power modulo \(\ell_0\). Then the subset \(\{b_{\gamma_0}, b_{\sigma_0}\} \subset B_N^\epsilon\) generates \(B_N^\epsilon\) as a \(R_N^\epsilon\)-module.

**Proof.** We give the proof in the case \(\ell_1 = p\); the case \(\ell_1 \neq p\) is exactly analogous, changing ‘ordinary’ to ‘finite-flat’ everywhere. Because \(B_{N, \text{ord}}^\epsilon\) surjects onto \(B_N^\epsilon\) and by Nakayama’s lemma, it is enough to show that the images \(b_{\text{ord,} \gamma_0}, b_{\text{ord,} \sigma_0}\) of \(b_{\text{ord,} \gamma_0}, b_{\text{ord,} \sigma_0}\) in \(B_{N, \text{ord}}^\epsilon / pB_{N, \text{ord}}^\epsilon\) generate \(B_{N, \text{ord}}^\epsilon / pB_{N, \text{ord}}^\epsilon\).

Using \(b_1, \tilde{b}_1\) defined in \[3.10\] and the lemmas there, we know that \(\{b_{\text{ord,} \gamma_0}, \tilde{b}_{\text{ord,} \gamma_1}\}\) is a basis for \(B_{N, \text{ord}}^\epsilon / pB_{N, \text{ord}}^\epsilon\) and \(b_1(b_{\text{ord,} \gamma_1}) = b_1(b_{\text{ord,} \gamma_1}) = \partial_{1j}\) for \(j = 0, 1\). Hence it is enough to show that \(b_1(b_{\text{ord,} \sigma_0}) \neq 0\). As in the proof of Lemma \[3.10.3\] the fact that \(\omega(\sigma_0) = 1\) implies that \(b_1(b_{\text{ord,} \sigma_0}) = \tilde{b}_1(\sigma_0)\). Because \(\ell_1\) is not a \(p\)-th power modulo \(\ell_0\), class field theory implies that \(\tilde{b}_1(\sigma_0) \neq 0\). \(\square\)

**Proposition 6.2.2.** Assume that \(\ell_1\) is not a \(p\)-th power modulo \(\ell_0\). Then
\[b_{\gamma_0} c_{\gamma_1}, b_{\gamma_1} c_{\gamma_0}, b_{\gamma_1} c_{\gamma_0} \in J^{\text{min}}^2.\]
If, in addition, \(\ell_1 \equiv 1 \pmod{p}\) and \(\ell_0\) is not a \(p\)-th power modulo \(\ell_1\), then \(b_{\gamma_0} c_{\gamma_1} \in J^{\text{min}}^2\) as well.
The proof for \( b_{\sigma_0}, c_{\gamma_0}, b_{\gamma_0}, c_{\gamma_0} \) is just as in Lemma 3.2.5. If we prove that \( b_{\gamma_0}, c_{\gamma_0} \in J^{\min}_{1} \), then we get \( b_{\sigma_0}, c_{\gamma_0} \in J^{\min}_{1} \) in the second statement by symmetry. So it suffices to prove \( b_{\gamma_0}, c_{\gamma_0} \in J^{\min}_{2} \).

Let \( X = a_{\sigma_0} - \ell_0 \) and \( W = a_{\gamma_0} - 1 \), and note that \( X, W \in J^{\min}_{1} \). From the \((1,1)\)-coordinate of the equation \( V^{\min}_{p_N}(\sigma_0, \gamma_0) = 0 \) defined in (3.4.2), we see that \( XW + b_{\sigma_0}c_{\gamma_0} = 0 \). In particular, \( b_{\sigma_0}c_{\gamma_0} \in J^{\min}_{2} \).

By Lemma 6.2.1 we know that \( b_{\gamma_1} \) is in the \( R^{\circ}_{N} \)-linear span of \( b_{\sigma_0} \) and \( b_{\gamma_0} \). Because both \( b_{\sigma_0}, c_{\gamma_0} \) and \( b_{\gamma_0}, c_{\gamma_0} \) lie in \( J^{\min}_{2} \), so does \( b_{\gamma_1}, c_{\gamma_0} \).

### 6.3. One interesting prime

Now we assume that \( \ell_0 \equiv 1 \pmod{p} \) and \( \ell_1 \not\equiv 1 \pmod{p} \) (including the possibility that \( \ell_1 = p \)). There is a natural surjective homomorphism \( \mathbb{T}^\circ_N \rightarrow \mathbb{T}_{\ell_0} \) by restricting to forms that are old at \( \ell_1 \).

**Theorem 6.3.1.** Assume that \( \ell_0 \equiv 1 \pmod{p} \), that \( \ell_1 \not\equiv 1 \pmod{p} \), and that \( \ell_1 \) is not a \( p \)-th power modulo \( \ell_0 \). Then the natural map \( \mathbb{T}^\circ_N \rightarrow \mathbb{T}_{\ell_0} \) is an isomorphism. In particular, \( I^\circ \) is principal, \( \mathbb{T}^\circ_N \) and \( \mathbb{T}^{\circ, 0}_N \) are complete intersections, and there are no newforms in \( S_{2}(N)_{\text{Eis}} \).

**Proof.** Just as in the proof of Lemma 5.1.1, it suffices to show that the map \( \mathbb{R}^\circ_N \rightarrow \mathbb{T}_{\ell_0} \) is an isomorphism. By Lemma 4.2.1 there is an isomorphism

\[
\mathbb{R}^\circ_N \approx \mathbb{Z}_p\langle Y_0 \rangle / \langle Y_0^2, (\ell_0 - 1)Y_0 \rangle,
\]

where the image of \( J^{\min}_{1} \) is the ideal generated by \( Y_0 \). This implies that \( J^{\min}_{2} \subset J^{\text{red}} \) and that there is an isomorphism

\[
\mathbb{Z}_p / (\ell_0 - 1)\mathbb{Z}_p \approx J^{\min}_{1} / J^{\text{red}}, \quad 1 \mapsto Y_0.
\]

On the other hand, we know that \( J^{\text{red}} \) is generated by the set \( \{b_{\gamma_0}, c_{\gamma_0}, b_{\gamma_1}, c_{\gamma_0}\} \) by Lemma 3.9.8 and the surjection (5.2.4). By Proposition 6.2.2 we see that this set is contained in \( J^{\min}_{2} \). Hence \( J^{\text{red}} \subset J^{\min}_{2} \), and so \( J^{\text{red}} = J^{\min}_{2} \).

Now we have \( \# J^{\min}_{1} / J^{\min}_{2} = p^\epsilon_{\epsilon_0}(\ell_0 - 1) \) and, just as in Proposition 5.2.1, the numerical criterion implies that \( \mathbb{R}^\circ_N \rightarrow \mathbb{T}_{\ell_0} \) is an isomorphism. \( \square \)

**Remark 6.3.2.** The assumption that \( \ell_1 \) is not a \( p \)-th power modulo \( \ell_0 \) is necessary: see the examples in (1.9.2).

### 6.4. Two interesting primes

Now we consider the case where \( \ell_i \equiv 1 \pmod{p} \) for \( i = 0,1 \).

**Theorem 6.4.1.** Let \( N = \ell_0 \ell_1 \) and \( \epsilon = (-1, -1) \). Assume that \( \ell_i \equiv 1 \pmod{p} \) for \( i = 0,1 \) and assume that neither prime is a \( p \)-th power modulo the other. Then the map \( \mathbb{R}^\circ_N \rightarrow \mathbb{T}^\circ_N \) is an isomorphism of complete intersection rings augmented over \( \mathbb{Z}_p \), and there is an isomorphism

\[
I^\circ / I^{\epsilon_2} \approx \mathbb{Z}_p / (\ell_0 - 1)\mathbb{Z}_p \oplus \mathbb{Z}_p / (\ell_1 - 1)\mathbb{Z}_p.
\]

**Proof.** By Lemma 4.2.3 we see that there is an isomorphism

\[
\mathbb{R}^\circ_N \approx \mathbb{Z}_p\langle Y_0, Y_1 \rangle / \langle Y_0^2, Y_0Y_1, Y_1^2, (\ell_0 - 1)Y_0, (\ell_1 - 1)Y_1 \rangle
\]

and that the image of \( J^{\min}_{1} \) is the ideal generated by \( (Y_0, Y_1) \). In particular \( J^{\min}_{1} \subset J^{\text{red}} \) and

\[
J^{\min}_{1} / J^{\text{red}} \approx \mathbb{Z}_p / (\ell_0 - 1)\mathbb{Z}_p \oplus \mathbb{Z}_p / (\ell_1 - 1)\mathbb{Z}_p.
\]
Moreover, by Proposition 6.2.2 and Lemma 3.9.8 we see that $J^{\text{red}} \subset J^{\text{min}}^2$ so we have

$$J^{\text{min}} / J^{\text{min}}^2 = J^{\text{min}} / J^{\text{red}} \cong \mathbb{Z}_p / (\ell_0 - 1)\mathbb{Z}_p \oplus \mathbb{Z}_p / (\ell_1 - 1)\mathbb{Z}_p.$$ 

In particular, $\# J^{\text{min}} / J^{\text{min}}^2 = p^{r_p(\ell_0 - 1) + r_p(\ell_1 - 1)}$.

