Some remarks on arithmetic physics *

V. S. Varadarajan
Department of Mathematics, University of California,
Los Angeles, CA 90095–1555, USA.
e–mail: vsv@math.ucla.edu

To Dr. C. R. Rao, with affection and admiration

There have been some recent speculations on connections between quantum theory and modern number theory. At the boldest level these suggest that there are two ways of viewing the quantum world, the usual and the arithmetic, which are in some sense complementary. At a more conservative level they suggest that there is much mathematical interest in examining structures which are important in quantum theory and analyze to what extent they make sense when the real and complex fields are replaced by the more unconventional fields and rings, like finite or nonarchimedean fields and adele rings, that arise in number theory. This paper explores some aspects of these questions.

0. Dedication

It was over 40 years ago, in 1956 to be exact, that I traveled from my home town of Madras in Southern India to Calcutta, almost a thousand miles away, to join the Indian Statistical Institute as a research scholar. I was a young lad of nineteen, eager to do research in probability theory. Dr. C. R. Rao was the central figure in the Institute at that time, and his dedication to work and

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concern for young workers made a deep impression on me. From then on his example has been a constant source of inspiration for me, as I am sure, it has been for his countless number of students and colleagues.

I remember especially vividly an occasion during the early 1980's, when I had been struggling in vain for a number of months to solve certain problems of reduction theory of matrices depending on parameters. I went to visit Dr. Rao in Pittsburgh for a few days and I saw how he was working literally all the time. Inspired by his example, I returned to Los Angeles and stepped up the intensity of my efforts. This led to a breakthrough in the problem and resulted in one of the best things I have ever done in my whole scientific career. Even today the memory of that visit and the subsequent happenings give me great pleasure.

It is therefore with great happiness and affection that I dedicate this small contribution to him, while wishing him many more years of good health and scientific activity.

1. Introduction.

Arithmetic physics, or better, arithmetic quantum theory, is a term referring to a collection of ideas and partial results, loosely held together, suggesting that there are deep connections between the worlds of quantum physics and number theory, and that one should try to discover and develop these connections. At one extreme is the modest idea that one should try to see if some of the mathematical structures arising in quantum theory make sense over fields and rings other than $\mathbf{R}$, such as the field of p–adic numbers $\mathbf{Q}_p$, or the ring $\mathbf{A}(\mathbf{Q})$ of adeles over the rationals $\mathbf{Q}$. The point here is not to try to develop the alternative theories as substitute models for the actual physical world, but rather to search for results that would extend what we already know over $\mathbf{R}$ or $\mathbf{C}$ and present them under a unified scheme.

One basis for this suggestion is the simple fact that all experimental calculations are essentially discrete and so can be modeled by mathematical structures that are over $\mathbf{Q}$. The theories over $\mathbf{R}$ are thus idealizations that are more convenient than essential and reflect the fact that the field of real numbers is a completion of the field of rational numbers. But there are other completions of the reals, namely the fields $\mathbf{Q}_p$, and it is clear that under suitable circumstances a large finite quantum system may be thought of as an approximation to a system defined over $\mathbf{Q}_p$. If we continue this line of thought further and want to make sure that no single $\mathbf{Q}_p$ is given a privileged role, it becomes necessary to consider *all* the completions of $\mathbf{Q}$ on the same footing, which means working over the ring of adeles $\mathbf{A}(\mathbf{Q})$.

At the other extreme are bold speculations that push forward the hypothesis that the exploration of the structure of quantum theories by replacing $\mathbf{R}$ by $\mathbf{Q}_p$ and $\mathbf{A}(\mathbf{Q})$ is not just a pleasant exercise but is *essential.* I quote the following remarks of Manin from his beautiful and inspiring paper[1].

> *On the fundamental level our world is neither real nor p–adic; it is adelic. For some reasons, reflecting the physical nature of our kind of living matter (e.g. the fact that we are built of massive particles), we tend to project the adelic picture onto its real side. We can equally well spiritually project it upon its non–Archimedean side and calculate most important things arithmetically.*
>
> *The relation between "real" and "arithmetical" pictures of the world is that of complementarity, like the relation between conjugate observables in quantum mechanics.*

In the following sections I shall briefly expand on these themes and try to describe some of the results that have been obtained from this perspective.

