5. SPINORS

5.1. Prologue. E. Cartan classified simple Lie algebras over $\mathbf{C}$ in his thesis in 1894, a classification that is nowadays done through the (Dynkin) diagrams. In 1913 he classified the irreducible finite dimensional representations of these algebras[1]. For any simple Lie algebra $\mathfrak{g}$ Cartan's construction yields an irreducible representation canonically associated to each node of its diagram. These are the so-called *fundamental representations* in terms of which all irreducible representations of $\mathfrak{g}$ can be constructed using $\otimes$ and subrepresentations. Indeed, if $\pi_j (1 \leq j \leq \ell)$ are the fundamental representations and m_j are integers ≥ 0, and if v_j is the highest vector of π_j, then the subrepresentation of

$$\pi = \pi_1^{\otimes m_1} \otimes \ldots \otimes \pi_\ell^{\otimes m_\ell}$$

generated by

$$v = v_1^{\otimes m_1} \otimes \ldots \otimes v_\ell^{\otimes m_\ell}$$

is irreducible with highest vector v, and every irreducible module is obtained in this manner uniquely. As is well-known, Harish-Chandra and Chevalley (independently) developed around 1950 a general method for obtaining the irreducible representations without relying on case by case considerations as Cartan did.

If $\mathfrak{g} = \mathfrak{sl}(\ell + 1)$ and $V = \mathbf{C}^{\ell+1}$, then the fundamental module π_j is $\Lambda^j(V)$, and all irreducible modules can be obtained by decomposing the tensor algebra over the defining representation V. Similarly, for the symplectic Lie algebras, the decomposition of the tensors over the defining representation gives all the irreducible modules. But Cartan noticed that this is not the case for the orthogonal Lie algebras. For these the fundamental representations corresponding to the right extreme node(s) (the nodes of higher norm are to the left) could not be obtained from the tensors over the defining representation. Thus for $\mathfrak{so}(2\ell)$ with $\ell \geq 2$, there are two of these, denoted by $S^{\pm}$, of dimension $2^{\ell-1}$, and for $\mathfrak{so}(2\ell+1)$ with $\ell \geq 1$, there is one such, denoted by S, of dimension 2^{ℓ}. These are the so-called *spin representations*; the $S^{\pm}$ are also referred to as *semi-spin representations.* The case $\mathfrak{so}(3)$ is the simplest. In this case the defining representation is SO(3) and its universal cover is SL(2). The tensors over the defining representation yield only the odd dimensional irreducibles; the spin representation is the 2-dimensional representation $D^{1/2} = \mathbf{2}$ of SL(2). The weights of the tensor representations are integers while $D^{1/2}$ has the weights $\pm 1/2$, revealing clearly why it cannot be obtained from the tensors. However $D^{1/2}$ generates all representations; the representation of highest weight $j/2$ (j an integer ≥ 0) is the j-fold symmetric product of $D^{1/2}$, namely $\mathrm{Symm}^{\otimes j} D^{1/2}$. In particular the vector representation of SO(3) is $\mathrm{Symm}^{\otimes 2} D^{1/2}$. In the other low dimensional cases the spin representations are as follows.

SO(4): Here the diagram consists of 2 unconnected nodes; the Lie algebra $\mathfrak{so}(4)$ is not simple but semisimple and splits as the direct sum of two $\mathfrak{so}(3)$'s. The group SO(4) is not simply connected and SL(2)$\times$SL(2) is its universal cover. The spin representations are the representations $D^{1/2,0} = \mathbf{2} \times \mathbf{1}$ and $D^{0,1/2} = \mathbf{1} \times \mathbf{2}$. The defining *vector* representation is $D^{1/2,0} \times D^{0,1/2}$.

SO(5): Here the diagram is the same as the one for Sp(4). The group SO(5) is not simply connected but Sp(4), which is simply connected, is therefore the universal cover of SO(5). The defining representation $\mathbf{4}$ is the spin representation. The representation $\Lambda^2 \mathbf{4}$ is of dimension 6 and contains the trivial representation, namely the line defined by the element that corresponds to the invariant symplectic form in $\mathbf{4}$. The quotient representation is 5-dimensional and is the defining representation for SO(5).

SO(6): We have come across this in our discussion of the Klein quadric. The diagrams for $\mathfrak{so}(6)$ and $\mathfrak{sl}(4)$ are the same and so the universal covering group for SO(6) is SL(4). The spin representations are the defining representation $\mathbf{4}$ of SL(4) and its dual $\mathbf{4}^*$, corresponding to the two extreme nodes of the diagram. The defining representation for SO(6) is $\Lambda^2 \mathbf{4} \simeq \Lambda^2 \mathbf{4}^*$.

SO(8): This case is of special interest. The diagram has 3 extreme nodes and the group $\mathfrak{S}_3$ of permutations in 3 symbols acts transitively on it. This means that $\mathfrak{S}_3$ is the group of automorphisms of SO(8) modulo the group of inner automorphisms, and so $\mathfrak{S}_3$ acts on the set of irreducible modules also. The *vector representation* $\mathbf{8}$ *as well as the spin representations* $\mathbf{8}^{\pm}$ *are all of dimension* 8 and S_3 permutes them. Thus it is immaterial which of them is identified with the vector or the spin representations. This is the famous *principle of triality.* There is an *octonionic model* for this case which makes explicit the principle of triality[8,13].

Dirac's equation of the electron and Clifford algebras. The definition given above of the spin representations does not motivate them at all. Indeed, at the time of their discovery by Cartan, the spin representations were not called by that name; that came about only after Dirac's sensational discovery around 1930 of the spin representation and the Clifford algebra in dimension 4, on which he based the relativistic equation of the electron bearing his name. This circumstance led to the general representations discovered by Cartan being named spin representations. The elements of the spaces on which the spin representations act were then called *spinors.* The fact that the spin representation cannot be obtained from tensors meant that the Dirac operator in quantum field theory must act on *spinor fields* rather than tensor fields. Since Dirac was concerned only with special relativity and so with *flat* Minkowski spacetime, there was no conceptual difficulty in defining the spinor fields there. But when one goes to *curved spacetime*, the spin modules of the orthogonal groups at each spacetime point form a structure which will exist in a global sense only when certain topological obstructions (cohomology classes) vanish. The structure is the so-called *spin structure* and the manifolds for which a spin structure exists are called *spin manifolds.* It is only on spin manifolds that one can formulate the global Dirac and Weyl equations.

Coming back to Dirac's discovery, his starting point was the Klein-Gordon equation

$$(\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2)\varphi = -m^2\varphi \qquad \left(\partial_\mu = \frac{\partial}{\partial x^\mu}\right)$$

where φ is the wave function of the particle (electron) and m is its mass. This equation is of course relativistically invariant. However Dirac was dissatisfied with it primarily because it was of the second order. He felt that the equation should be of the first order in time and hence, as all coordinates are on equal footing in special relativity, it should be of the first order in all coordinate variables. Translation invariance meant that the differential operator should be of the form

$$D = \sum_\mu \gamma_\mu \partial_\mu$$

where the γ_μ are constants. To maintain relativistic invariance Dirac postulated that

$$D^2 = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2 \qquad (1)$$

and so his equation took the form

$$D\varphi = \pm im\varphi.$$

Here the factor i can also be understood from the principle that only the $i\partial_\mu$ are self adjoint in quantum mechanics. Now a simple calculation shows that no *scalar* γ_μ can be found satisfying (1); the polynomial $X_0^2 - X_1^2 - X_2^2 - X_3^2$ is irreducible. Indeed, the γ_μ must satisfy the equations

$$\gamma_\mu^2 = \varepsilon_\mu, \qquad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 0(\mu \neq \nu) \quad (\varepsilon_0 = 1, \varepsilon_i = -1, i = 1, 2, 3) \qquad (2)$$

and so the γ_μ cannot be scalars. But Dirac was not stopped by this difficulty and asked if he could find *matrices* γ_μ satisfying (2). He found the answer to be yes. In fact he made the discovery that there is a solution to (2) where the γ_μ are 4×4 matrices, and that this solution is unique up to similarity in the sense that any other solution (γ'_μ) of degree 4 is of the form $(T\gamma_\mu T^{-1})$ where T is an invertible 4×4 matrix; even more, solutions occur only in degrees $4k$ for some integer $k \geq 1$ and are similar (in the above sense) to a direct sum of k copies of a solution in degree 4.

Because the γ_μ are 4×4 matrices, the wave function φ *cannot be a scalar anymore*; it has to have 4 components and Dirac realized that these extra components describe *some internal structure* of the electron. In this case he showed that they indeed encode the *spin* of the electron.

It is not immediately obvious that there is a natural action of the Lorentz group on the space of 4-component functions on spacetime, with respect to which the Dirac operator is invariant. To see this clearly, let $g = (\ell_{\mu\nu})$ be an element of the Lorentz group. Then it is immediate that

$$D \circ g^{-1} = g^{-1} \circ D', \quad D' = \gamma'_\mu \partial_\mu, \qquad \gamma'_\mu = \sum_\nu \ell_{\mu\nu}\gamma_\nu.$$

Since

$$D'^2 = (g \circ D \circ g^{-1})^2 = D^2$$

it follows that

$$\gamma'_\mu = S(g)\gamma_\mu S(g)^{-1}$$

for all μ, $S(g)$ being an invertible 4×4 matrix determined uniquely up to a scalar multiple. Thus

$$S : g \longmapsto S(g)$$

is a *projective* representation of the Lorentz group and can be viewed as an ordinary representation of the universal covering group of the Lorentz group, namely $H = \mathrm{SL}(2, \mathbf{C})$. The action of H on the 4-component functions is thus

$$\psi \longmapsto \psi^g := S(g)\psi \circ g^{-1}$$

and the Dirac operator is invariant under this action:

$$D\psi^g = (D\psi)^g.$$

From the algebraic point of view one has to introduce the universal algebra C over $\mathbf{C}$ generated by the symbols γ_μ with relations (2) and study its representations. If we work over $\mathbf{C}$ we can forget the signs $\pm$ and take the relations between the γ_μ in the form

$$\gamma_\mu^2 = 1,\ \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 0\ (\mu \neq \nu).$$

Dirac's result is then essentially that C has a unique irreducible representation, which is in dimension 4, and that any representation is a sum of copies of this one. Moreover, there is an action S of the group H on this representation space that is compatible with the action of the Lorentz group as automorphisms of C. S is the spin representation.

The Clifford algebra, as the algebra over $\mathbf{R}$ with n generators

$$e_1, e_2, \ldots, e_n$$

and relations

$$e_r^2 = -1, \qquad e_re_s + e_se_r = 0\ (r \neq s)$$

goes back to a paper of Clifford[2] in 1878 where it is viewed as a generalization of the quaternion algebra (for $n = 2$ it *is* the quaternion algebra). Their deeper significance became clear only after Dirac's discovery[3] of the spin representation, but only in dimensions 3 and 4. In 1935, R. Brauer and H. Weyl wrote a seminal paper[4] in which they studied various questions concerning the spinors and spin representations over the real and complex field but in arbitrary dimensions and in the definite and Minkowski signatures. The geometric aspects of spinors were treated by Cartan in a book[5] published in 1938. The general algebraic study of spinors in *arbitrary* fields was carried out by C. Chevalley in his book[6]. The theory of spinors in arbitrary dimensions but for positive definite quadratic forms was developed in a famous paper of Atiyah, Bott, and Shapiro[7] where they carried out many applications. In recent years, with the increasing interest of physicists in higher dimensional spacetimes, spinors in arbitrary dimensions and *arbitrary signatures* have come to the foreground.

The foundation of the theory of spinors lies in the theory of Clifford algebras and their representations. We do this in §2. In §3 we take up the theory of spin groups and spin representations; the key here is to view the spin group as embedded in the group of units of the even part of the Clifford algebra and to view the spin representations as modules for it. In §4 we study reality questions concerning the spin modules which are critical for applications in physics. Here we follow Deligne[8] and obtain the basic results as consequences of the theory of super Brauer groups, generalizing the classical theory of the ordinary Brauer group of a field. The theory is developed over an arbitrary field of characteristic 0 but we also include a shorter treatment based on[7] in which the main results on the reality of the spin representations are obtained more quickly. The concern in §5 is with pairings between spin modules and the vector and other exterior modules of the orthogonal group. The last section is an appendix where we discuss some well-known properties of orthogonal groups including Cartan's theorem that the reflections generate the orthogonal groups.

Our treatment leans heavily on that of Deligne[8]. One of its highlights is the study of the Clifford algebras and their representations from the point of view of the super category. This makes the entire theory extremely transparent. For those who are familiar with the physicists' language and formalism the paper of Regge[9] is a useful reference.

5.2. Clifford algebras and their representations. Tensors are objects functorially associated to a vector space. If V is a finite dimensional vector space and

$$T^{r,s} = V^{*\otimes r} \otimes V^{\otimes s}$$

then the elements of $T^{r,s}$ are the tensors of rank (r, s). V is regarded as a module for $\mathrm{GL}(V)$ and then $T^{r,s}$ becomes also a module for $\mathrm{GL}(V)$. Spinors on the other hand are in a much more subtle relationship with the basic vector space. In the first place, the spinor space is attached only to a vector space with a metric. Let us define a *quadratic vector space* to be a pair (V, Q) where V is a finite dimensional vector space over a field k of characteristic 0 and Q a nondegenerate quadratic form. Here a quadratic form is a function such that

$$Q(x) = \Phi(x, x)$$

where Φ is a symmetric bilinear form, with nondegeneracy of Q defined as the nondegeneracy of Φ. Thus

$$Q(x + y) = Q(x) + Q(y) + 2\Phi(x, y).$$

Notice that our convention, which is the usual one, differs from that of Deligne[8] where he writes Φ for our 2Φ. A quadratic subspace of a quadratic vector space (V, Q) is a pair (W, Q_W) where W is a subspace of V and Q_W is the restriction of Q to W, with the assumption that Q_W is nondegenerate. For quadratic vector spaces $(V, Q), (V', Q')$ let us define the quadratic vector space $(V \oplus V', Q \oplus Q')$ by

$$(Q \oplus Q')(x + x') = Q(x) + Q(x')(x \in V, x' \in V').$$

Notice that V and V' are orthogonal in $V \oplus V'$. Thus for a quadratic subspace W of V we have $V = W \oplus W^\perp$ as quadratic vector spaces. Given a quadratic vector space (V, Q) or V in brief, we have the orthogonal group $\mathrm{O}(V)$, the subgroup of $\mathrm{GL}(V)$ preserving Q, and its subgroup $\mathrm{SO}(V)$ of elements of determinant 1. If $k = \mathbf{C}$ and $\dim(V) \geq 3$, the group $\mathrm{SO}(V)$ is not simply connected, and $\mathrm{Spin}(V)$ is its universal cover which is actually a double cover. The spinor spaces carry certain special irreducible representations of Spin (V). Thus, when the space V undergoes a transformation $\in \mathrm{SO}(V)$ and $g^\sim$ is an element of $\mathrm{Spin}(V)$ above g, the spinor space undergoes the transformation corresponding to $g^\sim$. The spinor space is however *not* functorially attached to V. Indeed, when (V, Q) varies, the spinor spaces do not vary in a natural manner unless additional assumptions are made (existence of spin structures). This is the principal difficulty in dealing with spinors globally on manifolds. However, in this chapter we shall not treat global aspects of spinor fields on manifolds.

The *Clifford algebra* $C(V, Q) = C(V)$ of the quadratic vector space (V, Q) is defined as the associative algebra generated by the vectors in V with the relations

$$v^2 = Q(v)1 \qquad (v \in V).$$

The definition clearly imitates the Dirac definition (1) in dimension 4. The relations for the Clifford algebra are obviously equivalent to

$$xy + yx = 2\Phi(x, y)1 \qquad (x, y \in V).$$

Formally, let $T(V)$ be the tensor algebra over V, i.e.,

$$T(V) = \bigoplus_{r \geq 0} V^{\otimes r}$$

where $V^0 = k1$ and multiplication is $\otimes$. If

$$t_{x,y} = x \otimes y + y \otimes x - 2\Phi(x, y)1 \qquad (x, y \in V)$$

then

$$C(V) = T(V)/I$$

where I is the two-sided ideal generated by the elements $t_{x,y}$. If $(e_i)_{1\le i\le n}$ is a basis for V, then $C(V)$ is generated by the e_i and is the algebra with relations

$$e_ie_j + e_je_i = 2\Phi(e_i, e_j)\ (i, j = 1, 2, \ldots, n).$$

The tensor algebra $T(V)$ is graded by $\mathbf{Z}$ but this grading does not descend to $C(V)$ because the generators $t_{x,y}$ are *not* homogeneous. However if we consider the coarser $\mathbf{Z}_2$–grading of $T(V)$ where all elements spanned by tensors of even (odd) rank are regarded as even (odd), then the generators $t_{x,y}$ are even and so this grading descends to the Clifford algebra. Thus $C(V)$ is a *super algebra.* The point of view of super algebras may therefore be applied systematically to the Clifford algebras. Some of the more opaque features of classical treatments of Clifford algebras arise from an insistence on treating the Clifford algebra as an *ungraded* algebra. We shall see below that the natural map $V \longrightarrow C(V)$ is injective and so we may (and shall) identify V with its image in $C(V)$: $V \subset C(V)$ and the elements of V are odd.

Since $C(V)$ is determined by Q the subgroup of $\mathrm{GL}(V)$ preserving Q clearly acts on $C(V)$. This is the orthogonal group $\mathrm{O}(V)$ of the quadratic vector space V. For any element $g \in \mathrm{O}(V)$ the induced action on the tensor algebra T descends to an automorphism of $C(V)$.

