Unitary representations of super Lie groups

V. S. Varadarajan

Department of Mathematics
University of California, Los Angeles
CA 90095-1555
vsv@math.ucla.edu

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Abstract

The notion of symmetry plays a great role in quantum physics. Supersymmetry is an unusual and profound generalization of symmetry. It was discovered by the physicists in the early 1970's. Although there is still no experimental confirmation that nature is supersymmetric, the ideas of supersymmetry have played a tremendously important theoretical role in high energy physics. In particular, it seems to be widely accepted that a unified theory of all forces has to be supersymmetric.

In this series of three lectures I shall try to explain how to define the concept of a unitary representation of a super Lie group, and apply it to the classification of relativistic elementary particles in the presence of supersymmetry, leading to the concept of super multiplets.

Contents

Introduction

Unitary operators as quantum symmetries

• **States :** Points of $P(\mathcal{H})$, $\mathcal{H}$ a Hilbert space.

• **Symmetries :** Bijections of the space of states preserving transition probabilities.

• **Description of symmetries :** Wigner's theorem:

Symmetries are induced by unitary or anti unitary operators. The product of two anti unitary symmetries is a unitary symmetry.

Projective unitary representations of space time symmetry group as an expression of covariance

(The principle of covariance) *The description of the system should not depend on the observer.*

This puts in display an action of the spacetime group by symmetries. In QFT the symmetry group is the Poincaré group, in 2-dimensional field theory it is the Galilean group.

Theorem. *If the symmetry group G is a connected Lie group, covariance with respect to G is expressed by a unitary representation (UR) of G or at least a central extension of G by the circle group.*

Remark. If G is the simply connected Poincaré group there is no need to go to central extensions of G (Wigner). If G is the simply connected Galilean group, central extensions of G are unavoidable and introduce the so-called*mass superselection sectors* corresponding to the one parameter family of central extensions (Bargman).

Representations of Poincaré group

The requirement of Poincaré covariance is quite stringent. The simplest such systems arise when the UR of the Poincaré group is irreducible. The irreducible UR's (UIR) define **free particles.** The UIRs of the Poincaré group were first classified by Wigner by a method that goes back to Frobenius and later put in proper perspective by Mackey (Wigner's little group method = Mackey machine).

There are more UIRs than particles. One has to exclude the UIRs where the mass is imaginary (tachyons).

In systems where particle number is not conserved and particles are created and annihilated, the UR of the Poincaré group is not irreducible. QFT which tries to construct such systems is beset with problems which have been resolved only partially, and that too, only by ad hoc procedures (**renormalization.**)

Spacetime at small distances and times

In classical physics, and even in quantum mechanics, there is no necessity to question the use of flat Minkowskian geometry for spacetime, since gravitational forces are negligible in that scale. It is only when experiments began to probe extremely small distances that theories trying to understand and predict the experiments began to encounter serious conceptual difficulties. Physicists then began to look more closely into the structure of spacetime at ultrashort scales of distances and times. The **Planck scale** refers to distances of the order of

$$10^{-33} cm$$

and times of the order of

$$10^{-43} sec.$$

Riemann's vision of space at small distances

In his Göttingen inaugural lecture in 1854 Riemann speculated on the structure of space at small distances. Here is what he said:

Now it seems that the empirical notions on which the metric determinations of Space are based, the concept of a solid body and a light ray, lose their validity in the infinitely small; it is therefore quite definitely conceivable that the metric relations of Space in the infinitely small do not conform to the hypotheses of geometry; and in fact, one ought to assume this as soon as it permits a simpler way of explaining phenomena.

Göttingen inaugural address, 1854.

Remark. Here the phrase *do not conform to the hypotheses of geometry* (presumably) means the structure of space as a Riemannian manifold, or even, just a manifold.

Current views on structure of spacetime

• At the Planck scale, no measurements are possible and so conventional models can no longer be relied upon to furnish a true description of phenomena. String theory attempts to work in a framework where the smallest objects are not point-like but extended, i.e., strings or (more recently) membranes. Spacetime geometry at the Planck scale is thus almost surely non-commutative because there are no points. No one has so far constructed a convincing geometrical theory which is noncommutative but has the Riemann-Einstein geometry as a limit.

• Even at energies very much lower than the Planck scale, a better understanding of phenomena is obtained if we assume that the geometry of spacetime is described locally by a set of coordinates consisting of the usual ones supplemented by a set of anticommuting (Grassmann) coordinates.

Why Grassmann coordinates

The physicists got the idea that one should replace the classical manifold M by a manifold that admits (additional) Grassmann coordinates. The Grassman coordinates model the Pauli exclusion principle for the Fermions in an embryonic form. The theory of classical fields on such a manifold would then provide a basis for quantization that will yield the exterior algebras characteristic of quantum descriptions of Fermionic states. Such a manifold is nowadays called a super manifold.

The concept of supersymmetry

- Super symmetries are diffeomorphisms of super manifolds.

- The super Lie algebra of infinitesimal automorphisms of super manifolds.

- Super Poincaré and super conformal Lie algebras.

