FINITE APPROXIMATION OF WEYL SYSTEMS

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ABSTRACT. The functional analytic notion of approximation of Weyl systems, as introduced by Digernes and Varadarajan, is considered. It is shown that the Weyl system on any second countable locally compact abelian group can be approximated by suitably chosen finite Weyl systems (Weyl systems on finite abelian groups).

1. INTRODUCTION

There has in recent years been considerable interest in quantum theories that are analogous to the conventional one, but differ from it in some of their main features. We mention, without aiming at completeness, the following works: Finkelstein [5], and Chan, Finkelstein [2] on q-deformed quantum theories; Vladimirov [14], and Vladimirov, Volovich, Zelenov [15] on p-adic quantum mechanics; Varadarajan [12] on quantum kinematics over general locally compact abelian groups treated from the point of view of deformation and approximation. Quantum kinematics over finite abelian groups go back to Weyl [18], and Schwinger [9].

In this paper, we develop the point of view in [12] further, as we discuss approximations of quantum kinematics on locally compact abelian groups in more detail.

Our motivation for studying quantum models based on very general abelian groups does not arise solely, or even mainly, from any desire of generality. Rather it stems from the work of Schwinger [9] on the classification of finite quantum systems, and its variations treated in Husstad [4], strongly influenced by Digernes and Varadarajan, and [12]. In [9] and [4], as well as in [18], unitary representations of a finite abelian group G and its dual $\hat{G}$ satisfying Weyl commutation rules (= Weyl system) were studied, and it was shown that the conventional Weyl system associated to $\mathbb{R}^e$ may be approximated (as in Section 2) arbitrarily well by Weyl systems on finite abelian groups. The approximation scheme of Schwinger gave remarkable numerical results on the level of generators. Motivated by this, the validity of this approximation process was proved theoretically by Digernes, Varadarajan, Varadhan [3]. Consequently one could take the point of view that quantum theory associated with a finite abelian group is of much interest, and that the calculations over $\mathbb{R}^e$ are idealizations of the finite situation.

Our main Theorem 6.1 states that the Weyl system on any locally compact second countable abelian group is a limit of Weyl systems on finite abelian groups. This motivates dynamical considerations over other groups than $\mathbb{R}^e$, as they in this sense also give

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idealizations of finite quantum systems. Moreover, this scheme makes it possible to obtain numerical results for more unconventional quantum dynamical systems.

It turns out (cfr. the comments in [12]) that the Weyl system on $\mathbb{Z}/p^n\mathbb{Z}$ converges to that associated to the p-adic field $\mathbb{Q}_p$, as $n \longrightarrow \infty$. From this point of view one can for instance study 'harmonic oscillators', and 'coulomb' problems over local fields and rings. A path-integral formulation for vector spaces over division algebras over non-archimedean local fields has been established in Varadarajan [13].

The paper is organized as follows: In Section 2, Weyl systems and limits of such are defined. 'Continuity' results for duality and direct sum are presented, and the structure of finite Weyl systems is discussed.

In Section 3 we approximate any second countable locally compact abelian group G by elementary groups $H_N/K_n \simeq \mathbb{R}^e \oplus \mathbb{T}^{a_n} \oplus \mathbb{F}_N^n \oplus \mathbb{Z}^{b_N}$, H_N open compactly generated subgroup, while K_n is a compact subgroup for which G/K_n is the dual of a compactly generated group. The group $\mathbb{F}_N^n$ is finite abelian. This follows from the more general results of van Kampen [11], and Pontrjagin [7]. Let $N < n$. Here, $H_N \subset H_n$ which by [7] in particular induces an injection $\iota_N^n : \mathbb{Z}^{b_N} \longrightarrow \mathbb{Z}^{b_n}$, whereas the natural map $G/K_N \longleftarrow G/K_n$ induces an injection $\widehat{\kappa_N^n} : \mathbb{Z}^{a_N} \longrightarrow \mathbb{Z}^{a_n}$. In the resulting mixed inductive/projective limit description of G, induced maps (for which ι_N^n, and κ_N^n are two of the matrix coefficients) can be taken to be semi-aligned, but not diagonal in general.

Section 4 is used to define finite abelian groups G_n (and maps), candidates for approximating the Weyl system on G. We do a two-step approximation in the sense that we first take the diagonal H_n/K_n, and then construct G_n based on H_n/K_n and the matrix coefficients ι_i^n and $\widehat{\kappa_i^n}$, for all $i < n$. We choose to treat circle parts essentially as integer parts through Fourier transforms. In effect, the embedded finite translation on $l^2(\mathbb{Z}^{a_n})$, and the embedded finite multiplication by character on $L^2(\mathbb{T}^{a_n})$, are intertwined by the *non-finite* Fourier transform $\mathcal{F}^n : l^2(\mathbb{Z}^{a_n}) \to L^2(\mathbb{T}^{a_n})$.

The space of Schwartz-Bruhat [1] functions $\mathcal{S}(G)$, functions which live on the elementary group H_n/K_n for some n, is introduced in Section 5 to deal with the analysis. The key point is that $\mathcal{S}(G)$ is invariant under the standard Weyl system. This moves our calculations to H_n/K_n, and immediately shows that the Weyl system on G can be approximated by Weyl systems on elementary groups. In this section, we get around problems with semi-alignment: Given simple tensors in $\mathcal{S}(N)$ for which we control the support of their images in the discrete space $l^2(\mathbb{Z}^{a_N} \oplus \mathbb{F}_N^N \oplus \mathbb{Z}^{b_N})$, we find that we also control the support of their images in $l^2(\mathbb{Z}^{a_n} \oplus \mathbb{F}_n^n \oplus \mathbb{Z}^{b_n})$ for $n > N$ (Lemma 5.3). Their support is governed by the maps ι_N^n and $\widehat{\kappa_N^n}$.

In Section 6, the general approximation result is proved. First, pointwise convergence of characters is verified. The important point is that in $\mathbb{Z}$-directions, the approximation is exact from some n, and by semi-alignment we control the coordinate in these directions. In $\mathbb{T}$-directions we have no such control, but the result follows as the approximation in $\mathbb{T}$-directions is uniform. The strong convergence of projections follows directly from the support control of Section 5. The remaining statements essentially follow from pointwise convergence of characters, and the support control Lemma 5.3. For the sake of completeness, we conclude with a proof for the conventional case $\mathbb{R}^e$. Finally, some applications to local fields and rings are mentioned.

2. Limits of Weyl Systems

Let G be a second countable locally compact abelian group, with $\hat{G}$ as its Pontrjagin dual. The Weyl representations V and U of G and $\hat{G}$, respectively, are, for $x \in G$ and $\gamma \in \hat{G}$, given by

$$\begin{aligned} (V(x)f)(y) &= f(y-x), \\ (U(\gamma)f)(y) &= \langle y, \gamma \rangle f(y), \quad f \in L^2(G), \quad y \in G. \end{aligned}$$

This pair of strongly continuous unitary representations satisfies the *Weyl relations*;

$$U(\gamma)V(x) = \langle x, \gamma \rangle V(x) U(\gamma) \quad x \in G, \quad \gamma \in \hat{G}. \tag{1}$$

The pair (V, U) is called the *standard Weyl system* on G. The standard Weyl system is irreducible; the resulting projective unitary representation of $G \oplus \hat{G}$ has no non-trivial invariant subspaces in $L^2(G)$.

Definition 2.1. (Limit of Weyl systems) Let $\{G_n\}_{n=1}^\infty$, G be second countable, locally compact abelian groups with associated standard Weyl systems $\{(V_n, U_n)\}$ and (V, U). Then we say that the sequence $\{G_n\}$ converges to G in the sense of Weyl systems (or that (V, U) on G is the limit of (V_n, U_n) on G_n) if the following conditions are satisfied:

i) There is a Hilbert space $\mathfrak{H}$ and isometries $I_n : L^2(G_n) \longrightarrow \mathfrak{H}$, $I : L^2(G) \longrightarrow \mathfrak{H}$, such that $P_n \longrightarrow P$ strongly. Here, P_n and P are the orthogonal projections on $I_n(L^2(G_n))$ and $I(L^2(G))$, respectively.

ii) Setting

$$U'(\omega) = \begin{cases} IU(\omega)I^{-1} & \text{on } I(L^2(G)) \\ \text{identity} & \text{on } I(L^2(G))^\perp \end{cases} \qquad \omega \in \hat{G},$$

and defining V', U'_n and V'_n similarly, there are, for each $x \in G$, $\gamma \in \hat{G}$, sequences $\{x^n\}$ and $\{\gamma^n\}$ such that $x^n \in G_n$, $\gamma^n \in \hat{G}_n$, and

$$U'_n(\gamma^n) \longrightarrow U'(\gamma), \quad V'_n(x^n) \longrightarrow V'(x)$$

strongly.

If the conditions in Definition 2.1 are satisfied, we easily see that $\langle x^n, \gamma^n \rangle \longrightarrow \langle x, \gamma \rangle$, so pointwise convergence of characters is necessary for Weyl convergence.

Assume that the standard Weyl system on G is a limit of the standard Weyl systems on G_n. The Stone-von Neumann-Mackey Theorem says that up to multiplicity and unitary equivalence, the Weyl relations for G have a unique solution. Thus, in the natural sense we can approximate any Weyl system on G (= any other solution of (1)) by Weyl systems on G_n. In this paper, we exclusively work with standard Weyl systems.

2.1. **Duality.** The standard Weyl system $(\hat{V}, \hat{U})$ on $\hat{G}$ (identify G and its bidual) is connected to the standard Weyl system (V, U) on G through the Fourier transform; $\hat{V} = \mathcal{F}U\mathcal{F}^{-1}$ and $\hat{U} = \mathcal{F}V\mathcal{F}^{-1}$. The Fourier transform $\mathcal{F} : L^2(G) \longrightarrow L^2(\hat{G})$ is for suitable f given by $(\mathcal{F}f)(\gamma) = \int_{x\in G} f(x)\langle -x, \gamma\rangle dx$, $\gamma \in \hat{G}$.

Proposition 2.2. *If G is a limit (Definition 2.1) for G_n, then $\hat{G}$ is a limit for $\hat{G}_n$ in the sense of Weyl systems.*

Proof. Define $\hat{I}_n : L^2(\hat{G}_n) \longrightarrow \mathfrak{H}$ and $\hat{I} : L^2(\hat{G}) \longrightarrow \mathfrak{H}$ by $\hat{I}_n = I_n\mathcal{F}_n^{-1}$ and $\hat{I} = I\mathcal{F}^{-1}$. By construction, $\hat{P}_n = P_n$ and $\hat{P} = P$. Let $x \in G$. Then we easily see that $\hat{I}_n\hat{U}_n(x^n)\hat{I}_n^{-1} = I_nV_n(x^n)I_n^{-1}$ on $P_n(\mathfrak{H})$ and $\hat{I}\hat{U}(x)\hat{I}^{-1} = IV(x)I^{-1}$ on $P(\mathfrak{H})$. Similar formulas for $\hat{V}$ prove the proposition. □

2.2. **Direct Sum.** If G is decomposable, say the direct sum of two subgroups, $G = G_1 \oplus G_2$, then the standard Weyl system of G can be identified with the sum of the standard Weyl systems (V_{G_j}, U_{G_j}) on G_j, $j = 1,\ 2$. This means, for $x = (x_1, x_2) \in G$, $L^2(G) \simeq L^2(G_1) \otimes L^2(G_2)$ and $V_G(x) \simeq V_{G_1}(x_1) \otimes V_{G_2}(x_2)$. Similar relations hold for U. This extends to finite index sets.