The definition of the Clifford algebra is compatible with base change; if $k \subset k'$ and $V_{k'} := k' \otimes_k V$, then

$$C(V_{k'}) = C(V)_{k'} := k' \otimes_k C(V).$$

Actually the notions of quadratic vector spaces and Clifford algebras defined above may be extended to the case when k is any commutative ring with unit element in which 2 is invertible. The compatibility with base change remains valid in this general context. We shall however be concerned only with the case when k is a field of characteristic 0.

By an orthonormal (ON) basis for V we mean a basis (e_i) such that

$$\Phi(e_i, e_j) = \delta_{ij}.$$

If we only have the above for $i \ne j$ we speak of an orthogonal basis; in this case $Q(e_i) \ne 0$ and $e_ie_j + e_je_i = 2Q(e_i)\delta_{ij}$. For such a basis, if k is algebraically closed, there is always an ON basis. So in this case there is essentially only one Clifford algebra C_m for each dimension m. If k is not algebraically closed, there are many Clifford algebras. For instance let $k = \mathbf{R}$. Then any quadratic vector space (V, Q) over $\mathbf{R}$ is isomorphic to $\mathbf{R}^{p,q}$ where p, q are integers ≥ 0, and $\mathbf{R}^{p,q}$ is the vector space $\mathbf{R}^{p+q}$ with the metric

$$Q(x) = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2.$$

The numbers p, q are invariants of (V, Q) and we refer to either (p, q) or $p - q$ as the *signature* of V or Q. Thus, for $k = \mathbf{R}$, we have, as ungraded algebras,

$$C(\mathbf{R}^{0,1}) \simeq \mathbf{C}, \quad C(\mathbf{R}^{1,0}) \simeq \mathbf{R} \oplus \mathbf{R} \quad C(\mathbf{R}^{0,2}) \simeq \mathbf{H} \quad C(\mathbf{R}^{1,1}) \simeq M^2(\mathbf{R})$$

where $\mathbf{H}$ is the algebra of quaternions and $M^2(\mathbf{R})$ is the 2×2 matrix algebra over $\mathbf{R}$.

Basic elementary properties. Some of the basic elementary properties of Clifford algebras are as follows. For general k, $C(V)$ has dimension $2^{\dim(V)}$, and if $\dim(V) = n$, then the elements

$$1, e_I = e_{i_1} e_{i_2} \ldots e_{i_r} \qquad (I = \{i_1, \ldots, i_r\}\ i_1 < \ldots < i_r,\ 1 \leq r \leq n)$$

form a basis for $C(V)$. If we change Q to $-Q$, we obtain $C(V)^{\mathrm{opp}}$, the algebra opposite to $C(V)$:

$$C(V, -Q) \simeq C(V)^{\mathrm{opp}}. \tag{3}$$

Notice here that we are speaking of opposite algebras in the super category. Let V, V' be quadratic vector spaces. We then have the important relation

$$C(V \oplus V') = C(V) \otimes C(V') \tag{4}$$

as super algebras, *the tensor product being taken in the category of super algebras.* We remark that this relation is *not true* if the tensor product algebra is the usual one in ungraded algebras; indeed, as V and V' are orthogonal, their elements *anticommute* in $C(V \oplus V')$ but in the ordinary tensor product they will have to commute. This is again an indication that it is essential to treat the Clifford algebras as objects in the category of super algebras.

We shall first establish (4). If A is an associative algebra with unit and (W, R) is a quadratic vector space, then in order that a linear map $L(W \longrightarrow A)$ extend to a map $C(W) \longrightarrow A$ it is necessary and sufficient that $L(w)^2 = R(w)1$ for all $w \in A$, and that for A a super algebra, this is a map of super algebras if $L(w)$ is odd for all $w \in W$. Let

$$(W, R) = (V, Q) \oplus (V', Q'), \quad A = C(V) \otimes C(V'), \quad L(v \oplus v') = v \otimes 1 + 1 \otimes v'.$$

Since v, v' are odd, $(1 \otimes v')(v \otimes 1) = -v \otimes v'$, and so we have

$$(v \otimes 1 + 1 \otimes v')^2 = R(v \oplus v')1$$

so that L extends to a map of $C(V \oplus V')$ into $C(V) \otimes C(V')$. To set up the inverse map note that the inclusions $V, V' \subset V \oplus V'$ give even maps h, h' of $C(V), C(V') \longrightarrow C(V \oplus V')$ and hence a linear map $a \otimes a' \longmapsto h(a)h'(a')$ of $C(V) \otimes C(V')$ into $C(V \oplus V')$. Since h, h' preserve parity, this map will be a morphism of super algebras if for $a, b \in C(V)$ and $a', b' \in C(V')$ we can show that

$$h(b)h'(a') = (-1)^{p(b)p(a')} h'(a')h(b).$$

This comes down to showing that for $v_i \in V, v'_j \in V'$ we have

$$v_1 \ldots v_r v'_1 \ldots v'_s = (-1)^{rs} v'_1 \ldots v'_s v_1 \ldots v_r$$

in $C(V \oplus V')$. This is obvious since, by definition, v_i and v'_j anticommute in $V \oplus V'$. It is trivial to check that the two maps thus constructed are inverses of each other; indeed, the compositions in either order are the identities at the level of the vectors and so are the identities everywhere. Thus (4) is proved.

At this stage we can conclude that $C(V)$ has dimension 2^n where $n = \dim(V)$. In fact, if V has dimension 1 and v is nonzero in V with $Q(v) = a \neq 0$, then $C(V)$ is the span of 1 and v so that it has dimension 2; for arbitrary V of dimension n it follows from (4) that $C(V)$ has dimension 2^n. In particular, if $(e_i)_{1 \leq i \leq n}$ is a basis of V, then

$$1, e_I = e_{i_1} e_{i_2} \ldots e_{i_r} \qquad (I = \{i_1, \ldots, i_r\} \; i_1 < \ldots < i_r, \; 1 \leq r \leq n)$$

form a basis for $C(V)$. This implies at once that the natural map $V \longrightarrow C(V)$ is injective so that we shall assume from now on that $V \subset C(V)$.

We shall now prove (3). The identity map of V lifts to a morphism of $T(V)$ onto $C(V)^{\mathrm{opp}}$ as super algebras. We claim that this lift vanishes on the kernel of $T(V) \longrightarrow C(V^-)$ where we write V^- for $(V, -Q)$. It is enough to show that for $x \in V$, the image of $x \otimes x + Q(x)1$ in $C(V)^{\mathrm{opp}}$ is 0. But this image is the element $-x^2 + Q(x)1$ in $C(V)$ and so is 0. Thus we have a surjective morphism $C(V^-) \longrightarrow C(V)^{\mathrm{opp}}$. Since the dimensions are equal this is an isomorphism.

The Clifford algebra and the exterior algebra. The Clifford algebra is filtered in a natural way because the tensor algebra which sits above it is filtered by rank of tensors. Thus $C = C(V)$ acquires the filtration (C_r) where C_r is the span of elements of the form $v_1 \dots v_s$ where $v_i \in V$ and $s \le r$. Let C^{gr} be the associated graded algebra. Clearly $C_1^{\text{gr}} = V$. If $v \in V$, then $v^2 \in C_0$ and so $v^2 = 0$ in C^{gr}. Hence we have a homomorphism of C^{gr} onto the exterior algebra $\Lambda(V)$ preserving degrees, which is an isomorphism because both spaces have dimension $2^{\dim(V)}$. Thus

$$C^{\text{gr}} \simeq \Lambda(V) \qquad \text{(as graded algebras)}.$$

It is possible to construct a map in the reverse direction going from the exterior algebra to the Clifford algebra, the so-called skewsymmetrizer map

$$\lambda : v_1 \wedge \ldots \wedge v_r \longmapsto \frac{1}{r!} \sum_{\sigma} \varepsilon(\sigma) v_{\sigma(1)} \ldots v_{\sigma(r)}$$

where the sum is over all permutations σ of $\{1, 2, \dots, r\}$, $\varepsilon(\sigma)$ is the sign of σ, and the elements on the right side are multiplied as elements of $C(V)$. Indeed, the right side above is skewsymmetric in the v_i and so by the universality of the exterior power, the map λ is well-defined. If we choose a basis (e_i) of V such that the e_i are mutually orthogonal, the elements $e_{i_1} \dots e_{i_r}$ are clearly in the range of λ so that λ is surjective, showing that

$$\lambda : \Lambda(V) \simeq C(V)$$

is a linear isomorphism. If we follow λ by the map from C^r to C^{gr} we obtain the isomorphism of $\Lambda(V)$ with C^{gr} that inverts the earlier isomorphism. The definition of λ makes it clear that it commutes with the action of $\mathrm{O}(V)$ on both sides. Now $\Lambda(V)$ is the universal enveloping algebra of V treated as a purely odd Lie super algebra, and so λ is analogous to the symmetrizer isomorphism of the symmetric algebra of a Lie algebra with its universal enveloping algebra.

Center and super center. For any super algebra A its *super center* $\mathrm{sctr}(V)$ is the sub super algebra whose homogeneous elements x are defined by

$$xy - (-1)^{p(x)p(y)} yx = 0 \qquad (y \in A).$$

This can be very different from the center $\mathrm{ctr}(V)$ of A regarded as an ungraded algebra. Notice that both $\mathrm{sctr}(V)$ and $\mathrm{ctr}(V)$ are themselves super algebras.

Proposition 5.2.1. *We have the following.*

(i) $\mathrm{sctr}(C(V)) = k1$.

(ii) $\mathrm{ctr}(C(V)) = k1$ *if* $\dim(V)$ *is even.*

(iii) *If* $\dim(V) = 2m+1$ *is odd then* $\mathrm{ctr}(C(V))$ *is a super algebra of dimension* $1|1$*; if* ε *is a nonzero odd element of it, then* $\varepsilon^2 = a \in k \setminus (0)$ *and* $\mathrm{ctr}(V) =$

$k[\varepsilon]$. In particular it is a super algebra all of whose nonzero homogeneous elements are invertible and whose super center is k. If $(e_i)_{0 \le i \le 2m}$ is an orthogonal basis for V, then we can take $\varepsilon = e_0 e_1 \dots e_{2m}$. If further $e_i^2 = \pm 1$, then $\varepsilon^2 = (-1)^{m+q} 1$ where q is the number of i's for which $e_i^2 = -1$.

Proof. Select an orthogonal basis $(e_i)_{1 \le i \le n}$ for V. If $I \subset \{1, 2, \dots, n\}$ is nonempty, then

$$e_I e_j = \alpha_{I,j} e_j e_I \text{ where } \alpha_{I,j} = \begin{cases} -(-1)^{|I|} & (j \in I) \\ (-1)^{|I|} & (j \notin I). \end{cases}$$

Let $x = \sum_I a_I e_I$ be a homogeneous element in the super center of $C(V)$ where the sum is over I with parity of I being same as $p(x)$. The above formulae and the relations $x e_j = (-1)^{p(x)} e_j x$ imply, remembering that e_j is invertible,

$$(\alpha_{I,j} - (-1)^{p(x)}) a_I = 0.$$

If we choose $j \in I$, then, $\alpha_{I,j} = -(-1)^{p(x)}$, showing that $a_I = 0$. This proves (i). To prove (ii) let x above be in the center. We now have $x e_j = e_j x$ for all j. Then, as before,

$$(\alpha_{I,j} - 1) a_I = 0.$$

So $a_I = 0$ whenever we can find a j such that $\alpha_{I,j} = -1$. Thus $a_I = 0$ except when $\dim(V) = 2m+1$ is odd and $I = \{0, 1, \dots, 2m\}$. In this case $\varepsilon = e_0 e_1 \dots e_{2m}$ commutes with all the e_j and so lies in $\mathrm{ctr}(V)$. Hence $\mathrm{ctr}(V) = k[\varepsilon]$. A simple calculation shows that

$$\varepsilon^2 = (-1)^m Q(e_0) \dots Q(e_{2m})$$

from which the remaining assertions follow at once.

Remark. The center of $C(V)$ when V has odd dimension is an example of a super division algebra. A *super division algebra* is a super algebra whose nonzero *homogeneous* elements are invertible. If $a \in k$ is nonzero, then $k[\varepsilon]$ with ε odd and $\varepsilon^2 = a1$ is a super division algebra since ε is invertible with inverse $a^{-1}\varepsilon$.

Proposition 5.2.2. *Let $\dim(V) = 2m+1$ be odd and let $D = \mathrm{ctr}(V)$. Then*

$$C(V) = C(V)^+ D \simeq C(V)^+ \otimes D$$

as super algebras. Moreover, let $e_0 \in V$ be such that $Q(e_0) \neq 0$, $W = e_0^\perp$, and Q' be the quadratic form $-Q(e_0) Q_W$ on W; let $W' = (W, Q')$. Then

$$C(V)^+ \simeq C(W')$$

as ungraded algebras.

Proof. Let $(e_i)_{0 \le i \le 2m}$ be an orthogonal basis for V so that $e_1, \ldots, e_{2m}$ is an orthogonal basis for W. Let $\varepsilon = e_0 \ldots e_{2m}$ so that $D = k[\varepsilon]$. In the proof r, s vary from 1 to $2m$. Write $f_r = e_0 e_r$. Then $f_r f_s = -Q(e_0) e_r e_s$ so that the f_r generate $C(V)^+$. If $\gamma_p \in C(V)^+$ is the product of the $e_j (j \neq p)$ in some order, $\gamma_p \varepsilon = c e_p$ where $c \neq 0$, and so D and $C(V)^+$ generate $C(V)$. By looking at dimensions we then have the first isomorphism. For the second note that $f_r f_s + f_s f_r = 0$ when $r \neq s$ and $f_r^2 = -Q(e_0) Q(e_r)$, showing that the f_r generate the Clifford algebra over W'.

Structure of Clifford algebras over algebraically closed fields. We shall now examine the structure of $C(V)$ and $C(V)^+$ when k is algebraically closed. Representations of $C(V)$ are morphisms into $\mathbf{End}(U)$ where U is a super vector space.

The even dimensional case. The basic result is the following.

Theorem 5.2.3. *Let k be algebraically closed. If $\dim(V) = 2m$ is even, $C(V)$ is isomorphic to a full matrix super algebra. More precisely,*

$$C(V) \simeq \mathbf{End}(S) \qquad \dim(S) = 2^{m-1} | 2^{m-1}.$$

This result is true even if k is not algebraically closed provided $(V, Q) \simeq (V_1, Q_1) \oplus (V_1, -Q_1)$.

This is a consequence of the following theorem.

Theorem 5.2.4. *Suppose that k is arbitrary and $V = U \oplus U^*$ where U is a vector space with dual U^*. Let*

$$Q(u + u^*) = \langle u, u^* \rangle \qquad (u \in U, u^* \in U^*).$$

Let $S = \Lambda U^*$ be the exterior algebra over U^*, viewed as a super algebra in the usual manner. Then S is a $C(V)$–module for the actions of U and U^* given by

$$\mu(u^*) : \ell \longmapsto u^* \wedge \ell, \qquad \partial(u) : \ell \longmapsto \partial(u)\ell \qquad (\ell \in S)$$

where $\partial(u)$ is the odd derivation of S that is characterized by $\partial(u)(u^*) = \langle u, u^* \rangle$. Moreover the map $C(V) \longrightarrow \mathbf{End}(S)$ defined by this representation is an isomorphism.

Theorem 5.2.4 $\Longrightarrow$ Theorem 5.2.3. If k is algebraically closed we can find an ON basis $(e_j)_{1\le j\le 2m}$. If $f_r^{\pm} = 2^{-1/2}[e_r \pm i e_{m+r}](1 \le r \le m)$, then

$$\Phi(f_r^{\pm}, f_s^{\pm}) = 0, \qquad \Phi(f_r^{\pm}, f_s^{\mp}) = \delta_{rs}. \tag{$*$}$$

Let $U^{\pm}$ be the subspaces spanned by $(f_r^{\pm})$. We take $U = U^+$ and identify U^- with U^* in such a way that

$$\langle u^+, u^- \rangle = 2\Phi(u^+, u^-) \qquad (u^{\pm} \in U^{\pm}).$$

Then

$$Q(u^+ + u^-) = \langle u^+, u^- \rangle$$

for $u^{\pm} \in U^{\pm}$, and we can apply Theorem 5.2.4. If k is not algebraically closed but $(V, Q) = (V_1, Q_1) \oplus (V_1, -Q_1)$, we can find a basis $(e_j)_{1\le j\le 2m}$ for V such that the e_j are mutually orthogonal, $(e_j)_{1\le j\le m}$ span $V_1 \oplus 0$ while $(e_{m+j})_{1\le j\le m}$ span $0 \oplus V_1$, and $Q(e_j) = -Q(e_{m+j}) = a_j \neq 0$. Let $f_r^+ = e_r + e_{m+r}, f_r^- = (2a_r)^{-1}(e_r - e_{m+r})$. Then the relations $(*)$ are again satisfied and so the argument can be completed as before.

Proof of Theorem 5.2.4. It is clear that $\mu(u^*)^2 = 0$. On the other hand $\partial(u)^2$ is an even derivation which annihilates all u^* and so is 0 also. We regard S as $\mathbf{Z}_2$–graded in the obvious manner. It is a simple calculation that

$$\mu(u^*)\partial(u) + \partial(u)\mu(u^*) = \langle u, u^* \rangle 1 \qquad (u \in U, u^* \in U^*).$$

Indeed, for $g \in S$, by the derivation property, $\partial(u)\mu(u^*)g = \partial(u)(u^*g) = \langle u, u^*\rangle g - \mu(u^*)\partial(u)g$ which gives the above relation. This implies at once that

$$(\partial(u) + \mu(u^*))^2 = Q(u + u^*)1$$

showing that

$$r : u + u^* \longmapsto \partial(u) + \mu(u^*)$$

extends to a representation of $C(V)$ in S. Notice that the elements of V act as odd operators in S and so r is a morphism of $C(V)$ into $\mathbf{End}(S)$.