Now the numerical criterion of Proposition 5.2.1 implies that $R_N^\epsilon \to T_N^\epsilon$ is an isomorphism of complete intersection augmented $\mathbb{Z}_p$-algebras. It follows that $I^\epsilon = J^{\text{min}}$, and so the description of $I^\epsilon / I^\epsilon^2$ also follows.

\[ \square \]

**Remark 6.4.2.** Again, the assumptions are necessary. See the examples in §1.9.3

Let $T_N^\epsilon, \text{old}$ be the quotient of $T_N^\epsilon$ that acts faithfully on oldforms.

**Theorem 6.4.3.** Let $N = \ell_0 \ell_1$ and $\epsilon = (-1, -1)$ and assume that $\ell_i \equiv 1 \pmod{p}$ for $i = 0, 1$. Then $T_N^\epsilon, \text{old}$ is non-Gorenstein. In particular, if neither prime $\ell_i$ is a $p$-th power modulo the other, then there are newforms in $S_2(N)^{\epsilon, \text{Eis}}$.

**Proof.** Note that there are no newforms in $S_2(N)^{\epsilon, \text{Eis}}$ if and only if the map $T_N^\epsilon \to T_N^\epsilon, \text{old}$ is an isomorphism. Hence the second statement follows from the first statement and Theorem 6.4.1.

Note that, for either prime $\ell_i$, the composite map $T_N^\epsilon \to T_{\ell_i} \to \mathbb{Z}_p$ is the augmentation map of $T_N^\epsilon$ (sending $T_q$ to $q + 1$). Hence there is a map

$$T_N^\epsilon \to T_{\ell_0} \times_{\mathbb{Z}_p} T_{\ell_1}.$$ 

We claim that this map factors through an isomorphism $T_N^\epsilon, \text{old} \cong T_{\ell_0} \times_{\mathbb{Z}_p} T_{\ell_1}$. To see this, let $a_i = \ker(T_N^\epsilon \to T_{\ell_i})$. By duality, $a_i$ is the annihilator of the image of $M_2(\ell_i)^{\epsilon, \text{Eis}}$ in $M_2(N)^{\epsilon, \text{Eis}}$. It follows that $a_0 \cap a_1$ is the annihilator of all oldforms, which is the kernel of $T_N^\epsilon \to T_N^\epsilon, \text{old}$. The claim follows from Lemma C.2.1.

Using this isomorphism and Mazur’s results (4.1.1) on the structure of $T_{\ell_i}$, it is then a simple computation to see that

$$T_N^\epsilon, \text{old} / p T_N^\epsilon, \text{old} \cong \mathbb{F}_p[y_0, y_1] / (y_0^{e_0}, y_1^{e_1}, y_0 y_1)$$

for some $e_i > 0$. We see that $\text{Soc}(T_N^\epsilon, \text{old} / p T_N^\epsilon, \text{old}) = \mathbb{F}_p y_0^{e_0 - 1} \oplus \mathbb{F}_p y_1^{e_1 - 1}$. By Lemma C.1.3, $T_N^\epsilon, \text{old}$ is not Gorenstein.

\[ \square \]

### 7. Generators of the Eisenstein ideal

In this section, we prove Part (4) of Theorem 1.4.1 about the number of generators of the Eisenstein ideal, as well as Theorems 1.6.1 and 1.6.3 about specific generators.

#### 7.1. Determining the number of generators of $I^\epsilon$ when $\epsilon = (-1, 1, \ldots, 1)$.

In this subsection, we prove Part (4) of Theorem 1.4.1. Assume we are in the setting of that theorem, so $\epsilon = (-1, 1, \ldots, 1)$. Recall the fields $K_i$ of Definition 3.10.4.

**Theorem 7.1.1.** Assume that $\ell_i \equiv 1 \pmod{p}$ for $i = 1, \ldots, r$. The minimal number of generators of $I^\epsilon$ is $r + \delta$ where

$$\delta = \begin{cases} 1 & \text{if } \ell_0 \text{ splits completely in } K_i \text{ for } i = 1, \ldots, r \\ 0 & \text{otherwise.} \end{cases}$$
This immediately implies Part (4) of Theorem 7.1.1 by Lemma 5.1.1. For the rest of §7.1 we assume that \( r > 0 \) and \( \ell_i \equiv -1 \) (mod \( p \)) for \( i = 1, \ldots, r \), and we use \( \delta \) to refer to the integer \((7.1.2)\).

Note that, since \( m = J_{\text{min}} + pR_N \subset R_N^\ast \) is the maximal ideal, we have
\[
J_{\text{min}} / m_{\text{min}} \cong m / (p, m^2).
\]

By Nakayama’s lemma, the minimal number of generators of \( J_{\text{min}} \) is \( \dim_{\mathbb{F}_p} m / (p, m^2) \).

By Theorem 5.2.6, it suffices to show that \( \dim_{\mathbb{F}_p} m / (p, m^2) = r + \delta \), and this is what we will prove.

Recall the notation of §3.10 in particular, the class \( b_0 \in H^1(\mathbb{Z}[1/Np], \mathbb{F}_p(1)) \) and the representing cocycle \( \tilde{b}_0 \), as well as the classes \( c_0, \ldots, c_r \in H^1_p(\mathbb{Z}[1/Np], \mathbb{F}_p(-1)) \) (note that \( c_0 \) is only defined if \( \ell_0 \equiv \pm 1 \) (mod \( p \)). The starting point is the following proposition, which is proven in Appendix B.

**Proposition 7.1.3.** Let \( i \in \{1, \ldots, r\} \). Then \( \ell_0 \) splits completely in \( K_i \) if and only if \( \ell_0 \equiv 1 \) (mod \( p \)) and \( b_0 \cup c_i \) vanishes in \( H^2(\mathbb{Z}[1/Np], \mathbb{F}_p) \).

We can now prove one implication of Theorem 7.1.1.

**Proposition 7.1.4.** Suppose that the minimal number of generators of \( I^\ast \) is \( r + 1 \). Then \( \delta = 1 \).

**Proof.** By Theorem 5.2.6 we see that minimal number of generators of \( I^\ast \) is \( r + 1 \) if and only if \( \ell_0 \equiv 1 \) (mod \( p \)) and the images of the elements \( b_{\gamma_i}c_i \), for \( i = 1, \ldots, r \) in \( m / (p, m^2) \) are linearly independent. In particular, for each \( i \), the image of \( b_{\gamma_i}c_i \) in \( m / (p, m^2) \) is non-zero. Fix such an \( i \), and let (writing \( \mathbb{F}_p[\varepsilon] \) for \( \mathbb{F}_p[\varepsilon]/(\varepsilon^2) \))
\[
\alpha : R_N^\ast / (p, m^2) \to \mathbb{F}_p[\varepsilon]
\]
be a ring homomorphism sending \( b_{\gamma_i}c_i \) to \( \varepsilon \).

Let \( E = (\mathbb{F}_p[\varepsilon], \mathbb{F}_p, \varepsilon) \) be the \( \mathbb{F}_p[\varepsilon] \)-GMA with data \( (\mathbb{F}_p, \mathbb{F}_p, m) \) where \( m : \mathbb{F}_p \times \mathbb{F}_p \to \mathbb{F}_p[\varepsilon] \) is the map \((x, y) \mapsto xy\varepsilon\). By Lemma 3.10.3 we have a homomorphism of GMAs \( A : E_N^\ast \to E \) given by
\[
A = \begin{pmatrix}
\alpha \\
\tilde{b}_0 \\
\gamma_i \\
\alpha
\end{pmatrix}.
\]

Let \( D_A = \psi(A \circ \rho_N^\ast) : G_{Q,S} \to \mathbb{F}_p[\varepsilon] \) be the corresponding deformation of \( \tilde{D} \). Then \( D_A \) contributes a non-zero element to the tangent space \( t_\tilde{D} \) of \( R_{D} / pR_{\tilde{D}} \). Examining \cite{Bel12}, the image of \( D_A \) under \( \iota \) in the exact sequence of \cite{Bel12} Thm. A
\[
t_\tilde{D} \xrightarrow{\iota} H^1(\mathbb{F}_p(1)) \otimes_{\mathbb{F}_p} H^1(\mathbb{F}_p(-1)) \xrightarrow{\gamma^i \otimes c_i \mapsto (\gamma^i \cup c_i \cup \gamma^i)^+} H^2(\mathbb{F}_p) \oplus H^2(\mathbb{F}_p)
\]
is \( b_0 \otimes c_i \), and hence \( b_0 \cup c_i = 0 \). Since this is true for all \( i \), Proposition 7.1.3 implies that \( \delta = 1 \).

