

Symmetry and supersymmetry

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Abstract

The notion of symmetry plays a great role in quantum physics. Supersymmetry is an unusual and profound generalization of symmetry. It was discovered by the physicists in the early 1970's. Although there is still no experimental confirmation that nature is supersymmetric, the ideas of supersymmetry have played a tremendously important theoretical role in high energy physics. In particular it is widely accepted that a unified theory of all forces has to be supersymmetric.

In this talk, which will be very elementary, I shall try to explain what supersymmetry is, and describe the modifications we have to make in our conception of geometry, specifically the geometry of spacetime at small distances and times, to accommodate the requirements of supersymmetry.

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- Spacetime at small distances: from Riemann to Salam
- The concept of supersymmetry
- The super Poincaré group
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Quantum world and its symmetries

Quantum States

- I. **States** are unit vectors in a complex Hilbert space \mathcal{H} ; ϕ and $c\phi$ ($c \in \mathbf{C}$) represent the same state if $|c| = 1$.
- II. If ϕ is the state and we do an experiment to see if the state is ψ , then

$$|(\phi, \psi)|^2$$

is the **probability** that the experiment finds the system in state ψ .

By I. the states are the points of a projective space. Physicists call this the **superposition principle**.

This rule contains the entire statistical aspect of quantum theory and encodes at a fundamental level the fact that measurements interfere with the phenomena to be measured but that the results of measurement are in general only statistically determinate.

Note that the absolute value ensures that this quantity depends only on the states determined by ϕ, ψ , since it is unchanged by

$$\phi \longmapsto c\phi, \quad \psi \longmapsto c'\psi \quad (|c| = |c'| = 1).$$

Quantum Symmetries

Symmetries are bijections of the space of states preserving measurement probabilities. By the fundamental theorem of projective geometry these are therefore given by

$$\phi \longmapsto U\phi$$

where U is *unitary* or *antiunitary* :

$$(U\phi, U\psi) = (\phi, \psi) \quad \text{or} \quad (U\phi, U\psi) = (\psi, \phi).$$

Remark. The above result (due to Wigner) is the reason why quantum mechanics is *linear*. Weinberg and others have considered *nonlinear* dynamical systems on projective space in attempts to extend the scope of quantum mechanics.

Symmetry Groups

Internal symmetries. Quantum particles, unlike classical ones, have internal states which move with the particles. The internal symmetry group then characterizes the type of internal structure: spin, isospin, charm, etc. The internal groups are generally compact, $U(N)$ or $SU(N)$.

Spacetime symmetries. The description of the system should not depend on the observer. This puts in display an action of the spacetime group by symmetries (The *principle of covariance*). In QFT the symmetry group is the Poincaré group, in 2-dimensional field theory it is the Galilean group.

Theorem. *If the symmetry group G is a connected Lie group, covariance with respect to G is expressed by a unitary representation of G or at least a central extension of G by the circle group.*

Remark. If G is the Poincaré group there is no need to go to central extensions of G (Wigner). If G is the Galilean group, central extensions of G are unavoidable and introduce the so-called *mass superselection sectors* corresponding to the one parameter family of central extensions (Bargman).

Gauge symmetries. When spacetime and internal symmetries combine we get gauge symmetries. Physicists call Poincaré symmetry **global** and gauge symmetry **local**.

Relativistic systems

The requirement of Poincaré covariance is quite stringent. The simplest such systems arise when the UR of the Poincaré group is irreducible. These define **free particles**. The irreducible UR's of the Poincaré group were first classified by Wigner by a method that goes back to Frobenius and later put in proper perspective by Mackey (Wigner's little group method and the Mackey machine).

There are more irreducible UR's than particles. One has to exclude the UR's where the mass is imaginary or the energy is 0.

In systems where particle number is not conserved and particles are created and annihilated in reactions like



the UR of the Poincaré group is not irreducible. QFT which tries to construct such systems is beset with problems which have been resolved only partially, and that too, only by ad hoc procedures like **renormalization**.

**Spacetime geometry at small distances:
from Riemann to Salam**

What is the geometry of spacetime?