Super geometry

The fundamentals of super geometry are built like ordinary geometry. This is done at 3 levels:

- Infinitesimal
- Local
- Global

Super linear algebra

- Category of super vector spaces

- The isomorphism c_{UV}

$c_{UV} : U \otimes V \longrightarrow V \otimes U, \quad u \otimes v \longmapsto (-1)^{p(u)p(v)} v \otimes u$

- Action of $\mathfrak{S}_N$ on $V \otimes V \otimes \ldots \otimes V$

- **Super Lie algebras**

$$[\cdot, \cdot](1 + c_{\mathfrak{g}\mathfrak{g}}) = 0$$

$$[\cdot, [\cdot, \cdot]](1 + \sigma + \sigma^2) = 0$$

where σ is the automorphism of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ that corresponds to the permutation $(123) \longrightarrow (312)$.

The Berezinian

Let R be a super commutative algebra over a field k of characteristic 0 and $R^{p|q}$ be the free module of dimension $p|q$ over R. Let $\mathrm{GL}(p|q)(R)$ be the group of invertible even morphisms of $R^{p|q}$. Then the **Berezinian** is a morphism of $\mathrm{GL}(p|q)(R)$ into $R_0^\times$ (the group of units of the even part R_0 of R) given by

$$\mathrm{Ber}(x) = \det(A - BD^{-1}C)\det(D)^{-1}$$

where

$$x = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We have

$$\mathrm{Ber}(xy) = \mathrm{Ber}(x)\mathrm{Ber}(y).$$

This is the superversion of the determinant, discovered by **F. A. Berezin**, one of the pioneers of super algebra and super analysis. Since the entries of B and C are nilpotent, x is invertible if and only if A and D, whose entries are in the commutative ring R_0, are invertible.

Charts and local rings

The local coordinates are given by

$$x^1, x^2, \ldots, x^p, \theta^1, \theta^2, \ldots, \theta^q$$

where the x^i are the usual commuting coordinates and the θ^j are the grassmann coordinates:

$$\theta^k\theta^k + \theta^k\theta^j = 0.$$

The local ring is

$$C^\infty(x^1, x^2, \ldots, x^p)[\theta^1, \theta^2, \ldots, \theta^q].$$

The concept of a super manifold

A *super manifold* M of dimension $p|q$ is a smooth manifold $|M|$ of dimension p together with a sheaf $\mathcal{O}_M$ of super commuting algebras on $|M|$ that looks locally like

$$C^\infty(\mathbf{R}^p)[\theta^1, \theta^2, \ldots, \theta^q]$$

The intuitive picture of M is that of $|M|$ surrounded by a grassmannian cloud. **The cloud cannot be seen: in any measurement the odd variables will be 0 because they are nilpotent. Thus measurement sees only the underlying classical manifold $|M|$.** Nevertheless the presence of the cloud eventually has consequences that are striking.

Unlike classical geometry the local ring contains **nilpotents.** So the analogy is with a **Grothendieck scheme.**

Physicists refer to the sections of the structure sheaf as **superfields.**

Supersymmetries

A **Supersymmetry** is just a morphism between supermanifolds.

Example: The diffeomorphism

$$\mathbf{R}^{1|2} \simeq \mathbf{R}^{1|2} : t^1 \longmapsto t^1 + \theta^1\theta^2, \quad \theta^\alpha \longmapsto \theta^\alpha$$

is a typical supersymmetry. Note how the morphism interchanges odd and even variables. This is a basic example of how the grassmann cloud interacts with the classical manifold underlying the supermanifold.

Integration on super manifolds

Let

$$\theta^I = \theta^{i_1}\theta^{i_2}\dots\theta^{i_k} \qquad I = (i_\mu), \{i_1 < i_2 < \dots < i_k\}.$$

On

$$\Lambda = \mathbf{R}[\theta^1, \dots, \theta^q]$$

the integral is a linear map

$$a \longmapsto \int a d^q\theta$$

defined by

$$\int \theta^I d^q\theta = \begin{cases} 0 & \text{if } |I| < q \\ 1 & \text{if } I =: Q = \{1, 2, \dots, q\}. \end{cases}$$

Integration is also differentiation:

$$\int = \left(\frac{\partial}{\partial\theta^q}\right)\left(\frac{\partial}{\partial\theta^{q-1}}\right)\dots\left(\frac{\partial}{\partial\theta^1}\right).$$

In the local ring with coordinates x^i, θ^j,

$$\int s d^p x d^q\theta = \int s_Q d^p x \qquad (s = \sum_I s_I \theta^I).$$

The change of variables formula

For a morphism given locally as

$$\psi : (x, \theta) \longmapsto (y, \varphi)$$

we define the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial y}{\partial x} & -\frac{\partial y}{\partial \theta} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial \theta} \end{pmatrix}.$$

Then

$$\int s = \int \psi^*(s)\mathrm{Ber}(J\psi)$$

for compactly supported sections of the local ring. For arbitrary manifolds we use partitions of unity as in the classical case.

- This beautiful formula goes back to Berezin. The justification for the peculiar definition of integration in the anticommuting variables is the change of variables formula.

Super Lie groups and their super Lie algebras

Super Lie groups are group objects in the category of super manifolds. As in the theory of ordinary Lie groups one can define the super Lie algebra $\mathrm{Lie}(G)$ of a super Lie group G. The even part of a super Lie algebra is an ordinary Lie algebra and the super Lie algebra may be viewed as a supersymmetric enlargement of it.