The remainder of the proof of Theorem 7.1.1 relies on the following construction.

7.1.1. **Construction of a maximal first-order pseudodeformation.** Let \( H \) be the kernel of the map
\[
H^1_p(\mathbb{Z}[1/Np], \mathbb{F}_p(-1)) \to H^2(\mathbb{Z}[1/Np], \mathbb{F}_p) \oplus H^1(I_{\ell_0}, \mathbb{F}_p(-1)),
\]
\[
x \mapsto (b_0 \cup x, x|_{I_{\ell_0}}).
\]
Lemma 7.1.5. If $\ell_0 \equiv 1 \pmod{p}$ and $\delta = 0$, then $b_0 \cup c_i \neq 0$ for some $i$. In that case, there are elements $\alpha_j \in \mathbb{F}_p$ such that the set $\{c_j - \alpha_j c_i\}$ for $j \in \{1, \ldots, r\} \setminus \{i\}$ is a basis for $H$. Otherwise, the set $\{c_1, \ldots, c_r\}$ is a basis for $H$.

**Proof.** The first statement follows from Proposition 7.1.3. Recall that $c_i$ is a basis at $\ell_0$ if and only if $i = 0$, so $H$ is contained in the span of the linearly independent set $\{c_1, \ldots, c_r\}$. Since

$$\dim_{\mathbb{F}_p} H^2(\mathbb{Z}[1/Np], \mathbb{F}_p) = \begin{cases} 1 & \text{when } \ell_0 \equiv 1 \pmod{p}, \\ 0 & \text{when } \ell_0 \not\equiv 1 \pmod{p}, \end{cases}$$

the lemma follows. \qed

**Lemma 7.1.6.** If $\ell_0 \neq p$ and $h \in H$, the image $h|_{G_{\ell_0}} \in H^1(\mathbb{Q}_{\ell_0}, \mathbb{F}_p(-1))$ is zero.

**Proof.** If $\ell_0 \not\equiv \pm 1 \pmod{p}$, then $H^1(\mathbb{Q}_{\ell_0}, \mathbb{F}_p(-1)) = 0$. If $\ell_0 \equiv -1 \pmod{p}$, then $H^1(\mathbb{Q}_{\ell_0}, \mathbb{F}_p(-1)) = H^1(\mathbb{Q}_{\ell_0}, \mathbb{F}_p(1))$, and so this follows from Lemma B.1.3. Now assume $\ell_0 \equiv 1 \pmod{p}$. Then, since $h \cup b_0 = 0$ in $H^2(\mathbb{Z}[1/Np], \mathbb{F}_p)$, $b_0$ is ramified at $\ell_0$ and $h$ is unramified at $\ell_0$, Lemma B.1.3 implies that $h|_{G_{\ell_0}} = 0$. \qed

**Construction 7.1.7.** We construct a cocycle $C : G_{Q,S} \to H^*(1)$, where $H^* = \text{Hom}_G(H, \mathbb{F}_p)$ with trivial $G_{Q,S}$-action, and a cochain $F : G_{Q,S} \to H^*$ such that:

1. $C|_{G_p} = 0$.
2. if $\ell_0 \not\equiv p$, then $C|_{G_{\ell_0}}$ is a coboundary.
3. $dF = b_0 \sim C$.
4. $F|_{I_p} = 0$.
5. For any cocycle $\hat{h}$ whose cohomology class $h$ is in $H$, and any $\tau \in G_{Q,S}$ with $\omega(\tau) = 1$, we have $C(\tau)(h) = \hat{h}(\tau)$.

**Proof.** For any $G_{Q,S}$-module $M$, let

$$Z^1_{(p)}(\mathbb{Z}[1/Np], M) = \{(a, m) \in Z^1(\mathbb{Z}[1/Np], M) \times M \mid a(\tau) = (\tau - 1)m, \forall \tau \in G_p\}.$$

There is a surjection $Z^1_{(p)}(\mathbb{Z}[1/Np], \mathbb{F}_p(-1)) \to H^1_{(p)}(\mathbb{Z}[1/Np], \mathbb{F}_p(-1))$ sending $(a, m)$ to the class of $a$. Choose a linear section $s : H \to Z^1_{(p)}(\mathbb{Z}[1/Np], \mathbb{F}_p(-1))$, and write $s(h) = (s(h)_1, s(h)_2) \in Z^1(\mathbb{Z}[1/Np], \mathbb{F}_p(-1)) \times \mathbb{F}_p(-1)$.

Define an element $(C', x) \in C^1(\mathbb{Z}[1/Np], H^*(-1)) \times H^*(-1)$ by $C'(\tau)(h) = s(h)_1(\tau)$ and $x(h) = s(h)_2$ for $h \in H$. One observes $(C', x) \in Z^1_{(p)}(\mathbb{Z}[1/Np], H^*(-1))$. Then let $C = C' - dx$, so that $C|_{G_p} = 0$ and (1) holds. We also see that (5) holds, since the value $\hat{h}(\tau)$ is independent of the choice of cocycle. Computing with dual vector spaces, it is easy to see that $b_0 \cup C = 0$ in $H^2(\mathbb{Z}[1/Np], H^*)$ and that Lemma 7.1.6 implies (2).

Finally, to see (3) and (4), let $y$ be any cochain such that $dy = b_0 \sim C$. Note that the restriction map

$$H^1(\mathbb{Z}[1/Np], H^*) \to H^1(I_p, H^*)$$

is surjective, and that, since $H^*$ has trivial action, we may and do identify a cohomology class with its representing cocycle. Since $C|_{I_p} = 0$ and $dy = b_0 \sim C$, we see that $y|_{I_p} \in H^1(I_p, H^*)$. Hence there is a cocycle $y' \in H^1(\mathbb{Z}[1/Np], H^*)$ with $y'|_{I_p} = y|_{I_p}$. Letting $F = y - y'$, we have $dF = dy = b_0 \sim C$ and $F|_{I_p} = 0$. \qed
Let $M = H^* \oplus \mathbb{Z}/(p, \ell_0 - 1)$, and let $\mathbb{F}_p[M]$ be the vector space $\mathbb{F}_p \oplus M$ thought of as a local $\mathbb{F}_p$-algebra with square-zero maximal ideal $M$. We write elements of $\mathbb{F}_p[M]$ as triples $(x, y, z)$ with $x \in \mathbb{F}_p$, $y \in H^*$ and $z \in \mathbb{Z}/(p, \ell_0 - 1)\mathbb{Z}$.

Let $E_M$ be the $\mathbb{F}_p[M]$-GMA given by the data $(\mathbb{F}_p, H^*, m)$ where $m$ is the homomorphism

$$m : \mathbb{F}_p \otimes_{\mathbb{F}_p} H^* \cong H^* \xrightarrow{0 \to M} M \to \mathbb{F}_p[M].$$

Let $\rho_M : G_{Q,S} \to E_M^\times$ be the function

$$\rho_M(\tau) = \begin{pmatrix} \omega(\tau)(1, F(\tau), \log_{\ell_0}(\tau)) & \tilde{b}_0(\tau) \\ \omega(\tau)C(\tau) & (1, \tilde{b}_0(\tau)C(\tau) - F(\tau), -\log_{\ell_0}(\tau)) \end{pmatrix}.$$  

Then $\rho_M$ is a homomorphism by Construction 7.1.7. Let $D_M : G_{Q,S} \to \mathbb{F}_p[M]$ denote the pseudorepresentation $D_M := \psi(\rho_M)$.

**Lemma 7.1.9.** $\rho_M$ satisfies US$^N_M$.

**Proof.** As per Definition 3.8.1, we verify US$^N_M$ by proving that $\rho_M|_{G_p}$ is finite-flat if $\ell_0 \neq p$, and that $\rho_M|_{G_\ell}$ satisfies condition US$^\ell_M$ for all $\ell \mid N$.

If $\ell_0 \neq p$, $\rho_M|_{G_p}$ is finite-flat: For this, we make frequent use of the notion of a Cayley-Hamilton module, developed in [WWE17a, §2.6].