Proposition 2.3. *If (V^j, U^j) on G^j is the limit of (V_n^j, U_n^j) on G_n^j for j in a finite set, then (V, U) on $G = \oplus_j G^j$ is the limit of (V_n, U_n) on $G_n = \oplus_j G_n^j$.*

Proof. The result follows from strong continuity of tensor products for uniformly bounded sequences of operators. Let us give some details in the case of two summands: Define $I_n = I_n^1 \otimes I_n^2$, and $I = I^1 \otimes I^2$, both acting on $\mathfrak{H} := \mathfrak{H}^1 \otimes \mathfrak{H}^2$. Then $P_n = P_n^1 \otimes P_n^2$ and $P = P^1 \otimes P^2$. Likewise, for $x = (x_1, x_2) \in G$, put $x^n = (x_1^n, x_2^n)$. Applied to a simple tensor $\psi = \psi_1 \otimes \psi_2 \in \mathfrak{H}$, $V_n'(x^n)\psi = V_n'(x_1^n)\psi_1 \otimes V_n'(x_2^n)\psi_2$ and $V'(x)\psi = V'(x_1)\psi_1 \otimes V'(x_2)\psi_2$. As P_n^j and $V_n'(x_j^n)$ are uniformly bounded in norm, we get the expected convergence. The arguments are the same for U when we take $\gamma^n = (\gamma_1^n, \gamma_2^n)$ for $\gamma = (\gamma_1, \gamma_2) \in \hat{G} = \hat{G}_1 \oplus \hat{G}_2$. □

2.3. **Finite Weyl systems.** Let $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ be the prime expansion of n (p_j are different primes while r_j are non-negative integers). Then, $\mathbb{Z}_n \simeq \mathbb{Z}_{p_1^{r_1}} \oplus \mathbb{Z}_{p_2^{r_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{r_k}}$, and $\mathbb{Z}_n$ is indecomposable if and only if n is a prime power.

So, recalling the direct sum construction in the previous paragraph, the standard Weyl system (V, U) on $\mathbb{Z}_n$ is indecomposable in this geometrical sense, precisely when $n = p^r$ is a prime power. By their structure theory, namely as direct sums of finite cyclic groups, we can build the Weyl system on any finite abelian group from these geometrically indecomposable finite Weyl systems. Schwinger [9] started constructing this theory of finite degree of freedom. Finite quantum systems were also studied by Šťovíček and Tolar [10], and later in [4].

3. Structure of Second Countable Locally Compact Abelian Groups

Recall structure theorems on l.c.a. groups: Pontrjagin [7], Section 39, Theorem 51 proves that any compactly generated group is of the form $\mathbb{R}^e \oplus C \oplus \mathbb{Z}^b$, where C is a compact abelian group ('compactly generated' means 'generated by a compact neighbourhood of the identity'). In the same reference, Section 39, Proposition A, he proves that for any l.c.a. group G, and any compact set $K \subset G$, there is a compactly generated open subgroup H such that $K \subset H \subset G$. Moreover, the structure theorem of van Kampen [11], Theorem 2, says that any l.c.a. group G is of the form $\mathbb{R}^e \oplus G^1$, where G^1 contains a compact open subgroup K. For any other such decomposition, the exponent e is the same. Following Reiter [8], we say that G is a G^1-group if $e = 0$ in this decomposition. In particular, a compactly generated G^1-group is of the form $C \oplus \mathbb{Z}^b$, where C is compact.

If G is second countable and l.c.a., so is $\hat{G}$ and any subgroup and quotient of G. Likewise, the property of being a G^1-group is preserved under these operations.

We have not found Proposition 3.2 in any standard source in topological group theory. That proposition follows from the next lemma, which is probably also stated somewhere.

Lemma 3.1. *Let G^1 be a G^1-group, $H \subset G^1$ a compactly generated open subgroup, and let K' be a compactly generated open subgroup of $\widehat{G^1}$. If $K := (K')^{\perp} \subset H$ (the annihilator is taken in $\widehat{G^1}$), then $H/K \simeq \mathbb{T}^a \oplus \mathbb{F} \oplus \mathbb{Z}^b$, an elementary group ($\mathbb{F}$ is a finite abelian group).*

Proof. Use the structure theorem of Pontrjagin [7], Section 39, Theorem 51, for compactly generated groups on both H and K'. By duality, $G^1/K \simeq \mathbb{T}^a \oplus D$, where D is a discrete abelian group. Moreover, as K is compact, $H/K \simeq C/K \oplus \mathbb{Z}^b$, where C/K is a compact group. As $H \subset G^1$ is open, $H/K \subset G^1/K$ is open, and there is an open injection $C/K \oplus \mathbb{Z}^b \longrightarrow \mathbb{T}^a \oplus D$. In particular, the compact open subgroup $C/K \oplus \{0\}$ must map to a compact open subgroup. As $\mathbb{T}^a$ has no open subgroups but itself, the image of $C/K \oplus \{0\}$ is of the form $\mathbb{T}^a \oplus F$, where $F \subset D$ is discrete, but also compact. Thus F is finite. □

Proposition 3.2. *Let G be a second countable locally compact abelian group. Then $G \simeq \mathbb{R}^e \oplus G^1$, where the following is true for the group G^1: There exists an increasing sequence of open subgroups $\{H_n\}_{n=1}^{\infty}$ such that $\cup H_n = G^1$, and a decreasing sequence of compact subgroups $\{K_n\}$, $H_1 \supset K_n \supset K_{n+1}$, such that $\cap K_n = \{0\}$. Moreover, $H_n/K_n \simeq \mathbb{T}^{a_n} \oplus \mathbb{F}_n \oplus \mathbb{Z}^{b_n}$, where $\mathbb{F}_n$ is a finite abelian group (a_n and b_n are non-negative integers).*

Proof. The first part follows from [11].

Any separable l.c.a. group can be written as a countable union of compact sets (take an open neighbourhood of 0 with compact closure, and translate this closure with elements from a countable dense subset of G^1). Thus, by [7], Section 39, Proposition A, and Theorem 51, there is $\{H''_n\}$ such that $\cup H''_n = G^1$, and H''_n is open and compactly generated. Let $H'_n = H''_1 + H''_2 + \cdots + H''_n$, this subgroup is also open and compactly generated. Then $H'_n \subset H'_{n+1}$. Likewise, construct $\{K'_m\}$ such that K'_m is open and compactly generated, and $K'_m \nearrow \widehat{G^1}$. Thus, $K_m := (K'_m)^{\perp}$ is a compact subgroup of G^1 such that $K_m \searrow \{0\}$. As K_1 is compact and $\{H'_n\}$ covers G^1, there is an integer N such that for $n > N$, $K_1 \subset H'_n$. Let $H_n := H'_{N+n}$. Then, by the previous lemma, we are done. □

The proof of Lemma 3.1 implies that $H_n/K_m \simeq \mathbb{T}^{a_m} \oplus \mathbb{F}^m_n \oplus \mathbb{Z}^{b_n}$, so we get elementary groups also if $n \neq m$. Here, $\mathbb{F}^n_n = \mathbb{F}_n$.

3.1. **Semi-alignment for G^1-groups.** Let G^1 be a second countable G^1-group with $\{H_n\}$ and $\{K_m\}$ from Proposition 3.2. Let $\pi_m : G^1 \longrightarrow G^1/K_m$ and $\pi_{lm} : G^1/K_m \longrightarrow G^1/K_l$ be the natural maps ($m > l$). We have $\pi_l = \pi_m \circ \pi_{lm}$, and the kernel of π_{lm} is the compact group K_l/K_m. From the proof of Lemma 3.1, there are subgroups $Z_n, C_n \subset G^1$, where $Z_n \simeq \mathbb{Z}^{b_n}$, C_n is compact, such that $H_n = C_n + Z_n$ (direct sum). Likewise, there are subgroups $T_m, D_m \subset G/K_m$ with $T_m \simeq \mathbb{T}^{a_m}$, D_m is discrete, such that $G/K_m = T_m + D_m$ (direct sum). For $m > l$ and $k > n$ we then have the following commuting diagram, which describes the structure of G^1 in terms of elementary groups; the top row gives G^1 as an inductive limit, while the right-most column describes G^1 as a projective limit:

$$
\begin{array}{ccccc}
C_n + Z_n & \subset & C_k + Z_k & \subset & G^1 \\
\big\downarrow \pi_m|_{H_n} & & \big\downarrow \pi_m|_{H_k} & & \big\downarrow \pi_m \\
T_m + F_n^m + Z_n & \subset & T_m + F_k^m + Z_k & \subset & T_m + D_m \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
T_l + F_n^l + Z_n & \subset & T_l + F_k^l + Z_k & \subset & T_l + D_l .
\end{array} \tag{2}
$$

All sums are direct. By the subgroup $Z_n \subset G^1/K_m$ we mean the isomorphic image of Z_n under π_m. The subgroups $F_n^l \simeq \mathbb{F}_n^l$ are finite. This setup follows from the comment following Proposition 3.2. *However, in this diagram we select a basis for all $\mathbb{Z}^{b_n}$-parts from that in the top row, and a basis on all $\mathbb{T}^{a_n}$-parts from that in the right-most column.* We make no particular choice for the finite parts. Thus, in this basis the inclusion $H_n/K_m \subset H_k/K_m$ is represented by a 3×3-matrix

$$
i_{nk}^m = \begin{pmatrix} \mathrm{id} & & \\ & \theta_{nk}^m & \phi_{nk}^m \\ & & \iota_n^k \end{pmatrix}, \tag{3}
$$

for some morphisms $\theta_{nk}^m : F_n^m \longrightarrow F_k^m$, $\phi_{nk}^m : Z_n \longrightarrow F_k^m$, and $\iota_n^k : Z_n \longrightarrow Z_k$. The empty places represent 0-maps. The 2×2-matrix in the lower right corner is triangular because Z_k has no finite subgroups.

Lemma 3.3. *In the preceeding matrix representation, θ_{nk}^m, ι_n^k are both injective.*

Proof. The map θ_{nk}^m is injective as the finite part is only mapped into the finite part.