History. Gol'fand-Likhtman and Volkov-Akulov discovered the minimal SUSY extension of the Poincaré algebra in the early 1970's. Wess-Zumino discovered a little later, in 1974, the first example of a **simple** super Lie algebra, namely the minimal SUSY extension of the conformal Lie algebra. In 1975 V. Kac formally defined super Lie algebras and carried out the super version of the Cartan-Killing classification of simple Lie algebras over $\mathbf{C}$.

Super Minkowski spacetime

Let $\mathfrak{t}_0$ be a flat Minkowski spacetime of signature $(1, n)$. By a **flat Minkowski superspacetime** is meant a super Lie group whose even part is $\mathfrak{t}_0$ identified with its group of translations. The correponding super Lie algebra $\mathfrak{t}$ has the grading

$$\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_1.$$

Physical interpretations lead to the requirement that the adjoint action of $\mathfrak{t}_0$ on $\mathfrak{t}_1$,

$$a, b \longmapsto [a, b] \qquad (a \in V_0, b \in V_1)$$

is a very special module, namely a **spin module.** In this case, at least when $\mathfrak{t}_1$ is irreducible, there is an essentially unique symmetric bilinear form

$$\mathfrak{t}_1 \otimes \mathfrak{t}_1 \longrightarrow \mathfrak{t}_0$$

If we choose this to be the supercommutator of odd elements we may regard $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_1$ as a super Lie algebra. The super Lie group T of $\mathfrak{t}$ is flat **super Minkowski spacetime.**

Super Poincaré group

The semi direct product

$$G = T \times' \mathrm{Spin}(1, n)$$

is a **super Poincaré group.**

SUSY Field theory

All of the machinery is now in place for introducing the-Lagrangians and doing SUSY field theory on flat super Minkowski spacetime: **Wess-Zumino** (SUSY electrodynamics) and **Ferrara-Zumino** (SUSY Yang-Mills).

The susy extension of Einstein spacetime is more complicated and was first done in 1976 by **Ferrara, Freedman**, and **van Nieuwenhuizen**, and a little later, by **Deser** and **Zumino**. It is called **supergravity.**

Classification of super particles

In quantum theory the unitary irreducible representations (UIR) of the Poincare group classify free elementary particles. In SUSY quantum theory, the UIR's of a super Poincaré group classify elementary super particles. Each super particle, when viewed as a UR of the underlying Poincaré group, is the direct sum of a collection of ordinary particles, called a **multiplet.** The members of a multiplet are called **partners** of each other.

Unlike the classical case, the positivity of energy is a consequence of supersymmetry.

The existence of the superpartners of the known particles is the biggest prediction of supersymmetry.

It may be hoped that the new super collider being readied at CERN will create the super partners of the usual elementary particles. This is not certain because one does not know exactly the scale at which supersymmetry is broken.

Unitary representations of super Lie groups and the imprimitivity theorem for even super homogeneous spaces

Summary

We introduce the concepts of unitary representations and systems of imprimitivity for super Lie groups, using the well known equivalence of the category of super Lie groups with the category of super Harish-Chandra pairs. We extend the Mackey theory of induced representations to the super context when the homogeneous space is purely even.

Super Lie groups and super Lie algebras

The word *super* indicates $\mathbf{Z}_2$-graded objects; the suffixes 0 and 1 denote the graded components of even and odd elements. p is the parity function defined on homogeneous nonzero elements.

For a super Lie algebra $\mathfrak{g}$ and homogeneous $X, Y \in \mathfrak{g}$,

$$\mathrm{ad}(X)(Y) = [X, Y].$$

Then the super Jacobi identity is equivalent to

$$\mathrm{ad}[X, Y] = \mathrm{ad}(X)\mathrm{ad}(Y) - (-1)^{p(X)p(Y)}\mathrm{ad}(Y)\mathrm{ad}(X)$$

A *super Lie group* is a group object in the category of super manifolds. For any super Lie group G and any manifold T,

$$T \longmapsto G(T), \qquad G(T) = \mathrm{Morph}(T, G)$$

is a contravariant functor into the category of groups which determines G up to isomorphism.

Super Harish-Chandra pairs

$(G_0, \mathfrak{g})$ is a *super Harish-Chandra pair* (SHP) if

- G_0 is a Lie group, $\mathfrak{g}$ is a super Lie algebra with $\mathfrak{g}_0 = \mathrm{Lie}(G_0)$

- $\mathfrak{g}$ is a G_0-module, and the action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is the differential of the action of G_0 on $\mathfrak{g}_1$.

For G a super Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$, and G_0 the classical Lie group underlying G, $(G_0, \mathfrak{g})$ is a SHP and we have an equivalence of categories

$$G \longmapsto (G_0, \mathfrak{g}).$$

- $G = R^{p|q}$, with addition.

- $G = \mathrm{GL}(p|q)$ with the pair $G_0 = \mathrm{GL}(p) \times \mathrm{GL}(q), \mathfrak{g} = \mathfrak{gl}(p|q)$.

- $G = \mathrm{SL}(p|q)$ with the pair $G_0 = (\mathrm{GL}(p) \times \mathrm{GL}(q))_1$, suffix 1 meaning determinant 1, and $\mathfrak{g} = \mathfrak{sl}(p|q)$ where $\mathfrak{s}$ means that the *super trace* is 0. Globally this is the sub super Lie group of elements with the *Berezinian* equal to 1.