If we read space instead of spacetime, this question is ancient, going back to Euclid. Gauss was the first person to understand the distinction between this question and the question of existence of noneuclidean geometries, later discovered by Bolyai and Lobachevsky. Riemann took this question to another level when he created Riemannian geometry, discovered the curvature tensor, and proved that for space to be euclidean it must have vanishing curvature (flat). Spacetime came in after Einstein discovered that neither space nor time has an objective existence for all observers, and that only spacetime has an invariant significance. Highlights of his discoveries include:

- I. If we can neglect gravitation, spacetime is flat and is an affine space whose symmetries form the Poincaré group, the semidirect product of the translation group and the Lorentz group. [This is a consequence of the constancy of the velocity of light in vacuum in all inertial frames.]
- II. Gravitation is a manifestation of the curvature of spacetime, and in matter-free regions spacetime is Ricci flat, i.e., the Ricci tensor is 0.

Spacetime at small distances and times

In classical physics, and even in quantum mechanics, there is no necessity to question the use of flat Minkowskian geometry for spacetime, since gravitational forces are negligible in that scale. It is only when experiments began to probe extremely small distances that theories trying to understand and predict the experiments began to encounter serious conceptual difficulties. Physicists then began to look more closely into the structure of spacetime at ultrashort scales of distances and times. The **Planck scale** refers to distances of the order of

$$10^{-33}cm$$

and times of the order of

$$10^{-43}sec.$$

Riemann's vision of space at small distances

In his Göttingen inaugural lecture Riemann speculated on the structure of space at small distances. Here is what he said:

Now it seems that the empirical notions on which the metric determinations of Space are based, the concept of a solid body and a light ray, lose their validity in the infinitely small; it is therefore quite definitely conceivable that the metric relations of Space in the infinitely small do not conform to the hypotheses of geometry; and in fact, one ought to assume this as soon as it permits a simpler way of explaining phenomena.

Göttingen inaugural address, 1854.

Current views on structure of spacetime

- At the Planck scale, no measurements are possible and so conventional models can no longer be relied upon to furnish a true description of phenomena. String theory attempts to work in a framework where the smallest objects are not point-like but extended, i.e., strings or (more recently) membranes. Spacetime geometry at the Planck scale is thus almost surely non-commutative because there are no points. No one has so far constructed a convincing geometrical theory which is noncommutative but has the Riemann-Einstein geometry as a limit.
- Even at energies very much lower than the Planck scale, a better understanding of phenomena is obtained if we assume that the geometry of spacetime is described locally by a set of coordinates consisting of the usual ones supplemented by a set of anticommuting (Grassmann) coordinates.

Why Grassmann coordinates

Typically, quantum systems arise by quantization of classical systems, a procedure in which the classical configuration space M of the particles is replaced by the Hilbert space of functions on M , and the classical physical observables are promoted to become operators (self adjoint) on this Hilbert space (Schrödinger picture). Already in the early days of quantum theory this procedure led to an oversupply of states when dealing with many-electron systems such as an atom. The **Pauli exclusion principle** cut down the number of states by demanding that the electrons have the exclusion property: **no two electrons can occupy the same quantum state**. In mathematical terms this means that if \mathcal{H} is the Hilbert space of one electron, the Hilbert space of an N -electron system is not the full tensor product

$$\mathcal{H}^{\otimes N}$$

but the exterior product

$$\Lambda^N(\mathcal{H}).$$

It was a great idea of the physicists, most notably of Salam and Strathdee, that one should replace the classical manifold M by a manifold that admits (additional) Grassmann coordinates. The theory of classical fields on such a manifold would then provide a basis for quantization that will yield the exterior algebras. Such a manifold is nowadays called a supermanifold.

The concept of supersymmetry

Bosons, fermions, and the spin-statistics theorem

The exclusion principle for the electrons is a special case of a fundamental dichotomy among elementary particles based on their *spin*. The *spin states* of a particle form an irreducible module for $SU(2)$ of dimension $2j + 1$ where $j \in (1/2)\mathbf{Z}$ is the spin. Particles with j integral (resp. half integral) are **bosons** (resp. **fermions**). Electrons, protons and neutrons are fermions while the photon is a boson. The Hilbert space of a system of N identical particles of the same spin j with a Hilbert space \mathcal{H} is either $\text{Symm}^N(\mathcal{H})$ or $\Lambda^N(\mathcal{H})$ according as j is integral or half-integral, i.e., according as the particle is a boson or fermion. This is the **spin-statistics theorem** which is a consequence of general principles of quantum field theory.

In SUSY quantum mechanics the Hilbert spaces always have a decomposition into fermionic and bosonic states. Mathematicians call such spaces \mathbf{Z}_2 -graded. In the grading

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$$

the elements of \mathcal{H}_0 (resp. \mathcal{H}_1) are called **even** (resp. **odd**).