Let $0 \neq z \in Z_n$. Let $[z]$ be the image of z in $Z_n/\operatorname{Ker}(\phi_{nk}^m) \simeq \operatorname{Im}(\phi_{nk}^m)$. As this quotient is a finite abelian group, there is some integer k such that $[kz] = k[z] = 0$. Consequently, $kz \in \operatorname{Ker}(\phi_{nk}^m)$. Thus, by injectivity of i_{nk}^m, $0 \neq i_{nk}^m(0, 0, kz) = (0, 0, \iota_n^k(kz))$, so $\iota_n^k(z) \neq 0$. Thus, ι_n^k is injective. □

Let us recall some generalities on dual groups. For l.c.a. groups $\{G_i\}$ (i runs over a finite set), set $G = \oplus_i G_i$. Then $\widehat{G} = \widehat{\oplus_i G_i} \simeq \oplus_i \widehat{G_i}$. Duality between G and $\widehat{G} := \oplus_i \widehat{G_i}$ is set up with

$$\langle (x_i), (\gamma_j) \rangle = \prod_i \langle x_i, \gamma_i \rangle_{G_i} \tag{4}$$

where for $x_i \in G_i$ and $\gamma_i \in \widehat{G_i}$, $\langle x_i, \gamma_i \rangle_{G_i}$ is some duality between G_i and $\widehat{G_i}$. Let $\alpha : G \longrightarrow X$ be a morphism between l.c.a. groups $G = \oplus_i G_i$ and $X = \oplus_j X_j$. Thus, $\alpha = (\alpha_{ij})$, where $\alpha_{ij} : G_j \longrightarrow X_i$ is a morphism. Then the dual map $\widehat{\alpha} : \widehat{X} \longrightarrow \widehat{G}$ (under (4)) has matrix representation $\widehat{\alpha} = (\widehat{\alpha_{ji}})$ in the natural dual basis on both $\widehat{X}$, and $\widehat{G}$. Here, $\widehat{\alpha_{ij}} : X_i \longrightarrow G_j$ is the dual map under $\langle \cdot, \cdot \rangle_{G_j}$, and $\langle \cdot, \cdot \rangle_{X_i}$. Thus, the rule is the same as the usual one for the adjoint in matrix algebras.

So, $\pi_{lm}|_{H_n/K_m} =: j^n_{lm} : H_n/K_m \longrightarrow H_n/K_l$ has matrix representation

$$j^n_{lm} = \begin{pmatrix} \kappa^m_l & \phi'^n_{lm} & \\ & \theta'^n_{lm} & \\ & & \mathrm{id} \end{pmatrix}. \tag{5}$$

The reason is that the dual of j^n_{lm} is an open injection; of the same type as i^m_{nk}. Using (4):

Corollary 3.4. *In this matrix representation for j^n_{lm}, $\kappa^m_l : T_m \longrightarrow T_l$ and $\theta'^n_{lm} : F^m_n \longrightarrow F^l_n$ are both (non-zero) surjective. The morphism $\phi'^n_{lm} : F^m_n \longrightarrow T_l$ could be 0.*

The reason why ι^k_n, and κ^m_l do not depend on m and n, respectively, is that all diagrams in (2) are commutative. In fact, ι^k_n represents the inclusion of Z_n in H_n into Z_k in H_k (top row). Similarly, κ^m_l is the matrix coefficient in π_{lm} mapping T_m onto T_l (right most column).

Later, we need some additional technical properties on the description of G^1.

Lemma 3.5. *Let $i : \mathbb{Z}^b \longrightarrow \mathbb{Z}^{b'}$ be an injection. Then there is a complemented submodule B such that $i(\mathbb{Z}^b) \subset B \subset \mathbb{Z}^{b'}$, $B \simeq \mathbb{Z}^b$.*

Proof. Let B be those $z \in \mathbb{Z}^{b'}$ for which $nz \in i(\mathbb{Z}^b)$ for some $n \in \mathbb{Z}$. This is the torsion closure of the image of i. It is easy to see that B is a submodule. Let non-zero $n_j u_j \in i(\mathbb{Z}^b)$ for $n_j \in \mathbb{Z}, u_j \in B$, where j runs over a finite set. If $\{n_j u_j\}$ is dependent over $\mathbb{Z}$, then $\{u_j\}$ is also dependent over $\mathbb{Z}$. Conversely, as $i(\mathbb{Z}^b) \subset B$, the rank of B over $\mathbb{Z}$ equals the rank of $i(\mathbb{Z}^b)$ over $\mathbb{Z}$. As i is injective, this rank is b. Moreover, it follows from the definition of B that the quotient $\mathbb{Z}^{b'}/B$ is torsion free. This implies that B is a direct summand (with b generators). □

Lemma 3.6. *Let $\mathbb{Z}^{b_1} \xrightarrow{\iota_1} \mathbb{Z}^{b_2} \cdots \mathbb{Z}^{b_n} \xrightarrow{\iota_n} \mathbb{Z}^{b_{n+1}} \xrightarrow{\iota_{n+1}} \ldots$ where $\{\iota_n\}$ consists of injections. Then we can choose the basis on each $\mathbb{Z}^{b_n}$ such that for any pair $m > n$, for $\iota^m_n := \iota_{m-1} \circ \cdots \circ \iota_n$, $\iota^m_n(\mathbb{Z}^{b_n}) \subset \mathbb{Z}^{b_n} \oplus \{0\}^{b_m - b_n}$; maps into the first b_n factors.*

Proof. Choose a basis on $\mathbb{Z}^{b_1}$. Then use Lemma 3.5 to find complemented $\iota_1(\mathbb{Z}^{b_1}) \subset B_1 \subset \mathbb{Z}^{b_2}$. Now it is clear that we can choose a basis as wanted on $\mathbb{Z}^{b_2}$. Again from Lemma 3.5 we find complemented $\iota_2(\mathbb{Z}^{b_2}) \subset B_2 \subset \mathbb{Z}^{b_3}$. In particular, $\iota_2(B_1) \subset B_2$. Use Lemma 3.5

once more to find complemented $\iota_2(B_1) \subset B_2^1 \subset B_2$. Now we can obviously choose a basis on $\mathbb{Z}^{b_3}$ where $\mathbb{Z}^{b_2}$ is mapped into the first b_2 coordinates, while $\mathbb{Z}^{b_1}$ is mapped into the first b_1 coordinates of $\mathbb{Z}^{b_3}$. The proof is completed by continuing this construction. $\square$

Define the elementary groups $E_{nm} := \mathbb{T}^{a_m} \oplus \mathbb{F}_n^m \oplus \mathbb{Z}^{b_n}$, $\mathbb{F}_n^n = \mathbb{F}_n$ and $E_n := E_{nn}$. Then $H_n/K_m \simeq E_{nm}$.

Let us return to Diagram (2). Using this, we can assume that the induced injection (same notation) $i_{nk}^m : E_{nm} \longrightarrow E_{km}$ has a matrix representation like (3), and the induced surjection $j_{lm}^n : E_{nm} \longrightarrow E_{nl}$ has a matrix representation like (5).

As $\iota_n^k : \mathbb{Z}^{b_n} \longrightarrow \mathbb{Z}^{b_k}$ is given as in Lemma 3.6, we apply that lemma to $\{\mathbb{Z}^{b_n}\}$. We use this change of basis on $\mathbb{Z}^{b_n}$ in any E_{nl}. Thus, the matrix representation of i_{nk}^m is of the same type as before. As a dual condition (use (4) between $\mathbb{Z}^{a_n}$ and $\mathbb{T}^{a_n}$), we make the kernel of κ_l^m contain $\mathbb{T}^{a_m - a_l} \oplus \{0\}^{a_l}$. The annihilator of the image of a morphism is the kernel of the dual map. The observation that relates all $\mathbb{Z}^{b_n}$ (in H_n/K_l) to H_n is crucial at this point. Ordering problems would otherwise occur for the two-dimensional array (those $(k,l) \in \mathbb{Z}^2$ for which $k, l > 0$), and we could not get the analogue of Lemma 3.6.

Assume that $n > N$ are positive integers. Rename $i_{Nn} := i_{Nn}^n$ and $j_{Nn} := j_{Nn}^N$. The next result summarizes our discussion:

Proposition 3.7. (*semi-alignment*) *Let G^1 be a G^1-group with structure given by Proposition 3.2. Then $H_N/K_n \simeq E_{Nn} = \mathbb{T}^{a_n} \oplus \mathbb{F}_N^n \oplus \mathbb{Z}^{b_N}$, where for $n > N$ the induced injection $i_{Nn} : E_{Nn} \longrightarrow E_n$, and the induced surjection $j_{Nn} : E_{Nn} \longrightarrow E_N$ can be assumed to have matrix representations*

$$i_{Nn} = \begin{pmatrix} \mathrm{id} & & \\ & \theta_N^n & \phi_N^n \\ & & \iota_N^n \end{pmatrix} \quad \textit{and} \quad j_{Nn} = \begin{pmatrix} \kappa_N^n & \phi_N'^n & \\ & \theta_N'^n & \\ & & \mathrm{id} \end{pmatrix}.$$

The maps $\theta_N^n : \mathbb{F}_N^n \longrightarrow \mathbb{F}_n$ and $\iota_N^n : \mathbb{Z}^{b_N} \longrightarrow \mathbb{Z}^{b_n}$ are injective, and $\iota_N^n(\mathbb{Z}^{b_N}) \subset \mathbb{Z}^{b_N} \oplus \{0\}^{b_n - b_N}$. Furthermore, $\theta_N'^n : \mathbb{F}_N^n \longrightarrow \mathbb{F}_N$ and $\kappa_N^n : \mathbb{T}^{a_n} \longrightarrow \mathbb{T}^{a_N}$ are surjective. Also, the kernel of κ_N^n contains $\mathbb{T}^{a_n - a_N} \oplus \{0\}^{a_N}$. The morphisms $\phi_N^n : \mathbb{Z}^{b_n} \longrightarrow \mathbb{F}_n$ and $\phi_N'^n : \mathbb{F}_N^n \longrightarrow \mathbb{T}^{a_N}$ could be 0.

As ϕ_N^n and $\phi_N'^n$ in general are non-zero, the finite parts may intertwine in a non-trivial way. This causes technical problems in the rest of this paper.

3.2. **The dual system.** This paragraph contains definitions.

From standard topological group theory, $\mathbb{Z}^{a_n} \oplus \mathbb{F}_N^n \oplus \mathbb{T}^{b_N} \simeq \widehat{H_N/K_n} \simeq K_n^\perp / H_N^\perp$. The sequence $\{K_n^\perp\} \nearrow \widehat{G^1}$ consists of open subgroups while the elements of $\{H_N^\perp\} \searrow \{0\}$ are compact subgroups. So, the (non-unique) sequences $\{H_N\}$ and $\{K_n\}$ single out a special decomposition for the dual group $\widehat{G^1}$.

Fix a duality $\langle \cdot, \cdot \rangle$ between G^1 and $\widehat{G^1}$. Let $x \in H_N$ and $\gamma \in K_n^\perp$. Then

$$\langle x + K_n, \gamma + H_N^\perp \rangle_{Nn} = \langle x, \gamma \rangle \tag{6}$$

(well) defines the duality $\langle \cdot, \cdot \rangle_{Nn}$ between H_N/K_n and $\widehat{H_N/K_n} = K_n^\perp / H_N^\perp$.