2. Quantum systems over finite fields and rings: a brief overview.

The idea of considering quantum systems over fields and rings other than $\mathbf{R}$ has a long history and goes back to the origins of quantum theory itself. I shall make some brief remarks about the evolution of these ideas. My aim is not to be complete but to show that there has been considerable interest in this theme for a long time.

Weyl (1928). The idea of considering quantum systems in which the configuration space is replaced by a finite abelian group first appears in the famous book[2] of Hermann Weyl. Recall that quantum theory prescribes that the mathematical quantities representing the position and momentum of a particle in one dimension should be operators q and p on the Hilbert space of states of the system satisfying the Heisenberg commutation rules (with $\hbar = 1$)

$$[p, q] = -iI$$

The structure of this remarkable relation is such that the Hilbert space on which the operators p and q act has to be infinite dimensional and further that q and p have to be unbounded. Weyl, who preferred to work with bounded rather than unbounded operators, replaced q and p by the unitary groups they generate, and introduced the commutation rules between these unitary groups that are formally equivalent to the Heisenberg commutation rules. Let

$$U(x) = e^{ixq}, \qquad V(\xi) = e^{i\xi p} \qquad (x, \xi \in \mathbf{R})$$

so that U and V are unitary representations of the additive group $\mathbf{R}$. Then we get the commutation rule of Weyl which is formally equivalent to that of Heisenberg, namely,

$$U(x)V(\xi) = e^{-ix\xi}V(\xi)U(x) \qquad (x, \xi \in \mathbf{R})$$

In the standard Schrödinger model for quantum mechanics the Hilbert space is $L^2(\mathbf{R})$ and U, V are defined by

$$(U(x)f)(\tau) = e^{i\tau x}f(\tau), \qquad (V(\xi)f)(\tau) = f(\tau + \xi)$$

It is easy to check that the pair (U, V) acts irreducibly on $L^2(\mathbf{R})$.

If I am not mistaken, it was Weyl who formulated the uniqueness question associated to the pair of unitary representations satisfying the Weyl commutation rule in the following manner: is it true that such a pair (U, V) of unitary representations, under the further assumption of irreducibility, is equivalent to the pair defined by the standard model? Weyl's discussion did not lead to a rigorous proof of the uniqueness; the first proofs were given by Stone and Von Neumann (independently)[3] very soon after Weyl's formulation. But in the course of his attempts to understand this question Weyl had a very remarkable idea. He realized that his commutation rules could be formulated in much greater generality, in fact *for any pair of abelian groups in duality.* To make precise Weyl's idea let us introduce a definition.

Definition. Let A and B be abelian groups in duality through a bicharacter $(\cdot\ ,\ \cdot)$. Then a Weyl system for (A, B) is a pair of unitary representations U (of A) and V (of B) such that

$$U(x)V(\xi) = (x, \xi)^{-1}V(\xi)U(x) \qquad ((x, \xi) \in A \times B)$$

We recall that a bicharacter for (A, B) is a map of $A \times B$ into the multiplicative group of complex numbers of absolute value 1 which is a character in each argument. A and B are said to be in duality with respect to $(\cdot, \cdot)$ if the map of A into the character group $\widehat{B}$ of B induced by the bicharacter is an isomorphism. If we are dealing with topological groups, everything has to be modified with the addition of appropriate continuity conditions in the obvious manner. In particular the unitary representations are to be continuous. A very general situation arises when A and B are separable locally compact (abelian) groups; then one can construct a standard irreducible Weyl system in complete analogy with the case of $\mathbf{R}$. However the case when A and B are *infinite dimensional* is also of great interest in quantum field theory. In fact, already in Dirac's theory of the interaction of matter with the electromagnetic field, the classical electromagnetic field was expanded as a Fourier series, and the Fourier coefficients, which are infinitely many, were regarded as the position coordinates of the field with their time derivatives as the momenta; their quantization followed Heisenberg's prescription but now for *infinitely many* q's and p's.