We shall now prove that r is surjective as a morphism of ungraded algebras; this is enough to conclude that r is an isomorphism of super algebras since $\dim(C(V)) = 2^{2\dim(U^*)} = \dim(\mathbf{End}(S))$ where all dimensions are of the ungraded vector spaces. Now, if A is an associative algebra of endomorphisms of a vector space acting irreducibly on it, and its commutant, namely the algebra of endomorphisms commuting with A, is the algebra of scalars $k1$, then by Wedderburn's theorem, A is the algebra of all endomorphisms of the vector space in question. We shall now prove that r is irreducible and has scalar commutant. Let (u_i) be a basis of U and u_j^* the dual basis of U^*.

The proof of the irreducibility of r depends on the fact that if L is a subspace of S invariant under all $\partial(u)$, then $1 \in L$. If L is $k1$ this assertion in trivial; otherwise let $g \in L$ be not a scalar; then, replacing g by a suitable multiple of it we can write

$$g = u_I^* + \sum_{J \neq I, |J| \leq |I|} a_J u_J^* \qquad I = \{i_1, \ldots, i_p\}, \quad p \geq 1.$$

As

$$\partial(u_{i_p}) \ldots \partial(u_{i_1}) g = 1$$

we see that $1 \in L$. If now L is invariant under $C(V)$, applying the operators $\mu(u^*)$ to 1 we see that $L = S$. Thus S is irreducible. Let T be an endomorphism of S commuting with r. The proof that T is a scalar depends on the fact that the vector $1 \in S$, which is annihilated by all $\partial(u)$, is characterized (projectively) by this property. For this it suffices to show that if $g \in S$ has no constant term, then for some $u \in U$ we must have $\partial(u)g \neq 0$. If $g = \sum_{|I| \geq p} a_I u_I^*$ where $p \geq 1$ and some a_J with $|J| = p$ is nonzero, then $\partial(u_j)g \neq 0$ for $j \in J$. This said, since $\partial(u_i)T1 = T\partial(u_i)1 = 0$ we see that $T1 = c1$ for some $c \in k$. So replacing T by $T - c1$ we may assume that $T1 = 0$. We shall now prove that $T = 0$. Let $T1 = v^*$. Then, as T commutes with all the $\mu(u^*)$, we have, $Tu^* = u^* \wedge v^*$ for all $u^* \in U^*$. So it is a question of proving that v^* is 0. Since T commutes with $\partial(u)$ we have, for all $u \in U$,

$$\partial(u)Tu^* = T\langle u, u^* \rangle = \langle u, u^* \rangle v^*$$

while we also have

$$\partial(u)Tu^* = \partial(u)(u^* \wedge v^*) = \langle u, u^* \rangle v^* - u^* \wedge \partial(u)v^*.$$

Hence

$$u^* \wedge \partial(u)v^* = 0 \qquad (u \in U, u^* \in U^*).$$

Fixing u and writing $w^* = \partial(u)v^*$, we see that $u^* \wedge w^* = 0$ for all $u^* \in U^*$. A simple argument shows that the only elements that are killed by $\mu(u^*)$ for all $u^* \in U^*$ are the multiples of the element of the highest degree in S†. But $w^* = \partial(u)v^*$ is definitely a linear combination of elements of degree $< \dim(U^*)$. Hence $\partial(u)v^* = 0$. As u is arbitrary, we must have $v^* = c1$ for some constant c. Then $Tu^* = cu^*$ for

† This is dual to the earlier characterization of $k1$ as the common null space of all the $\partial(u)$.

all u^* and as $T1 = 0$ we must have $c = 0$ so that $T = 0$. This finishes the proof that r maps $C(V)$ onto $\mathbf{End}(S)$.

Remark 1. Write $V = U \oplus U^*$ as a direct sum of $V_i = U_i \oplus U_i^* (i = 1, 2)$ where $\dim(U_i) \neq 0$. Then

$$C(V) \simeq C(V_1) \otimes C(V_2)$$

while an easy calculation shows that

$$r = r_1 \otimes r_2$$

where r_i is the representation of $C(V_i)$ defined above. Induction on m then reduces the surjectivity of r to the case when $\dim(U) = \dim(U^*) = 1$ where it is clear from an explicit calculation. The proof given here, although longer, reveals the structure of S in terms of the operators of multiplication and differentiation which are analogous to the creation and annihilation operators in Fock space. In fact the analogy goes deeper and is discussed in the next remark.

Remark 2: The analogy with the Schrödinger representation. There is an analogy of the Clifford algebra with the Heisenberg algebra which makes the representation r the *fermionic analogue to the Schrödinger representation.* If V is an even vector space with a symplectic form Φ then the Heisenberg algebra $H(V)$ associated to (V, Φ) is the algebra generated by the commutation rules

$$xy - yx = 2\Phi(x, y)1 \qquad (x, y \in V). \tag{H}$$

For any symplectic Φ we can *always* write $V = U \oplus U^*$ with Φ vanishing on $U \times U$ and $U^* \times U^*$ and $2\Phi(u, u^*) = \langle u, u^* \rangle$. The algebraic representation of $H(V)$ is constructed on the *symmetric algebra* $\mathrm{Symm}(U^*)$ with u^* acting as the operator of multiplication by u^* and u acting as the (even) derivation $\partial(u)$. The splitting $V = U \oplus U^*$ is usually called a *polarization* of V. The commutation rule (H) is the bosonic analogue of the fermionic rule

$$xy + yx = 2\Phi(x, y)1 \tag{C}$$

which defines the Clifford algebra. The analogy with the Clifford situation is now obvious. Unlike in the bosonic case, the polarization does not always exist in the fermionic case but will exist if k is algebraically closed. The vector 1 is called the *Clifford vacuum* by physicists. Notice that it is canonical *only after a polarization is chosen.* Indeed, there can be *no distinguished line in* S; otherwise S would be attached *functorially* to V and there would be no need to consider spin structures.

Remark 3. For any field k the quadratic vector spaces of the form $(V_1, Q_1) \oplus (V_1, -Q_1)$ are called *hyperbolic.* When k is real these are precisely the quadratic vector spaces $\mathbf{R}^{m,m}$ of signature 0.

From the fact that the Clifford algebra of an even dimensional quadratic space is a full matrix super algebra follows its simplicity. Recall the classical definition that an algebra is *simple* if it has no proper nonzero two-sided ideal. It is classical that full matrix algebras are simple. We have, from the theorems above, the following corollary.

Corollary 5.2.5. *For arbitrary k, if V is even dimensional, then $C(V)$ is simple as an ungraded algebra.*

Proof. $C(V)$ is simple if it stays simple when we pass to the algebraic closure $\overline{k}$ of k. So we may assume that k is algebraically closed. The result then follows from the fact that the ungraded Clifford algebra is a full matrix algebra.

Classically, the algebra $E(V)$ of all endomorphisms of a vector space V has the property that V is its only simple module and all its modules are direct sums of copies of V, so that any module is of the form $V \otimes W$ for W a vector space. We wish to extend this result to the super algebra $\mathbf{End}(V)$ of any super vector space. In particular such a result would give a description of all modules of a Clifford algebra $C(V)$ for V even dimensional and k algebraically closed.

We consider finite dimensional modules of finite dimensional super algebras. Submodules are defined by invariant sub super vector spaces. If A, B are super algebras and V, W are modules for A and B respectively, then $V \otimes W$ is a module for $A \otimes B$ by the action

$$a \otimes b : v \otimes w \longmapsto (-1)^{p(b)p(v)} av \otimes bw.$$

In particular, if $B = k$, $V \otimes W$ is a module for A where A acts only on the first factor. Imitating the classical case we shall say that a super algebra A is *semisimple* if all its modules are completely reducible, i.e., direct sums of simple modules. Here, by a simple module for a super algebra we mean an irreducible module, namely one with no nontrivial proper submodule. If a module for A is a sum of simple modules it is then a *direct sum* of simple modules; indeed, if $V = \sum_j V_j$ where the V_j are simple submodules, and (U_i) is a maximal subfamily of linearly independent members of the family (V_j), and if $U = \oplus U_i \neq V$, then for some j, we must have $V_j \not\subset U$, so that, by the simplicity of V_j, $V_j \cap U = 0$, contradicting the maximality of (U_i). In particular a quotient of a direct sum of simple modules is a direct sum of simple modules. Now any module is a sum of cyclic modules generated by homogeneous elements, and a cyclic module is a quotient of the module defined by the left regular representation. Hence A is semisimple if and only if the left regular representation of A is completely reducible, and then any module is a direct sum of simple modules that occur in the decomposition of the left regular representation.

With an eye for later use let us discuss some basic facts about semisimplicity and base change. The basic fact is that if A is a super algebra, M is a module for A, and k'/k is a Galois extension (possibly of infinite degree), then M is semisimple for A if and only if $M' := k' \otimes_k M$ is semisimple for $A' := k' \otimes_k A$. This is proved exactly as in the classical case. In physics we need this only when $k = \mathbf{R}$ and $k' = \mathbf{C}$. For the sake of completeness we sketch the argument. Let $G = \mathrm{Gal}(k'/k)$. Then elements of G operate in the usual manner ($c \otimes m \mapsto c^g \otimes m$) on M' and the action preserves parity. To prove that the semisimplicity of M implies that of M' we may assume that M is simple. If $L' \subset M'$ is a simple submodule for A', then $\sum_{g \in G} L'^g$ is G–invariant and so is of the form $k' \otimes_k L$ where $L \subset M$ is a submodule. So $L = M$, showing that M' is semisimple, being a span of the simple modules L'^g. In the reverse direction it is a question of showing that if $L'_1 \subset M'$ is a G-invariant submodule, there exists a G-invariant complementary submodule L'_2. It is enough to find an even map $f \in \mathbf{End}_{k'}(M')$ commuting with A and G such that

$$f(M') \subset L'_1, \qquad f(\ell') = \ell' \text{ for all } \ell' \in L'_1. \tag{$*$}$$

We can then take L'_2 to be the kernel of f. By the semisimplicity of M' we can find even f_1 satisfying $(*)$ and commuting with A; indeed, if L''_2 is a complementary submodule to L'_1, we can take f_1 to be the projection $M \longrightarrow L'_1 \bmod L''_2$. Now f_1 is *defined over a finite Galois extension* k''/k and so if $H = \mathrm{Gal}(k''/k)$ and

$$f = \frac{1}{|H|} \sum_{h \in H} h f_1 h^{-1},$$

then f commutes with A and H and satisfies $(*)$. But, if $g \in G$ and h is the restriction of g to k'', then $gfg^{-1} = hfh^{-1} = f$ and so we are done. In particular, applying this result to the left regular representation of A we see that A is semisimple if and only if A' is semisimple.

It is also useful to make the following remark. Let A be a super algebra and S a module for A. Suppose M is a direct sum of copies of S. Then $M \simeq S \otimes W$ where W is a purely even vector space. To see this write $M = \oplus_{1 \le i \le r} M_i$ where $t_i : S \longrightarrow M_i$ is an isomorphism. Let W be a purely even vector space of dimension r with basis $(w_i)_{1 \le i \le r}$. Then the map

$$t : \sum_{1 \le i \le r} u_i \otimes w_i \longmapsto \sum_{1 \le i \le r} t_i(u_i)$$

is an isomorphism of $S \otimes W$ with M.

For any super vector space V, recall that ΠV is the super vector space with the same underlying vector space but with reversed parities, i.e., $(\Pi V)_0 = V_1, (\Pi V)_1 = V_0$. If V is a module for a super algebra A, so is ΠV. If V is simple, so is ΠV. Notice that the identity map $V \longrightarrow \Pi V$ is *not* a morphism in the super category since it is *parity reversing.* One can also view ΠV as $V \otimes k^{0|1}$. Let

$$E(V) = \mathbf{End}(V)$$

for any super vector space V. If $\dim V_i > 0$ $(i = 0, 1)$, then $E(V)^+$, the even part of $E(V)$, is isomorphic to the algebra of all endomorphisms of the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

and so is isomorphic to $E(V_0) \oplus E(V_1)$, and its center is isomorphic to $k \oplus k$. In particular, the center of $E(V)^+$ has two characters $\chi_i (i = 0, 1)$ where the notation is such that χ_1 is $(c_1, c_2) \longmapsto c_i$. So on V the center of $E(V)^+$ acts through (χ_1, χ_2) while on ΠV it acts through (χ_2, χ_1).

Proposition 5.2.6. *For k arbitrary the super algebra $E(V)$ has precisely two simple modules, namely V and ΠV. Every module for $E(V)$ is a direct sum of copies of either V or ΠV. In particular, $E(V)$ is semisimple and any module for $E(V)$ is of the form $V \otimes W$ where W is a super vector space.*

Proof. The ungraded algebra $E(V)$ is a full matrix algebra and it is classical that it is simple, V is its only simple module up to isomorphism, and any module is a direct sum of copies of V. The proposition extends these results to the super case where the same results are true except that we have to allow for parity reversal.

Let W be a simple module for $E(V)$. Since $E(V)$ is simple as an ungraded algebra, W is faithful, i.e., the kernel of $E(V)$ acting on W is 0. We first show that W is simple for $E(V)$ regarded as an ungraded algebra. Indeed, let U be a subspace stable under the ungraded $E(V)$. If $u = u_0 + u_1 \in U$ with $u_i \in W_i$, and we write any element of $E(V)$ as

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

then for $gu = v = v_0 + v_1$ we have $v_0 = Au_0 + Bu_1, v_1 = Cu_0 + Du_1$. Taking $A = I, B = C = D = 0$ we see that $u_0, u_1 \in U$. Hence U has to be graded and so $U = 0$ or V. Hence we have an isomorphism $t(W \longrightarrow V)$ as ungraded modules for the ungraded $E(V)$. Write $t = t_0 + t_1$ where $p(t_i) = i$. Then $t_0 a + t_1 a = a t_0 + a t_1$ for all $a \in E(V)$. As $p(at_0) = p(t_0 a) = p(a)$ and $p(at_1) = p(t_1 a) = 1 + p(a)$ we see that $at_0 = t_0 a$ and $at_1 = t_1 a$. If $t_0 \neq 0$, then t_0 is a nonzero element of $\mathrm{Hom}_{E(V)}(W, V)$ as super modules and so, by the simplicity of V and W, we must have that t_0 is an isomorphism. Thus $W \simeq V$. If $t_1 \neq 0$, then $t_1 \in \mathrm{Hom}(W, \Pi V)$ and we argue as before that $W \simeq \Pi V$. We have thus proved that a simple $E(V)$-module is isomorphic to either V or ΠV.

It now remains to prove that an arbitrary module for $E(V)$ is a direct sum of simple modules. As we have already observed, it is enough to do this for the left regular representation. Now there is an isomorphism

$$V \otimes V^* \simeq E(V), \qquad v \otimes v^* \longmapsto R_{v,v^*} : w \longmapsto v^*(w)v$$

of super vector spaces. If $L \in E(V)$, it is trivial to verify that $R_{Lv,v^*} = LR_{v,v^*}$, and so the above isomorphism takes $L \otimes 1$ to left multiplication by L in $E(V)$. Thus it is a question of decomposing $V \otimes V^*$ as a $E(V)$-module for the action $L \longmapsto L \otimes 1$. Clearly $V \otimes V^* = \oplus_{e^*} V \otimes ke^*$ where e^* runs through a homogeneous basis for V^*. The map $v \longmapsto v \otimes e^*$ is an isomorphism of the action of $E(V)$ on V with the action of $E(V)$ on $V \otimes ke^*$. But this map is even for e^* even and odd for e^* odd. So the action of $E(V)$ on $V \otimes ke^*$ is isomorphic to V for even e^* and to ΠV for odd e^*. Hence the left regular representation of $E(V)$ is a direct sum of r copies of V and s copies of ΠV if $\dim(V) = r|s$. The direct sum of r copies of V is isomorphic to $V \otimes W_0$ where W_0 is purely even of dimension r. Since $\Pi V \simeq V \otimes k^{0|1}$ the direct sum of s copies of ΠV is isomorphic to $V \otimes W_1$ where W_1 is a purely odd vector space of dimension s. Hence the left regular representation is isomorphic to $V \otimes W$ where $W = W_0 \oplus W_1$.

Theorem 5.2.7. *Let V be an even dimensional quadratic vector space. Then the Clifford algebra $C(V)$ is semisimple. Assume that either k is algebraically closed or k is arbitrary but V is hyperbolic. Then $C(V) \simeq \mathbf{End}(S)$, $C(V)$ has exactly two simple modules $S, \Pi S$, and any module for $C(V)$ is isomorphic to $S \otimes W$ where W is a super vector space.*

Proof. By Theorem 3 we know that $C(V)$ is isomorphic to $\mathbf{End}(S)$. The result is now immediate from the proposition above.

In which the vector space is odd dimensional. We shall now extend the above results to the case when V has odd dimension. Let D be the super division algebra $k[\varepsilon]$ where ε is odd and $\varepsilon^2 = 1$. We first rewrite Proposition 2 as follows.