Define the *standard* duality $\langle\cdot,\cdot\rangle_s$ between $E_{Nn} = \mathbb{T}^{a_n} \oplus \mathbb{F}_N^n \oplus \mathbb{Z}^{b_N}$ and $\widehat{E_{Nn}} = \mathbb{Z}^{a_n} \oplus \mathbb{F}_N^n \oplus \mathbb{T}^{b_N}$ (dual basis) as follows: Let $x = (t, f, z) = ((t_i), (f_j), (z_k)) \in E_{Nn}$, and $\gamma = (u, g, s) = ((u_i), (g_j), (s_k)) \in \widehat{E_{Nn}}$. Then

$$\langle x, \gamma\rangle_s = \langle t, u\rangle_{\mathbb{T}^{a_n}} \langle f, g\rangle_{\mathbb{F}_N^n} \langle z, s\rangle_{\mathbb{Z}^{b_n}} = \prod_i t_i^{u_i} \prod_j e^{2\pi i \frac{f_j g_j}{n_j}} \prod_k s_k^{z_k}, \tag{7}$$

$\mathbb{F}_N^n = \oplus_j \mathbb{Z}_{n_j}$ from the structure theorem of finite abelian groups.

In the previous section we found an isomorphism $H_N/K_n \simeq E_{Nn}$ giving Proposition 3.7. Let

$$x + K_n \in H_N/K_n \longrightarrow x_{Nn} \in E_{Nn}, \quad \text{in particular} \quad x_n := x_{nn} \tag{8}$$

denote this isomorphism. There is an isomorphism between $K_n^\perp/H_N^\perp$ and $\widehat{E_{Nn}}$,

$$\gamma + H_N^\perp \in K_n^\perp/H_N^\perp \longrightarrow \gamma_{Nn} \in \widehat{E_{Nn}}, \quad \text{in particular} \quad \gamma_n := \gamma_{nn}, \tag{9}$$

such that for $x \in x + K_n$, and $\gamma \in \gamma + H_N^\perp$,

$$\langle x, \gamma\rangle = \langle x + K_n, \gamma + H_N^\perp\rangle_{Nn} = \langle x_{Nn}, \gamma_{Nn}\rangle_s. \tag{10}$$

This is because there is only one dual pairing modulo automorphisms (for any isomorphism $K_n^\perp/H_N^\perp \simeq \widehat{E_{Nn}}$, (10) defines some dual pairing, compose this isomorphism with the appropriate automorphism).

Assume $N < n$. Then $\widehat{i_{Nn}}$ (under the standard dual pairing) is the surjection induced from the natural map $K_n^\perp/H_n^\perp \longrightarrow K_n^\perp/H_N^\perp$, and $\widehat{j_{Nn}}$ is the injection induced from the inclusion $K_N^\perp/H_N^\perp \subset K_n^\perp/H_N^\perp$: Let $x \in H_N$, $\gamma \in K_N^\perp$. Then $\langle x + K_N, \gamma + H_N^\perp\rangle_{NN} = \langle x + K_n, \gamma + H_N^\perp\rangle_{Nn} = \langle x + K_n, \gamma + H_n^\perp\rangle_{nn}$ from (6). Thus, the dual of the inclusion $H_N/K_n \subset H_n/K_n$ is the natural map $K_n^\perp/H_N^\perp \longleftarrow K_n^\perp/H_n^\perp$, and the other way around for the original natural map. As the standard dual pairing in particular is of the form (4), we have in the dual basis,

$$\widehat{i_{Nn}} = \begin{pmatrix} \mathrm{id} & & \\ & \widehat{\theta_N^n} & \\ & \widehat{\phi_N^n} & \widehat{\iota_N^n} \end{pmatrix} \quad \text{and} \quad \widehat{j_{Nn}} = \begin{pmatrix} \widehat{\kappa_N^n} & & \\ \widehat{\phi_N'^n} & \widehat{\theta_N'^n} & \\ & & \mathrm{id} \end{pmatrix}.$$

The meaning of $\widehat{\theta_N^n}$ etc. is clear from the definition of the standard dual pairing.

The dual map of κ_N^n, $\widehat{\kappa_N^n}$, is an injection $\mathbb{Z}^{a_N} \xrightarrow{\widehat{\kappa_N^n}} \mathbb{Z}^{a_n}$. The form of the standard dual pairing shows that $\widehat{\kappa_N^n}(\mathbb{Z}^{a_N}) \subset \mathbb{Z}^{a_N} \oplus \{0\}^{a_n - a_N}$. The maps $\{\iota_N^n\}$ and $\{\widehat{\kappa_N^n}\}$ are used in the definitions of the next section.

3.3. **The inclusion $L^2(H_N/K_n) \longrightarrow L^2(G^1)$.** Later, we apply the structure theory on the level of functions. Let us still work with G^1. As H_N/K_n is needed to describe the relation between H_n/K_n and H_N/K_N, we incorporate H_N/K_n in the analysis of this situation. Define for any positive integers N, n the linear map

$$L^2(H_N/K_n) \xrightarrow{S_N^n} L^2(G^1) \quad \text{by} \quad (S_N^n\Phi)(x) = \begin{cases} \Phi(x+K_n) & \text{if } x \in H_N \\ 0 & \text{otherwise} \end{cases}.$$

The norm of $S_N^n\Phi$ is finite as H_N is an open subgroup and K_n is a compact subgroup (explained in Paragraph 3.3.1).

Let G^1 have some fixed Haar measure. Then there is a unique Haar measure on H_N/K_n such that S_N^n is an isometry. It turns out that we do not need to know more about these measures for the main approximation result, Section 6. Nevertheless, the next paragraph gives an explicit description of these measures, and for convenience we will use these measures in the rest of this paper. To simplify the notation we set $S_n := S_n^n$.

3.3.1. *Measures.* If B is a subgroup of the abelian group A, we say that $(A, B, A/B)$ is a Weil triple if A, B, and A/B have Haar measures satisfying Weil formula, symbolically written $d_{A/B} \cdot d_B = d_A$. If K is a compact group, by normalized measure, we mean the Haar measure on K such that total measure of K is one.

We make the following choice: H_N has restricted measure as an open subgroup of G^1, K_n has normalized measure, and $(H_N, K_n, H_N/K_n)$ is a Weil triple. This defines the correct measure on H_N/K_n:

$$\begin{aligned}\int_{G^1} |(S_N^n f)(x)|^2 dx = \int_{H_N} |(S_N^n f)(x)|^2 dx &= \int_{H_N/K_n} |f(x+K_n)|^2 d(x+K_n) \cdot \mathrm{meas}_{K_n}(K_n) \\ &= \int_{H_N/K_n} |f(x+K_n)|^2 d(x+K_n).\end{aligned}$$

We omit the proof of the next lemma as this description is not strictly necessary for Theorem 6.1. By counting measure, we mean counting measure with point weight one.

Lemma 3.8. *Let G^1 be a G^1-group with structure $\{(H_N, K_n)\}$, $H_N/K_n \simeq E_{Nn} = \mathbb{T}^{a_n} \oplus \mathbb{F}_N^n \oplus \mathbb{Z}^{b_N}$, and $\mathbb{F}_n^n =: \mathbb{F}_n$.*

- *i) There is a Haar measure on G^1 such that the map S_1 is isometric when E_1 has the product measure where $\mathbb{Z}^{b_1}$ has counting measure, and both $\mathbb{T}^{a_1}$, and $\mathbb{F}_1$ have normalized measures.*
- *ii) If G^1 has the measure of i), for any positive integer n, S_n is isometric when E_n has the product measure where $\mathbb{Z}^{b_n}$ has counting measure, and $\mathbb{T}^{a_n}$ has normalized measure. The measure on $\mathbb{F}_n$ has the same point weight as the point weight on $\mathbb{F}_1^n$ when $\mathbb{F}_1^n$ has normalized measure.*

Part *ii)* of this lemma is illustrated in the examples of Section 7.

4. Setup

Let $G = \mathbb{R}^e \oplus G^1$, where G^1 is a second countable G^1-group. Recall Definition 2.1. The purpose of this section is to define finite approximands for (U, V) on G. Because G is a finite direct sum, we use the construction in Proposition 2.3. We first introduce some notation which will be explained below. Here, n is an odd positive integer throughout this section, and for any such odd positive integer we define n° through $n = 2n^\circ + 1$:

$$\begin{aligned}
\text{Finite abelian group} \quad & G_n = \mathbb{Z}_n^e \oplus G_n^1, \\
\text{Isometry} \quad & I_n = R_n^e \otimes I_n^1, \\
\text{Group element} \quad & y^n = (r^n, x^n) \in G_n, \\
\text{Dual group element} \quad & \beta^n = (d^n, \beta^n) \in \widehat{G_n} = \mathbb{Z}_n^e \oplus \widehat{G_n^1}.
\end{aligned}$$

For an odd positive integer j, $\mathbb{Z}_j = \{-j^\circ, -j^\circ + 1, \dots, -1, 0, 1, \dots, j^\circ\}$, considered as a finite cyclic group. The e-th power of $\mathbb{Z}_j$ is denoted by $\mathbb{Z}_j^e$. Moreover, self-duality is set up with $\langle k, l\rangle = e^{2\pi i k l/j}$ for $k, l \in \mathbb{Z}_j$.

4.1. **Schwinger embedding.** The real part $\mathbb{R}^e$ is handled by the groups and scalings ϵ_n of Schwinger [9]. We follow Schwinger as we use $\langle x, y\rangle = e^{ixy}$ for $x, y \in \mathbb{R}$. Then we get the same scalings ϵ_n as he used. The finite group has already been taken as $\mathbb{Z}_n^e$.

4.1.1. *The maps* R_n^e. Let $R_n^e : l^2(\mathbb{Z}_n^e) \longrightarrow L^2(\mathbb{R}^e)$ be the e times tensor map of the operator $R_n : l^2(\mathbb{Z}_n) \longrightarrow L^2(\mathbb{R})$, where for $k \in \mathbb{Z}_n$, $\mathbb{I}_{\{k\}} \longrightarrow (\epsilon_n)^{-1/2}\mathbb{I}_{[(k-1/2)\epsilon_n, (k+1/2)\epsilon_n)} = (\epsilon_n)^{-1/2}\mathbb{I}_{I_n^k}$, where $\epsilon_n = \sqrt{2\pi/n}$. Characteristic function for the Borel set E is denoted by $\mathbb{I}_E$.

4.1.2. *Group element* r^n *and dual group element* d^n. Let $|\cdot|_\infty$ denote the sup-norm $|r|_\infty = \max_{i=1,\dots,j} |r_i|$ for $r = (r_i)$. For $r = (r_i) \in \mathbb{R}^e$, we approximate by $r^n \in \mathbb{Z}_n^e$ in the following way: If $|r|_\infty \le (n^\circ + 1/2)\epsilon_n$, then define $r^n = (r_1^n, \dots, r_e^n) \in \mathbb{Z}_n^e$, where r_i^n is the unique integer such that $r_i \in [(r_i^n - 1/2)\epsilon_n, (r_i^n + 1/2)\epsilon_n)$. Otherwise, r^n is by definition 0. We identify $\mathbb{R}^e$ and its dual group, and the approximation of $d \in \mathbb{R}^e$ is given by the same procedure as that for r.