For an arbitrary but not necessarily locally compact pair (A, B) neither the existence nor the uniqueness of irreducible Weyl systems is obvious. But Weyl considered the case when

$$A = B = \mathbf{Z}_N := \mathbf{Z}/N\mathbf{Z}, \quad (a, b) = e^{2i\pi ab/N} \qquad (a, b \in \mathbf{Z})$$

and proved in this case the uniqueness of the irreducible Weyl systems. He then discussed in a heuristic fashion how one can identify $\mathbf{Z}_N$ with a grid in $\mathbf{R}$ in such a manner that when N goes to ∞ the mesh of the grid goes to 0 and the operators of the Weyl system for $(\mathbf{Z}_N, \mathbf{Z}_N)$ converge to the operators of the Weyl system for $(\mathbf{R}, \mathbf{R})$.

Schwinger (1960). Weyl's ideas were revisited by Julian Schwinger when he examined the foundations of quantum kinematics in a series of beautiful papers in the late 1950's and early 1960's and then expanded on them in a book[4]. Schwinger's work went beyond Weyl's and brought out new aspects of the situation. First he emphasized the fact that the finite systems were of interest in their own way, and not merely as approximations to the continuum systems. Indeed, it was for the finite systems that he introduced his famous *algebra of measurements* which is a complex $*$–algebra whose hermitian elements represent the physical quantities of the system[4a]. They are nowadays known as *Schwinger algebras* and have been studied intensively[4b]. Second, he made the approximation process involving Weyl systems much more transparent (although he refrained from giving a general definition). Finally, he suggested, implicitly if not explicitly, that for large N the approximation will be close *not merely in the kinematic sense but also dynamically*. I shall now explain briefly these contributions of Schwinger.

Classification of finite Weyl systems. Schwinger had already undertaken in[4] a detailed treatment of the kinematics of finite quantum systems and from his point of view the Weyl systems associated to finite abelian groups furnish the most important examples of finite systems. The spectra of the representations U and V define maximal observables and the Weyl commutation rules imply that these are conjugate observables–when one of them is measured with complete accuracy, all values of the other are equally likely. He then noticed that the classification of finite abelian groups gives a classification of finite Weyl systems. In this way he arrived at the principle that the Weyl systems associated to $A = B = \mathbf{Z}_p$ where p runs over all the primes are the building blocks. Curiously this enumeration is incomplete and one has to include the cases[5] where $A = B = \mathbf{Z}_{p^r}$ where p is as before a prime but r is any integer ≥ 1.

Approximation of the Weyl system for $A = B = \mathbf{R}$ by that for $A = B = \mathbf{Z}_N$. The idea is to identify $\mathbf{Z}_N$ with a grid in $\mathbf{R}$. This can be done also for $\mathbf{R}^d$ with $\mathbf{Z}_N^d$ as the approximating abelian group for any $d \geq 1$ but we shall treat here only the case $d = 1$. This approximation is also at the basis of the very useful theory of fast Fourier transforms. Let

$$L_N = \left\{ r\varepsilon \middle| r = 0, \pm 1, \pm 2, \ldots, \pm (N-1)/2 \right\} \qquad \varepsilon = \left(\frac{2\pi}{N} \right)^{1/2}$$

where N is an odd integer. The map that sends the equivalence class $[r]$ of r mod N to $r\varepsilon$ is an identification of $\mathbf{Z}_N$ with the grid L_N. The Hilbert space $L^2(\mathbf{Z}_N)$ is imbedded in $L^2(\mathbf{R})$ by sending the delta function at $[r]$ to the function which is the characteristic function of the interval $[(r - 1/2)\varepsilon, (r + 1/2)\varepsilon]$ multiplied by $\varepsilon^{-1/2}$. Now one can introduce the position operator q_N as the operator of multiplication by the function $[r] \mapsto r\varepsilon$. For the momentum operator p_N Schwinger had the real insight and originality to define it as the *Fourier transform (on the finite group $\mathbf{Z}_N$) of* q_N; actually the approach via Weyl systems shows that this is the only way to define p_N. Notice that p_N is now *not a local difference operator on the grid but a global operator,* more like a pseudo difference operator if I may use that expression in analogy with a pseudo differential operator.