Super Poincaré groups

To say that a super Lie group $G = (G_0, \mathfrak{g})$ is a *super Poincaré group* means the following.

- G_0 is the semidirect product $\mathbf{R}^{1,D-1} \times' \mathrm{Spin}(1, D-1)$ where $\mathrm{Spin}(1, D-1)$ is the 2-fold cover of the classical connected orthogonal group of signature $(1, D-1)$
- The G_0-module $\mathfrak{g}_1$ is a direct sum (over $\mathbf{C}$) of spin modules
- $\mathbf{R}^{1,D-1}$ acts trivially on $\mathfrak{g}_1$
- $[\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathbf{R}^{1,D-1}$

  The last condition means there is a non zero $\mathrm{Spin}(1, D-1)$-equivariant symmetric bilinear map from $\mathfrak{g}_1 \times \mathfrak{g}_1$ to $\mathbf{R}^{1,D-1}$. *Such a map always exists and is projectively unique if $\mathfrak{g}_1$ is irreducible*, and, upto a change of sign, the image of the map $\mathfrak{g}_1 \times \mathfrak{g}_1$ is inside the forward light cone (minus the origin) (*positivity of energy*).

N usually denotes the number of irreducible components of $\mathfrak{g}_1$. If $N > 1$ we speak of *N-extended supersymmetry*. In this case positivity of the energy is an added requirement.

- $S := T_0 \times' \mathfrak{g}_1$ is a *super Minkowski spacetime* and $G = \mathrm{Aut}(S)$.

SUSY quantum mechanics

Super Hilbert space (SHS)

- A SHS is a super vector space $\mathcal{H}$ over $\mathbf{C}$, with $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$.
- $\mathcal{H}$ is a Hilbert space, the $\mathcal{H}_i$ are closed and orthogonal

Super adjoints

- If T is an operator of $\mathcal{H}$ with adjoint T^*, its *super adjoint* $T^\dagger$ is defined as follows:

$$p(T^\dagger) = p(T), \qquad T^\dagger = \begin{cases} T^* & \text{if } T \text{ is even} \\ \\ -iT^* & \text{if } T \text{ is odd.} \end{cases}$$

A Hamiltonian H is *supersymmetric* if $H = X^2$ where X is an odd operator.

- *Supersymmetry makes the energy positive.*

Self adjointness and symmetry

For $\mathcal{H}$ infinite dimensional, Hamiltonians are usually *unbounded* operators. Indeed, most Hamiltonians are differential operators, and so are defined only on functions which are sufficiently smooth. The property of self adjointness is more than just demanding symmetry. The fact that it is self adjointness that implies a spectral resolution means that *Hamiltonians have to be self adjoint, not merely symmetric.* Heuristically, self adjointness for a differential operator means that the boundary conditions are correctly posed. A *core* for a symmetric operator is a dense subspace on which it is defined, with the property that *there is a unique self adjoint extension.* For most (atomic) quantum Hamiltonians the smooth functions with compact support is a core; this is a famous result of Kato. For any operator A, $D(A)$ is its domain of definition.

Lemma. *Let H be self adjoint on $\mathcal{H}$, $U(t) = e^{itH}$, and $\mathcal{B} \subset D(H)$ a dense U-invariant subspace. Then $\mathcal{B}$ is a core for H. If X is a symmetric operator such that $X\mathcal{B} \subset D(X)$ and $H = X^2$ on $\mathcal{B}$, for example if $X\mathcal{B} \subset \mathcal{B}$, then X is essentially self adjoint on $\mathcal{B}$ and $\overline{X}^2 = H$. In particular, $\overline{X}^{1/2} = H \geq 0$.*

UR's of super Lie groups (finite dimensional)

A *finite dimensional* unitary representation (UR) of a super Lie group $G = (G_0, \mathfrak{g})$ is a pair (π, γ) where π is an *even* representation of G_0 in a SHS $\mathcal{H}$, and γ is a super representation of $\mathfrak{g}$ in $\mathcal{H}$ such that

- $\gamma\big|_{\mathfrak{g}_0} = d\pi$,
- $\gamma(gX) = \pi(g)\gamma(X)\pi(g)^{-1} \qquad (g \in G_0, X \in \mathfrak{g})$
- (*unitarity condition*) $\gamma(X)^\dagger = -\gamma(X) \qquad (X \in \mathfrak{g})$

Let

$$\zeta = e^{-i\pi/4}, \qquad \rho(X) = \zeta\gamma(X).$$

Then finite dimensional UR's are pairs (π, ρ) with

- π an even UR of G_0 in a SHS $\mathcal{H}$
- $\rho : \mathfrak{g}_1 \longrightarrow \mathbf{End}(\mathcal{H})_1$ linear and self adjoint
- $-id\pi([X, Y]) = \rho(X)\rho(Y) + \rho(Y)\rho(X)$ for $X, Y \in \mathfrak{g}_1$
- $\rho(gX) = \pi(g)\rho(X)\pi(g)^{-1}$ for $g \in G_0, X \in \mathfrak{g}_1$.