Let $E_M'$ be the $\mathbb{F}_p[M]$-subalgebra of $E_M$ given by $E' = \big( \mathbb{F}_p[M] : \mathbb{F}_p, 0, 0, 0 \big)$. Since $C|_{G_p} = 0$, we see that the action of $G_p$ on $E_M$ via $\rho_M$ factors through $E'_M$. Hence $(E'_M, \rho_M|_{G_p})$ is a Cayley-Hamilton representation, and $E_M$ is a faithful Cayley-Hamilton module of $(E'_M, \rho_M|_{G_p})$; by [WWE17a, Thm. 2.6.3], it is enough to show that $(E'_M, \rho_M|_{G_p})$ is finite-flat.

Consider the extension $E_{b_0}$ defined by $\tilde{b}_0$:

$$0 \to \mathbb{F}_p(1) \to E_{b_0} \to \mathbb{F}_p \to 0,$$

which is finite-flat by Kummer theory. Let $W_0 = \mathbb{F}_p[M]$ and $W_1 = \mathbb{F}_p[M]$ with $G_p$ acting by the characters $\omega(1, F, \log_{\ell_0})$ and $(1, -F, -\log_{\ell_0})$, respectively. Since $F|_{\ell_0}$ and $\log_{\ell_0}|_{\ell_0}$ are zero, $W_0$ and $W_1$ are finite-flat. We have exact sequences of $\mathbb{F}_p[M]|_{G_p}$-modules

$$0 \to M(1) \to W_0 \to \mathbb{F}_p(1) \to 0, \quad 0 \to M \to W_1 \to \mathbb{F}_p \to 0.$$ 

Let $l : \mathbb{F}_p \to M$ be an injective linear map. This induces a injection $\mathbb{F}_p(1) \to W_0$ of $\mathbb{F}_p[M]|_{G_p}$-modules. Taking the pushout of $E_{b_0}$ by this injection, we obtain an exact sequence

$$0 \to W_0 \to E_{b_0,0} \to \mathbb{F}_p \to 0.$$ 

Pulling back this sequence by $W_1 \to \mathbb{F}_p$, we obtain an exact sequence

$$0 \to W_0 \to E_{b_0,1} \to W_1 \to 0.$$ 

Following [WWE17a, Appendix C], we see that $E_{b_0,0,1}$ is finite-flat and that there is an isomorphism $E_{b_0,0,1} \cong \mathbb{F}_p[M]^{\oplus 2}$ under which the action of $G_p$ is given by

$$\begin{pmatrix} \omega(1, F, \log_{\ell_0}) & (0, \tilde{b}_0 \cdot l(1)) \\ 0 & (1, -F, -\log_{\ell_0}) \end{pmatrix} : G_p \to \text{GL}_2(\mathbb{F}_p[M]).$$ 

We now use this isomorphism $E_{b_0,0,1} \cong \mathbb{F}_p[M]^{\oplus 2}$ as an identification.
We have an injective $\mathbb{F}_p[M]$-algebra homomorphism $l': E'_M \to \text{End}_{\mathbb{F}_p[M]}(E_{b_0,\omega,1}) = M_{2 \times 2}(\mathbb{F}_p[M])$ given by

$$l' = \begin{pmatrix} \text{id}_{\mathbb{F}_p[M]} & l \\ 0 & \text{id}_{\mathbb{F}_p[M]} \end{pmatrix}.$$ 

By (7.1.10), we see that action of $G_p$-action on $E_{b_0,\omega,1}$ factors through $l'$. In other words, $E_{b_0,\omega,1}$ is a faithful Cayley-Hamilton module of $(E'_M, \rho_M|_{G_p})$. Since $E_{b_0,\omega,1}$ is finite-flat, $(E'_M, \rho_M|_{G_p})$ is finite-flat by [WWE17a, Thm. 2.6.3].

**Proof.** By Lemma 7.1.9, we have $E'_M$ is US$_0^{-1}$: This follows from Proposition 3.7.5 and Construction 7.1.7.

If $\ell_0 \equiv 1 \pmod{p}$, then $\rho_M|_{G_{t_0}}$ is ordinary: This follows from Proposition 3.7.5 and Construction 7.1.7.

If $\ell_0 \equiv 1 \pmod{p}$, then $\rho_M|_{G_{t_0}}$ is US$_0^{-1}$: By assumption, we have $M = H^*$ and $\log \ell_0 = 0$, and we will simply write elements of $\mathbb{F}_p[M]$ as pairs $(x, y)$ with $x \in \mathbb{F}_p$ and $y \in H^*$. Since $\text{C}_G G_{t_0}$ is a coboundary, there is an $z \in H^*$ such that $C(\tau) = (\omega^{-1}(\tau) - 1)z$ for all $\tau \in G_{t_0}$.

Let $\rho'_M : G_{t_0} \to E_{M}^{\times}$ the composite of $\rho_M$ with the automorphism $E_M \cong E_M$ given by conjugation by $(\frac{1}{z} \frac{1}{1}) \in E_M^{\times}$. By explicit computation, we see that

$$\rho'_M = \begin{pmatrix} \omega(1, F_a) \\ \omega(C - (\omega^{-1} - 1)z) \\ (1, F_d) \end{pmatrix},$$

where $F_a = F - \omega^{-1}b_0z$ and $F_d = \tilde{b}_0C + F + \omega \tilde{b}_0z$; in particular, the $(2, 1)$-coordinate of $\rho'_M|_{G_{t_0}}$ is zero. This implies that $F_a|_{G_{t_0}}, F_d|_{G_{t_0}} : G_{t_0} \to H^*$ are homomorphisms. Therefore $\ell_0 \not\equiv 0, 1 \pmod{p}$ and $H^*$ has exponent $p$, they are unramified.

For any $(\sigma, \tau) \in G_{t_0} \times I_{t_0}$, we compute that

$$V_{\rho'_M}^{-1}(\sigma, \tau) = \begin{pmatrix} \varepsilon & * \\ 0 & * \end{pmatrix} \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} = 0$$

where $\varepsilon \in M$. Equivalently, $V_{\rho'_M}^{-1} = 0$. A similar computation shows that $V_{\rho'_M}^{-1}(\sigma, \tau) = 0$ for $(\sigma, \tau) \in I_{t_0} \times G_{t_0}$.

If $\ell | N$ and $\ell \neq \ell_0$, then $\rho_M|_{G_{t}}$ is US$_0^{-1}$: In this case we have $\ell \equiv -1 \pmod{p}$, and hence $\omega|_{G_{t}} = \lambda(-1)$. Since $\ell \neq \ell_0$, we have $b_0|_{G_{t}} = 0$, so $b_0|_{G_{t}} = 0$ by Lemma 6.1.1. Hence there exists $a \in \mathbb{F}_p$ such that $b_0(\tau) = (\omega(\tau) - 1)a$ for all $\tau \in G_{t}$. Exclusively as in the previous step, we can show that $V_{\rho'_M}^{-1}(\sigma, \tau) = 0$ for all $(\sigma, \tau) \in G_{t} \times I_{t} \cup I_{t} \times G_{t}$ by conjugating $\rho_M$ by $(\frac{1}{a} \frac{1}{1}) \in E_{M}^{\times}$. \qed

7.1.2. End of the proof. We will show that $D_M$ is, in a sense, the universal US$_N^{-1}$ first-order deformation of $D$.

**Proposition 7.1.11.** The pseudodeformation $D_M$ of $D$ induces an isomorphism $R_N^{\ast}/(p, m^2) \cong \mathbb{F}_p[M]$.

**Proof.** By Lemma 7.1.9, $\rho_M$ is US$_N$, so $D_M$ is also US$_N$ by Definition 3.8.1 and there is an induced map $E'_N \to E_M$. This gives us a local ring morphism $R_N^{\ast} \to \mathbb{F}_p[M]$, and any such morphism factors through a morphism $R_N^{\ast}/(p, m^2) \to \mathbb{F}_p[M]$. 

Let \( f \) denote the restriction \( \mathfrak{m}/(p, \mathfrak{m}^2) \to M \). It suffices to show that \( f \) is an isomorphism.

Assume that the GMA structure on \( E_N^* \) is chosen so that \( E_N^* \to E_M^* \) is a morphism of GMAs (such a GMA structure is known to exist by [WWE17a, Thm. 3.2.2]). By Theorem 5.2.6 we see that the elements \( b_{\omega c_i} \) for \( i = 1, \ldots, r \) together with the element \( d_{\gamma_0} - 1 \) generate \( \mathfrak{m}/(p, \mathfrak{m}^2) \), and, moreover, if \( \ell_0 \not\equiv 1 \pmod{p} \), the elements \( b_{\omega c_i} \) for \( i = 1, \ldots, r \) are a basis.

By construction, we see that \( f(b_{\omega c_i}) = (0, b_0(\gamma_0)C(\gamma_i), 0) = (0, C(\gamma_i), 0) \), and that \( f(d_{\gamma_0} - 1) = (0, 0, -\log \ell_0(\gamma_0)) \) (which is non-zero if \( \ell_0 \equiv 1 \pmod{p} \)). By Lemma 7.1.13 below, \( f \) is surjective.