Schwinger gave a treatment of the behavior of this approximation which was more detailed than that of Weyl and even suggested that the states of the continuum system be restricted to those for which this approximation procedure is uniform in some sense. His work raised the question whether the above approximation was good dynamically also. Thus, if we take a reasonably simple

Hamiltonian such as the oscillator,

$$H = (1/2)(p^2 + q^2)$$

then one should ask if the corresponding dynamical group could be approximated closely by the dynamical group generated by the finite Hamiltonian

$$H_N = (1/2)(p_N^2 + q_N^2)$$

for large N. Numerical calculations[6] showed that this is correct and that the approximation is unexpectedly good even for relatively small values of N. In fact, a very strong dynamical limit theorem[7] can be proved for Hamiltonians

$$H = (1/2)(p^2 + V(q))$$

where the potential V goes to ∞ when $|q| \to \infty$. The condition on V guarantees that the energy spectrum is discrete. The generalization of these results to the case of mixed spectrum remains open.

Beltrametti (1971), Nambu (1987). In the Weyl–Schwinger theory the structures are still over **R**. Now in a general theory the fields enter in at least two places–once when we decide to build space time as a vector space over this field, and second when we introduce the carrier space of all the values of physical fields and functions. In view of the well known divergences that occur in the conventional models of space time it is an attractive idea to examine what the possibilities for a quantum field theory are when finite fields, rings, and other algebraic structures are allowed to replace the field of real and complex numbers. One of the earlier treatments of this question of the microstructure of space time goes back to Beltrametti [8]; there are later treatments of similar questions[9]. Nambu[10] has examined this question more recently but so far there has been no systematic effort to develop a quantum field theory in such a context, for instance a quantum field theory over a Riemann surface over a finite field.

Weil (1961). The most profound discussion of Weyl systems over general locally compact abelian groups is due to Andre Weil. In a pair of epoch-making papers[11] he examined Weyl systems when A is any locally compact abelian group and $B = \widehat{A}$ is its dual group. Weil considered the case when A is a finite dimensional vector space over a local field (e.g., a p–adic field) or a free module over the ring adeles over a global field such as a field of algebraic numbers. He applied his theory to reinterpret the Siegel theory of quadratic forms over global fields and his work may be regarded as the quantum theory of the quadratic form. Weil's work has had a profound influence on modern number theory and has provided a deep link between number theory and representation theory.

3. Convergence of Weyl systems.

The remarks made above on the approximation of Weyl systems over $\mathbf{R}$ by those of $\mathbf{Z}_N$ suggest that it is desirable to have a formal definition of convergence of Weyl systems. This is not difficult to do[5]. If then A is any separable locally compact abelian group, there is a sequence of finite abelian groups A_N such that the Weyl system associated to $(A, \widehat{A})$ is the limit of the Weyl systems associated to $(A_N, \widehat{A_N})$.[12] For instance the Weyl systems associated to the field $\mathbf{Q}_p$ of p–adic numbers is the limit of Weyl systems associated to $\mathbf{Z}_{p^r}$.

4. Quantization and Schrödinger theory over nonarchimedean fields.

Deformation of observable algebras.

The first question is whether we can view quantum theory over nonarchimedean fields from the point of view of *deformation quantization.* The simplest situation is the following. Let K be a local field of characteristic different from 2 and let $X = K \times K$. We write $S(X)$ for the Schwartz–Bruhat space of X, namely the space of compactly supported locally constant functions with complex values on X. There is no structure of a Poisson algebra on $S(X)$ but at least there is the structure of an associative algebra on $S(X)$, namely the one coming from pointwise multiplication. One can ask at least whether this algebra has nontrivial, for instance, nonabelian, deformations. The answer is no, at least if one interprets deformations in the usual formal sense. However, one may ask whether there is a topological space T and a point $t_0 \in T$ such that there are associative algebra structures $f, g \mapsto f \cdot_t g$ on $S(X)$ for each $t \in T$ which are nonabelian, such that