We turn to the G^1-group part of the set up. Here, the idea is to use the structure theory of the previous section to get hold of finite abelian groups G_n^1. This set up will relate to the standard duality (7).

4.2. **The groups** G_n^1. Let $\mathbb{F}_{kmn}$, k, m, n odd positive integers, be given by $\mathbb{F}_{kmn} = \mathbb{Z}_m^{a_n} \oplus \mathbb{F}_n \oplus \mathbb{Z}_k^{b_n}$. The numbers b_n, a_n and the groups $\mathbb{F}_n$ come from the elementary group structure of G^1, Proposition 3.2; $H_n/K_n \simeq E_n = \mathbb{T}^{a_n} \oplus \mathbb{F}_n \oplus \mathbb{Z}^{b_n}$ for some choice of $\{H_n\}$ and $\{K_n\}$. For j odd, let F_j^b consist of those $k \in \mathbb{Z}^b$ for which $|k|_\infty \le j^\circ$, the j-cube in $\mathbb{Z}^b$ centered in origo. Recall that we in Section 3 found $\iota_i^n : \mathbb{Z}^{b_i} \longrightarrow \mathbb{Z}^{b_n}$ for $i < n$. Let k_n be the smallest odd integer such that $\iota_i^n(F_n^{b_i}) \subset F_{k_n}^{b_n}$ for all $i < n$. Equivalently, k_n is the smallest odd integer such that $|k|_\infty \le n^\circ$ implies $|\iota_i^n(k)|_\infty \le k_n^\circ$ for any $i < n$. Likewise, as $\widehat{\kappa_i^n} : \mathbb{Z}^{a_i} \longrightarrow \mathbb{Z}^{a_n}$ for $i < n$, let m_n be the smallest odd integer such that $\widehat{\kappa_i^n}(F_n^{a_i}) \subset F_{m_n}^{a_n}$ for all $i < n$. For n odd define $G_n^1 := \mathbb{F}_{k_n m_n n} = \mathbb{Z}_{m_n}^{a_n} \oplus \mathbb{F}_n \oplus \mathbb{Z}_{k_n}^{b_n}$.

4.3. **The embeddings I_n^1.** We start by constructing $I_{kmn} : l^2(\mathbb{F}_{kmn}) \longrightarrow L^2(G^1)$. These maps are defined through

$$l^2(\mathbb{F}_{kmn}) \simeq l^2(\mathbb{Z}_m^{a_n}) \otimes l^2(\mathbb{F}_n) \otimes l^2(\mathbb{Z}_k^{b_n})$$
$$\xrightarrow{T_m^{a_n} \otimes \mathrm{id} \otimes Z_k^{b_n}} L^2(\mathbb{T}^{a_n}) \otimes l^2(\mathbb{F}_n) \otimes l^2(\mathbb{Z}^{b_n}) \simeq L^2(H_n/K_n) \xrightarrow{S_n} L^2(G^1).$$

Here we identify I_{kmn} with $S_n\left(T_m^{a_n} \otimes \mathrm{id} \otimes Z_k^{b_n}\right)$, S_n is the lift of Paragraph 3.3. Finally, let $I_n^1 := I_{k_n m_n n}$.

The measures on $\mathbb{Z}^{b_n}$, $\mathbb{T}^{a_n}$, and $\mathbb{F}_n$ are those of Lemma 3.8. The maps $Z_k^{b_n}$ and $T_m^{a_n}$ are defined below in order to be isometric when $\mathbb{Z}_k^{b_n}$ has the usual counting measure, while $\mathbb{Z}_m^{a_n}$ has normalized measure.

4.3.1. *The maps $Z_k^{b_n}$.* The embeddings $Z_k^{b_n} : l^2(\mathbb{Z}_k^{b_n}) \longrightarrow l^2(\mathbb{Z}^{b_n})$ are defined in the obvious way by sending $\mathbb{I}_{\{i\}}$ to $\mathbb{I}_{\{i\}}$ in $l^2(\mathbb{Z}^{b_n})$ for $i \in \mathbb{Z}_k^{b_n}$.

4.3.2. *The maps $T_m^{a_n}$.* Let $T_m^{a_n} = \mathcal{F}^n Z_m^{a_n} \left(\mathcal{F}_{mn}\right)^{-1}$, where $\mathcal{F}_{mn} : l^2(\mathbb{Z}_m^{a_n}) \longrightarrow l^2(\mathbb{Z}_m^{a_n})$ and $\mathcal{F}^n : l^2(\mathbb{Z}^{a_n}) \longrightarrow L^2(\mathbb{T}^{a_n})$ are Fourier transforms. This is a variation of the approach in Proposition 2.2.

4.4. **Approximate group element in G^1.** Given $x \in G^1$, we associate to it $x^{kmn} \in \mathbb{F}_{kmn}$. As $G^1 = \cup H_n$, $x \in H_{W_x}$ for some smallest integer W_x. For $n \geq W_x$, surject (recall Equation 8) $x \longrightarrow x + K_n = x_n = (t_n, f_n, z_n) \in E_n$. Then let $x^{kmn} = (t^{mn}, f_n, z^{kn})$ if $n \geq W_x$, and 0 otherwise, where the elements $z^{kn} \in \mathbb{Z}_k^{b_n}$ and $t^{mn} \in \mathbb{Z}_m^{a_n}$ are defined below. Finally, let $x^n := x^{k_n m_n n}$, $t^n := t^{m_n n}$ and $z^n := z^{k_n n}$.

4.4.1. *The integer part z^{kn}.* If $z_n \in F_k^{b_n}$, let $z_n =: z^{kn} \in \mathbb{Z}_k^{b_n}$. Otherwise, put z^{kn} to 0.

4.4.2. *The circle part t^{mn}.* Here we apply root functions. Parametrize the circle $\mathbb{T}$ by $z = e^{2\pi i\theta}$, $\theta \in [-1/2, 1/2)$. For our $t_n = (t_{n,j}) \in \mathbb{T}^{a_n}$, let $t^{mn} = (t_j^{mn}) \in \mathbb{Z}_m^{a_n}$, where t_j^{mn} is the unique element in $\mathbb{Z}_m$ such that $\theta_{n,j} \in [(t_j^{mn} - 1/2)/m, (t_j^{mn} + 1/2)/m)$ for $t_{n,j} = e^{2\pi i\theta_{n,j}}$.

4.5. **Approximate character in $\widehat{G^1}$.** Recall the dual construction and definitions of Paragraph 3.2. Therefore, for $\gamma \in \widehat{G^1}$, $\gamma^{kmn} \in \mathbb{Z}_m^{a_n} \oplus \mathbb{F}_n \oplus \mathbb{Z}_k^{b_n}$ (dual basis) is chosen by the same procedure as for the group element case. Let us fix some notation: For $n \geq \hat{W}_\gamma$ ($\hat{W}_\gamma$ chosen analogous to W_x), $\gamma_n = (u_n, g_n, s_n) \in \widehat{E_n}$ (recall Equation 9). Then $\gamma^{kmn} = (u^{mn}, g_n, s^{kn})$ if $n \geq \hat{W}_\gamma$, and 0 otherwise. Here the circle part $s^{kn} \in \mathbb{Z}_k^{b_n}$ and the integer part $u^{mn} \in \mathbb{Z}_m^{a_n}$ are defined by the procedures in the last paragraph (with reverse notation). Finally, we set $\gamma^n := \gamma^{k_n m_n n}$, $u^n := u^{m_n n}$ and $s^n := s^{k_n n}$.

5. The Space of Schwartz-Bruhat Functions

Let G be a l.c.a. group. Recall from Section 3 the existence of pairs (H, K), where H/K is elementary, H is an open subgroup, and $K \subset H$ is a compact subgroup.

Bruhat [1] introduces the Schwartz-Bruhat space of functions on G, $\mathcal{S}(G)$, as those complex valued functions which have support in some H, and are locally constant on some corresponding K. Thus Φ is naturally defined on the elementary group H/K, here Φ should look like an ordinary Schwartz function. This means: Let P be a polynomial function on H/K, and D a translation invariant differential operator. Then $\Phi \in \mathcal{S}(H/K)$ if Φ is smooth, and all the seminorms $||P \cdot Df||_\infty$ are finite. Alternatively, this can be formulated by the tensor product of Grothendieck for locally convex spaces (Schwartz spaces are nuclear). We use T_{HK} to denote

$$\Phi \in L^2(G^1) \longrightarrow T_{HK}\Phi \in L^2(H/K), \quad (T_{HK}\Phi)(x+K) = \Phi(x), \quad x \in x+K \in H/K.$$

Notice that T_{HK} is the inverse of S_N^n (Paragraph 3.3) for $H = H_N$ and $K = K_n$ on the range of S_N^n. If $H \subset H'$ and $K' \subset K$, then Φ Schwartz-Bruhat on (H,K) implies Φ Schwartz-Bruhat on (H',K') as well. There are 'large' enough pairs (H,K) for $\mathcal{S}(G)$ to be dense in $L^2(G)$, and the Fourier transform leaves this space invariant; if Φ is Schwartz-Bruhat on (H,K), then $\hat{\Phi}$ is Schwartz-Bruhat on $(K^\perp, H^\perp)$.

As G by [11] is of the form $\mathbb{R}^e \oplus G^1$, where G^1 is a G^1-group, the definition of $\mathcal{S}(G^1)$ is really what is new in this extension of Schwartz functions.

Let G^1 be a second countable G^1-group. Use Proposition 3.2 to find $\{H_N\}$, $\{K_n\}$. It suffices to define $\mathcal{S}(G^1)$ on the pairs $\{(H_n,K_n)\}$:

Lemma 5.1. *Let G^1, $\{H_n\}$ and $\{K_n\}$ satisfy the conclusions of Proposition 3.2. If (H,K) is some other pair in the definition of $\mathcal{S}(G^1)$, then there is a non-negative integer n such that $H_n \supset H$ and $K_n \subset K$.*

Proof. First, $H/K \simeq \mathbb{T}^a \oplus \mathbb{F} \oplus \mathbb{Z}^b$, so, as the quotient is compactly generated and K is compact, H itself is compactly generated (the pre-image of a compact set is compact as K is compact). If C is compact and generates H, then $G^1 = \cup H_n$ covers C, and $C \subset H_N$ for some integer N. Thus, $H \subset H_N$. Next, $K^\perp \subset \widehat{G^1}$ is open while $H^\perp \subset K^\perp \subset \widehat{G^1}$ is compact. Moreover, $K^\perp/H^\perp \simeq \widehat{H/K}$, so $K^\perp$ is compactly generated, and the same reasoning as before gives $K^\perp \subset K_M^\perp$ and $K_M \subset K$. So the claim follows with n as the larger of N and M. □

We use $\Phi \in \mathcal{S}(n)$ to denote $\Phi \in \mathcal{S}(G^1)$ supported in H_n/K_n. Notice that $\mathcal{S}(n) \subset \mathcal{S}(n')$ when $n < n'$. Also, put $\Phi^n := T_{H_nK_n}\Phi$ for $\Phi \in \mathcal{S}(n)$. Let V^n and U^n denote the standard Weyl system on $L^2(H_n/K_n)$.