Now we count dimensions. By Theorem 5.2.6 and Proposition 7.1.4, we have

\[
\dim_{\mathbb{F}}(\mathfrak{m}/(p, \mathfrak{m}^2)) = \begin{cases} 
 r & \text{if } \delta = 0 \\
 r + 1 & \text{if } \delta = 1.
\end{cases}
\]

By Lemma 7.1.5, we have

\[
\dim_{\mathbb{F}}(M) = \begin{cases} 
 r & \text{if } \delta = 0 \\
 r + 1 & \text{if } \delta = 1.
\end{cases}
\]

Since \( f \) is surjective, this implies that \( f \) is an isomorphism in all cases. \( \square \)

**Lemma 7.1.13.** Let \( \tau_1, \ldots, \tau_r \in G_{Q,S} \) be any elements such that:

- \( \omega(\tau_i) = 1 \) for \( i = 1, \ldots, r \), and
- \( \tilde{c}_j(\tau_i) = \partial_{ij} \) for all \( 1 \leq i, j \leq r \).

If \( \delta = 1 \) or \( \ell_0 \not\equiv 1 \pmod{p} \), then the set \( \{C(\tau_i) : i = 1, \ldots, r\} \) is a basis for \( H^* \). Otherwise \( b_0 \cup c_j \not= 0 \) for some \( j \) and the set \( \{C(\tau_i) : i = 1, \ldots, r, i \not= j\} \) is a basis for \( H^* \).

**Proof.** Indeed, if \( c_j - \alpha c_k \in H \) for some \( \alpha \in \mathbb{F}_p \) and \( j, k \in \{1, \ldots, r\} \), then by Construction 7.1.7(5) we have

\[
C(\tau_i)(c_j - \alpha c_k) = \tilde{c}_j(\tau_i) - \alpha \tilde{c}_k(\tau_i) = \partial_{ij} - \alpha \partial_{ik}.
\]

Using the explicit basis of \( H \) constructed in Lemma 7.1.5 the lemma follows. \( \square \)

**Proof of Theorem 7.1.1.** By Proposition 7.1.11, we have \( \mathfrak{m}/(p, \mathfrak{m}^2) \cong M \), and the dimension of \( M \) is given by \( (7.1.12) \). This completes the proof. \( \square \)

**7.2. Good sets of primes in the case \( \epsilon = (-1, 1, \ldots, 1) \).** In this section, we prove Theorem 1.6.1 in precise form. Assume that \( \epsilon = (-1, 1, \ldots, 1) \), and, as in §5 order the primes \( \ell_i \) so that \( \ell_i \equiv -1 \pmod{p} \) for \( i = 1, \ldots, s \) and \( \ell_i \not\equiv -1 \pmod{p} \) for \( s < i \leq r \). We use the number fields \( K_i \) set up in Definition 3.10.4

**Definition 7.2.1.** Consider an ordered set of primes \( Q' = \{q_0, q_1, \ldots, q_s\} \) disjoint from the primes dividing \( N \) and satisfying the following conditions:

1. \( q_0 \not\equiv 1 \pmod{p} \), and
2. \( q_0 \) not a \( p \)-th power modulo \( \ell_0 \);

and, for \( i = 1, \ldots, s \),

3. \( q_i \equiv 1 \pmod{p} \),
4. \( \ell_0 \) is not a \( p \)-th power modulo \( q_i \),
5. \( q_i \) does not split completely in \( K_i \), and
6. \( q_i \) does split completely in each \( K_j \) for \( j = 1, \ldots, s \) with \( j \not= i \).
In the following cases, the described ordered subset $Q$ of $Q'$ is called a good set of primes for $(N, p, \epsilon)$:

- if $\delta = 1$, $Q := Q'$,
- if $\delta = 0$ and $t_0 \equiv 1 \pmod{p}$, then $Q := Q' \setminus \{q_j\}$ for an index $j > 0$ such that $b_0 \cup c_j \neq 0$,
- if $t_0 \not\equiv 1 \pmod{p}$, then $Q := Q' \setminus \{q_0\}$.

Note that, by Chebotarev density, there is an infinite set of primes $q_0$ satisfying (1)-(2) and, for each $i$, there is an infinite set of primes $q_i$ satisfying (3)-(6). Note that when $p \nmid N$ and $t_0 \equiv 1 \pmod{p}$, it is possible that $p \in Q$.

**Theorem 7.2.2.** Let $Q$ be a good set of primes for $(N, p, \epsilon)$. Then $\{T_q - (q + 1) | q \in Q\} \subset T_N^*$ is a minimal set of generators for $I^*$.

**Proof.** We freely refer to $\rho_M$ and related objects in this proof (see (7.1.8)). Let $J$ be the index set of $Q$ (i.e. $J = \{0, \ldots, s\}$, $J = \{0, \ldots, s\} \setminus \{j\}$ or $J = \{1, \ldots, s\}$ in the three cases of Definition 7.2.1, respectively).

By Theorem 6.4.1, Proposition 7.1.11, and Nakayama’s lemma, it suffices to show that the projection $\Upsilon(q)$ of $T_q - (q + 1)$ under $T_N^* \sim \sim R_N^* \rightarrow F_p[M]$ comprise a basis $\{\Upsilon(q)\}_{q \in Q}$ of $M$. The conditions (1)-(6) on $Q$ have been chosen so that:

1. If $0 \in J$ and $q_0 \neq p$, then $\omega(F_{q_0}) \neq 1$ and $\log_{t_0}(F_{q_0}) \neq 0$. This follows from (1) and (2).
2. $\omega(F_{q_0}) = 1$ for $i \in J$ with $i > 0$. This follows from condition (3).
3. $\tilde{b}_0(F_{q_0}) \neq 0$ for $i \in J$ with $i > 0$. This follows from (4) by class field theory.
4. $\{C(F_{q_0}) : i \in J, i > 0\}$ is a basis for $H^*$. This follows from Lemma 7.1.13 by (ii), (5), and (6).

When $q_i \neq p$, it is clear that $\Upsilon(q_i) = \text{Tr}_{p}[F_{q_i} - (q_i + 1)]$, and we calculate:

(a) By (ii), $\Upsilon(q_i) = (0, \tilde{b}_0(F_{q_i}) \cdot C(F_{q_i}), 0) \in F_p[M]$ for $i \in J$ with $i > 0$. By (iii) and (iv), these elements form a basis of $H^*$.

(b) If $0 \in J$ and $q_0 \neq p$, then $\Upsilon(q_0) \in F_p[M]$ lies in $M$ and projects via $M \rightarrow Z/(p, t_0 - 1)$ to $(\omega(F_{q_0}) - 1) \tilde{b}_0(F_{q_0})$. This is non-zero, by (i).

(c) If $0 \in J$ and $q_0 = p$, we claim that $\Upsilon(p) \in F_p[M]$ lies in $M$ and maps to $\log_{t_0} \neq 0$ under the summand projection $M \rightarrow Z/(p, t_0 - 1)$. This follows from the same argument as in Case $q_0 = p$ of the proof in (7.3) but is simpler. \qed

**Remark 7.2.3.** The reader will note that, in this proof, our conditions are used to ensure that a certain matrix is diagonal with non-zero diagonal entries. Of course, the necessary and sufficient condition is simply that this same matrix is invertible.

### 7.3. Good pairs of primes in the case $\epsilon = (-1, -1)$

In this subsection, we prove Theorem 1.6.3. We assume we are in the setting of Theorem 6.4.1.

**Proof of Theorem 1.6.3.** By Theorem 6.4.1 and Nakayama’s lemma, $T_N$ is generated by $\{T_q - (q_i + 1)\}_{i=0,1}$ if and only if their images $\{\Upsilon(q_i)\}_{i=0,1}$ via $T_N^* \sim \sim R_N^*/(p, m^2)$ are a basis of $m/(p, m^2)$. We see in the proof of Theorem 6.4.1 that $J^\text{red} = J^\text{min} - \{p\} \subset R_N^*$, there are isomorphisms

$$R_N^*/(p, m^2) \sim \sim R_N^*/(p) \cong F_p[Y_0, Y_1]/(Y_0^2, Y_0Y_1, Y_1^2), \quad m/(p, m^2) \sim \sim (Y_0, Y_1),$$

which we use as identifications. Then $D_N$ pulls back to the pseudorepresentation $D = \psi(\cdot^{-1} \otimes (-)) : G_{Q, S} \rightarrow R_N^*/(p)$, where, for particular choices of $\log_{t_1}$,

$$G_{Q, S} \ni \tau \mapsto \langle \tau \rangle := 1 + \log_{t_0}(\tau)Y_0 + \log_{t_1}(\tau)Y_1.$$
We see that if \( q_i \neq p \), then \( \Upsilon(q_i) = \text{Tr}_D(Fr_{q_i}) \) \(-\) \((q_i + 1)\).