Theorem 5.2.8. *Let* $\dim(V) = 2m+1$ *and let* k *be algebraically closed. Then* $\mathrm{ctr}(C(V)) \simeq D$ *and, for some purely even vector space* S_0 *of dimension* 2^m*,*

$$C(V) \simeq \mathrm{End}(S_0) \otimes D, \qquad C(V)^+ \simeq \mathrm{End}(S_0) \qquad (\dim(S_0) = 2^m).$$

Proof. If $(e_i)_{0 \le i \le 2m}$ is an ON basis for V and $\varepsilon = i^m e_0 e_1 \ldots e_{2m}$ where $i = (-1)^{1/2}$, then ε is odd, $\varepsilon^2 = 1$, and $\mathrm{ctr}(C(V)) = D = k[\varepsilon]$, by Proposition 1. The theorem is now immediate from Proposition 2 since $C(V)^+$ is isomorphic to the ungraded Clifford algebra in even dimension $2m$ and so is a full matrix algebra in dimension 2^m.

Let k be arbitrary and let U be an even vector space of dimension r. Write $E(U) = \mathrm{End}(U)$. Let A be the super algebra $E(U) \otimes D$ so that the even part A^+ of A is isomorphic to $E(U)$. We construct a simple (super) module S for A as follows. $S = U \oplus U$ where $S_0 = U \oplus 0$ and $S_1 = 0 \oplus U$ (or vice versa). $E(U)$ acts diagonally and ε goes to the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It is obvious that S is simple. Notice that S is not simple for the ungraded algebra underlying A since the diagonal (as well as the anti-diagonal) are stable under A. S can be written as $U \otimes k^{1|1}$ where A^+ acts on the first factor and D on the second with ε acting on $k^{1|1}$ by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The action of D on $k^{1|1}$ is also isomorphic to the left regular representation of D on itself.

Proposition 5.2.9. *Let* k *be arbitrary. Then,* S *is the unique simple module for* $A = E(U) \otimes D$ *where* U *is a purely even vector space over* k*. Any simple module for* A *is a direct sum of copies of* S *and so is isomorphic to* $S \otimes W$ *where* W *is a purely even vector space.*

From this is we get the following theorem.

Theorem 5.2.10. *If* V *is an odd dimensional quadratic vector space, then* $C(V)$ *is semisimple. For* k *algebraically closed,* $C(V) \simeq \mathrm{End}(S_0) \otimes D$ *has a unique simple module* $S = S_0 \otimes D$ *up to isomorphism; and any module of* $C(V)$ *is isomorphic to* $S \otimes W$ *where* W *is a purely even vector space.*

Proof. Theorem 10 follows from Proposition 9 and Theorem 8. It is therefore enough to prove Proposition 9.

Let T be a simple module for A. As $A^+ \simeq E(U)$, we have $T_0 \simeq aU, T_1 \simeq bU$ as A^+-modules for suitable integers $a, b \ge 0$. But the action of ε commutes with that of A^+ and $\varepsilon^2 = 1$, so that $\varepsilon(T_0 \longrightarrow T_1)$ is an isomorphism of A^+–modules. Hence we must have $a = b \geq 1$. But if R is a submodule of T_0, $R \oplus \varepsilon R$ is stable for A and so it has to equal T. Thus $a = b = 1$, showing that we can take $T_0 = T_1 = U$ and ε as $(x, y) \longmapsto (y, x)$. But then $T = S$. To prove that any module for A is a direct sum of copies of S it is enough (as we have seen already) to do this for the left regular representation. If $A^+ = \oplus_{1 \leq j \leq r} A_j$ where A_j as a left A^+–module is isomorphic to U, it is clear that $A_j \otimes D$ is isomorphic to $U \otimes k^{1|1} \simeq S$ as a module for A, and $A = \oplus_j (A_j \otimes D)$.

Representations of $C(V)^+$. We now obtain the representation theory of $C(V)^+$ over algebraically closed fields. Since this is an ungraded algebra the theory is classical and not super.

Theorem 5.2.11. *For any k, $C(V)^+$ is semisimple. Let k be algebraically closed. If $\dim(V) = 2m + 1$, $C(V)^+ \simeq \mathbf{End}(S_0)$ where S_0 is a purely even vector space of dimension 2^m, and so $C(V)^+$ has a unique simple module S_0. Let $\dim(V) = 2m$, let $C(V) \simeq \mathbf{End}(S)$ where $\dim(S) = 2^{m-1}|2^{m-1}$, and define $S^\pm$ to be the even and odd subspaces of S; then $C(V)^+ \simeq \mathbf{End}(S^+) \oplus \mathbf{End}(S^-)$. It has exactly two simple modules, namely $S^\pm$, with $\mathrm{End}(S^\pm)$ acting as 0 on $S^\mp$, its center is isomorphic to $k \oplus k$, and every module is isomorphic to a direct sum of copies of $S^\pm$.*

Proof. Clear.

Center of the even part of the Clifford algebra of an even dimensional quadratic space. For later use we shall describe the center of $C(V)^+$ when V is of even dimension D and k arbitrary. Let $(e_i)_{1 \leq i \leq D}$ be an orthogonal basis. Let

$$e_{D+1} = e_1 e_2 \ldots e_D.$$

We have

$$e_{D+1} e_i = -e_i e_{D+1}.$$

Then

$$\mathrm{ctr}\ (C(V)^+) = k \oplus k e_{D+1}.$$

Moreover, if the e_i are orthonormal, then

$$e_{D+1}^2 = (-1)^{D/2}.$$

It is in fact enough to verify the description of the center over $\overline{k}$ and so we may assume that k is algebraically closed. We may then replace each e_i by a suitable multiple so that the basis becomes orthonormal. Since e_{D+1} anticommutes with all e_i, it commutes with all e_ie_j and hence lies in the center of $C(V)^+$. If $a = \sum_{|I| \text{ even}} a_I e_I$ lies in the center of $C(V)^+$ and $0 < |I| < D$, then writing $I = \{i_1, \ldots, i_{2r}\}$, we use the fact that e_I anticommutes with $e_{i_1}e_s$ if $s \notin I$ to conclude that $a_I = 0$. Thus $a \in k1 \oplus e_{D+1}$.

5.3. Spin groups and spin representations. In this section we shall define the spin groups and the spin representations associated to real and complex quadratic vector spaces V. We treat first the case when $k = \mathbf{C}$ and then the case when $k = \mathbf{R}$.

Summary. The spin group $\mathrm{Spin}(V)$ for a *complex* V is defined as the universal cover of $\mathrm{SO}(V)$ if $\dim(V) \geq 3$. As the fundamental group of $\mathrm{SO}(V)$ is $\mathbf{Z}_2$ when $\dim(V) \geq 3$ it follows that in this case $\mathrm{Spin}(V)$ is a double cover of $\mathrm{SO}(V)$. If $\dim(V) = 1$ it is defined as $\mathbf{Z}_2$. For $\dim(V) = 2$, if we take a basis $\{x, y\}$ such that $\Phi(x,x) = \Phi(y,y) = 0$ and $\Phi(x,y) = 1$, then $\mathrm{SO}(V)$ is easily seen to be isomorphic to $\mathbf{C}^\times$ through the map

$$t \longmapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

The fundamental group of $\mathbf{C}^\times$ is $\mathbf{Z}$ and so $\mathrm{SO}(V)$ in this case has a unique double cover which is defined as $\mathrm{Spin}(V)$. For any V we put $C = C(V)$ for the Clifford algebra of V and $C^+ = C(V)^+$ its even part. We shall obtain for all V a natural imbedding of $\mathrm{Spin}(V)$ inside C^+ as a complex algebraic group which lies as a double cover of $\mathrm{SO}(V)$; this double cover is unique if $\dim(V) \geq 3$. So modules for C^+ may be viewed by restriction as modules for $\mathrm{Spin}(V)$. The key property of the imbedding is that the restriction map gives a bijection between simple C^+-modules and *certain* irreducible $\mathrm{Spin}(V)$-modules. *These are precisely the spin and semi-spin representations.* Thus the spin modules are the irreducible modules for C^+, or, as we shall call them, *Clifford modules.* The algebra C^+ is semisimple and so the restriction of any module for it to $\mathrm{Spin}(V)$ is a direct sum of spin modules. These are called *spinorial modules* of $\mathrm{Spin}(V)$.

Suppose now that V is a *real* quadratic vector space. If $V = \mathbf{R}^{p,q}$, we denote $\mathrm{SO}(V)$ by $\mathrm{SO}(p,q)$; this group does not change if p and q are interchanged and so we may assume that $0 \leq p \leq q$. If $p = 0$ then $\mathrm{SO}(p,q)$ is connected; if $p \geq 1$, it has 2 connected components (see the Appendix). As usual we denote the identity component of any topological group H by H^0. Let $V_{\mathbf{C}}$ be the complexification of V. Then the algebraic group $\mathrm{Spin}(V_{\mathbf{C}})$ is defined over $\mathbf{R}$, and so it makes sense to speak of the group of its real points. This is by definition $\mathrm{Spin}(V)$ and we have an exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V)^0 \longrightarrow 1.$$

If $\dim(V) = 1$, $\mathrm{Spin}(V) = \{\pm 1\}$. If V has signature $(1,1)$, then $\mathrm{Spin}(V)$ has two connected components. In all other cases it is connected and forms a double cover of $\mathrm{SO}(V)^0$. If $\dim(V) \geq 2$ is of signature $(p,q) \neq (1,1)$, then for $\min(p,q) \leq 1$, $\mathrm{SO}(p,q)$ has $\mathbf{Z}_2$ or $\mathbf{Z}$ as its fundamental group, and so has a unique double cover; and $\mathrm{Spin}(V)$ is that double cover. If p, q are both ≥ 2, then $\mathrm{Spin}(p,q)$ is characterized as the unique double cover which induces a double cover of both $\mathrm{SO}(p)$ and $\mathrm{SO}(q)$. Finally, if $\dim(V) \geq 3$, $\mathrm{Spin}(V)$ is the universal cover of $\mathrm{SO}(V)^0$ if and only if $\min(p,q) \leq 1$.

The relationship between the spin modules and modules for C^+ persists in the real case. The spinorial modules are the restriction to $\mathrm{Spin}(V)$ of C^+-modules. One can also describe them as modules of $\mathrm{Spin}(V)$ which are direct sums of the complex spin modules when we complexify.

Spin groups in the complex case. Let V be a *complex* quadratic vector space. A motivation for expecting an imbedding of the spin group inside $C^\times$ may be given as follows. If $g \in \mathrm{O}(V)$, then g lifts to an automorphism of C which preserves parity. If V has even dimension, $C = \mathbf{End}(S)$, and so this automorphism is induced by an invertible homogeneous element $a(g)$ of $C = \mathbf{End}(S)$, uniquely determined up to a scalar multiple. It turns out that this element is even or odd according as $\det(g) = \pm 1$. Hence we have a projective representation of $\mathrm{SO}(V)$ which can be lifted to an ordinary representation of $\mathrm{Spin}(V)$ (at least when $\dim(V) \geq 3$), and hence to a map of $\mathrm{Spin}(V)$ into $C^{+\times}$. It turns out that this map is an imbedding, and further that such an imbedding can be constructed when the dimension of V is odd also. Infinitesimally this means that there will be an imbedding of $\mathfrak{so}(V)$ inside C_L^+ where C_L^+ is the Lie algebra whose elements are those in C^+ with bracket $[a,b] = ab - ba$. We shall first construct this Lie algebra imbedding and then exponentiate it to get the imbedding $\mathrm{Spin}(V) \hookrightarrow C^{+\times}$.

To begin with we work over $k = \mathbf{R}$ or $\mathbf{C}$. It is thus natural to introduce the *even Clifford group* Γ^+ defined by

$$\Gamma^+ = \{u \in C^{+\times} \mid uVu^{-1} \subset V\}$$

where $C^{+\times}$ is the group of invertible elements of C^+. Γ^+ is a closed (real or complex) Lie subgroup of $C^{+\times}$. For each $u \in \Gamma^+$ we have an action

$$\alpha(u) : v \longmapsto uvu^{-1} \qquad (v \in V)$$

on V. Since

$$Q(uvu^{-1})1 = (uvu^{-1})^2 = uv^2u^{-1} = Q(v)1$$

we have

$$\alpha : \Gamma^+ \longrightarrow \mathrm{O}(V)$$

with kernel as the centralizer in $C^{+\times}$ of C, i.e., $k^\times$.

If A is any finite dimensional associative algebra over k, the Lie algebra of $A^\times$ is A_L where A_L is the Lie algebra whose underlying vector space is A with the bracket defined by $[a, b]_L = ab - ba(a, b \in A)$. Moreover, the exponential map from A_L into $A^\times$ is given by the usual exponential series:

$$\exp(a) = e^a = \sum_{n \geq 0} \frac{a^n}{n!} \qquad (a \in A)$$

so that the Lie algebra of $C^{+\times}$ is C_L^+. Thus, Lie (Γ^+), the Lie algebra of Γ^+, is given by

$$\mathrm{Lie}(\Gamma^+) = \{u \in C^+ \mid uv - vu \in V \text{ for all } v \in V\}.$$

For the map α from Γ^+ into $O(V)$ the differential $d\alpha$ is given by

$$d\alpha(u)(v) = uv - vu \qquad (u \in \mathrm{Lie}(\Gamma^+), v \in V).$$

Clearly $d\alpha$ maps $\mathrm{Lie}(\Gamma^+)$ into $\mathfrak{so}(V)$ with kernel as the centralizer in C^+ of C, i.e., k.

We now claim that $d\alpha$ is surjective. To prove this it is convenient to recall that the orthogonal Lie algebra is the span of the momenta in its 2-planes. First let $k = \mathbf{C}$. Then there is an ON basis (e_i), the elements of the orthogonal Lie algebra are precisely the skewsymmetric matrices, and the matrices

$$M_{e_i, e_j} := E_{ij} - E_{ji} \quad (i < j),$$

where E_{ij} are the usual matrix units, form a basis for $\mathfrak{so}(V)$. The M_{e_i,e_j} are the infinitesimal generators of the group of rotations in the (e_i, e_j)-plane with the matrices

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Now a simple calculation shows that

$$M_{e_i, e_j} v = \Phi(e_j, v)e_i - \Phi(e_i, v)e_j.$$

So, if we define, for any two $x, y \in V$,

$$M_{x,y} v = \Phi(y, v)x - \Phi(x, v)y \qquad (v \in V),$$

then $M_{x,y}$ is bilinear in x and y and so the $M_{x,y} \in \mathfrak{so}(V)$ for all $x, y \in V$ and span it. The definition of $M_{x,y}$ makes sense for $k = \mathbf{R}$ also and it is clear that the $M_{x,y}$ span $\mathfrak{so}(V)$ in this case also. As the $M_{x,y}$ are bilinear and skew symmetric in x and y we see that there is a unique linear isomorphism of $\Lambda^2(V)$ with $\mathfrak{so}(V)$ that maps $x \wedge y$ to $M_{x,y}$:

$$\Lambda^2(V) \simeq \mathfrak{so}(V), \qquad x \wedge y \longmapsto M_{x,y}.$$

For $x, y \in V$, a simple calculation shows that

$$d\alpha(xy)(v) = xyv - vxy = 2M_{x,y}v \in V \qquad (v \in V).$$

Hence $xy \in \mathrm{Lie}(\Gamma^+)$ and $d\alpha(xy) = 2M_{x,y}$. The surjectivity of $d\alpha$ is now clear. Note that xy is the infinitesimal generator of the one-parameter group $\exp(txy)$ which must lie in Γ^+ since $xy \in \mathrm{Lie}(\Gamma^+)$. We have an exact sequence of Lie algebras

$$0 \longrightarrow k \longrightarrow \mathrm{Lie}(\Gamma^+) \xrightarrow{d\alpha} \mathfrak{so}(V) \longrightarrow 0 \tag{5}$$

where k is contained in the center of Lie (Γ^+). We now recall the following standard result from the theory of semisimple Lie algebras.

Lemma 5.3.1. *Let $\mathfrak{g}$ be a Lie algebra over k, $\mathfrak{c}$ a subspace of the center of $\mathfrak{g}$, such that $\mathfrak{h} := \mathfrak{g}/\mathfrak{c}$ is semisimple. Then $\mathfrak{c}$ is precisely the center of $\mathfrak{g}$, $\mathfrak{g}_1 := [\mathfrak{g}, \mathfrak{g}]$ is a Lie ideal of $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{g}_1$ is a direct product of Lie algebras. Moreover $\mathfrak{g}_1$ is the unique Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{h}$ and $\mathfrak{g}_1 = [\mathfrak{g}_1, \mathfrak{g}_1]$. In particular there is a unique Lie algebra injection γ of $\mathfrak{h}$ into $\mathfrak{g}$ inverting the map $\mathfrak{g} \longrightarrow \mathfrak{h}$, and its image is $\mathfrak{g}_1$.*

Proof. Since center of $\mathfrak{h}$ is 0 it is immediate that $\mathfrak{c}$ is precisely the center of $\mathfrak{g}$. For $X, Y \in \mathfrak{g}$, $[X, Y]$ depends only on the images of X, Y in $\mathfrak{h}$ and so we have an action of $\mathfrak{h}$ on $\mathfrak{g}$ which is trivial *precisely* on $\mathfrak{c}$. As $\mathfrak{h}$ is semisimple it follows that there is a *unique* subspace $\mathfrak{h}'$ of $\mathfrak{g}$ complementary to $\mathfrak{c}$ which is stable under $\mathfrak{h}$. Clearly $\mathfrak{h}'$ is a Lie ideal, $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{h}'$ is a direct product, and, as $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$, it follows that $\mathfrak{h}'$ coincides with $\mathfrak{g}_1 = [\mathfrak{g}_1, \mathfrak{g}_1]$. If $\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{h}$, then $\mathfrak{a} \longrightarrow \mathfrak{h}$ is an isomorphism so that $\mathfrak{a}$ is stable under the action of $\mathfrak{h}$ and hence $\mathfrak{a} = \mathfrak{g}_1$.