We can then take

$$\gamma(X) = d\pi(X_0) + \zeta^{-1}\rho(X_1) \text{ for } X \in \mathfrak{g}.$$

Infinite dimensional UR's of super Lie groups

The main (apparent) obstruction in transferring the definition to the infinite dimensional case is that the operators $\rho(X)$ will in general be unbounded. We make the following definition and discuss possible variants.

For any UR λ of G_0, $C^\infty(\lambda)$ is the space of smooth vectors for λ. Then UR's of $(G_0, \mathfrak{g})$ are pairs (π, ρ) with

- π an even UR of G_0 in $\mathcal{H}$
- $\rho : \mathfrak{g}_1 \longrightarrow \mathbf{End}(C^\infty(\pi))_1$ linear and symmetric
- $-id\pi([X,Y]) = \rho(X)\rho(Y) + \rho(Y)\rho(X)$ for $X, Y \in \mathfrak{g}_1$
- $\rho(gX) = \pi(g)\rho(X)\pi(g)^{-1}$ for $g \in G_0, X \in \mathfrak{g}_1$.

Then

$$X \longmapsto d\pi(X_0) + \zeta^{-1}\rho(X) \qquad (X \in \mathfrak{g})$$

is a super representation of $\mathfrak{g}$ in $C^\infty(\pi)_1$.

Variants

It may appear that the choice of $C^\infty(\pi)$ in the above definition, while natural and canonical, is still somewhat arbitrary; for instance we could have chosen the space of *analytic vectors* for π in its place. It turns out that all such choices are essentially equivalent in the sense that for any variant of the above definition which uses a different subspace than $C^\infty(\pi)$, the operators $\rho(X)$ can be extended all the way to $C^\infty(\pi)$ so that we obtain a UR in the above sense, and moreover, the $\rho(X)$ will all be self adjoint with $C^\infty(\pi)$ as a core.

UR's of $(G_0, \mathfrak{g})$ are systems $(\pi_0, \rho, \mathcal{B})$ with

- $\mathcal{B}$ a dense π-invariant super linear subspace of $\mathcal{H}$, $\mathcal{B} \subset D(d\pi(Z))$ for all $Z \in [\mathfrak{g}_1, \mathfrak{g}_1]$
- ρ a linear map of $\mathfrak{g}_1$ into $\mathbf{Hom}(\mathcal{B}, \mathcal{H})_1$ such that
  - (i) $\rho(X)$ is symmetric for all $X \in \mathfrak{g}_1$
  - (ii) $\rho(gX) = \pi(g)\rho(X)\pi(g)^{-1}$ for all $g \in G_0, X \in \mathfrak{g}_1$
  - (iii) $-id\pi([X, Y]) = \rho(X)\rho(Y) + \rho(Y)\rho(X)$ for $X, Y \in \mathfrak{g}_1$
  - (iv) $\rho(X)\mathcal{B} \subset D(\rho(Y))$ for all $X, Y \in \mathfrak{g}_1$

Equivalence of the various definitions

The basic result on the concept of a UR of a super Lie group is the following theorem.

Theorem. *Let* $(\pi, \rho, \mathcal{B})$ *be a* UR*. Then*

(i) For any $X \in \mathfrak{g}_1$*,* $\rho(X)$ *is essentially self adjoint on* $\mathcal{B}$ *and* $C^\infty(\pi) \subset D(\overline{\rho(X)})$

(ii) Let $\overline{\rho}(X) = \overline{\rho(X)}\big|_{C^\infty(\pi)}$ *for* $X \in \mathfrak{g}_1$*. Then* $(\pi_0, \overline{\rho})$ *is a* UR *in the first sense*

If $(\pi_0, \rho', \mathcal{H})$ *is a* UR *such that* $\mathcal{B} \subset D(\overline{\rho'(X)})$ *and* $\overline{\rho'(X)}$ *restricts to* $\rho(X)$ *on* $\mathcal{B}$ *for all* $X \in \mathfrak{g}_1$*, then* $\rho' = \overline{\rho}$*.*

This theorem makes it clear that the concept of a UR of a super Lie group is a viable one even in the infinite dimensional case.

Variant for analytic vectors

There is a variant using analytic vectors. Let $C^\omega(\pi)$ be the subspace of analytic vectors for π. We then have

- For any UR (π, ρ), the $\rho(X)(X \in \mathfrak{g}_1)$ map $C^\omega(\pi)$ into itself and so

  $$\gamma : X \longmapsto d\pi(X_0) + \zeta^{-1}\rho(X_1)$$

  is a super representation of $\mathfrak{g}$ in $C^\omega(\pi)$

Let G_0 be connected, π an even UR of G_0, and $\mathcal{B} \subset C^\omega(\pi)$ a dense subspace on which we have a super representation γ of $\mathfrak{g}$ such that $\gamma(Z)$ is a restriction of $d\pi(Z)$ to $\mathcal{B}$ for $Z \in \mathfrak{g}_0$ and $\rho(X) = \zeta\gamma(X)$ is symmetric on $\mathcal{B}$ for $X \in \mathfrak{g}_1$. Then

- $\rho(X)$ is essentially self adjoint on $\mathcal{B}$ and $C^\infty(\pi) \subset D(\overline{\rho(X)})$ for $X \in \mathfrak{g}_1$

- $(\pi, \overline{\rho})$ is a UR if we define $\overline{\rho}(X) = \overline{\rho(X)}\big|_{C^\infty(\pi)}$

- Uniqueness as in the Theorem above.