Lemma 5.2. *Assume $x \in G^1$, $\gamma \in \widehat{G^1}$ and $\Phi \in \mathcal{S}(G^1)$. Then we can find an n such that Φ, $V(x)\Phi$ and $U(\gamma)\Phi$ are all contained in $\mathcal{S}(n)$, $(V(x)\Phi)^n = V^n(x+K_n)\Phi^n$, and $(U(\gamma)\Phi)^n = U^n(\gamma + H_n^\perp)\Phi^n$.*

Proof. Assume $\Phi \in \mathcal{S}(n')$. Since $H_n \nearrow G^1$, x is contained in some $H_{n''}$, and by the group property of any H_n it follows that $V(x)\Phi \in \mathcal{S}(\max\{n',n''\})$. Replacing x by $x+k$, for $k \in K_n$, clearly makes no difference. That multiplication by character is locally constant is a special case of Weil [16], Proposition 2. Here, the proof is straightforward: As $\cup K_n^\perp = \widehat{G^1}$, $\gamma \in K_n^\perp$ for some n. Thus, multiplication by γ is automatically locally constant on K_n. Moreover, since Φ is supported in H_n, multiplication by γ or $\gamma + \gamma'$, for $\gamma' \in H_n^\perp$, gives the same result. The largest n from the two parts of the proof gives the desired result as translation and multiplication by character is invariant for the Schwartz space of any elementary group. □

Using $I_n = S_n$, it follows easily from this lemma that G^1 is a limit of the elementary groups H_n/K_n in the sense of Weyl systems.

5.1. **Exploring $\mathcal{S}(G^1)$.** Assume that G^1 is a second countable G^1-group. For $n > N$, if $\Phi \in \mathcal{S}(N)$, then $\Phi \in \mathcal{S}(n)$. We need more on the relationship between Φ^n and Φ^N. Recall from Section 3 the induced maps $E_N \xleftarrow{j_{Nn}} E_{Nn} \xrightarrow{i_{Nn}} E_n$. From their construction,

$$\Phi^n = \begin{cases} \Phi^N \circ j_{Nn} \circ (i_{Nn})^{-1} & \text{on} \quad i_{Nn}(E_{Nn}) \\ 0 & \text{otherwise.} \end{cases}$$

We easily get this by passing through H_N/K_n. Notice that scaling factors would enter without the measure considerations of Paragraph 3.3.1. Let $\Phi \in \mathcal{S}(N)$, where in addition $\Phi^N = \Phi^N_{\mathbb{T}} \Phi^N_{\mathbb{F}} \Phi^N_{\mathbb{Z}}$ is a simple tensor; $\Phi^N_{\mathbb{T}} \in \mathcal{S}(\mathbb{T}^{a_N})$, $\Phi^N_{\mathbb{F}} \in \mathcal{S}(\mathbb{F}_N)$, and $\Phi^N_{\mathbb{Z}} \in \mathcal{S}(\mathbb{Z}^{b_N})$. Then by Proposition 3.7, for $(t, f, z) \in E_n = \mathbb{T}^{a_n} \oplus \mathbb{F}_n \oplus \mathbb{Z}^{b_n}$ and $n > N$ (primed coordinates are in E_{Nn}),

$$\begin{aligned} \Phi^n(t,f,z) = &\begin{cases} \Phi^N_{\mathbb{T}}(\kappa^n_N(t) + \phi'^n_N(f')) & \text{if} \quad z = \iota^n_N(z') \text{ and } f = \phi^n_N(z') + \theta^n_N(f') \\ 0 & \text{otherwise} \end{cases} \\ \times &\begin{cases} \Phi^N_{\mathbb{F}}(\theta'^n_N(f')) & \text{if} \quad z = \iota^n_N(z') \text{ and } f = \phi^n_N(z') + \theta^n_N(f') \\ 0 & \text{otherwise} \end{cases} \\ \times &\begin{cases} \Phi^N_{\mathbb{Z}}(z') & \text{if} \quad z = \iota^n_N(z') \\ 0 & \text{otherwise} \end{cases} . \end{aligned} \tag{11}$$

This is a product of one function in all three coordinates, one function in $\mathbb{F}_n$ and $\mathbb{Z}^{b_n}$, and one function in $\mathbb{Z}^{b_n}$ alone. We see the same by calculating $j_{Nn} \circ i^{-1}_{Nn}$ (on the image of i_{Nn}). This matrix is upper triangular.

Recall the Fourier transform $\mathcal{F}^n$ on $\mathbb{Z}^{a_n}$. Let $D_n = \mathbb{Z}^{a_n} \oplus \mathbb{F}_n \oplus \mathbb{Z}^{b_n}$. We also denote $\mathcal{F}^n \otimes \mathrm{id} \otimes \mathrm{id}$ acting on $l^2(D_n)$ by $\mathcal{F}^n$. We generally skip all $\otimes$ and any id in the notation for tensor product of operators. So, for instance, we use the same symbol for an operator and its amplification by identity operators.

Lemma 5.3. *Let $\Psi^N_{\mathbb{T}} = (\mathcal{F}^N)^{-1}\Phi^N_{\mathbb{T}}$, n any integer larger than N, and $\Psi^n = (\mathcal{F}^n)^{-1}\Phi^n \in l^2(D_n)$. Then*

$$\Psi^n = \Psi^n_{\mathbb{T}} \cdot \Lambda_n \cdot \Phi^n_{\mathbb{Z}},$$

where $\Psi^n_{\mathbb{T}} = \Psi^N_{\mathbb{T}} \circ [\widehat{\kappa^n_N}]^{-1}$ (and 0 outside range of $\widehat{\kappa^n_N}$) is a function of $\mathbb{Z}^{a_n}$, $\Phi^n_{\mathbb{Z}} = \Phi^N_{\mathbb{Z}} \circ [\iota^n_N]^{-1}$ (and 0 outside range of ι^n_N) is a function of $\mathbb{Z}^{b_n}$, while Λ_n depends on all three coordinates.

Proof. From (11) only the first part in the product formula for Φ^n, $\Xi^n := \Phi^N_{\mathbb{T}}(\kappa^n_N(t) + \phi'^n_N(f'))$ if $z = \iota^n_N(z')$, $f = \phi^n_N(z') + \theta^n_N(f')$, and 0 otherwise, depends on $\mathbb{T}^{a_n}$. Let $q = (k, f, z) \in D_n$. First, if f and/or z is outside the range of i_{Nn}, Ξ^n is obviously 0. This settles the 'otherwise' case for the requirements for f and z. If not, we are left with

Fourier transform in $\mathbb{T}^{a_n}$-direction of the function $\Phi^N_{\mathbb{T}}(\kappa^N_n(t)+F^n_N(z,f))$, where $F^n_N(z,f) := \phi'^n_N(f')$ to show clearer that f' also depends on z. Thus, we take the inverse Fourier transform of a translated function that is locally constant on $\mathrm{Ker}(\kappa^n_N)$. Then the Weil formula shows that the Fourier transform is supported in the range of $\widehat{\kappa^n_N}$, and translation goes to multiplication:

$$\begin{aligned}((\mathcal{F}^n)^{-1}\Xi^n)(q) &= \int_{t\in\mathbb{T}^{a_n}} \Phi^N_{\mathbb{T}}(\kappa^n_N(t) + F^n_N(f,z))\langle k,t\rangle dt \\ &= \int_{\tilde{t}\in\mathbb{T}^{a_n}/\,\mathrm{Ker}\,\kappa^n_N} \Phi^N_{\mathbb{T}}(\kappa^n_N(\tilde{t}) + F^n_N(f,z))\langle k,\tilde{t}\rangle d\tilde{t} \int_{t'\in\mathrm{Ker}\,\kappa^n_N} \langle k,t'\rangle dt'.\end{aligned}$$

In the last line, κ^n_N also denotes the map (isomorphism) from the quotient $\mathbb{T}^{a_n}/\,\mathrm{Ker}\,\kappa^n_N$ to $\mathbb{T}^{a_N}$. From Paragraph 3.3.1, $\mathrm{Ker}(\kappa^n_N)$ has normalized measure as this compact group is the circle part of K_N/K_n. Then as $\mathbb{T}^{a_n}$ has normalized measure, also the quotient $\mathbb{T}^{a_N}$ has normalized measure. The integral in t' is non-zero, and then it always has value 1, if and only if k is in the annihilator of $\mathrm{Ker}(\kappa^n_N)$, which means that $k \in \mathrm{Im}(\widehat{\kappa^n_N})$. That takes care of rest of the 'otherwise' claims, and gives for $k = \widehat{\kappa^n_N}(k')$

$$\begin{aligned}((\mathcal{F}^n)^{-1}\Xi^n)(q) &= \int_{\tilde{t}} \Phi^N_{\mathbb{T}}(\kappa^n_N(\tilde{t}) + F^n_N(f,z))\langle \widehat{\kappa^n_N}(k'),\tilde{t}\rangle d\tilde{t} \\ &= \int_{\tilde{t}} \Phi^N_{\mathbb{T}}(\kappa^n_N(\tilde{t}) + F^n_N(f,z))\langle k',\kappa^n_N(\tilde{t})\rangle d\tilde{t} \\ &= \int_{u\in\mathbb{T}^{a_N}} \Phi^N_{\mathbb{T}}(u)\langle k',u\rangle du \cdot \langle -k', F^n_N(f,z)\rangle \\ &= \Psi^n_{\mathbb{T}}(k') \cdot \langle -k', F^n_N(f,z)\rangle.\end{aligned}$$

Combined with (11), the result follows. □

Without the measure considerations of Paragraph 3.3.1, again scaling factors enter (but can be taken into Λ_n).

This lemma has the following important consequence: Let Supp denote the support of a function, then Lemma 5.3 shows that $\mathrm{Supp}\,\Psi^n \subset \widehat{\kappa^n_N}(\mathrm{Supp}\,\Psi^N_{\mathbb{T}}) \oplus \mathbb{F}_n \oplus \iota^n_N(\mathrm{Supp}\,\Phi^N_{\mathbb{Z}})$. This motivates the choice of the size parameters k_n and m_n in Section 4.

6. Main Approximation Result

Theorem 6.1. *The second countable locally compact abelian group G is a limit, in the sense of Weyl systems (Definition 2.1), of the finite abelian groups $\{G_n\}_{n\ odd}$ defined in Section 4 .*

Proof. Propositions 2.3, 3.2, and Propositions 6.3, 6.4 below. □

6.1. **The G^1-case.**

Proposition 6.2. *Let $x \in G^1$, and $\gamma \in \widehat{G^1}$. Then $\langle x^n, \gamma^n\rangle_{G^1_n} \longrightarrow \langle x, \gamma\rangle_{G^1}$ as $n \longrightarrow \infty$ (n odd). Here, $x^n \in G^1_n$ and $\gamma^n \in \widehat{G^1_n}$ are as in Section 4.*

Proof. For $n \geq W$, the larger of W_x and $\hat{W}_\gamma$ (as in Section 4), by Equation 10, $\langle x, \gamma\rangle = \langle x + K_n, \gamma + H_n^\perp\rangle_{H_n/K_n} = \langle x_n, \gamma_n\rangle_s$. Thus, for $n \geq W$

$$\begin{aligned} \left|\langle x^n, \gamma^n\rangle_{G^1_n} - \langle x, \gamma\rangle_{G^1}\right| &= |\langle (t^n, f_n, z^n), (u^n, g_n, s^n)\rangle_{G^1_n} - \langle (t_n, f_n, z_n), (u_n, g_n, s_n)\rangle_s| \\ &\leq \quad |\langle z^n, s^n\rangle - \langle z_n, s_n\rangle_{\mathbb{Z}^{b_n}}| \qquad (12) \\ &\quad + |\langle t^n, u^n\rangle - \langle t_n, u_n\rangle_{\mathbb{T}^{a_n}}|. \qquad (13) \end{aligned}$$

For the inequality, write out the characters as products, then use triangle inequalities, and the fact that all numbers involved have absolute value one.