The quadratic subalgebra. We return to the exact sequence (5). Since $xy + yx$ is a scalar, we have

$$d\alpha((1/4)(xy - yx)) = d\alpha((1/2)xy) = M_{x,y} \qquad (x, y \in V).$$

Let us therefore define

$$C^2 = \text{linear span of } xy - yx \qquad (x, y \in V) \qquad C^2 \subset \mathrm{Lie}(\Gamma^+).$$

Then $d\alpha$ maps C^2 onto $\mathfrak{so}(V)$. Of course $C^2 = 0$ if $\dim(V) = 1$. If x and y are orthogonal, then $xy - yx = 2xy$ from which it follows easily that C^2 is the span of the $e_r e_s$ $(r < s)$ for *any* orthogonal basis (e_i) of V (orthogonal bases exist always, they may not be orthonormal). We claim that C^2 is a *Lie subalgebra* of C_L^+ which may be called the *quadratic subalgebra* of C_L^+. Since $2xy = xy - yx +$ a scalar, we have, for $x, y, z, t \in V$,

$$[xy - yx, zt - tz] = 4[xy, zt] = 8\left(-\Phi(t,x)zy + \Phi(t,y)zx - \Phi(z,x)yt + \Phi(y,z)xt\right)$$

after a simple calculation. For $u, v \in V$ we have $uv - vu = 2uv - 2\Phi(u,v)$ so that

$$uv \equiv \Phi(u,v) \pmod{C^2}.$$

Substituting in the preceding equation we find that

$$[xy - yx, zt - tz] \equiv 8([yz, xt] + [yt, zx]) \equiv 0 \pmod{C^2}.$$

We now claim that $k1$ and C^2 are linearly independent. In fact, if (e_i) is an orthogonal basis, the elements $1, e_i e_j$ $(i < j)$ are linearly independent, proving the claim. Since $d\alpha$ maps C^2 onto $\mathfrak{so}(V)$ it follows that

$$\mathrm{Lie}(\Gamma^+) = k \oplus C^2, \qquad d\alpha : C^2 \simeq \mathfrak{so}(V)$$

and the map

$$\gamma : M_{x,y} \longmapsto (1/4)(xy - yx) \qquad (x, y \in V) \tag{6}$$

splits the exact sequence (5), i.e., it is a Lie algebra injection of $\mathfrak{so}(V)$ into Lie (Γ^+) such that

$$d\alpha \circ \gamma = \mathrm{id} \text{ on } \mathfrak{so}(V).$$

We have

$$\gamma(\mathfrak{so}(V)) = C^2.$$

Theorem 5.3.2. *If* $\dim(V) \geq 3$*, then* $C^2 = [C^2, C^2]$ *is the unique subalgebra of* $\mathrm{Lie}(\Gamma^+)$ *isomorphic to* $\mathfrak{so}(V)$*, and* γ *the only Lie algebra map splitting* (5)*. If further* $k = \mathbf{C}$ *and* G *is the complex analytic subgroup of* Γ^+ *determined by* C^2*, then* (G, α) *is a double cover of* $\mathrm{SO}(V)$ *and hence* $G \simeq \mathrm{Spin}(V)$*. In this case* G *is the unique connected subgroup of* Γ^+ *covering* $\mathrm{SO}(V)$*.*

Proof. If $\dim(V) \geq 3$, $\mathfrak{so}(V)$ is semisimple, and so it follows from the Lemma that the exact sequence (5) splits *uniquely* and

$$\gamma(\mathfrak{so}(V)) = C^2 = [C^2, C^2].$$

Let $k = \mathbf{C}$. The fact that G is the unique connected subgroup of Γ^+ covering $\mathrm{SO}(V)$ follows from the corresponding uniqueness of C^2. It remains to show that G is a double cover. If $x, y \in V$ are orthonormal we have $(xy)^2 = -1$ and $xy = (1/2)(xy - yx)$ so that

$$a(t) := \exp\{t(xy - yx)/2\} = \exp(txy) = (\cos t)\ 1 + (\sin t)\ xy$$

showing that the curve $t \longmapsto (\cos t)\ 1 + (\sin t)\ xy$ lies in G. Taking $t = \pi$ we see that $-1 \in G$. Hence G is a nontrivial cover of $\mathrm{SO}(V)$. But the universal cover of $\mathrm{SO}(V)$ is its only nontrivial cover and so $G \simeq \mathrm{Spin}(V)$. This finishes the proof.

Explicit description of complex spin group. Let $k = \mathbf{C}$. We shall see now that we can do much better and obtain a very explicit description of G and also take care of the cases when $\dim(V) \leq 2$. This however requires some preparation. We introduce the *full Clifford group* Γ defined as follows.

$$\Gamma = \{u \in C^\times \cap (C^+ \cup C^-) \mid uVu^{-1} \subset V\}.$$

Clearly

$$\Gamma = (\Gamma \cap C^+) \cup (\Gamma \cap C^-), \qquad \Gamma \cap C^+ = \Gamma^+.$$

We now extend the action α of Γ^+ on V to an action α of Γ on V by

$$\alpha(u)(x) = (-1)^{p(u)} uxu^{-1} \qquad (u \in \Gamma, x \in V).$$

As in the case of Γ^+ it is checked that α is a homomorphism from Γ to $\mathrm{O}(V)$.

Proposition 5.3.3. *We have an exact sequence*

$$1 \longrightarrow \mathbf{C}^\times 1 \longrightarrow \Gamma \xrightarrow{\alpha} \mathrm{O}(V) \longrightarrow 1.$$

Moreover $\alpha^{-1}(\mathrm{SO}(V)) = \Gamma^+$ *and*

$$1 \longrightarrow \mathbf{C}^\times 1 \longrightarrow \Gamma^+ \xrightarrow{\alpha} \mathrm{SO}(V) \longrightarrow 1$$

is exact.

Proof. If $v \in V$ and $Q(v) = 1$, we assert that $v \in \Gamma^-$ and $\alpha(v)$ is the reflection in the hyperplane orthogonal to v. In fact, $v^2 = 1$ so that $v^{-1} = v$, and, for $w \in V$, $\alpha(v)(w) = -vwv^{-1} = -vwv = w - 2\Phi(v, w)v$. By a classical theorem of E. Cartan (see the Appendix for a proof) any element of $\mathrm{O}(V)$ is a product of reflections in hyperplanes orthogonal to unit vectors. Hence α maps Γ onto $\mathrm{O}(V)$.

If $\alpha(u) = 1$, then u lies in the *super center* of C and so is a scalar. This proves the first assertion. By Cartan's result, any element of $\mathrm{SO}(V)$ is a product of an *even* number of reflections and so, if G' is the group of all elements of the form $v_1 \ldots v_{2r}$ where the v_i are unit vectors, then $G' \subset \Gamma^+$ and α maps G' onto $\mathrm{SO}(V)$. We first show that $\alpha(\Gamma^+) = \mathrm{SO}(V)$. In fact, if the image of Γ^+ is more than $\mathrm{SO}(V)$ it must be all of $\mathrm{O}(V)$ and so for any unit vector $v \in V$, $\alpha(v)$ must also be of the form $\alpha(u)$ for some $u \in \Gamma^+$. As the kernel of α is $\mathbf{C}^\times 1$ which is in C^+, it follows that $v = cu$ where c is a scalar, hence that $v \in \Gamma^+$ which is a contradiction. If $u \in \Gamma$ and $\alpha(u) \in \mathrm{SO}(V)$, then there is $u' \in \Gamma^+$ such that $\alpha(u') = \alpha(u)$ and so $u = cu'$ where c is a scalar, showing that $u \in \Gamma^+$ already. This finishes the proof.

Let us now introduce the unique antiautomorphism β of the ungraded Clifford algebra which is the identity on V, called the principal or canonical antiautomorphism. Thus

$$\beta(x_1 \ldots x_r) = x_r \ldots x_1 \qquad (x_i \in V). \tag{7}$$

Thus β preserves parity. We then have the following theorem which gives the explicit description of $\mathrm{Spin}(V)$ as embedded in $C^{+\times}$ for all dimensions.

Theorem 5.3.4. *The map $x \longmapsto x\beta(x)$ is a homomorphism of Γ into $\mathbf{C}^\times 1$. Let G be the kernel of its restriction to Γ^+.*

(i) *If $\dim(V) = 1$, then $G = \{\pm 1\}$.*

(ii) *If $\dim(V) \geq 2$, then G is the analytic subgroup of $C^{+\times}$ defined by C^2 and (G, α) is a double cover of $\mathrm{SO}(V)$.*

In particular,

$$\mathrm{Spin}(V) \simeq G = \{x \in C^{+\times} \mid xVx^{-1} \subset V,\ x\beta(x) = 1\}. \tag{8}$$

Proof. Given $x \in \Gamma$ we can, by Cartan's theorem, find unit vectors $v_j \in V$ such that $\alpha(x) = \alpha(v_1) \ldots \alpha(v_r)$ and so $x = cv_1 \ldots v_r$ for a nonzero constant c. But then

$$x\beta(x) = c^2 v_1 \ldots v_r v_r \ldots v_1 = c^2$$

so that $x\beta(x) \in \mathbf{C}^\times 1$. If $x, y \in C^{+\times}$, then

$$x\beta(x)(y\beta(y)) = x(y\beta(y))\beta(x) = xy\beta(xy).$$

Hence $x \longmapsto x\beta(x)$ is a homomorphism of Γ into $\mathbf{C}^\times 1$. Let G be the kernel of the restriction to Γ^+ of this homomorphism.

If $\dim(V) = 1$ and e is a basis of V, then C^+ is $\mathbf{C}$ so that $\Gamma^+ = \mathbf{C}^\times$. Hence $x\beta(x) = x^2$ for $x \in \mathbf{C}^\times$ and so $G = \{\pm 1\}$.

Let now $\dim(V) \geq 2$. If $g \in \mathrm{SO}(V)$ we can find $u \in \Gamma^+$ such that $\alpha(u) = g$. If $c \in \mathbf{C}^\times$ is such that $u\beta(u) = c^2 1$ it follows that for $v = c^{-1}u \in \Gamma^+$ and $\alpha(v) = \alpha(u) = g$. We thus see that α maps G onto $\mathrm{SO}(V)$. If $u \in G$ and $\alpha(u) = 1$, then u is a scalar and so, as $u\beta(u) = u^2 = 1$, we see that $u = \pm 1$. Since $\pm 1 \in G$ it follows that α maps G onto $\mathrm{SO}(V)$ with kernel $\{\pm 1\}$. We shall now prove that G is connected; this will show that it is a double cover of $\mathrm{SO}(V)$. For this it is enough to show that $-1 \in G^0$. If $x, y \in V$ are orthogonal, we have, for all $t \in \mathbf{C}$,

$$\beta(\exp(txy)) = \sum_{n \geq 0} \frac{t^n}{n!} \beta((xy)^n) = \sum_{n \geq 0} \frac{t^n}{n!} (yx)^n = \exp(tyx).$$

Hence, for all $t \in \mathbf{C}$,

$$\exp(txy)\beta(\exp(txy)) = \exp(txy)\exp(tyx) = \exp(txy)\exp(-txy) = 1.$$

Thus $\exp(txy)$ lies in G^0 for all $t \in \mathbf{C}$. If x, y are orthonormal, then $(xy)^2 = -1$ and so we have

$$\exp(txy) = (\cos t)\, 1 + (\sin t)\, xy$$

as we have seen already. Therefore $-1 = \exp(\pi xy) \in G^0$. Hence G is a double cover of $\mathrm{SO}(V)$, thus isomorphic to $\mathrm{Spin}(V)$.

The fact that G is the analytic subgroup of Γ^+ defined by C^2 when $\dim(V) \geq 3$ already follows from Theorem 2. So we need only consider the case $\dim(V) = 2$. Let $x, y \in V$ be orthonormal. Then $\exp(txy) \in G$ for all $t \in \mathbf{C}$. But

$$\exp(txy) = \exp(t(xy - yx)/2)$$

so that $xy - yx \in \mathrm{Lie}(G)$. Hence $\mathrm{Lie}(G) = \mathbf{C}xy = \mathbf{C}(xy - yx) = C^2$. Since it is a connected group of dimension 1 it follows that it is identical with the image of the one-parameter group $t \longmapsto \exp(txy)$.

We write $\mathrm{Spin}(V)$ for G.

Proposition 5.3.5. *Let V be arbitrary. Then*

$$\mathrm{Spin}(V) = \big\{x = v_1 v_2 \ldots v_{2r},\ v_i \in V,\ Q(v_i) = 1\big\}. \tag{9}$$

Proof. The right side of the formula above describes a group which is contained in $\mathrm{Spin}(V)$ and its image by α is the whole of $\mathrm{SO}(V)$ by Cartan's theorem. It contains -1 since $-1 = (-u)u$ where $u \in V$ with $Q(u) = 1$. So it is equal to $\mathrm{Spin}(V)$.

Spin groups for real orthogonal groups. We now take up spin groups over the reals. Let V be a quadratic vector space over $\mathbf{R}$. Let $V_{\mathbf{C}}$ be its complexification. Then there is a unique conjugation $x \longmapsto^{\mathrm{conj}}$ on the Clifford algebra $C(V_{\mathbf{C}})$ that extends the conjugation on $V_{\mathbf{C}}$, whose fixed points are the elements of $C(V)$. This conjugation commutes with β and so leaves $\mathrm{Spin}(V_{\mathbf{C}})$ invariant. The corresponding subgroup of $\mathrm{Spin}(V_{\mathbf{C}})$ of fixed points for the conjugation is a real algebraic Lie group, namely, the group of real points of $\mathrm{Spin}(V_{\mathbf{C}})$. It is by definition $\mathrm{Spin}(V)$:

$$\mathrm{Spin}(V) = \big\{x \in \mathrm{Spin}(V_{\mathbf{C}}), x = x^{\mathrm{conj}}\big\}. \tag{10}$$

Clearly $-1 \in \mathrm{Spin}(V)$ always. If $\dim(V) = 1$, we have

$$\mathrm{Spin}(V) = \{\pm 1\}.$$

Lemma 5.3.6. *Let $\dim(V) \geq 2$ and let $x, y \in V$ be mutually orthogonal and $Q(x), Q(y) = \pm 1$. Then $e^{txy} \in \mathrm{Spin}(V)^0$ for all real t. Let J_{xy} be the element of $\mathrm{SO}(V)$ which is -1 on the plane spanned by x, y and $+1$ on the orthogonal complement of this plane. Then*

$$e^{\pi xy} = -1 \quad (Q(x)Q(y) > 0) \qquad \alpha(e^{(i\pi/2)xy}) = J_{xy} \quad (Q(x)Q(y) < 0).$$

In the second case $e^{i\pi xy} = -1$.

Proof. We have already seen that e^{txy} lies in $\mathrm{Spin}(V_{\mathbf{C}})$ for all complex t. Hence for t real it lies in $\mathrm{Spin}(V)$ and hence in $\mathrm{Spin}(V)^0$. Suppose that $Q(x)Q(y) > 0$. Then $(xy)^2 = -1$ and so

$$e^{txy} = (\cos t)\, 1 + (\sin t)\, xy$$

for real t. Taking $t = \pi$ we get the first relation. Let now $Q(x)Q(y) < 0$. We have

$$\alpha(e^{itxy}) = e^{itd\alpha(xy)} = e^{2itM_{x,y}}.$$

Since $Q(x)Q(y) < 0$, the matrix of $M_{x,y}$ on the complex plane spanned by x and y is

$$\pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

from which it follows that $\alpha(e^{itxy})$ is 1 on the complex plane orthogonal to x and y, while on the complex span of x and y it has the matrix

$$\cos 2t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pm i \sin 2t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Taking $t = \pi/2$ we get the second relation. Since $(xy)^2 = 1$ we have,

$$e^{txy} = (\cosh t)\ 1 + (\sinh t)\ xy$$

for all complex t, so that $e^{i\pi xy} = -1$.

Theorem 5.3.7. *Let V be a real quadratic vector space and let $\mathrm{Spin}(V)$ be the group of real points of $\mathrm{Spin}(V_{\mathbf{C}})$. Then:*

(i) *If $\dim(V) = 1$ then $\mathrm{Spin}(V) = \{\pm 1\}$.*

(ii) *If $\dim(V) \geq 2$, $\mathrm{Spin}(V)$ always maps onto $\mathrm{SO}(V)^0$. It is connected except when $\dim(V) = 2$ and V is indefinite. In this exceptional case*

$$(\mathrm{Spin}(V), \mathrm{SO}(V)^0, \alpha) \simeq (\mathbf{R}^\times, \sigma) \qquad (\sigma(u) = u^2).$$

(iii) *In all other cases $\mathrm{Spin}(V)$ is connected and is a double cover of $\mathrm{SO}(V)^0$. If $V = \mathbf{R}^{p,q}$, then $\mathrm{Spin}(p,q) := \mathrm{Spin}(V)$ is characterized as the unique double cover of $\mathrm{SO}(V)^0$ when one of $p, q \leq 2$, and as the unique double cover which is nontrivial over both $\mathrm{SO}(p)$ and $\mathrm{SO}(q)$, when $p, q \geq 2$. In particular, $\mathrm{Spin}(V)$ is the universal cover of $\mathrm{SO}(V)^0$ if and only if $\dim(V) \geq 3$ and $p = 0, 1$.*

Proof. We need only check (ii) and (iii). The Lie algebra map

$$d\alpha : (1/4)(xy - yx) \longmapsto M_{x,y}$$

maps $\mathrm{Lie}(\Gamma^+)$ onto $\mathfrak{so}(V)$. So, α maps $\mathrm{Spin}(V)^0$ onto $\mathrm{SO}(V)^0$, and $\mathrm{Spin}(V)$ into $\mathrm{SO}(V)$ with kernel $\{\pm 1\}$. Since the group $\mathrm{SO}(V)$ remains the same if we interchange p and q we may suppose that $V = \mathbf{R}_{p,q}$ where $0 \leq p \leq q$ and $p + q \geq 2$.