The category of UR's of a super Lie group

A morphism

$$(\pi, \rho) \longrightarrow (\pi', \rho')$$

is a bounded operator

$$A : \mathcal{H} \longrightarrow \mathcal{H}'$$

such that

$$A\pi_0 = \pi_0' A, \qquad A\rho = \rho' A.$$

Note that the first condition implies that

$$A(C^\infty(\pi)) \subset C^\infty(\pi')$$

so that the second condition makes sense.

- For any UR (π, ρ) and any closed π-invariant subspace $\mathcal{M}$, the invariance of $\mathcal{M}^\infty = \mathcal{M} \cap C^\infty(\pi)$ under all $\rho(X)$ is equivalent to its invariance under all the spectral projections of all the $\rho(X)$.

- The UR (π, ρ) is *irreducible* if the only self morphisms are scalars, or equivalently, if the only closed super linear subspaces invariant under π and the spectral projections of the $\rho(X)$ are 0 and the whole Hilbert space

Systems of imprimitivity

Let $G = (G_0, \mathfrak{g})$ act on M a super manifold.

- $\mathcal{D}(M)$=the super algebra of compactly supported smooth sections of $\mathcal{O}_M$

Then G_0 acts on the super algebra $\mathcal{D}(M)$. We have an *anti-morphism* λ of $\mathfrak{g}$ into the super Lie algebra of vector fields on M.

- $\lambda(X)$ = vector field on M corresponding to $X \in \mathfrak{g}_1$

A *super system of imprimitivity* for G based on M is (tentatively) a UR (π, ρ) of G together with a representation B of $\mathcal{D}(M)$ in $C^\infty(\pi)$ by bounded operators such that

- $\pi(g)B(s)\pi(g)^{-1} = B(g{\cdot}s) \qquad (g \in G_0, s \in \mathcal{O}(M))$

- $[\rho(X), B(s)] = B(\lambda(X)s) \qquad (X \in \mathfrak{g}_1, s \in \mathcal{O}(M))$

The main application is when M is transitive, i.e., when

$$M \simeq G/H, \qquad H = (H_0, \mathfrak{h}).$$

Special subgroups and imprimitivity systems

The sub super Lie group $H = (H_0, \mathfrak{h})$ of $G = (G_0, \mathfrak{g})$ is *special* if $\mathfrak{h}_1 = \mathfrak{g}_1$. The quotient super manifold $\Omega = G/H \simeq G_0/H_0$ is then purely even. Since $\lambda(X)$ is odd for $X \in \mathfrak{g}_1$, it follows that $\lambda(X) = 0$ on Ω. Hence $\rho(X)$ commutes with $B(s)$ for all $X \in \mathfrak{g}_1$ and $s \in \mathcal{D}(M)$. In this special case we require that B is a $*$-representation. We then have the following *precise* description.

A *system of imprimitivity* based on Ω is a system (π, ρ^π, P) where (π, ρ^π) is a UR of $(G_0, \mathfrak{g})$, P an even projection valued measure on Ω such that (π, P) is a classical system of imprimitivity and the projections of P commute with the spectral projections of $\overline{\rho(X)}$ for all $x \in \mathfrak{g}_1$ $(P \leftrightarrow \rho^\pi)$.

The super imprimitivity theorem. *There is a natural equivalence of categories from* UR*'s of* $(H_0, \mathfrak{h})$ *to special systems of imprimitivity on* Ω*.*

Explicit construction

Let $(\sigma, \rho^\sigma, \mathcal{K})$ be a UR of $(H_0, \mathfrak{h})$. We assume that Ω has a G_0-invariant measure (this is unnecessary but makes life easier) $d\Omega$. Then π is left translation acting on the Hilbert space $\mathcal{H}$ of functions from G_0 to $\mathcal{K}$ such that

- $f(x\xi) = \sigma(\xi)^{-1} f(x) \qquad (x \in G_0, \xi \in H_0)$
- $||f||^2_{\mathcal{H}} = \int_\Omega ||f(x)||^2_{\mathcal{K}} d\Omega < \infty$

Then the space $\mathcal{B}$ of smooth vectors for π which are compactly supported mod H_0 is also the space of smooth f's with compact support mod H. Moreover, for $f \in \mathcal{B}$, one can show that $f(x) \in C^\infty(\sigma)$ for all $x \in G_0$. We now define ρ^π as follows:

$$(\rho^\pi(X)f)(x) = \rho^\sigma(x^{-1}X)f(x) \qquad (X \in \mathfrak{g}_1).$$

Then

$$(\pi, \rho^\pi, \mathcal{B})$$

defines a UR of $(G_0, \mathfrak{g})$. If P is the natural projection valued measure in $\mathcal{H}$ based on Ω, then

$$(\pi, \rho^\pi, P)$$

is the special system of imprimitivity associated with (σ, ρ^σ) in the above equivalence of categories.

Unitary representations of super semidirect products

Summary

We show that the unitary irreducible representations of super semidirect products can be classified by a generalization of the classical little group method to the super context. We apply this theory to the classification of super particles and the description of their multiplet structure.

UIR's of a classical semidirect product (SDP)

Projective UIR's of the underlying symmetry group G classify elementary particles.