**Case** \( q_0, q_1 \neq p \). One computes that the matrix expressing \( \{\Upsilon(q_0), \Upsilon(q_1)\} \) in the basis \( \{Y_0, Y_1\} \) of \( \mathfrak{m}/(p, \mathfrak{m}^2) \cong (Y_0, Y_1) \) is

\[
\begin{pmatrix}
(q_0 - 1) \log_{\ell_0} q_0 & (q_1 - 1) \log_{\ell_0} q_1 \\
(q_0 - 1) \log_{\ell_1} q_0 & (q_1 - 1) \log_{\ell_1} q_1
\end{pmatrix} \in M_2(\mathbb{F}_p),
\]

which completes the proof.

**Case** \( q_0 = p \). We note that the images of \( T_p - (p + 1) \) and \( U_p - 1 \) in \( I'/\mathfrak{m}I' \) are equal, so we may replace \( T_p - (p + 1) \) by \( U_p - 1 \) in the statement. We recall from Step 3 of the proof of Proposition 4.1.1 that \( U_p \) is the image under \( R_{\sigma_N}^* \to \mathbb{T}_N^* \) of \( \frac{1}{x^2}(x\text{Tr}(\rho_N')(\sigma_p) - \text{Tr}(\rho_N')((\sigma\tau_p))) \), where \( \tau \in I_p \) is such that \( \omega(\tau) \neq 1 \) and \( x = \kappa_{\text{cyc}}(\tau) \). We compute that

\[
\Upsilon(p) = \frac{1}{x - 1}(x\text{Tr}(\sigma_p) - \text{Tr}(\sigma\tau_p)) - 1 = \log_{\ell_0}(p)Y_0 + \log_{\ell_1}(p)Y_1.
\]

Thus, the matrix expressing \( \{\Upsilon(p), \Upsilon(q_1)\} \) in the basis \( \{Y_0, Y_1\} \) of \( \mathfrak{m}/(p, \mathfrak{m}^2) \) is

\[
\begin{pmatrix}
\log_{\ell_0} p & (q_1 - 1) \log_{\ell_0} q_1 \\
\log_{\ell_1} p & (q_1 - 1) \log_{\ell_1} q_1
\end{pmatrix} \in M_2(\mathbb{F}_p).
\]

**Appendix A. Comparison with the Hecke algebra containing \( U_{\ell} \)**

In order to compare our results with existing results and conjectures, in this appendix we consider a Hecke algebra that contains the \( w_{\ell} \) operators rather than the \( w_{\ell} \) operators. We prove comparison results between Eisenstein completions of this algebra and the Eisenstein completions \( \mathbb{T}_N \) studied in this paper. Throughout this appendix, we drop the subscripts ‘\( \mathcal{N} \)’ on all Hecke algebras to avoid cumbersome notation.

Recall that we have the normalization map of Lemma 2.3.1

\[
\mathbb{T}^* \hookrightarrow \mathbb{Z}_p \oplus \bigoplus_{f \in \Sigma} \mathcal{O}_f,
\]

where \( \Sigma, \mathcal{O}_f \) were defined there. For each \( f \in \Sigma \), there is a unique pair \( (N_f, \tilde{f}) \) of a divisor \( N_f \) of \( N \) and a newform \( \tilde{f} \) of level \( N_f \) such that \( a_q(f) = a_q(\tilde{f}) \) for all primes \( q \) not dividing \( N_f \) and \( a_{\ell}(\tilde{f}) = -\epsilon_{\ell} \) for primes \( \ell \) dividing \( N_f \). For this \( \tilde{f} \), we have \( a_q(\tilde{f}) \equiv 1 + q \pmod{m_f} \) for all \( q \nmid N_f \).

**A.1. Oldforms and stabilizations.** Just as in [2.1.5] if \( \ell \mid N \) and \( f \in S_2(N/\ell; \mathbb{Z}_p) \) is an eigenform for all \( T_n \) with \( (n, N/\ell) = 1 \), then there are two ways to stabilize \( f \) to be a \( U_{\ell} \)-eigenform in \( S_2(N; \mathbb{Z}_p) \). Let \( \alpha_{\ell}(f), \beta_{\ell}(f) \) denote the roots of \( x^2 - a_{\ell}(f)x + \ell \). Then \( f_{\alpha_{\ell}}(z) = f(z) - \beta_{\ell}(f)f(\ell z) \) and \( f_{\beta_{\ell}}(z) = f(z) - \alpha_{\ell}(f)f(\ell z) \) satisfy \( U_{\ell} f_{\alpha_{\ell}} = \alpha_{\ell}(f)f_{\alpha_{\ell}} \) and \( U_{\ell} f_{\beta_{\ell}} = \beta_{\ell}(f)f_{\beta_{\ell}} \). Note that, unlike in [2.1.5] it may happen that \( \alpha_{\ell}(f) \equiv \beta_{\ell}(f) \pmod{p} \).

**A.2. The case** \( p \nmid N \). For this section, assume \( p \nmid N \). Let \( \mathbb{T}^*_U \) and \( \mathbb{T}^*_U^0 \) be the \( \mathbb{Z}_p \)-subalgebras of

\[
\text{End}_{\mathbb{Z}_p}(M_2(N; \mathbb{Z}_p)) \quad \text{and} \quad \text{End}_{\mathbb{Z}_p}(S_2(N; \mathbb{Z}_p)),
\]

respectively, generated by the Hecke operators \( T_{\ell} \) for \( \ell \nmid N \) and \( U_{\ell} \) for \( \ell \mid N \). These are semi-simple commutative algebras (see [CF98] for the semi-simplicity).
For each \( \epsilon \in \mathcal{E} \) as in \([1,3.1]\) we let \( I_\epsilon^U \subset T'_U \) be the ideal generated by the set

\[
\{ T_q - (q + 1), \; U_\ell - \ell^\frac{q + 1}{2} : q \nmid N, \; \ell \mid N \text{ primes} \}.
\]

Note that \( I_\epsilon^U \) is the annihilator of a certain stabilization of the Eisenstein series \( E_{2.1} \) (but generally not \( E_{2,N} \)). Let \( T'_U \) and \( T^0_\epsilon U \) denote the completions of \( T'_U \) and \( T^0_U \) respectively, at the maximal ideal \( (p, I^U_\epsilon) \subset T'_U \). Let \( m_I^U \subset T_U^0 \) and \( m_\epsilon^{0,U} \subset T^0_\epsilon U \) be the maximal ideals.

**A.2.1. The normalization of \( T'_U \).** Since \( T'_U \) and \( T^0_\epsilon U \) are semi-simple, the standard description of prime ideals in terms of eigenforms allows us to describe their normalizations, just as for \( T' \). For newforms \( f \), we know that \( U_\ell f = -w_\ell f \) for all \( \ell \mid N \). For oldforms, we can use the stabilization formulas from \([2.1.5]\) and \([A.1]\) to describe the eigenforms for \( T'_U \) in terms of the set \( \Sigma \). We write down the result of this description explicitly in Lemma \([A.2.1]\).

We require the following notation. Let \( L_N = \{ \ell \mid N : \ell \equiv 1 \pmod{p} \} \). For each \( f \in \Sigma \) and each \( \ell \mid N \), let \( \alpha_\ell(f) \) and \( \beta_\ell(f) \) be the roots of \( x^2 - \alpha_\ell(f)x + \ell \). Assume that \( \alpha_\ell(f) \equiv \ell^{\frac{q + 1}{2}} \pmod{p} \) and let \( L_f = \{ \ell \mid \frac{N_f}{N_f} : \ell \equiv 1 \pmod{p} \} \). Let \( \tilde{\mathcal{O}}_f \) be the extension of \( \mathcal{O}_f \) generated by \( \alpha_\ell(f) \) and \( \beta_\ell(f) \). If \( \ell \not\equiv 1 \pmod{p} \), then the congruence condition determines \( \alpha_\ell(f) \) (and \( \beta_\ell(f) \)) uniquely, and \( \tilde{\mathcal{O}}_f = \mathcal{O}_f \); in this case, only stabilizations of \( \tilde{f}_\alpha \) can appear in the completion \( S_3(N; \mathbb{Z}_p) \otimes_{T'_U} T^0_\epsilon U \).

If \( \ell \equiv 1 \pmod{p} \), then we label the two roots arbitrarily (in this situation, below, we will use the two roots symmetrically), and \( \tilde{\mathcal{O}}_f \) may be a quadratic extension of \( \mathcal{O}_f \); in this case the stabilizations of both \( \tilde{f}_\alpha \) and \( \tilde{f}_\beta \) can appear in the completion \( S_3(N; \mathbb{Z}_p) \otimes_{T'_U} T^0_\epsilon U \).