Supergeometry

The fundamentals of **supergeometry** are built like ordinary geometry. This is done at 3 levels:

- Infinitesimal
- Local
- Global

Super linear algebra

Infinitesimal geometry is linear. Super linear algebra is just linear algebra over super vector spaces, namely vector spaces which are \mathbf{Z}_2 -graded. The transition from linear to super linear is based on the rule of signs: in any classical formula where two odd elements are interchanged, one should introduce a minus sign. For instance, a super algebra is commutative in the super category, i.e., supercommutative, if

$$ab = \begin{cases} ba & \text{if at least one of } a, b \text{ is even} \\ -ba & \text{if both } a \text{ and } b \text{ are odd.} \end{cases}$$

This rule also has its origins in QFT where the fermi fields obey anticommutation rules. The super commutator is defined by

$$[a, b] = \begin{cases} ab - ba & \text{if at least one of } a, b \text{ is even} \\ ab + ba & \text{if both } a \text{ and } b \text{ are odd.} \end{cases}$$

The supercommutator obeys a super Jacobi identity and leads to the definition of Lie super algebras. Note that the anticommutator of odd elements

$$a, b \longmapsto [a, b] = ab + ba$$

is symmetric and equivariant with respect to the commutator action of the even elements.

The Berezinian

Let R be a super commutative algebra over a field k of characteristic 0 and $R^{p|q}$ be the free module of dimension $p|q$ over R . Let $\mathrm{GL}(p|q)(R)$ be the group of invertible even morphisms of $R^{p|q}$. Then the **Berezinian** is a morphism of $\mathrm{GL}(p|q)(R)$ into R_0^\times (the group of units of the even part R_0 of R) given by

$$\mathrm{Ber}(x) = \det(A - BD^{-1}C) \det(D)^{-1}$$

where

$$x = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We have

$$\mathrm{Ber}(xy) = \mathrm{Ber}(x)\mathrm{Ber}(y).$$

This is the superversion of the determinant, discovered by **F. A. Berezin**, one of the pioneers of super algebra and super analysis. Since the entries of B and C are nilpotent, x is invertible if and only if A and D , whose entries are in the commutative ring R_0 , are invertible.

Charts and local rings

The local coordinates are given by

$$x^1, x^2, \dots, x^p, \theta^1, \theta^2, \dots, \theta^q$$

where the x^i are the usual commuting coordinates and the θ^j are the grassmann coordinates:

$$\theta^k \theta^k + \theta^k \theta^j = 0.$$

The local ring is

$$C^\infty(x^1, x^2, \dots, x^p)[\theta^1, \theta^2, \dots, \theta^q].$$

The concept of a super manifold

A *super manifold* M of dimension $p|q$ is a smooth manifold $|M|$ of dimension p together with a sheaf \mathcal{O}_M of super commuting algebras on $|M|$ that looks locally like

$$C^\infty(\mathbf{R}^p)[\theta^1, \theta^2, \dots, \theta^q]$$

The intuitive picture of M is that of $|M|$ surrounded by a grassmannian cloud. **The cloud cannot be seen: in any measurement the odd variables will be 0 because they are nilpotent. Thus measurement sees only the underlying classical manifold $|M|$.** Nevertheless the presence of the cloud eventually has consequences that are striking.

Unlike classical geometry the local ring contains **nilpotents**. So the analogy is with a **Grothendieck scheme**.

Physicists refer to the sections of the structure sheaf as **superfields**.

Supersymmetries

A **Supersymmetry** is just a morphism between supermanifolds.

Example: The diffeomorphism

$$\mathbf{R}^{1|2} \simeq \mathbf{R}^{1|2} : t^1 \longmapsto t^1 + \theta^1 \theta^2, \quad \theta^\alpha \longmapsto \theta^\alpha$$

is a typical supersymmetry. Note how the morphism interchanges odd and even variables. This is how the grassmann cloud interacts with the classical manifold underlying the supermanifold.