First assume that $p = 0$. Then $\mathrm{SO}(V)$ is already connected. We can then find mutually orthogonal $x, y \in V$ with $Q(x) = Q(y) = -1$ and so, by the Lemma above, $-1 \in \mathrm{Spin}(V)^0$. This proves that $\mathrm{Spin}(V)$ is connected and is a double cover of $\mathrm{SO}(V)$.

Let $1 \le p \le q$. We shall first prove that $\mathrm{Spin}(V)$ maps into (hence onto) $\mathrm{SO}(V)^0$. Suppose that this is not true. Then the image of $\mathrm{Spin}(V)$ under α is the whole of $\mathrm{SO}(V)$. In particular, if $x, y \in V$ are mutually orthogonal and $Q(x) = 1,\ Q(y) = -1$, there is $u \in \mathrm{Spin}(V)$ such that $\alpha(u) = J_{xy}$. By the Lemma above $\alpha(e^{(i\pi/2)xy}) = J_{xy}$ also, and so $u = \pm e^{(i\pi/2)xy}$. This means that $e^{(i\pi/2)xy} \in \mathrm{Spin}(V)$ and so must be equal to its conjugate. But its conjugate is $e^{(-i\pi/2)xy}$ which is its inverse and so we must have $e^{i\pi xy} = 1$, contradicting the Lemma.

Assume now that we are not in the exceptional case (ii). Then $q \ge 2$ and so we can find mutually orthogonal $x, y \in V$ such that $Q(x) = Q(y) = -1$. The argument for proving that $\mathrm{Spin}(V)$ is a double cover for $\mathrm{SO}(V)^0$ then proceeds as in the definite case.

Suppose now that we are in the exceptional case (ii). Then this last argument does not apply. In this case let $x, y \in V$ be mutually orthogonal and $Q(x) = 1,\ Q(y) = -1$. Then $(xy)^2 = 1$ and $\mathrm{Spin}(V_{\mathbf{C}})$ coincides with the image of the one-parameter group e^{txy} for $t \in \mathbf{C}$. But $e^{txy} = (\cosh t)\, 1 + (\sinh t)\, xy$ and such an element lies in $\mathrm{Spin}(V)$ if and only if $\cosh t,\ \sinh t$ are both real. Thus

$$\mathrm{Spin}(V) = \{\pm a(t) \mid t \in \mathbf{R}\} \qquad a(t) = \cosh t\ 1 + \sinh t\ xy.$$

On the other hand,

$$\alpha(\pm a(t)) = e^{2tM_{x,y}} = (\cosh 2t)\ 1 + (\sinh 2t)\ M_{x,y}$$

so that $\mathrm{SO}(V)^0$ is the group of all matrices of the form

$$m(t) = \begin{pmatrix} \cosh 2t & \sinh 2t \\ \sinh 2t & \cosh 2t \end{pmatrix} \qquad (t \in \mathbf{R}).$$

This is isomorphic to $\mathbf{R}_+^\times$ through the map $m(t) \longmapsto e^{2t}$, while $\mathrm{Spin}(V) \simeq \mathbf{R}^\times$ through the map $\pm a(t) \longmapsto \pm e^t$. The assertion (ii) now follows at once.

It remains only to characterize the double cover when V is not exceptional. If $p = 0$, the fundamental group of $\mathrm{SO}(V)^0$ is $\mathbf{Z}$ when $q = 2$ and $\mathbf{Z}_2$ when $q \ge 3$; if $p = 1$, the fundamental group of $\mathrm{SO}(V)^0$ is $\mathbf{Z}_2$ for $q \ge 2$. Hence the double cover of $\mathrm{SO}(V)^0$ is unique in these cases without any further qualification. We shall now show that when $2 \le p \le q$, $\mathrm{Spin}(p, q)$ is the unique double cover of $S_0 = \mathrm{SO}(p, q)^0$ with the property described. If S is a double cover of S_0, the preimages L_p, L_q of $\mathrm{SO}(p), \mathrm{SO}(q)$ are compact and for $L_r (r = p, q)$ there are only two possibilities: either (i) it is connected and a double cover of $\mathrm{SO}(r)$ or (ii) it has two connected components and $L_r^0 \simeq SO(r)$. We must show that L_p, L_q have property (i) and

Spin(p, q) is the unique double cover possessing this property for both SO(p) and SO(q).

To this end we need a little preparation. Let $\mathfrak{g}$ be a real semisimple Lie algebra and let G_0 be a connected real Lie group with finite center (for instance a matrix group) and Lie algebra $\mathfrak{g}$. Let us consider the category $\mathcal{G}$ of pairs (G, π) where G is a connected real semisimple Lie group and π a *finite* covering map $G \longrightarrow G_0$; G then has finite center also. Morphisms $f : (G_1, \pi_1) \longrightarrow (G_2, \pi_2)$ are finite covering maps compatible with the $\pi_i(i = 1, 2)$. We generally suppress the maps π in the discussion below. If $G_i(i = 1, 2)$ are two objects in $\mathcal{G}$ there is a third group G that covers both G_i finitely, for instance the fiber product $G_1 \times_{G_0} G_2$. Any G in $\mathcal{G}$ has maximal compact subgroups; these are all connected and mutually conjugate, and all of them contain the center of G. If $f : G_1 \longrightarrow G_2$ and K_i is a maximal compact of G_i, then $f(K_1)$ (resp. $f^{-1}(K_2)$) is a maximal compact of G_2 (resp. G_1). Fix a maximal compact K_0 of G_0. Then for each G in $\mathcal{G}$ the preimage K of K_0 is a maximal compact of G and the maps $G_1 \longrightarrow G_2$ give maps $K_1 \longrightarrow K_2$ with the kernels of the maps being the same. Suppose now that $G, G_i(i = 1, 2)$ are in $\mathcal{G}$ and $G \longrightarrow G_i$ with kernel $F_i(i = 1, 2)$. It follows form our remarks above that to prove that there is a map $G_1 \longrightarrow G_2$ it is enough to prove that there is a map $K_1 \longrightarrow K_2$. For the existence of a map $K_1 \longrightarrow K_2$ it is clearly necessary and sufficient that $F_1 \subset F_2$.

In our case $G_0 = \mathrm{SO}(p, q)^0, K_0 = SO(p) \times \mathrm{SO}(q)$. Then Spin$(p, q)$ is in the category $\mathcal{G}$ and $K_{p,q}$, the preimage of K_0, is a maximal compact of it. Since both p and q are ≥ 2, it follows from the lemma that -1 lies in the connected component of the preimages of both SO(p) and SO(q). So if K_r is the preimage of SO$(r)(r = p, q)$, then $K_r \longrightarrow \mathrm{SO}(r)$ is a double cover. Let G_1 be a double cover of G_0 with preimages $L_p, L_q, L_{p,q}$ of $\mathrm{SO}(p), \mathrm{SO}(q), K_0$ with the property that L_r is connected and $L_r \longrightarrow \mathrm{SO}(r)$ is a double cover. We must show that there is a map $G_1 \longrightarrow \mathrm{Spin}(p, q)$ above G_0. By our remarks above this comes down to showing that there is a map $L_{p,q} \longrightarrow K_{p,q}$ above K_0. Since the fundamental group of SO(r) for $r \geq 2$ is $\mathbf{Z}$ for $r = 2$ and $\mathbf{Z}_2$ for $r \geq 3$, SO(r) has a unique double cover and so we have isomorphisms $L_r \simeq K_r$ above SO(r) for $r = p, q$.

The Lie algebra of K_0 is the direct product of the Lie algebras of SO(p) and SO(q). This implies that L_p, L_q, as well as K_p, K_q, commute with each other and $L_{p,q} = L_p L_q, K_{p.q} = K_p K_q$. Let $M_{p,q} = \mathrm{Spin}(p) \times \mathrm{Spin}(q)$. Then we have unique maps $M_{p,q} \longrightarrow L_{p,q}, K_{p,q}$ with $\mathrm{Spin}(r) \simeq L_r, K_r, (r = p, q)$. To show that we have an isomorphism $L_{p,q} \simeq K_{p.q}$ it is enough to show that the kernels of $M_{p,q} \longrightarrow L_{p,q}, K_{p,q}$ are the same. The kernel of $M_{p,q} \longrightarrow K_0$ is $\mathbf{Z}_2 \times \mathbf{Z}_2$. Since $\mathrm{Spin}(r) \simeq L_r, K_r$ it follows that the kernels of $M_{p,q} \longrightarrow L_{p,q}, K_{p,q}$, which are both nontrivial, have the property that their intersections with $\mathrm{Spin}(p) \times 1$ and $1 \times \mathrm{Spin}(q)$

are trivial. But $\mathbf{Z}_2 \times \mathbf{Z}_2$ has only one nontrivial subgroup that has trivial intersection with both of its factors, namely the diagonal. The uniqueness of this subgroup gives the map $L_{p,q} \simeq K_{p,q}$ that we want. This finishes the proof.

Remark. The above discussion also gives a description of $K_{p,q}$, the maximal compact of $\mathrm{Spin}(p,q)^0$. Let us write ε_r for the nontrivial element in the kernel of $K_r \longrightarrow \mathrm{SO}(r)(r = p, q)$. Then

$$K_{p,q} = K_p \times K_q/Z \qquad Z = \{1, (\varepsilon_p, \varepsilon_q)\}.$$

Thus a map of $K + p \times K_q$ factors through to $K_{p,q}$ if and only if it maps ε_p and ε_q to the same element.

We shall now obtain the analog of Proposition 5 in the real case.

Proposition 5.3.8. *For $p, q \geq 0$ we have*

$$\mathrm{Spin}(p,q) = \{v_1 \dots v_{2a} w_1 \dots w_{2b} \mid v_i, w_j \in V, Q(v_i) = 1, Q(w_j) = -1\}. \qquad (10)$$

Proof. By the results of Cartan[5] (see the Appendix) we know that the elements of $\mathrm{SO}(p,q)^0$ are exactly the products of an even number of space-like reflections and an even number of time-like reflections; here a reflection in a hyperplane orthogonal to a vector $v \in V$ with $Q(v) = \pm 1$ is space-like or time-like according as $Q(v) = +1$ or -1. It is then clear that the right side of (10) is a group which is mapped by α *onto* $\mathrm{SO}(p,q)^0$. As it contains -1 the result follows at once.

Spin representations as Clifford modules. We consider the following situation. A is a finite dimensional associative algebra over the field k which is either $\mathbf{R}$ or $\mathbf{C}$. Let $A^\times$ be the group of invertible elements of A. Then $A^\times$ is a Lie group over k and its Lie algebra is A_L which is A with the bracket $[a,b] = ab - ba$. The exponential map is the usual one:

$$\exp(a) = e^a = \sum_{n \geq 0} \frac{a^n}{n!}.$$

Let $\mathfrak{g} \subset A_L$ be a Lie algebra and G the corresponding analytic subgroup of $A^\times$. *We assume that A is generated as an associative algebra by the elements of* $\mathfrak{g}$. The exponential map $\mathfrak{g} \longrightarrow G$ is the restriction of the exponential map from A_L to $A^\times$. A finite dimensional representation $\rho(r)$ of $\mathfrak{g}(G)$ is said to be of A-type if there is a representation $\mu(m)$ of A such that $\mu(m)$ restricts to $\rho(r)$ on $\mathfrak{g}(G)$. Since $\mathfrak{g} \subset A$ and generates A as an associative algebra, we have a surjective map $\mathcal{U}(\mathfrak{g}) \longrightarrow A$, where $\mathcal{U}(\mathfrak{g}) \supset \mathfrak{g}$ is the universal enveloping algebra of $\mathfrak{g}$, which is the identity on $\mathfrak{g}$. So the representations of $\mathfrak{g}$ of A-type, which are just the A-modules, are precisely those whose extensions to $\mathcal{U}(\mathfrak{g})$ factor through the map $\mathcal{U}(\mathfrak{g}) \longrightarrow A$. We now have the following elementary result.

Proposition 5.3.9. *Let $\mathfrak{g}$ generate A as an associative algebra. If r is a representation of G of A-type, then $\rho = dr$ is a representation of $\mathfrak{g}$ of A-type and every representation of $\mathfrak{g}$ of A-type is uniquely obtained in this manner. Restriction to G is thus a fully faithful functor from the category of A-modules to the category of modules for G of A-type.*

Proof. Given r and its extension m, we have, for $\rho = dr$ the formula $\rho(a) = (d/dt)_{t=0}(r(e^{ta}))(a \in \mathfrak{g})$. Let $\mu(b) = (d/dt)_{t=0}(m(e^{tb}))(b \in A)$. Since $m(e^{tb}) = e^{tm(b)}$ it follows that $\mu(b) = m(b)$ while obviously μ extends ρ. Hence ρ is of A-type. Conversely, let ρ be of A-type and μ a representation of A that restricts to ρ on $\mathfrak{g}$. Let r be the restriction of μ to G. Then, for $a \in \mathfrak{g}$, we have, $(dr)(a) = (d/dt)_{t=0}(\mu(e^{ta})) = \mu(a) = \rho(a)$. Hence r is of A-type and $dr = \rho$. Since $\mathfrak{g}$ generates A as an associative algebra, it is clear that the extensions $m(\mu)$ are unique, and it is obvious that restriction is a fully faithful functor.

The imbedding

$$\gamma : \mathfrak{so}(V) \longrightarrow C_L^+, \qquad M_{x,y} \longmapsto (1/2)(xy - yx)$$

has the property that its image generates C^+ as an associative algebra. Indeed, if (e_i) is an ON basis for V, $\gamma(M_{e_i,e_j}) = e_i e_j$ and these generate C^+. Hence the conditions of the above proposition are satisfied with $G = \mathrm{Spin}(V), \mathfrak{g} = \mathfrak{so}(V)$ (identified with its image under γ) and $A = C^+$. By a *Clifford module* we mean any module for $\mathfrak{so}(V)$ or $\mathrm{Spin}(V)$, which is the restriction to $\mathrm{Spin}(V)$ or $\mathfrak{so}(V)$ of a module for C^+. Since we know the modules for C^+, all Clifford modules are known. These are, in the even dimensional case, direct sums of $S^\pm$, and in the odd dimensional case, direct sums of S.

Identification of the Clifford modules with the spin modules. We shall now identify $S^\pm$ and S as the spin modules. In the discussion below we shall have to use the structure theory of the orthogonal algebras. For details of this theory see[12], Chapter 4.

$\dim(V) = 2m$: We take a basis $(e_i)_{1 \le i \le 2m}$ for V such that the matrix of the quadratic form is

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Thus

$$\Phi(e_r, e_{m+r}) = 1 \qquad (1 \le r \le m)$$

and all other scalar products between the e's are zero. In what follows we use $r, s, r', \ldots$ as indices varying between 1 and m. The matrices of $\mathfrak{so}(V)$ are those of the form

$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \qquad (B^t = -B,\ C^t = -C)$$

where A, B, C, D are $m \times m$ matrices. In the usual structure theory of this classical algebra the Cartan subalgebra is the set of diagonal matrices $\simeq \mathbf{C}^m$ via

$$(a_1, \ldots, a_m) \longmapsto \text{ diag } (a_1, \ldots, a_n, -a_1, \ldots, -a_m).$$

We write E'_{ij} for the usual matrix units of the $m \times m$ matrix algebra and define

$$E_{ij} = \begin{pmatrix} E'_{ij} & 0 \\ 0 & -E'_{ji} \end{pmatrix}, \qquad F_{pq} = \begin{pmatrix} 0 & E'_{pq} - E'_{qp} \\ 0 & 0 \end{pmatrix}, \qquad G_{pq} = \begin{pmatrix} 0 & 0 \\ E'_{pq} - E'_{qp} & 0 \end{pmatrix}.$$

Then the E_{ij}, F_{pq}, G_{pq} are the root vectors with corresponding roots $a_i - a_j, a_p + a_q, -(a_p + a_q)$. For the positive system of root vectors we choose

$$E_{ij}(i < j), \quad F_{pq}(p < q).$$

Writing $M_{t,u}$ for M_{e_t,e_u}, it is easy to check that

$$M_{r,m+s} = E_{rs}, \quad M_{r,s} = F_{rs}, \quad M_{m+r,m+s} = G_{rs}, \quad M_{r,m+r} = E_{rr}.$$

Thus the positive root vectors are

$$M_{r,m+s}\ (r < s), \quad M_{r,s}\ (r < s).$$

The linear functions corresponding to the fundamental weights at the right extreme nodes of the Dynkin diagram are

$$\delta^{\pm} := (1/2)(a_1 + a_2 + \ldots + a_{m-1} \pm a_m).$$

Since the $\pm a_i$ are the weights of the defining representation in $\mathbf{C}^{2m}$, the weights of the tensor representations are those of the form

$$k_1 a_1 + k_2 a_2 + \ldots + k_m a_m$$

where the k_i are *integers*, and so it is clear that the irreducible representations with highest weights $\delta^\pm$ cannot occur in the tensors; this was Cartan's observation. We shall now show that the representations with highest weights $\delta^\pm$ are none other than the C^+–modules $S^\pm$ viewed as modules for $\mathfrak{so}(V)$ through the injection $\mathfrak{so}(V) \hookrightarrow C_L^+$.