Let us estimate (12). Here, $z_n = \iota^n_W(z_W)$ by semi-alignment Proposition 3.7. Observe that $k^\circ_n \geq |z_n|_\infty$ for all n larger than some L: Let L be such that $|z_W|_\infty \leq n^\circ$ for $n \geq L$. Then by construction in Section 4, $|\iota^n_W(z_W)|_\infty \leq k^\circ_n$. Consequently, for n larger than L (taken $> W$), $z^n = z_n = \iota^n_W(z_W)$. Again by Proposition 3.7, $\iota^n_W(\mathbb{Z}^{b_W})$ is contained in the first b_W factors of $\mathbb{Z}^{b_n}$. Thus $\langle z_n, s_n\rangle = \prod_{l=1}^{b_W} e^{2\pi i \iota^n_W(z_W)_l \theta_{n,l}}$ where $s_n = (s_{n,l})$, $s_{n,l} = e^{2\pi i\theta_{n,l}}$. Likewise, for this large n, $\langle z^n, s^n\rangle = \prod_{l=1}^{b_W} e^{2\pi i \iota^n_W(z_W)_l \theta^n_{n,l}}$ where $s^n = (s^n_l)$, $s^n_l/k_n = \theta^n_{n,l}$. Then we calculate

$$(12) \leq \sum_{l=1}^{b_W} |1 - e^{2\pi i \iota^n_W(z_W)_l \delta^l_n}|$$

as $\theta^n_{n,l} = \theta_{n,l} + \delta^l_n$ where $|\delta^l_n| \leq 1/(2k_n)$. Moreover, (12) tends to 0 because

$$|\iota^n_W(z_W)_l \delta^l_n| \leq b_W |z_W|_\infty k^\circ_n/(n^\circ 2k_n) \leq b_W|z_W|_\infty/(2n^\circ).$$

The estimate on $\iota^n_W(z_W)_l$ comes from the fact that ι^n_W is a $\mathbb{Z}$-module map between two free $\mathbb{Z}$-modules. In the standard bases, ι^n_W has integer coefficient matrix representation (a^l_j). By the construction of k_n, this is easily seen to imply $|a^l_j| \leq k^\circ_n/n^\circ$. Consequently, $|\iota^n_W(z_n)_l| \leq b_W \max_j |z_{W,j}| k^\circ_n/n^\circ$ follows from this matrix representation.

Finally, (13) is estimated as (12) because m_n is constructed similar to k_n. □

Let us settle some matters of notation for the projections of Section 4: $P_n : L^2(G^1) \longrightarrow I^1_n(l^2(G^1_n))$, $P^{\mathbb{T}}_n : L^2(\mathbb{T}^{a_n}) \longrightarrow T^{a_n}_{m_n}(l^2(\mathbb{Z}^{a_n}_{m_n}))$, and $P^{\mathbb{Z}}_n : L^2(\mathbb{Z}^{b_n}) \longrightarrow Z^{b_n}_{k_n}(l^2(\mathbb{Z}^{b_n}_{k_n}))$.

Proposition 6.3. *The second countable G^1-group G^1 is a limit, in the sense of Weyl systems, of the finite abelian groups $\{G^1_n\}_{n\ odd}$ of Section 4 .*

Proof. Let $\Phi \in \mathcal{S}(N)$, N some positive integer, where $\Phi^N = \Phi^N_{\mathbb{T}}\Phi^N_{\mathbb{F}}\Phi^N_{\mathbb{Z}}$ is a simple tensor as in Paragraph 5.1. Moreover, $\Phi^N_{\mathbb{Z}}$ is taken with finite support, and $\Phi^N_{\mathbb{T}}$ is a trigonometric polynomial.

The linear span of the chosen Φ is dense in $L^2(G^1)$: Through $\Phi \longrightarrow \Phi^N$, the linear span of all the $L^2(E_N)$ is dense in $L^2(G^1)$. Moreover, it is easy to see that the linear span of simple tensors with finite support are dense in the discrete space $l^2(D_N)$. Applying

$\mathcal{F}^N$ gives the desired density in $L^2(E_N)$ as the Fourier transform $\mathcal{F}^N$ on $\mathbb{Z}^{a_N}$ takes finite support functions to trigonometric polynomials, and the other way around. Finally, the resulting functions are in $\mathcal{S}(E_N)$. Thus, we really work in a smaller space than $\mathcal{S}(G^1)$.

Convergence of projections. As $P_n\Phi \in \mathcal{S}(n)$, $(P_n\Phi)^n = P_n^{\mathbb{T}} P_n^{\mathbb{Z}} \Phi^n$ from the definition of P_n (the linear map S_n is the inverse of $T_{H_nK_n}$), and

$$||P_n\Phi - \Phi||_{G^1} = ||P_n^{\mathbb{T}} P_n^{\mathbb{Z}} \Phi^n - \Phi^n||_{E_n}$$

for $n > N$. Let $C^n = F_{m_n}^{a_n} \oplus \mathbb{F}_n \oplus F_{k_n}^{b_n}$. By the construction in Section 4 (Ψ^n is as in Paragraph 5.1), $P_n^{\mathbb{T}} P_n^{\mathbb{Z}} \Phi^n = \mathcal{F}^n \mathbb{I}_{C^n} \Psi^n$. As $\Phi_{\mathbb{Z}}^N$ and $\Psi_{\mathbb{T}}^N$ have finite support, there is a Q (taken larger than N) such that for $n > Q$, $\mathrm{Supp}(\Phi_{\mathbb{Z}}^N) \subset F_n^{b_N}$ and $\mathrm{Supp}(\Psi_{\mathbb{T}}^N) \subset F_n^{a_N}$. Then, by definition of k_n and m_n, and Lemma 5.3, $\mathrm{Supp}(\Psi^n) \subset C^n$. Thus, $P_n\Phi = \Phi$ for $n > Q$, and strong convergence of projection has been proven.

Convergence of the V's. Let $x \in G^1$ as in Proposition 6.2. Because of Lemma 5.2, for $n > M$, which is the larger of N and W (W as in Proposition 6.2), $V(x)\Phi \in \mathcal{S}(n)$, and $(V(x)\Phi)^n = V^n(x_n)\Phi^n$. Here (Equation 8) $x_n = (t_n, f_n, z_n) \in E_n$. We agree on a notation where V (V_n') denotes translation (finite embedded translation), and the argument tells us what group is involved. Then $(V(x)\Phi)^n = V(t_n)V(f_n)V(z_n)\Phi^n$ for $n > M$. Moreover, by construction of the isometry S_n, $V_n'(x^n)\Phi \in \mathcal{S}(n)$ for $n > N$, and $(V_n'(x^n)\Phi)^n = V_n'(t^n)V_n(f_n)V_n'(z^n)\Phi^n$, where for instance $V_n'(t^n)$ is the embedding of $V_n(t^n)$ through the map $T_{m_n}^{a_n}$. Thus, as $V(f_n)$ is unitary, for $n > M$,

$$\begin{aligned} ||V(x)\Phi - V_n'(x^n)\Phi||_{G^1} &= ||V(z_n)V(t_n)\Phi^n - V_n'(z^n)V_n'(t^n)\Phi^n||_{E_n} \\ &= ||V(z_n)\hat{U}(t_n)\Psi^n - V_n'(z^n)\hat{U}_n'(t^n)\Psi^n||_{D_n}. \end{aligned} \tag{14}$$

From the definition in Section 4, $V_n'(t^n) = \mathcal{F}^n \hat{U}_n'(t^n)(\mathcal{F}^n)^{-1}$, where $\hat{U}_n(t^n) = U_n(t^n)$ as we identify $l^2(\mathbb{Z}_{m_n}^{a_n})$ with itself under the finite Fourier transform. Moreover, $\hat{U}(t_n)$ operates through multiplication by $\langle \cdot, t_n\rangle$ on $l^2(\mathbb{Z}^{a_n})$. Let L (now taken $> M$) be so large that $z_M \subset F_n^{b_M}$ for $n > L$. Then from Section 4, $z_n = \iota_M^n(z_M) \subset F_{k_n}^{b_n}$, and $z^n = z_n$. So, for $n > L$, the operator $V_n'(z^n)$ acts as $V(z_n)$ on $\hat{U}_n'(t^n)\Psi^n$ if both $\hat{U}_n'(t^n)\Psi^n$ and its translate by z_n is in the range of the nth projection $P_n^{\mathbb{Z}}$. As the support of $\hat{U}_n'(t^n)\Psi^n$ in $\mathbb{Z}^{b_n}$-direction is restricted by the support of $\Phi_{\mathbb{Z}}^n$, we can find J (taken $> L$ and $> Q$) such that for $n > J$, $\mathrm{Supp}(V(z_L)\Phi_{\mathbb{Z}}^L) \subset F_n^{b_L}$, consequently $\mathrm{Supp}(V(z_n)\Phi_{\mathbb{Z}}^n) \subset F_{k_n}^{b_n}$. So, for $n > J$,

$$(14) = ||\hat{U}(t_n)\Psi^n - \hat{U}_n'(t^n)\Psi^n||_{D_n}$$

as $V(z_n)$ is unitary. For $n > J$ and $q = (k, u, v) \in D_n$, $[\hat{U}(t_n) - \hat{U}_n'(t^n)]\Psi_{\mathbb{T}}^n\Lambda_n(q) = 0$ for k not in $F_{m_n}^{a_n}$, and otherwise

$$[\hat{U}(t_n) - \hat{U}_n'(t^n)]\Psi_{\mathbb{T}}^n\Lambda_n(q) = \Psi_{\mathbb{T}}^n\Lambda_n(q)\left[\langle k, t^n\rangle - \langle k, t_n\rangle\right].$$