Integration on super manifolds

Let

$$\theta^I = \theta^{i_1} \theta^{i_2} \dots \theta^{i_k} \quad I = (i_\mu), \{i_1 < i_2 < \dots < i_k\}.$$

On

$$\Lambda = \mathbf{R}[\theta^1, \dots, \theta^q]$$

the integral is a linear map

$$a \longmapsto \int a d^q \theta$$

defined by

$$\int \theta^I d^q \theta = \begin{cases} 0 & \text{if } |I| < q \\ 1 & \text{if } I =: Q = \{1, 2, \dots, q\}. \end{cases}$$

Integration is also differentiation:

$$\int = \left(\frac{\partial}{\partial \theta^q} \right) \left(\frac{\partial}{\partial \theta^{q-1}} \right) \dots \left(\frac{\partial}{\partial \theta^1} \right).$$

In the local ring with coordinates x^i, θ^j ,

$$\int s d^p x d^q \theta = \int s_Q d^p x \quad (s = \sum_I s_I \theta^I).$$

The change of variables formula

For a morphism given locally as

$$\psi : (x, \theta) \longmapsto (y, \varphi)$$

we define the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial y}{\partial x} & -\frac{\partial y}{\partial \theta} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial \theta} \end{pmatrix}.$$

Then

$$\int s = \int \psi^*(s) \text{Ber}(J\psi)$$

for compactly supported sections of the local ring. For arbitrary manifolds we use partitions of unity as in the classical case.

- This beautiful formula goes back to Berezin. The justification for the peculiar definition of integration in the anticommuting variables is the change of variables formula.

Super spacetime and super Poincaré group

Super Lie groups and their super Lie algebras

Super Lie groups are group objects in the category of super manifolds. As in the theory of ordinary Lie groups one can define the super Lie algebra $\text{Lie}(G)$ of a super Lie group G . The even part of a super Lie algebra is an ordinary Lie algebra and the super Lie algebra may be viewed as a supersymmetric enlargement of it.

History. Gol'fand-Likhtman and Volkov-Akulov discovered the minimal SUSY extension of the Poincaré algebra in the early 1970's. Wess-Zumino discovered a little later, in 1974, the first example of a **simple** super Lie algebra, namely the minimal SUSY extension of the conformal Lie algebra. In 1975 V. Kac formally defined super Lie algebras and carried out the super version of the Cartan-Killing classification of simple Lie algebras over \mathbf{C} .

Super Minkowski spacetime

Let \mathfrak{t}_0 be a flat Minkowski spacetime of signature $(1, n)$. By a **flat Minkowski superspacetime** is meant a super Lie group whose even part is \mathfrak{t}_0 identified with its group of translations. The corresponding super Lie algebra \mathfrak{t} has the grading

$$\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_1.$$

Physical interpretations lead to the requirement that the adjoint action of \mathfrak{t}_0 on \mathfrak{t}_1 ,

$$a, b \longmapsto [a, b] \quad (a \in V_0, b \in V_1)$$

is a very special module, namely a **spin module**. In this case, at least when \mathfrak{t}_1 is irreducible, there is an essentially unique symmetric bilinear form

$$\mathfrak{t}_1 \otimes \mathfrak{t}_1 \longrightarrow \mathfrak{t}_0$$

If we choose this to be the supercommutator of odd elements we may regard $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_1$ as a super Lie algebra. The super Lie group T of \mathfrak{t} is flat **super Minkowski spacetime**.

Super Poincaré group

The semi direct product

$$G = T \times' \text{Spin}(1, n)$$

is a **super Poincaré group**.

SUSY Field theory

All of the machinery is now in place for introducing the Lagrangians and doing SUSY field theory on flat super Minkowski spacetime: **Wess-Zumino** (SUSY electrodynamics) and **Ferrara-Zumino** (SUSY Yang-Mills).

The SUSY extension of Einstein spacetime is more complicated and was first done in 1976 by **Ferrara, Freedman**, and **van Nieuwenhuizen**, and a little later, by **Deser** and **Zumino**. It is called **supergravity**.

- Zumino proved in 1979 that a nonlinear σ model (in which the fields take values in a Riemannian target space) is supersymmetric if and only if the target space is a Kähler manifold.

Superparticles

Classification of super particles

In quantum theory the unitary irreducible representations (UIR) of the Poincaré group classify free elementary particles. In SUSY quantum theory, the UIR's of a super Poincaré group classify elementary super particles. Each super particle, when viewed as a UR of the underlying Poincaré group, is the direct sum of a collection of ordinary particles, called a **multiplet**. The members of a multiplet are called **partners** of each other.

Unlike the classical case, the positivity of energy is a consequence of supersymmetry.

The existence of the superpartners of the known particles is the biggest prediction of supersymmetry.

It is the hope of many that the new super collider being readied at CERN will create the super partners of the usual elementary particles. This is not certain because one does not know exactly the scale at which supersymmetry is broken.

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