The representation of the full algebra C as described in Theorem 2.4 acts on ΛU^* where U^* is the span of the e_{m+s}. The duality between U, the span of the e_r and U^* is given by $\langle e_r, e_{m+s}\rangle = 2\delta_{rs}$. The action of C^+ is through even elements and so preserves the even and odd parts of ΛU^*. We shall show that these are separately irreducible and are equivalent to the representations with highest weights $\delta^\pm$. To decompose ΛU^* we find all the vectors that are killed by the positive root vectors. It will turn out that these are the vectors in the span of 1 and e_{2m}. So 1 generates the even part and e_{2m} the odd part; the respective weights are δ^+, δ^- and so the claim would be proved.

The action of C on ΛU^* is as follows:

$$e_r : u^* \longmapsto \partial(e_r)(u^*), \qquad e_{m+r} : u^* \longmapsto e_{m+r} \wedge u^*.$$

The injection γ takes $M_{x,y}$ to $(1/4)(xy - yx)$ and so we have

$$\gamma(M_{r,m+r}) = (1/2)e_r e_{m+r} - (1/2),\ \gamma(M_{t,u}) = (1/2)e_t e_u\ (1 \le t, u \le 2m).$$

We now have

$$\gamma(M_{r,m+r})1 = 1/2, \quad \gamma(M_{r,m+r})e_{2m} = ((1/2) - \delta_{rm})e_{2m}.$$

Let us now determine all vectors v killed by

$$\gamma(M_{r,m+s}), \qquad \gamma(M_{r,s})\ (r < s).$$

As $\mathrm{diag}(a_1, \ldots, a_m, -a_1, \ldots, -a_m) = \sum_r a_r M_{r,m+r}$ we see that 1 has weight δ^+ while e_{2m} has weight δ^-. Since 1 is obviously killed by the positive root vectors we may suppose that v has no constant term and has the form

$$v = \sum_{|I| \ge 1} c_I e_{m+I}.$$

We know that v is killed by all $\partial(e_{j_1})\partial(e_{j_2})(1 \le j_1 < j_2 \le m)$. If we apply $\partial(e_{j_1})\partial(e_{j_2})$ to a term e_{m+I} with $|I| \ge 2$, we get $e_{m+I'}$ if I contains $\{j_1, j_2\}$ where $I' = I \setminus \{j_1, j_2\}$, or 0 otherwise, from which it is clear that $c_I = 0$. So

$$v = \sum_j c_j e_{m+j}.$$

Since $\gamma(M_{r,m+s})v = 0$ for $r < s$ we conclude that $c_r = 0$ for $r < m$.

$\dim(V) = 2m + 1$: We take a basis $(e_t)_{0 \le t \le 2m}$ with the $e_t (1 \le t \le 2m)$ as above and e_0 a vector of norm 1 orthogonal to them. If $f_t = ie_0 e_t$, then $f_s f_t = e_s e_t$ and so C^+ is generated by the (f_t) with the same relations as the e_t. This gives the fact already established that C^+ is the full ungraded Clifford algebra in dimension $2m$ and so is a full matrix algebra. It has thus a unique simple module S. We wish to identify it with the irreducible module with highest weight δ corresponding to the right extreme node of the diagram of $\mathfrak{so}(V)$. We take the module S for C^+ to be $\Lambda(F)$ where F is the span of the f_{m+s}, with f_r acting as $\partial(f_r)$ and f_{m+r} acting as multiplication by f_{m+r}. Then, as $e_t e_u = f_t f_u$, γ is given by

$$M_{r,s} \mapsto (1/2) f_r f_s \; M_{m+r,m+s} \mapsto (1/2) f_{m+r} f_{m+s}, \; M_{r,m+r} \mapsto (1/2) f_r f_{m+r} - (1/2)$$

while

$$M_{0,s} \mapsto (\text{-}i/2) f_s, \qquad M_{0,m+s} \mapsto (\text{-}i/2) f_{m+s}.$$

We take as the positive system the roots $a_r - a_s, -(a_r + a_s), -a_r$ so that the positive root vectors are

$$M_{m+r,s} \; (r < s), \qquad M_{r,s} \; (r < s), \qquad M_{0,r}.$$

It is easy to show, as in the previous example, that 1 is of weight δ and is killed by

$$\gamma(M_{r,s}) \; (r \neq s), \qquad \gamma(M_{m+r,s})$$

so that it generates the simple module of highest weight δ. To prove that this is all of S it is enough to show that the only vectors killed by all the positive root vectors are the multiples of 1. Now if v is such a vector, the argument of the previous case shows that $v = a1 + bf_{2m}$. But then $\partial(f_m)v = b = 0$. This finishes the proof.

Let V be a complex quadratic vector space of dimension D. Then for D odd the spin module has dimension $2^{\frac{D-1}{2}}$ while for D even the semispin modules have dimension $2^{\frac{D}{2}}$. Combining both we see that

$$\text{dimension of the spin module(s)} \; = 2^{[\frac{D+1}{2}]-1} \qquad (D \ge 1) \tag{11}$$

in all cases where $[x]$ is the largest integer $\le x$.

Remark. The identification of the spin modules with Clifford modules has a very important consequence. If V is a quadratic space and W a quadratic subspace, it is obvious that the restriction of a $C(V)^+$–module to $C(W)^+$ splits as a direct sum of simple modules and so the restriction of a spinorial module for $\mathrm{Spin}(V)$ to $\mathrm{Spin}(W)$ is spinorial. There are many situations like this occuring in physics and one can explicitly write down some of these "branching rules"[8].

Centers of the complex and real spin groups. We shall now determine the centers of the spin groups, both in the complex and real case. Let us first consider the case of a complex quadratic space V of dimension $D \geq 3$. If D is odd, $\mathrm{SO}(V)$ has trivial center and so is the adjoint group, and its fundamental group is $\mathbf{Z}_2$. As $\mathrm{Spin}(V)$ is the universal cover of $\mathrm{SO}(V)$, its center is $\mathbf{Z}_2$.

In even dimensions the determination of the center of $\mathrm{Spin}(V)$ is more delicate. If V above has even dimension $D = 2m$, the center of $\mathrm{SO}(V)$ is $\mathbf{Z}_2$, consisting of $\pm I$, I being the identity endomorphism of V. Its preimage in $\mathrm{Spin}(V)$, say Z, is the center of $\mathrm{Spin}(V)$, and is a group with 4 elements, hence is either $\mathbf{Z}_4$ or $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. We shall now determine in terms of D when these two possibilities occur. For this we need to use the obvious fact that the center of $\mathrm{Spin}(V)$ is the subgroup that lies in the center of $C(V)^+$. We have already determined the center of $C(V)^+$. If $(e_i)_{1 \leq i \leq D}$ is an orthonormal basis and $e_{D+1} = e_1 e_2 \ldots e_D$, then the center of $C(V)^+$ is spanned by 1 and e_{D+1}. Now $e_{D+1}^2 = (-1)^m$, e_{D+1} anticommutes with all e_i, and $\beta(e_{D+1}) = (-1)^m e_{D+1}$, so that $x = a + b e_{D+1}$ lies in the spin group if and only if $xVx^{-1} \subset V$ and $x\beta(x) = 1$. The second condition reduces to $a^2 + b^2 = 1,\ ab(1 + (-1)^m) = 0$, while the first condition, on using the fact that $x^{-1} = \beta(x)$, reduces to $ab(1 - (-1)^m) = 0$. Hence we must have $ab = 0, a^2 + b^2 = 1$, showing that

$$\mathrm{center}(\mathrm{Spin}(V)) = \{\pm 1, \pm e_{D+1}\}.$$

If m is even, $e_{D+1}^2 = 1$ and so the center is $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. For m odd we have $e_{D=1}^2 = -1$ and so the center is $\mathbf{Z}_4$ generated by $\pm e_{D+1}$. Thus,

$$\mathrm{center}(\mathrm{Spin}(V)) \simeq \begin{cases} \mathbf{Z}_2 & \text{if } D = 2k+1 \\ \mathbf{Z}_4 & \text{if } D = 4k+2 \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 & \text{if } D = 4k. \end{cases}$$

Suppose now that V is a *real* quadratic vector space of D. If D is odd it is immediate that the center of $\mathrm{Spin}(V)$ is $\{\pm 1\} \simeq \mathbf{Z}_2$. Let now D be even and let $V = \mathbf{R}^{a,b}$ where $a \leq b$ and $a + b = D$. If a, b are both odd, $-I \notin \mathrm{SO}(a) \times \mathrm{SO}(b)$ and so the center of $\mathrm{SO}(V)^0$ is trivial. This means that the center of $\mathrm{Spin}(V)$ is $\{\pm 1\} \simeq \mathbf{Z}_2$. Suppose that both a and b are even. Then $-I \in \mathrm{SO}(a) \times \mathrm{SO}(b)$ and so the center of $\mathrm{Spin}(V)^0$ consists of $\pm I$. Hence the center of $\mathrm{Spin}(V)$ has 4 elements and so coincides with Z, the center of $\mathrm{Spin}(V_{\mathbf{C}})$. Thus we have the following:

$$\text{center of } \mathrm{Spin}(\mathbf{R}^{a,b}) \simeq \begin{cases} \mathbf{Z}_2 & \text{if } D = 2k+1 \text{ or } D = 2k, a, b \text{ odd} \\ \mathbf{Z}_4 & \text{if } D = 2k, a, b \text{ even.} \end{cases}$$

5.4. Reality of spin modules. For applications to physics, the theory of spin modules over $\mathbf{C}$, is not enough; one needs the theory over $\mathbf{R}$. Representation theory over $\mathbf{R}$ is a little more subtle than the usual theory over $\mathbf{C}$ because Schur's lemma takes a more complicated form. If V is a real vector space and $A \subset \mathrm{End}_{\mathbf{R}}(V)$ is an algebra acting irreducibly on V, the commutant A' of A, namely the algebra of elements of $\mathrm{End}_{\mathbf{R}}(V)$ commuting with A, is a *division algebra.* Indeed, if $R \in A'$, the kernel and image of R are submodules and so each is either 0 or V. So, if $R \neq 0$, then both are 0 and so R is bijective, hence invertible, and $R^{-1} \in A'$. Now $\mathbf{R}, \mathbf{C}, \mathbf{H}$ are all division algebras over $\mathbf{R}$, $\mathbf{H}$ being the algebra of quaternions. Examples can be given to show that all three arise as commutants of simple modules of $\mathbf{R}$–algebras. For instance, if A denotes anyone of these, it is a simple module for the left regular representation, and its commutant is isomorphic to $A^{\mathrm{opp}} \simeq A$. A classical theorem of Frobenius asserts that these are the only (associative) division algebras over $\mathbf{R}$. So simple modules for a real algebra may be classified into 3 types according to the division algebra arising as the commutants in their simple modules. The main goal of this section is to determine the types of the simple modules for the even parts of the Clifford algebras of real quadratic vector spaces. The main result is that the types are governed by the *signature of the quadratic space* mod 8. This is the first of two beautiful periodicity theorems that we shall discuss in this and the next section.

It is not difficult to see that the types depend on the signature. Indeed, if we replace V by $V \oplus W$ where W is hyperbolic, then $C(V \oplus W) \simeq C(V) \otimes C(W)$ and $C(W)$ is a full endomorphism super algebra of a super vector space U. One can show that the simple modules for $C(V)$ and $C(V \oplus W)$ are S and $S \otimes U$ and the commutants are the same. Hence the types for $C(V)$ and $C(V \oplus W)$ are the same. Since two spaces V_1, V_2 have the same signature if and only if we can write $V_i = V \oplus W_i$ for $i = 1, 2$ where the W_i are hyperbolic, it is immediate that the types of $C(V_1)$ and $C(V_2)$ are the same. A little more work is needed to come down to the even parts. However one needs a much closer look to see that there is a periodicity mod 8 here.

We shall actually work over an arbitrary field k of characteristic 0 and specialize to $k = \mathbf{R}$ only at the very end. All algebras considered in this section are finite dimensional with unit elements and all modules are finite dimensional. $\overline{k} \supset k$ is the algebraic closure of k.

The Brauer group of a field. If A is an associative algebra over k and M is a module for A, we write A_M for the image of A in $\mathrm{End}_k(M)$. If M is simple, then the commutant $D = A'_M$ of A in M is a division algebra as we have seen above. However, unlike the case when k is algebraically closed this division algebra need not be k. The classical theorem of Wedderburn asserts that A_M is the commutant of D, i.e.,

$$A_M = \mathrm{End}_D(M).$$

We can reformulate this as follows. The definition $m \cdot d = dm (m \in M, d \in D)$ converts M into a right vector space over D^{opp}, the division algebra opposite to D. Let $(m_i)_{1 \le i \le r}$ be a D^{opp}-basis for M. If we write, for any $a \in A_M$, $am_j = \sum_i m_i \cdot a_{ij}$, then the map

$$a \longmapsto (a_{ij})$$

is an isomorphism of A_M with the algebra $M^r(D^{\mathrm{opp}})$ of all matrices with entries from D^{opp}:

$$A_M \simeq M^r(D^{\mathrm{opp}}) \simeq M^r(k) \otimes D^{\mathrm{opp}}.$$

Here, for any field k', $M^r(k')$ is the full matrix algebra over k'.

The classical theory of the Brauer group is well-known and we shall now give a quick summary of its basic results. We shall not prove these here since we shall prove their super versions here. Given an associative algebra A over k and a field $k' \supset k$ we define

$$A_{k'} = k' \otimes_k A.$$

We shall say that A is *central simple* (CS) if $A_{\overline{k}}$ is isomorphic to a full matrix algebra:

$$A \text{ CS } \iff A_{\overline{k}} \simeq M^r(\overline{k}).$$

Since

$$M^r(k') \otimes M^s(k') \simeq M^{rs}(k')$$

it follows that if A, B are CS algebras so is $A \otimes B$. Since

$$M^r(k')^{\mathrm{opp}} \simeq M^r(k')$$

it follows that for A a CS algebra, A^{opp} is also a CS algebra. The basic facts about CS algebras are summarized in the following proposition. Recall that for an algebra A over k and a module M for it, M is called *semisimple* if it is a direct sum of simple modules. M is semisimple if and only if $\overline{M} := M \otimes_k \overline{k}$ is semisimple for $A_{\overline{k}} = \overline{k} \otimes_k A$. A itself is called *semisimple* if all its modules are semisimple. This will be the case if A, viewed as a module for itself by left action, is semisimple. Also we have an action of $A \otimes A^{\mathrm{opp}}$ on A given by the morphism t from $A \otimes A^{\mathrm{opp}}$ into $\mathrm{End}_k(A)$ defined as follows:

$$t(a \otimes b) : x \longmapsto axb \qquad (a, x \in A, b \in A^{\mathrm{opp}}).$$

Proposition 5.4.1. *The following are equivalent.*

(i) *A is* CS.

(ii) $t : A \otimes A^{\mathrm{opp}} \simeq \mathrm{End}_k(A)$.

(iii) $\mathrm{ctr}(A) = k$ *and A is semisimple.*

(iv) $A = M^r(k) \otimes K$ *where K is a division algebra with* $\mathrm{ctr}(K) = k$.

(v) $\mathrm{ctr}(A) = k$ *and A has no proper nonzero two-sided ideal.*

In this case A has a unique simple module with commutant D and $A \simeq M^r(k) \otimes D^{\mathrm{opp}}$. Moreover, if M is any module for A and B is the commutant of A in M, then the natural map $A \longrightarrow \mathrm{End}_B(M)$ is an isomorphism:

$$A \simeq \mathrm{End}_B(M).$$

Finally, in (iv)*, K^{opp} is the commutant of A in its simple modules.*

An algebra A over k is *central* if its center is k, and *simple* if it has no nonzero two-sided ideal. Thus CS is the same as central and simple. Two central simple algebras over k are *similar* if the division algebras which are the commutants of their simple modules are isomorphic. This is the same as saying that they are both of the form $M^r(k) \otimes K$ for the same central division algebra K but possibly different r. Similarity is a coarser notion of equivalence than isomorphism since A and $M^r(k) \otimes A$ are always similar. Write $[A]$ for the similarity class of A. Since $M^r(k)$ has zero divisors as soon as $r > 1$, $M^r(k) \otimes K$ and K cannot both be division algebras unless $r = 1$, and so it follows that for central *division algebras* similarity and isomorphism coincide. Thus each similarity class contains a unique isomorphisms class of central division algebras. On the set of similarity classes we now define a multiplication, the so-called *Brauer multiplication*, by the rule

$$[A]\cdot[B] = [A \otimes B].$$

Since

$$(M^r(k) \otimes A) \otimes (M^s(k) \otimes B) = M^{rs}(k) \otimes (A \otimes B)$$

it follows that Brauer multiplication is well-defined. In particular, if E, F are two central division algebras, there is a central division algebra G such that $E \otimes F$ is the full matrix algebra over G, and

$$[E]\cdot[F] = [G].$$

The relations

$$[M^r(k) \otimes A] = [A], \qquad A \otimes B \simeq B \otimes A \qquad A \otimes A^{\mathrm{opp}} \simeq M^r(k)\ (r = \dim(A))$$

show that Brauer multiplication converts the set of similarity classes into a *commutative group* with $[k]$ as its identity element and $[A^{\mathrm{opp}}]$ as the inverse of $[A]$. This group is called the *Brauer group of the field k* and is denoted by $\mathrm{Br}(k)$. If k is algebraically closed, we have $\mathrm{Br}(k) = 1$ since every CS algebra over k is a full matrix algebra. For $k = \mathbf{R}$ we have

$$\mathrm{Br}(\mathbf{R}) = \mathbf{Z}_2.$$

In fact $\mathbf{R}$ and $\mathbf{H}$ are the only central division algebras over $\mathbf{R}$ (note that $\mathbf{C}$ as an $\mathbf{R}$-algebra is not central), and $\mathbf{H}$ is therefore isomorphic to its opposite. Hence the square of the class of $\mathbf{H}$ is 1. For our purposes we need a super version of Brauer's theory because the Clifford algebras are CS only in the super category. However the entire discussion above may be extended to the super case and will lead to a treatment of the Clifford modules from the perspective of the theory of the super Brauer group.