For the last expression to be non-zero, by Lemma 5.3, we must in particular have $k = \widehat{\kappa_N^n}(k_N)$ for some $k_N \in \mathbb{Z}^{a_N}$. As $\Psi_{\mathbb{T}}^N$ has finite support, the pointwise convergence of (13) in the proof of Proposition 6.2 can be made uniform on the $\mathbb{Z}^{a_N}$-support of $\Psi_{\mathbb{T}}^N$, consequently also on the $\mathbb{Z}^{a_n}$-support of $\Psi_{\mathbb{T}}^n$ (these two sets have the same number of elements). Thus, for $\epsilon > 0$ there is an R_ϵ such that for $n > R_\epsilon$, $|\langle k, t^n\rangle - \langle k, t_n\rangle| < \epsilon$ for all k in the support of the $\mathbb{Z}^{a_n}$-direction of $\Psi^n_{\mathbb{T}}$. Thus, by Fourier transforming back again, for $n > R_\epsilon$ (and $n > J$),

$$(14) < \epsilon \cdot ||\Psi^n||_{D_n} = \epsilon \cdot ||\Phi^n||_{E_n} = \epsilon \cdot ||\Phi||_{G^1}.$$

Convergence of the U's. The arguments are essentially as for translation, only the order of the steps is altered. Let $\gamma \in \widehat{G^1}$ as in Proposition 6.2. Again by Lemma 5.2, for $n > M$, $(U(\gamma)\Phi)^n = U^n(\gamma_n)\Phi^n$ where $\gamma_n = (u_n, g_n, s_n) \in \widehat{E_n}$. By preliminaries similar to those for translation, for $n > M$,

$$||U(\gamma)\Phi - U'_n(\gamma^n)\Phi||_{G^1} = ||U(s_n)\Phi^n_{\mathbb{Z}}\hat{V}(u_n)\Psi^n_{\mathbb{T}}\Lambda_n - U'_n(s^n)\Phi^n_{\mathbb{Z}}\hat{V}'_n(u^n)\Psi^n_{\mathbb{T}}\Lambda_n||_{D_n}. \tag{15}$$

With arguments as in the first part of the previous paragraph, there is $\hat{L}$ (taken $> M$ and $> Q$) such that $\hat{V}'_n(u^n)\Psi^n_{\mathbb{T}}\Lambda_n = \hat{V}(u_n)\Psi^n_{\mathbb{T}}\Lambda_n$ for $n > \hat{L}$, and

$$(15) = ||U(s_n)\Phi^n - U'_n(s^n)\Phi^n||_{E_n}.$$

Let $\tilde{q} = (v, u, l) \in E_n$. Then, for $n > \hat{L}$, and $l \in F^{b_n}_{k_n}$ (otherwise the expression below in 0), $[U(s_n) - U'_n(s^n)]\Phi^n(\tilde{q}) = [\langle l, s_n\rangle - \langle l, s^n\rangle]\Phi^n(\tilde{q})$. Now, we follow the procedure of the last part of the proof for translation. Again, (15) will be bounded by $\epsilon \cdot ||\Phi||_{G^1}$ for sufficiently large n. □

6.2. **Real numbers.** The set up is as in Section 4. For $f \in C(\mathbb{R})$, $f_{av}(k\epsilon_n)$, $k \in \mathbb{Z}_n$, is defined by $f_{av}(k\epsilon_n) = \epsilon_n^{-1}\int_{(k-1/2)\epsilon_n}^{(k+1/2)\epsilon_n} f(x)dx$, $\epsilon_n = \sqrt{2\pi/n}$.

Proposition 6.4. *The group $\mathbb{R}$ is a limit, in the sense of Weyl systems, for the finite cyclic groups $\mathbb{Z}_n$, where n runs through the odd positive integers.*

Proof. Let $f \in C_c(\mathbb{R})$ be a continuous function with compact support inside $[-B, B]$, $B > 0$. We consider the isometries $R_n : l^2(\mathbb{Z}_n) \longrightarrow L^2(\mathbb{R})$ defined in Section 4.1 with associated projections P_n as in Definition 2.1.

The fact that $P_n \longrightarrow \mathrm{id}$ strongly is probably well-known. Nevertheless, we give a proof based on uniform convergence. If $\epsilon_n < 1$, then $f_n := P_n f = \sum_{|k|\le n^\circ} f_{av}(k\epsilon_n)\cdot\mathbb{I}_{I^k_n}$ has support in $[-B-1, B+1] := I(B)$. Furthermore, for any x in this interval and $n^\circ\epsilon_n > B+1$, x lies in some I^k_n, thus $f_n(x) = f_{av}(k\epsilon_n)$. By the mean value theorem, $f_{av}(k\epsilon_n) = f(z)$ for some z, also in I^k_n. So, for any $\epsilon > 0$, just make n large enough for $|f(y) - f(x)| < \epsilon$ for any pair $x, y \in I^k_n$, for all k such that $I^k_n \cap I(B)$ is non-empty. Then $|f_n(x) - f(x)| < \epsilon$ for any x in $I(B)$. Hence $f_n \longrightarrow f$ uniformly, which suffices for the L^2-convergence.

The result for $V'_n \longrightarrow V$ follows very similarly.

As for $U'_n \longrightarrow U$, it is easily seen that it is enough to check that $U'_n(d^n)f_n$ gets close to the compression $(U(d)f)_n$. Here $d^n \in \mathbb{Z}_n$ approximates $d \in \mathbb{R}$ as in Section 4.1. So, $|(U(d)f)_n - U'_n(d^n)f_n| = \left|\sum_{|k|\le n^\circ}\epsilon_n^{-1}\int_{(k-1/2)\epsilon_n}^{(k+1/2)\epsilon_n}(e^{ixd} - e^{\frac{2\pi i}{n}kd^n})f(x)dx\cdot\mathbb{I}_{I^k_n}\right| \le \sum_{|k|\le n^\circ}$ $\| f \|_\infty\ \epsilon_n^{-1}\int_{(k-1/2)\epsilon_n}^{(k+1/2)\epsilon_n}|e^{ixd} - e^{i\epsilon_n k\epsilon_n d^n}|dx\cdot\mathbb{I}_{I^k_n}$. Since $(U(d)f)_n - U'_n(d^n)f_n$ has support in $I(B)$ when $\epsilon_n < 1$, consider the uniformly continuous function $G(x, y) = e^{ixy}$ on the compact strip $I(B) \times [d-1, d+1]$. So, given any $\epsilon > 0$, we can find a uniform n such that $|G(x, y) - G(x', y')| < \epsilon$ for any pair of points in each $I^k_n \times I^{y^n}_n$, for those

$k \in \mathbb{Z}_n$ for which $I_n^k \cap I(B)$ is non-empty. Thus, for any $z \in I(B)$, there is a k giving $|(U(d)f)_n(z) - (U'_n(d^n)f_n)(z)| < \| f \|_\infty (\epsilon_n)^{-1} \int_{(k-1/2)\epsilon_n}^{(k+1/2)\epsilon_n} \epsilon dx = \| f \|_\infty \epsilon$. As this works uniformly, the theorem is correct. □

7. Applications: p-adic Numbers and Rational Adeles

Let p be a prime number. Varadarajan mentions in [12] that the Weyl system on $\mathbb{Z}_{p^n}$, as $n \to \infty$, converges to the Weyl system associated to the field of p-adic numbers $\mathbb{Q}_p$. We construct examples of H_n and K_n in this case. These groups are interesting as they lead to phase spaces for p-adic quantum theories. Also, $\mathbb{Q}_p$ is the canonical example of a non-compact, non-discrete G^1-group.

7.1. **The p-adic numbers.** The field of p-adic numbers (see Gouvêa [6] for basic properties) is the completion of the rational numbers $\mathbb{Q}$ under the p-adic valuation $|\cdot|_p$. There is a (continuous) field structure on $\mathbb{Q}_p$ when p is a prime. In a coordinate representation, the p-adic numbers can be viewed as Laurent series x, $x = \sum_{l=n}^{\infty} x_l p^l$, where n is an integer and $x_l \in \{0, 1, \dots, p-1\}$. Under the natural addition and multiplication, such that the resulting series also has coefficients in this set, $\mathbb{Q}_p$ is a field. Furthermore, under $|\cdot|_p$, defined by $|x|_p = p^{-n}$ where x_n is the first non-zero coefficient in the series of x, these series are no longer just formal. In fact, $\mathbb{Q}_p$ is a complete metric space. As an abelian topological group w.r.t. addition, $\mathbb{Q}_p$ is self-dual. The compact open subgroup $p^n\mathcal{O}_p$ consists of those $x \in \mathbb{Q}_p$ for which $x_k = 0$ for all k smaller than the integer n. The p-adic integers $\mathcal{O}_p$ form the maximal subring of $\mathbb{Q}_p$.

Put $H_n = p^{-a_n}\mathcal{O}_p$ and $K_n = p^{b_n}\mathcal{O}_p$, where $a_n + b_n = n$ and the integers $a_n, b_n \longrightarrow \infty$ as $n \longrightarrow \infty$. Then $H_n \nearrow \mathbb{Q}_p$ while $K_n \searrow \{0\}$, and through multiplication by p^{a_n}, $H_n/K_n \simeq \mathbb{Z}_{p^n}$. Thus, by Theorem 6.1, $\mathbb{Q}_p$ is a limit of

$$G_n = \mathbb{Z}_{p^n} \quad (n \text{ odd})$$

in the sense of Weyl systems.

Notice that Schwartz-Bruhat functions on $\mathbb{Q}_p$ are locally constant functions with compact support.

The measure on $\mathbb{Z}_{p^n}$ from Paragraph 3.3.1, when $\mathbb{Q}_p$ has its usual Haar measure and total measure of $\mathcal{O}_p$ equals 1, is the following: Then measure$(\mathbb{Z}_{p^n})$ = measure$(H_n) = p^{a_n}$, so measure$(\{0\}) = p^{a_n - n} = p^{-b_n}$. It makes sense from the approximation point of view that the measure of a point goes to zero, while the total measure goes to infinity. If n is even and $a_n = n/2$, then $\mathbb{Z}_{p^n}$ has the self-dual Haar measure of our set up.

7.2. **Rational adeles.** The locally compact abelian ring $\mathcal{A}$ of adeles over $\mathbb{Q}$ (see Weil [17]) is defined as the product $\mathbb{R} \times \mathcal{A}_f$, where $\mathcal{A}_f$ is the group of finite adeles; the sequences $x = (x_p) \in \prod_{p \text{ prime}} \mathbb{Q}_p$ such that $x_p \in \mathcal{O}_p$ for all but finitely many places. These adeles define a locally compact ring under restricted product topology and pointwise addition and multiplication. Let

$$H_n = \mathbb{R} \oplus_{p \text{ prime} \leq n} p^{-n}\mathcal{O}_p \oplus_{p \text{ prime} > n} \mathcal{O}_p \quad \text{and} \quad K_n = \{0\} \oplus_{p \text{ prime} \leq n} p^{n}\mathcal{O}_p \oplus_{p \text{ prime} > n} \mathcal{O}_p.$$

Then each H_n is an open subgroup, and $H_n \nearrow \mathcal{A}$ while $K_n \subset H_1$ is compact, $K_n \searrow \{0\}$ and $H_n/K_n \simeq \mathbb{R} \oplus_{p \text{ prime} \leq n} \mathbb{Z}_{p^{2n}}$. Consequently, $\mathcal{A}$ is a limit of

$$G_n = \mathbb{Z}_n \oplus_{p \text{ prime} \leq n} \mathbb{Z}_{p^{2n}} \quad (n \text{ odd})$$

in the sense of Weyl systems.

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