**Lemma A.2.1.** The normalization of \( T'_U \) is the injection

\[
T'_U \hookrightarrow \left( \bigoplus_{L \subset L_N} \mathbb{Z}_p \right) \oplus \left( \bigoplus_{f \in \Sigma} \left( \bigoplus_{L \subset L_f} \tilde{\mathcal{O}}_f \right) \right)
\]

given by

\[
T_q \mapsto ((1 + q)_{L \subset L_N}, a_q(f)_{f \in \Sigma, L \subset L_f}) \text{ for all } q \nmid N,
\]

and sending \( U_\ell \) for \( \ell \mid N \) as follows. The factor \( T'_U \rightarrow \mathbb{Z}_p \) for \( L \subset L_N \) sends

\[
U_\ell \mapsto \begin{cases} 
\ell^\frac{q + 1}{2} & \text{if } \ell \mid N, \ell \not\in L, \\
\ell^{-\frac{q + 1}{2}} & \text{if } \ell \mid N, \ell \in L.
\end{cases}
\]

The factor \( T_U \rightarrow \tilde{\mathcal{O}}_f \) corresponding to \( f \in \Sigma \) and \( L \subset L_f \) sends

\[
U_\ell \mapsto u_\ell(f) := \begin{cases} 
-\epsilon_i & \text{if } \ell \mid N_f, \\
\alpha_\ell(f) & \text{if } \ell \mid N, \ell \not\mid N_f, \ell \not\in L, \\
\beta_\ell(f) & \text{if } \ell \mid N, \ell \mid N_f, \ell \in L.
\end{cases}
\]

The normalization of \( T^0_\epsilon U \) is the same, but without the factor \( \left( \bigoplus_{L \subset L_N} \mathbb{Z}_p \right) \), which corresponds to Eisenstein series.
A.2.2. Comparisons. We now compare the algebras $T_{U}^{0,\epsilon}$ and $T^{0,\epsilon}$. The following proposition gives a necessary and sufficient condition for the algebras to coincide.

**Proposition A.2.2.** Suppose that both of the following are true:

1. For each $f \in \Sigma$, we have $L_f = \emptyset$; and
2. $T_{U}^{0}$ is generated as a $\mathbb{Z}_{p}$-algebra by $\{T_q : q \nmid Np\}$.

Then $T_{U}^{0} = T^{0,0}$. Moreover, if one of these conditions is false, then $T_{U}^{0} \neq T^{0,0}$.

**Proof.** The first condition ensures that $T_{U}^{0}$ and $T^{0,0}$ have the same normalization, so it is certainly necessary. The second condition is true for $T^{0,0}$ by Proposition 4.1.1, so it is necessary. Furthermore, if we assume (1) and (2), then $T_{U}^{0}$ and $T^{0,0}$ are identified with the subalgebra of $\bigoplus_{f \in \Sigma} \mathcal{O}_f$ generated by $\{a_{q}(f) : q \nmid Np\}$. \qed

We now verify these conditions in certain cases considered in this paper.

**Proposition A.2.3.** Assume that $\ell, \not\equiv 1 \pmod{p}$ for $0 < i \leq r$ and assume that $\epsilon = (-1, 1, \ldots, 1)$. Then $T_{U}^{0} = T^{0,0}$.

**Proof.** We verify the conditions (1) and (2) of Proposition A.2.2.

To verify (1), assume, for a contradiction, that there is an $f \in \Sigma$ with $L_f \neq \emptyset$. By our assumptions on $\ell$, we must have $L_f = \{\ell_{0}\}$. Then the newform $\tilde{f} \in S_2(N_f; \mathbb{Q}_p)$ satisfies $a_{q}(\tilde{f}) \equiv 1 + q \pmod{p}$ for all $q \nmid N_f$ and $a_{q}(\tilde{f}) = -1$ for all $\ell \nmid N_f$ (since $\ell_{0} \nmid N_f$ by assumption). But this is impossible by a theorem of Ribet (see [BD14 Thm. 2.3(2)]), so (1) holds.

Just as in the proof of Proposition 4.1.1, we have a homomorphism $R_{N}^{\epsilon} \to T_{U}^{0,\epsilon}$ sending $\text{Tr}(\rho_{N}(\sigma_{\ell}))$ for each $f \in \Sigma$ with $L_f \neq \emptyset$ to $\mathbb{Z}_{p}$, and whose image is the subalgebra of $T_{U}^{0}$ generated by $\{T_q : q \nmid Np\}$. Note that, by (1), for each $f \in \Sigma$ we have $\ell_{0} \nmid N_f$, so $u_{\ell_{0}}(f) = 1$. This implies $U_{\ell_{0}} = 1$ in $T_{U}^{0}$. Hence to verify (2), we need only show that $U_{\ell}$ is in the image of $R_{N}^{\epsilon} \to T_{U}^{0,\epsilon}$ for all $\ell \nmid N$ with $\ell \neq \ell_{0}$.

Now fix such an $\ell$. Let $\tilde{U} \in R_{N}^{\epsilon}$ be the root of the polynomial $x^2 - \text{Tr}(\rho_{N}(\sigma_{\ell}))x + \ell$ such that $\tilde{U} + 1 \in \mathbb{Z}_{p}$; such a $\tilde{U}$ exists and is unique by Hensel’s Lemma. We claim that the image of $\tilde{U}$ in $T_{U}^{0,\epsilon}$ is $U_{\ell}$. To prove the claim, it suffices to show that $\tilde{U} \mapsto u_{\ell}(f)$ under the map $R_{N}^{\epsilon} \to \mathcal{O}_{f}$ for each $f \in \Sigma$.

First assume that $\ell \nmid N_{f}$. Then $u_{\ell}(f) = -1$ and, by [2.3.2], $\text{Tr}(\rho_{f}(\sigma_{\ell})) = -(\ell + 1)$. So $\tilde{U}$ is sent to a root of $x^2 + (\ell + 1)x + \ell = (x + 1)(x + \ell)$ that is congruent to $-1$ modulo $m_{f}$. Since $\ell \equiv -1 \pmod{p}$, we see that $\tilde{U}$ is sent to $-1$, which is $u_{\ell}(f)$.

Next assume that $\ell \mid N$ and $\ell \mid N_{f}$. Then $u_{\ell}(f) = a_{\ell}(f)$ and $\text{Tr}(\rho_{f}(Fr_{\ell})) = a_{\ell}(\tilde{f})$. Recall that $a_{\ell}(f)$ and $\beta(f)$ are the roots of $x^2 - a_{\ell}(f)x + \ell$ and that $a_{\ell}(f) \equiv \ell \pmod{m_{f}}$. We see that $\beta(f) = 1 \pmod{m_{f}}$, and so $\tilde{U}$ must be sent to $a_{\ell}(f)$. \qed

The proof of the first part of the following proposition is almost identical, but simpler, so we leave it to the reader. The second part is an application of Theorem 6.3.1.

**Proposition A.2.4.** Assume that $N = \ell_{0}f_{1}$, that $\ell_{1} \not\equiv 1 \pmod{p}$, and that $\epsilon = (-1, -1)$. Then $T_{U}^{0,\epsilon} = T_{U}^{0,\epsilon}$. If, in addition, $\ell_{1}$ is not a $p$-th power modulo $\ell_{0}$, then $T_{U}^{0,\epsilon}$ and $T_{U}^{0,\epsilon}$ are both identical to the Hecke algebra at level $\ell_{0}$ considered by Mazur.
A.3. The case $p | N$. In this section, we maintain the notation of the previous section, but we assume that $\ell_0 = p$ and that $\epsilon_0 = -1$ (for $0 < i \leq r, \epsilon_i$ is arbitrary).

We consider a variant $T^*_H$ of the Hecke algebra that is intermediate to $T^*$ and $T^*_U$. Namely, $T^*_H$ is the completion of the Hecke algebra generated by the $T_q$ for $q \nmid N$, together with $U_p$ and $w_i$, for $0 < i \leq r$, at the ideal generated by $p, T_q - (q + 1), U_p - 1$, and $w_i - \epsilon_i$. Note that, as in the case of $T^*\epsilon$, we have $w_i = \epsilon_i$ in $T^*_H$. For each $f \in \Sigma$, if $p \nmid N_f$, we let $\alpha_p(f) \in O_f$ be the (unique) unit root of $x^2 - \alpha_p(f)x + p$.