- $G =$ Galilean group

  The UIR's classify Schrödinger particles of mass $m > 0$ and spin j and give rise to *mass superselection sectors*.

- $G =$ Poincaré group

  All projective UR's are ordinary and the UIR's classify Dirac particles of mass $m > 0$ and spin j, *and* Weyl particles of mass $m = 0$ and helicity j.

$G = T_0 \times' L_0$, T_0 a real (f.d) vector space amd $L_0 \subset \mathrm{SL}(T_0)$ a closed subgroup. If P is the spectral measure of T_0 in a UR of G_0, its support is the *spectrum* of the UR. $O(\lambda)$ is the orbit of λ. We assume that the orbit space T_0^*/L_0 is smooth in the Borel sense.

For $\lambda \in T_0^*$, L_0^λ = stabilizer (little group) of λ in L_0^λ; $G_0^\lambda = T_0 L_0^\lambda$. A UR of G_0^λ is λ-*admissible* if T_0 acts as the character $e^{i\lambda}$.

$$\begin{aligned}&\text{UIR's of } G_0 \text{ with spectrum} = O(\lambda)\\ &\iff \text{admissible UIR's of } G_0^\lambda.\end{aligned}$$

Super semidirect products (SSDP) and super Poincaré groups

$(G_0, \mathfrak{g})$ is a *super semidirect product* if:

- $G_0 = T_0 \times' L_0$, T_0 a real (f.d) vector space, $L_0 \subset \mathrm{SL}(T_0)$ a closed subgroup

- T_0 acts trivially on $\mathfrak{g}_1$ and $[\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{t}_0 := \mathrm{Lie}(T_0)$

$(G_0, \mathfrak{g})$ is a *super Poincaré group* if:

- T_0 is a Minkowski space of signature $(1, D-1)$ and $L_0 = \mathrm{Spin}(1, D-1)$ (2-fold cover of $\mathrm{SO}(1, D-1)^0$)

- L_0 acts on $\mathfrak{g}_1$ spinorially, i.e., its complexification splits as a direct sum of spin modules.

- The odd commutator map $\mathfrak{g}_1 \times \mathfrak{g}_1 \longrightarrow \mathfrak{t}_0$ maps into the closure of the open forward light come (minus the origin) (*positivity of energy*)

Theorem *Given any spinorial module V there is a super Poincaré group $(G_0, \mathfrak{g})$ with $\mathfrak{g}_0 = \mathrm{Lie}(G_0), \mathfrak{g}_1 = V$. The bracket on $\mathfrak{g}_1$ is projectively unique if V is irreducible.*

UIR's of a SSDP

For $\lambda \in T_0^*$, $S^\lambda = (G_0^\lambda, \mathfrak{g}^\lambda)$ is the *little* (*super*) *group* at λ: $G_0^\lambda = T_0 L_0^\lambda$ and $\mathfrak{g}^\lambda = \mathfrak{t}_0 \oplus \mathfrak{l}_0^\lambda \oplus \mathfrak{g}_1$. It is a special sub super Lie group. For a UR (π, ρ^π) of $G = (G_0, \mathfrak{g})$, P is the spectral measure of $\pi\big|_{T_0}$. *Since T_0 acts trivially on $\mathfrak{g}_1$, the $\pi(t)$ commute with the $\rho^\pi(X)$.* If the UR is irreducible (UIR), then P is concentrated on an orbit.

If $\lambda \in T_0^*$, a UR of G is λ-*admissible* if $\pi(t) = e^{i\lambda(t)} I (t \in T_0)$. λ itself is *admissible* if there is a λ-admissible UR ($\Longleftrightarrow$ if there is a λ-admissible UIR).

$$T_0^+ = \left\{ \lambda \;\middle|\; \lambda \in T_0^*, \lambda \text{ admissible} \right\}.$$

Theorem. *For any $\lambda \in T_0^+$, the super imprimitivity theorem gives an equivalence of categories from the category of λ-admissible* UR*'s of S^λ with the* UR*'s of $(G_0, \mathfrak{g})$ whose spectra are contained in the orbit of λ. In particular, a* UIR *has spectrum in the orbit of λ if and only if λ is admissible, and then we have a bijection between the sets of equivalence classes of* UIR*'s of G and S^λ.*

Remark. Note the significant difference from the classical one in that there is a *selection rule* for the orbits: admissibility.

Admissibility as the positive energy condition

Let λ be admissible and (σ, ρ^σ) be a λ-admissible UIR for S^λ. Then

$$-id\sigma(Z) = \lambda(Z)I \qquad (Z \in \mathfrak{t}_0).$$

- $Q_\lambda(X) = (1/2)\lambda([X, X])$ is a L_0^λ-invariant quadratic form on $\mathfrak{g}_1$
- $\rho^\sigma(X)^2 = Q_\lambda(X)I$ on $C^\infty(\sigma)$

It follows, as $\rho^\sigma(X)$ is essentially self adjoint on $C^\infty(\sigma)$, that

- Q_λ is *nonnegative* and the ρ^σ are *bounded.*

Theorem. *Let $\lambda \in T_0^*$. Then the following are equivalent.*

(i) *λ is admissible*

(ii) *$Q_\lambda(X) \geq 0$ for all $X \in \mathfrak{g}_1$.*

We shall see that if G is a super Poincaré group, condition (ii) is essentially the condition that energy is positive. Hence we refer to (ii) as the *positive energy condition.* We shall sketch an outline of the proof assuming L_0^λ is connected. This is satisfied for super Poincaré groups.