(a) as $t \to 0, f \cdot_t g \to fg$

(b) $f \cdot_{t_0} g = fg$

Then the answer is yes. In fact, the Moyal–Weyl formula for $*$–product on $S(\mathbf{R}^2)$, the Schwartz space of $\mathbf{R}^2$, makes sense over $S(X)$ and defines a family of associative algebra structures parametrized by K having the properties described above. It would be of interest to examine if such $*$–products can be defined for the spaces $S(X)$ for other manifolds X over K[13].

Path integrals and Schrödinger theory

Let D be a division ring which is finite dimensional and central over K. Let V be a vector space of dimension $n < \infty$ over D with a norm $|\cdot|$ which is homogeneous and satisfies the ultrametric norm inequality

$$|u+v| \leq \max(|u|,|v|) \qquad (u, v \in V)$$

Using a nontrivial additive character on D one can define a Fourier transform $\mathcal{F}$ on the Schwartz–Bruhat space $S(V)$, which extends to a unitary isomorphism of $\mathcal{H} = L^2(V)$. One can define a family of (pseudodifferential!) operators on $\mathcal{H}$ by

$$\Delta_b = -\mathcal{F} M_{|x|^b} \mathcal{F}^{-1} \qquad (b > 0)$$

where $M_{|x|^b}$ is the operator of multiplication by $|x|^b$. Notice that if $D = \mathbf{R}$ and $b = 2$, then Δ coincides with the usual Laplacian. The Hamiltonians on $\mathcal{H}$ are then

$$H = -\Delta_b + U$$

where U is a Borel function. The theory of Kato (*Kato potentials*) goes through without difficulties and allows us to view H as an essentially self adjoint operator on the Schwartz–Bruhat space $S(V)$ of V under suitable conditions on U, for instance if U is locally L^2 and bounded at infinity, or if, after an affine transformation of V, U becomes a function of $< n$ variables of the type described just now. The adelic generalizations of these results also offer no difficulties[14a].

It is now possible to prove that we can obtain a path integral representation for the propagator of the dynamical group generated by this Hamiltonian. One has to go to *imaginary time* for getting a rigorous measure on path space. We restrict ourselves to the local situation. The probability measure on the path space is now not Wiener measure but an appropriate measure whose finite dimensional densities can be explicitly described. They are not gaussian but have Fourier transforms of the form

$$\varphi_{t_1,b} \otimes \varphi_{t_2-t_1,b} \otimes \ldots \otimes \varphi_{t_N-t_{N-1},b} \qquad (0 < t_1 < t_2 < \ldots < t_N)$$

where

$$\varphi_{t,b} = e^{-t|u|^b}$$

(One can verify directly that this is the characteristic function of a probability density.) There is an additional departure from the theory over $\mathbf{R}$ in that the measure is not defined on the space of continuous maps from $[0, \infty)$ to V but on the space of maps which are right continuous and have limits from the left everywhere[14b].

5. The Segal–Shale–Weil representation.

In the remainder of this note I shall discuss one of the deeper aspects of Weyl systems over locally compact abelian groups, namely the construction of the so called metaplectic representation due to Segal, Shale, and Weil. To begin with, let A be a separable locally compact abelian group, let $B = \widehat{A}$, and let (U, V) be a Weyl system for the pair $(A, \widehat{A})$. Write

$$G = A \times \widehat{A}$$

and suppose for simplicity that A (hence also $\widehat{A}$) has no 2–torsion. This means that $x \mapsto 2x$ is an automorphism of A; we write $x \mapsto 2^{-1}x$ for the inverse automorphism. Let