Just as in Lemma A.2.1, we can compute the normalization of $T^*_H$. It is the injective map

$$T^*_H \to \mathbb{Z}_p \oplus \left( \bigoplus_{f \in \Sigma} O_f \right)$$

sending $T_q$ to $(1 + q, a_q(f))$ for $q \nmid N$ and $U_p$ as follows. The component $T^*_H \to \mathbb{Z}_p$ sends $U_p$ to 1. The component $T^*_H \to O_f$ sends $U_p$ to $u_p(f)$ defined by

$$u_p(f) := \begin{cases} \frac{1}{\alpha_p(f)} & \text{if } p | N_f \\ \alpha_p(f) & \text{if } p \nmid N_f. \end{cases}$$

**Proposition A.3.1.** With the assumptions that $\ell_0 = p$ and $\epsilon_0 = -1$, we have $T^*_H = T^*\epsilon$ as subalgebras of $\mathbb{Z}_p \oplus \left( \bigoplus_{f \in \Sigma} O_f \right)$ and $T^*_{H,0} = T^*\epsilon,0$ as subalgebras of $\left( \bigoplus_{f \in \Sigma} O_f \right)$.

**Proof.** The proof is just as in the proof of Proposition A.2.3 so we will be brief. We have a map $R^*_H \to T^*_H$ and we need only show that $U_p$ is in the image of this map. Letting $\hat{U} \in R^*_H$ be the unique unit root of $x^2 - \text{Tr}(\rho^*_H(\sigma_p))x + p$, we see that $\hat{U}$ maps to $U_p$. \qed

**Corollary A.3.2.** Let $N = p\ell$ with $\ell \equiv 1 \pmod{p}$ and $\epsilon \equiv (-1,-1)$. Assume that $p$ is not a $p$-th power modulo $\ell$. Then the Eisenstein ideal of $T^*_H$ is generated by $U_p - 1$. In particular, $T^*_H$ and $T^{0,\epsilon}_H$ are Gorenstein.

**Proof.** Combine the previous proposition with Theorem 6.3.1 and Mazur’s good prime criterion (§1.1). \qed

**Appendix B. Computation of some cup products**

**B.1. Cohomology calculations.**

**Lemma B.1.1.** If $\ell \not\equiv 0, 1 \pmod{p}$, then the restriction map

$$H^1(\mathbb{Q}_\ell, \mathbb{F}_p(1)) \to H^1(I_r, \mathbb{F}_p(1))$$

is injective.

**Proof.** Under the isomorphisms of Kummer theory, this map corresponds to the map $\mathbb{Q}_\ell^\times \otimes \mathbb{F}_p \to \mathbb{Q}_p^\times \otimes \mathbb{F}_p$ induced by the inclusion. Since $\ell \not\equiv 0, 1 \pmod{p}$, $\mathbb{Q}_\ell^\times \otimes \mathbb{F}_p$ is generated by the class of $\ell$, which maps the class of $\ell$ in $\mathbb{Q}_p^\times \otimes \mathbb{F}_p$, which is nonzero. \qed

**Lemma B.1.2.** Let $N = \ell_0 \cdots \ell_r$ be squarefree and assume $p \nmid N$. Let $V = \{ i : p \mid (\ell_i - 1) \}$. The local restriction maps to induce an isomorphism

$$H^2(\mathbb{Z}[1/Np], \mathbb{F}_p) \iso \bigoplus_{i \in V} H^2(\mathbb{Q}_{\ell_i}, \mathbb{F}_p) \iso \bigoplus_{i \in V} H^2(\mathbb{Q}_{\ell_i}, \mathbb{F}_p).$$
of vector spaces of dimension \#V.

Proof. Just as in [WWE17c, Lem. 12.1.1], we know that

$$H^2(\mathbb{Z}[1/Np], \mathbb{F}_p) \to H^2(\mathbb{Q}_p, \mathbb{F}_p) \oplus \bigoplus_{i=1}^r H^2(\mathbb{Q}_\ell_i, \mathbb{F}_p)$$

is a surjection because $H^3_{(\ell)}(\mathbb{Z}[1/Np], \mathbb{F}_p) \cong H^0(\mathbb{Z}[1/Np], \mathbb{F}_p(1))^* = 0$. By Tate duality, $H^2(\mathbb{Q}_\ell, \mathbb{F}_p) = H^0(\mathbb{Q}_\ell, \mathbb{F}_p(1))^*$, which is one-dimensional if $\ell \equiv 1 \pmod{p}$ and zero otherwise. It remains to verify that $H^2(\mathbb{Z}[1/Np], \mathbb{F}_p)$ has the same dimension. This follows from the global Tate Euler characteristic computation of [WWE17c, Lem. 12.1.1].

The following is a consequence of Tate duality.

Lemma B.1.3. Assume that $\ell \equiv 1 \pmod{p}$ is prime. Then $H^1(\mathbb{Q}_\ell, \mathbb{F}_p(-1)) = H^1(\mathbb{Q}_\ell, \mathbb{F}_p(1))$. This cohomology group is 2-dimensional, the unramified subspace is I-dimensional, and the cup product pairing

$$\cup : H^1(\mathbb{Q}_\ell, \mathbb{F}_p(-1)) \times H^1(\mathbb{Q}_\ell, \mathbb{F}_p(1)) \to H^2(\mathbb{Q}_\ell, \mathbb{F}_p)$$

is non-degenerate and symplectic. In particular, the cup product of two unramified classes vanishes, and the cup product of a ramified class with a non-trivial unramified class does not vanish.

B.2. Proof of Proposition 7.1.3. By the description of the number fields $K_i$ in Definition 7.1.1, $\ell_0$ splits completely in $K_i$ if and only if $\ell_0 \equiv 1 \pmod{p}$ and the image $c_i|_{G_{\ell_0}}$ of $c_i$ in $H^1(\mathbb{Q}_{\ell_0}, \mathbb{F}_p(-1))$ is zero. Since $b_0|_{G_{\ell_0}}$ is ramified and $c_i|_{G_{\ell_0}}$ is unramified, Lemma B.1.3 implies that $c_i|_{G_{\ell_0}} = 0$ if and only if $b_0|_{G_{\ell_0}} \cup c_i|_{G_{\ell_0}} = 0$. By Lemma B.1.2 this happens if and only if $b_0 \cup c_i = 0$.

APPENDIX C. ALGEBRA

C.1. Some comments about Gorenstein defect. Let $(A, \mathfrak{m}_A, k)$ be a regular local ring, and let $(R, \mathfrak{m}_R)$ be a finite, flat, local $A$-algebra (in particular, $R$ is a Cohen-Macaulay ring).

For an $A$-module $M$, let $\bar{M} = M/\mathfrak{m}_A M$, and let $M^g = \text{Hom}_A(M, A)$. For a $k$-vector space $M$, let $M^g = \text{Hom}_k(M, k)$. For an $R$-module $M$, give $M^g$ the $R$-module structure given by $(r \cdot f)(x) = f(rx)$ for $f \in M^g$ and $r \in R$, and let $g_R(M) = \dim_k(M/\mathfrak{m}_R M)$ be the minimal number of generators of $M$.

Define the Gorenstein defect $\delta(R)$ of $R$ to be the integer $\delta(R) = g_R(R^g) - 1$. Then $R$ is Gorenstein if and only if $\delta(R) = 0$ (this is a definition of Gorenstein). Kilford and Wiese [KW08] define the Gorenstein defect of $R$ to be $\dim_k(\text{Soc}(\bar{R})) - 1$, where $\text{Soc}(\bar{R}) = \text{Ann}_{\bar{R}}(\mathfrak{m}_{\bar{R}})$. Our goal is Lemma C.1.3; these definitions amount to the same thing.

Lemma C.1.1. Assume that $A = k$. Then the canonical pairing $R \times R^g \to k$ induces a perfect pairing $\text{Ann}_R(\mathfrak{m}_R) \times R^g/\mathfrak{m}_R R^g \to k$. In particular, $\delta(R) = \dim_k(\text{Ann}_R(\mathfrak{m}_R)) - 1$.

Lemma C.1.2. There is a canonical isomorphism of $\bar{R}$-modules $\overline{R^g} \cong \bar{R}^g$.

Lemma C.1.3. $\delta(\bar{R}) = \delta(R) = \dim_k(\text{Soc}(\bar{R})) - 1$

This follows from Lemmas C.1.1 and C.1.2 which are standard linear algebra.
C.2. **Fiber products of commutative rings.** Note that the category of commutative rings has all limits. The underlying set of the limit of a diagram of commutative rings is the limit of the diagram of underlying sets.

**Lemma C.2.1.** Consider a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\pi_B} & B \\
\downarrow{\pi_C} & & \downarrow{\phi_B} \\
C & \xrightarrow{\phi_C} & D
\end{array}
\]

in the category of commutative rings. Assume that all the maps are surjective and that \( \ker(\pi_B) \cap \ker(\pi_C) = 0 \). Then the canonical map \( A \to B \times_D C \) is an isomorphism.

The proof is a diagram chase.

**References**


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