Clifford algebras associated to positive energy orbits

Let $\mathcal{C}_\lambda$ be the algebra generated by $\mathfrak{g}_1$ with the relations

$$X^2 = Q_\lambda(X)1(X \in \mathfrak{g}_1).$$

Even though Q_λ may have a nonzero radical we call $\mathcal{C}_\lambda$ the *Clifford algebra* of $(\mathfrak{g}_1, Q_\lambda)$. If

$$\mathfrak{g}_{1\lambda} := \mathfrak{g}_1/\text{rad } Q_\lambda$$

then Q_λ is strictly positive on $\mathfrak{g}_{1\lambda}$ and there is a natural map

$$\mathcal{C}_\lambda \longrightarrow \mathcal{C}_\lambda^\sim = \text{ Clifford algebra of } \mathfrak{g}_{1\lambda}$$

with kernel as the ideal generated by the radical of Q_λ.

We wish to build a UIR (σ, ρ) of the little group S^λ with

- ρ a representation of $\mathcal{C}_\lambda$ by *bounded operators*, $\rho(X)$ *self adjoint and odd* for all $X \in \mathfrak{g}_1$; ρ is called a *self adjoint representation.*
- σ is an even UR of L_0^λ such that

$$\sigma(t)\rho(X)\sigma(t)^{-1} = \rho(tX) \qquad (t \in L_0^\lambda, X \in \mathfrak{g}_1)$$

Simply connected little super groups

We shall assume that L_0^λ is simply connected. This is satisfied if G is a super Poincaré group and $D \geq 4$. Since Q_λ is L_0^λ-invariant we have a map

$$L_0^\lambda \longrightarrow \mathrm{SO}(\mathfrak{g}_{1\lambda})$$

which lifts to a map

$$L_0^\lambda \longrightarrow \mathrm{Spin}(\mathfrak{g}_{1\lambda}).$$

There is an *irreducible* self adjoint representation τ_λ of $\mathcal{C}_\lambda$, finite dimensional, unique if $\dim(\mathfrak{g}_{1\lambda})$ is odd, unique up to parity reversal otherwise. The spin representation of $\mathrm{Spin}(\mathfrak{g}_{1\lambda})$ lifts to an even UR κ_λ of L_0^λ, with

$$\kappa_\lambda(t)\tau_\lambda(X)\kappa_\lambda(t)^{-1} = \tau_\lambda(tX) \qquad (t \in L_0^\lambda, X \in \mathfrak{g}_1).$$

The assignment

$$r \longmapsto \theta_{r\lambda} = (\sigma, \rho), \qquad \sigma = e^{i\lambda} r \otimes \kappa_\lambda, \qquad \rho = 1 \otimes \tau_\lambda$$

is an equivalence of categories from the category of purely even UR's r of L_0^λ to the category of λ-admissible UR's of the little super group S^λ. It gives a bijection (up to equivalence) between UIR's of L_0^λ and UIR's of S^λ.

When the little group is only connected

If L_0^λ is connected but not simply connected, we assume that it is of the form

$$L_0^\lambda = A \times' T \qquad (A \text{ simply connected }, T \text{ a torus}).$$

Then there is a 2-fold cover

$$T^\sim \longrightarrow T$$

such that

$$L_0^\lambda \longrightarrow \mathrm{SO}(\mathfrak{g}_{1\lambda})$$

lifts to

$$p : L_0^\sim \longrightarrow \mathrm{Spin}(\mathfrak{g}_{1\lambda}), \quad L_0^\sim = A \times' T^\sim, \quad p(1,\xi) = -1$$

where ξ is the non trivial element in the kernel of $T^\sim \longrightarrow T$. We can lift the spin representation of $\mathrm{Spin}(\mathfrak{g}_{1\lambda})$ to a UR κ'_λ of $L_0^\sim$. If we take a character χ of $T^\sim$ with $\chi(\xi) = -1$, and view it as a character of $L^\sim$, then

$$\kappa_\lambda = \chi\kappa'_\lambda$$

takes $(1,\xi)$ to 1, hence may be viewed as a UR of L_0^λ. From now on the development is the same as before.

The fundamental multiplet

The theory now gives a bijection

$$r \longleftrightarrow \theta_{r\lambda} \longleftrightarrow \Theta_{r\lambda}$$

between UIR's r of L_0^λ and UIR's $\Theta_{r\lambda}$ of G with spectrum in the orbit of λ. The $\Theta_{r\lambda}$ represent the *super particles.* The corresponding UR's of G_0 are *not* irreducible and their irreducible constituents define the so-called *super multiplets.* The members of the multiplet are the ordinary particles that correspond to the orbit of λ and the irreducible constituents of $r \otimes \kappa_\lambda$. When r is the trivial representation we obtain the *fundamental multiplet.* They are the ordinary particles defined by the orbit of λ and the irreducible constituents of κ_λ. In the case of super Poincaré groups κ_λ can be explicitly determined and its decomposition into irredeucibles described (in principle). When $D = 4$ this was done using the R-group in the paper of Ferrara, Savoy, and Zumino.

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