$$W(a,b) = (a,b)^{1/2}U(a)V(b) \qquad ((a,b) \in G)$$

where the square root is defined by

$$(a,b)^{1/2} = (2^{-1}a, 2^{-1}b)^2$$

Then W satisfies the relations

$$W(a,b)W(a',b') = m((a,b),(a',b'))W(a+a',b+b')$$

where

$$m((a,b),(a',b')) = \frac{(a',b)^{1/2}}{(a,b')^{1/2}}$$

This means that W is a *projective unitary representation* with multiplier m. The uniqueness theorem for Weyl systems is then the statement that up to unitary equivalence there is only one irreducible projective unitary representation of G with multiplier m. As we mentioned earlier it was proved by Stone and Von Neumann[3] for $A = B = \mathbf{R}^d$ with the usual duality

$$(a,b) = e^{2\pi i(a\cdot b)}$$

This was later extended to all separable locally compact abelian A with $B = \widehat{A}$ by Mackey; for a detailed discussion of this result see[15].

Let $\mathcal{H}$ be the Hilbert space on which W acts. Let s be an automorphism of $A\times\widehat{A}$ which leaves m invariant; we shall call such an automorphism *symplectic.* Then the map

$$W^s : (a,b) \longmapsto W(s(a,b))$$

is again an irreducible projective unitary representation of G with the same multiplier, and so, by the uniqueness, there is a unitary operator U_s, unique up to a phase factor, such that

$$W(s(a,b)) = U_s W(a,b) U_s^{-1}$$

It is then immediate that

$$s \longmapsto U_s$$

induces a homomorphism (say u) of $S(G)$ into the projective unitary group of $\mathcal{H}$, such that for each $s \in S(G)$, U_s intertwines W and W^s.

Let us suppose that A is of local type, i.e., a vector space a over local field, or of adelic type, i.e., a free module over an adele ring. Then $S(G)$ is a separable locally compact group and one can show that u is continuous, so that we may choose the U_s to depend in a Borel manner on s. U is thus a projective unitary representation of $S(G)$ (notice the analogy with the construction of the spin representation of the orthogonal group). This is the famous *metaplectic representation*, first introduced by Weil, nowadays called the *Weil representation* or the *oscillator representation.* The metaplectic representation is not equivalent to an ordinary representation of the group $S(G)$. But Weil proved that it lifts to an ordinary unitary representation of a two–fold covering group $M(G)$ of $S(G)$, which he called the *metaplectic group.* It played a crucial role in Weil's work .

Although it does not seem to have arithmetic implications, it would be appropriate if I make some remarks on the work of Shale[16]. Inspired by Segal's work on quantum field theory, Shale considered Weyl systems for (A, A) where A is the additive group of an *infinite dimensional Hilbert space* in duality with itself by

$$(a,b) = e^{\frac{i}{2}\Im(a,b)}$$

In this case it turns out that the uniqueness fails. One can start with a reasonable generalization of the standard model in the finite dimensional case but the failure of the uniqueness theorem makes it necessary to restrict sharply the symplectic automorphisms in order that they may preserve the generalized standard model. Shale discovered what these restrictions should be and proved that the projective representation lifts to an ordinary representation of a 2–fold covering group of the restricted symplectic group.

It turns out that the Weil representation has a remarkable relation to the quantum harmonic oscillator. Let me restrict myself to the case when $A = \mathbf{R} = \widehat{A}$. Then $S(G)$ is just $SL(2, \mathbf{R})$ and $M(G)$ is the unique 2–fold covering group of $SL(2, \mathbf{R})$. The subgroup of rotations in $SL(2, \mathbf{R})$ is the dynamical group of the *classical* harmonic oscillator. Its preimage B in $M(G)$ is again isomorphic to the circle group and so can be regarded as a homomorphic image of $\mathbf{R}$. The restriction to B of the Weil representation thus gives rise to a unitary representation of $\mathbf{R}$. The remarkable fact is that it is just the dynamical group of the one dimensional *quantum* harmonic oscillator.

If we now replace $\mathbf{R}$ by $\mathbf{Q}_p$, we see that the restrictions of the Weil representation to the preimages of the subgroups in $SL(2, \mathbf{Q}_p)$ may be viewed as furnishing the quantizations of these subgroups regarded as classical dynamical systems over $\mathbf{Q}_p$. However this view makes time p–adic. The systematic exploration of these ideas will be postponed to a later occasion[17].

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