Central simple (CS) super algebras over a field. For any field k and any $u \in k^\times$ let $D = D_{k,u}$ be the super division algebra $k[\varepsilon]$ where ε is odd and $\varepsilon^2 = u$. It is obvious that the isomorphism class of $D_{k,u}$ depends only on the image of u in $k^\times/{k^\times}^2$. Clearly

$$D_{k,u}^{\mathrm{opp}} = D_{k,-u}.$$

In particular, if $D_k := D_{k,1}$, then $D^{\mathrm{opp}} = k[\varepsilon^0]$ where ε^0 is odd and ${\varepsilon^0}^2 = -1$. If k is algebraically closed, D_k is the only super division algebra apart from k. To see this let B be a super division algebra over k algebraically closed. If u is an odd nonzero element, it is invertible and so multiplication by u is an isomorphism of B_1 with B_0. But B_0 is an ordinary division algebra over k and so is k itself, so that $\dim(B_1) = 1$. As u^2 is nonzero and even, we have $u^2 = a1$, and so replacing u by $\varepsilon = a^{-1/2}u$, we see that $B = D_k$. If there is no ambiguity about k we write D for D_k. Because of this result we have

$$D \simeq D^{\mathrm{opp}} \qquad (k \text{ algebraically closed }).$$

In imitation of the classical case and guided by the Clifford algebras we define a super algebra A over k to be *central simple* (CS) if

$$A_{\overline{k}} \simeq M^{r|s}(\overline{k}) \text{ or } \simeq M^n \otimes D_{\overline{k}}. \qquad (\text{ CS })$$

From our results on Clifford algebras we see that the Clifford algebra $C(V)$ of a quadratic vector space over k is always central simple in the super category. We shall prove presently the super version of Proposition 1 that will allow us to define the notions of similarity for CS super algebras and of Brauer multiplication between them, and prove that this converts the set of similarity classes of CS super algebras over a field into a commutative group, the super Brauer group of the field. An explicit determination of the super Brauer group of $\mathbf{R}$ will then lead to the classification of types of simple Clifford modules over $\mathbf{R}$. This was first done by C. T. C. Wall[8].

We begin with some preparatory lemmas. If A, B are super algebras over k and V, W are modules for A, B respectively, recall $V \otimes W$ is a module for $A \otimes B$ if we define

$$(a \otimes b)(v \otimes w) = (-1)^{p(b)p(v)} av \otimes bw.$$

Let A be a sub super algebra of $\mathbf{End}_k(V)$. The *supercommutant* A' of A is the super algebra whose homogeneous elements x are defined by

$$ax = (-1)^{p(a)p(x)} xa \qquad (a \in A).$$

We must distinguish this from the super algebra, denoted by A'_u, which is the ordinary commutant, namely consisting of elements $x \in \mathbf{End}_k(V)$ such that $ax = xv$ for all $a \in A$. We often write A_u for A regarded as an ungraded algebra. Note however that A' and A'_u have the same even part. If A is a super algebra and V a super module for A, we write A_V for the image of A in $\mathbf{End}_k(V)$.

Lemma 5.4.2. *We have*

$$(A \otimes B)'_{V \otimes W} = A'_V \otimes B'_W.$$

Furthermore,

$$\mathrm{sctr}(A \otimes B) = \mathrm{sctr}(A) \otimes \mathrm{sctr}(B).$$

Proof. We may identify A and B with their images in the respective spaces of endomorphisms. It is an easy check that $A' \otimes B' \subset (A \otimes B)'$. We shall now prove the reverse inclusion. First we shall show that

$$(A \otimes 1)' = A' \otimes \mathbf{End}_k(W). \qquad (*)$$

Let $c = \sum_j a_j \otimes b_j \in (A \otimes 1)'$ where the b_j are linearly independent in $\mathbf{End}_k(W)$. Then $c(a \otimes 1) = (-1)^{p(c)p(a)}(a \otimes 1)c$ for a in A. Writing this out and observing that $p(c) = p(a_j) + p(b_j)$ for all j we get

$$\sum_j (-1)^{p(a)p(b_j)} \Big[aa_j - (-1)^{p(a)p(a_j)} a_j a \Big] \otimes b_j = 0.$$

The linear independence of the b_j implies that $a_j \in A'$ for all j, proving $(*)$. If now $c \in (A \otimes B)'$ we can write $c = \sum_j a_j \otimes b_j$ where the a_j are in A' and linearly independent. Proceeding as before but this time writing out the condition that $c \in (1 \otimes B)'$ we get $b_j \in B'$ for all j. Hence $c \in A' \otimes B'$. The second assertion is proved in a similar fashion.

Our next result is the Wedderburn theorem in the super context.

Lemma 5.4.3. *Let A a super algebra and V a semisimple module for A. Then, primes denoting commutants,*

$$A_V = A''_V.$$

Proof. We may assume that $A = A_V$. Let $v_j (1 \leq j \leq N)$ be homogeneous nonzero elements in V. It is enough to prove that if $L \in A''$, then there is $a \in A$ such that $av_j = Lv_j$ for all j. Consider first the case when $N = 1$. Since V is a direct sum of simple sub super modules it follows as in the classical case that any sub super module W has a complementary super module and hence there is a projection $V \longrightarrow W$, necessarily even, that lies in A'. Applying this to the submodule Av_1 we see that there is a projection $P(V \longrightarrow Av_1)$ that lies in A'. By assumption L commutes with P and so L leaves Av_1 invariant, i.e., $Lv_1 \in Av_1$. This proves the assertion for $N = 1$. Let now $N > 1$. Consider $V^N = V \otimes U$ where U is a super vector space with homogeneous basis $(e_j)_{1 \leq j \leq N}$ where e_j has the same parity as v_j. Then V^N, being the direct sum of the $V \otimes ke_j$, is semisimple, and so is itself semisimple. By Lemma 2, $(A \otimes 1)'' = A'' \otimes k$. Let $v = \sum_j v_j \otimes e_j$. Then v is even and by what has been proved above, given $L \in A''$ we can find $a \in A$ such that $(L \otimes 1)v = (a \otimes 1)v$, i.e.,

$$\sum Lv_j \otimes e_j = \sum av_j \otimes e_j.$$

This implies that $Lv_j = av_j$ for all j, finishing the proof.

Lemma 5.4.4. *If A is a super algebra and M a simple super module for A, then the super commutant of A_M is a super division algebra. If B is a super division algebra over k which is not purely even, and V is a super vector space, then*

$$\mathbf{End}_k(V) \otimes B \simeq \mathrm{End}_k(V') \otimes B$$

where V' is the ungraded vector space V and $\mathrm{End}_k(V')$ is the purely even algebra of all endomorphisms of V'. In particular

$$M^{r|s}(k) \otimes B \simeq M^{r+s} \otimes B.$$

Proof. Let L be a homogeneous element of A'_M. Then the kernel and image of L are sub super modules and the argument is then the same as in the classical Schur's lemma. For the second assertion we begin by regarding B as a module for B^{opp} by

$$b \cdot b' = (-1)^{p(b)p(b')} b'b.$$

Clearly the commutant of this module is B acting by left multiplication on itself. By Lemma 1 we therefore have, on $V \otimes B$,

$$(1 \otimes B^{\mathrm{opp}})' = \mathbf{End}_k(V) \otimes B.$$

Choose now $\eta \neq 0$ in B_1. Let v_i form a basis of V with $v_i(i \leq r)$ even and $v_i(i > r)$ odd. Then, the *even* elements

$$v_1 \otimes 1, \ldots, v_r \otimes 1, v_{r+1} \otimes \eta, \ldots, v_{r+s} \otimes \eta$$

form a B^{opp}-basis of $V \otimes B$. This implies that

$$V \otimes B \simeq V' \otimes B$$

as B^{opp}-modules. Since the commutant of $1 \otimes B^{\mathrm{opp}}$ in $V' \otimes B$ is $\mathrm{End}_k(V') \otimes B$ the result follows.

One can see easily from the definition that if A, B are CS super algebras, then so are $A \otimes B$ and A^{opp}. To see this write $M^{r|s} = M^{r|s}(\overline{k}), D = D_{\overline{k},1}, D_k = D_{k,1}$. We then have the following.

$$\begin{aligned} M^{r|s}(k) \otimes M^{p|q}(k) &\simeq M^{rp+sq|rq+sp}(k) \\ M^{r|s}(k) \otimes (M^n(k) \otimes D_k) &\simeq M^{nr|ns}(k) \otimes D_k \simeq M^{n(r+s)}(k) \otimes D_k \\ (M^m(k) \otimes D_k) \otimes (M^n \otimes D_k)^{\mathrm{opp}} &\simeq M^{mn}(k) \otimes M^{1|1}(k) \simeq M^{mn|mn}(k). \end{aligned}$$

Taking $\overline{k}$ instead of k and remembering that $D^{\mathrm{opp}} \simeq D$ we see that $A \otimes B$ is CS if A, B are CS. In the second relation we are using Lemma 4. The verification of the third comes down to seeing that

$$D_k \otimes D_k^{\mathrm{opp}} \simeq M^{1|1}.$$

This last relation is proved as follows. For any super algebra A, we have an action $t = t_A$ of $A \otimes A^{\mathrm{opp}}$ on A given by

$$t(a \otimes b)(x) = (-1)^{p(b)p(x)} axb \qquad (a, x \in A, b \in A^{\mathrm{opp}}).$$

Thus

$$t : A \otimes A^{\mathrm{opp}} \longrightarrow \mathbf{End}_k(A)$$

is a morphism of super algebras. In the special case when $A = D_k$ we can compute t explicitly and verify that it is an isomorphism. In the basis $1, \varepsilon$ for D_k we have, for the action of t,

$$\begin{aligned} t(1 \otimes 1) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad & t(1 \otimes \varepsilon) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ t(\varepsilon \otimes 1) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad & t(\varepsilon \otimes \varepsilon) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

So the result is true in this case. To see that the opposite of a CS super algebra is also CS we first prove that

$$\mathbf{End}_k(V)^{\mathrm{opp}} \simeq \mathbf{End}_k(V).$$

Let V^* be the dual of V and for $T \in \mathbf{End}_k(V)$ let us define $T^* \in \mathbf{End}_k(V^*)$ by

$$(T^* v^*)(v) = (-1)^{p(T^*)p(v^*)} v^*(Tv).$$

It is then easily checked that $p(T^*) = p(T)$ and

$$(T_1 T_2)^* = (-1)^{p(T_1)p(T_2)} T_2^* T_1^*$$

which proves that the map $T \longmapsto T^*$ is an isomorphism of $\mathbf{End}_k(V)$ with $\mathbf{End}_k(V^*)^{\mathrm{opp}}$. However we have, noncanonically, $V \simeq V^*$, and so

$$\mathbf{End}_k(V) \simeq \mathbf{End}_k(V^*)^{\mathrm{opp}} \simeq \mathbf{End}_k(V)^{\mathrm{opp}}.$$

Next, as $D \simeq D^{\mathrm{opp}}$ for k algebraically closed, we have

$$(M^n \otimes D)^{\mathrm{opp}} \simeq M^n \otimes D$$

where we are using the easily proved fact that $(A \otimes B)^{\mathrm{opp}} \simeq A^{\mathrm{opp}} \otimes B^{\mathrm{opp}}$.

We shall now prove the super version of Proposition 1. Recall that for a super algebra, the complete reducibility of all its modules is equivalent to the complete reducibility of the left regular representation, and that we have called such super algebras *semisimple.*

Proposition 5.4.5. *The following are equivalent.*

(i) *A is* CS.

(ii) $t : A \otimes A^{\mathrm{opp}} \longrightarrow \mathrm{End}_k(A)$ *is an isomorphism.*

(iii) $\mathrm{sctr}(A) = k$ *and the ungraded algebra A is semisimple.*

(iv) $\mathrm{sctr}(A) = k$ *and the super algebra A is semisimple.*

(v) $\mathrm{sctr}(A) = k$ *and A has no proper nonzero two-sided homogeneous ideal.*

(vi) $A = M^r(k) \otimes K$ *where K is a super division algebra with* $\mathrm{sctr}(K) = k$.

(vii) $\mathrm{sctr}(A) = k$ *and A has a faithful semisimple representation.*

Proof. (i) $\Longrightarrow$ (ii). Since the map t is already well-defined, the question of its being an isomorphism can be settled by working over $\overline{k}$. Hence we may assume that k is already algebraically closed. We consider two cases.

Case 1: $A \simeq M^{r|s}$. Let E_{ij} be the matrix units with respect to the usual homogeneous basis of $k^{r|s}$. Then

$$t(E_{ij} \otimes E_{q\ell}) : E_{mn} \longmapsto (-1)^{[p(q)+p(\ell)][p(m)+p(n)]}\delta_{jm}\delta_{nq}E_{i\ell},$$

and so $t(E_{ij} \otimes E_{q\ell})$ takes E_{jq} to $\pm E_{i\ell}$ and E_{mn} to 0 if $(m,n) \neq (j,q)$. This proves that the image of t is all of $\mathbf{End}_{\overline{k}}(A)$. Computing dimensions we see that t is an isomorphism.

Case 2: $A \simeq \mathrm{End}_k(V) \otimes D$ where V is a purely even vector space. We have already verified that t is an isomorphism when $V = k$, i.e., $A = D$. If we write $t_{A\otimes B}, t_A, t_B$ for the maps associated to $A \otimes B, A, B$, then a simple calculation shows (after the identifications $(A \otimes B) \otimes (A \otimes B)^{\mathrm{opp}} \simeq (A \otimes A^{\mathrm{opp}}) \otimes (B \otimes B^{\mathrm{opp}})$ and $t_{A\otimes B} \simeq t_A \otimes t_B$) that

$$t_{A\otimes B} = t_A \otimes t_B.$$

Hence the result for $A = \mathrm{End}_k(V) \otimes D$ follows from those for End_k and D.

(ii) $\Longrightarrow$ (iv). Let $x \in \mathrm{sctr}(A)$. Then $xa = (-1)^{p(x)p(a)}xa$ for all $a \in A$. We now assert that $x \otimes 1$ is in the super center of $A \otimes A^{\mathrm{opp}}$. In fact,

$$(x \otimes 1)(a \otimes b) = xa \otimes b = (-1)^{p(x)p(a)}ax \otimes b = (-1)^{p(x)p(a\otimes b)}(a \otimes b)(x \otimes 1)$$

proving our claim. So $x \otimes 1 \in k$, showing that $x \in k$. We must now show that the left regular representation of the super algebra A is completely reducible. Let L be a (graded) subspace of A stable and irreducible under left translations. Then, under our assumption (ii), the spaces $t(a \otimes b)[L] = Lb$ span A as b varies among the homogeneous elements of A. This means that the spaces Lb span A. Right multiplication by b is a map of L with Lb commuting with the left action and so Lb

is a quotient of L or ΠL according as b is even or odd, thus irreducible as a super module for the left regular representation. Thus A is the sum of simple sub super modules for the left action and hence A is semisimple.

(iv) $\Longrightarrow$ (ii). We begin by remarking that if L and M are simple nonzero sub super modules of A under the left action, then $M = Lb$ for some homogeneous $b \in A$ if and only if either $M \simeq L$ or $M \simeq \Pi L$. Indeed, if $M = Lb$ then left multiplication by b is a nonzero element of $\mathrm{Hom}_A(L, M)$ if b is even and $\mathrm{Hom}_A(\Pi L, M)$ if M is odd, and hence is an isomorphism. For the reverse result, write $A = L \oplus L_1 \ldots$ where $L, L_1, \ldots$ are simple sub super modules of A for the left action. Let $T(L \longrightarrow M)$ be a homogeneous linear isomorphism $L \simeq M$ as A–modules. Define T as 0 on $L_1, \ldots$. Then T is homogeneous and commutes with left action. If $T1 = b$, then b is homogeneous and $Ta = ab$. Hence $M = Lb$ as we wished to show.

This said, let us write $A = \oplus A_i$ where the A_i are simple sub super modules of A for the left action. We write $i \sim j$ if A_i is isomorphic under left action to A_j or ΠA_j. This is the same as saying, in view of our remark above, that for some homogeneous b, $A_j = A_i b$; and $\sim$ is an equivalence relation. Let $I, J, \ldots$ be the equivalence classes and $A_I = \oplus_{i \in I} A_i$. Each A_I is graded and A_I does not change if we start with another A_j with $i \sim j$. Moreover A_I is invariant under left as well as right multiplication by elements of A and so invariant under the action of $A \otimes A^{\mathrm{opp}}$. We now claim that each A_I is *irreducible* as a super module under the action of $A \otimes A^{\mathrm{opp}}$. To show this it is enough to prove that if M is a graded subspace of A_I stable and irreducible under the left action, then the subspaces Mb for homogeneous b span A_I. Now A_I is a sum of submodules all equivalent to A_i for some $i \in I$, and so M has to be equivalent to A_i also. So, by the remark made at the outset, $A_i = Mb_0$ for some homogeneous b_0; but then as the $A_i b$ span A_I it is clear that the Mb span A_I. Thus A_I is a simple module for $A \otimes A^{\mathrm{opp}}$. Since $A = \sum_I A_I$ it follows that the action of $A \otimes A^{\mathrm{opp}}$ on A is semisimple. So Lemma 3 is applicable to the image R of $A \otimes A^{\mathrm{opp}}$ in $\mathbf{End}_k(A)$. Let $T \in R'$ and $T1 = \ell$. The condition on T is that

$$t(a \otimes b)T = (-1)^{p(T)p(t(a \otimes b))} T t(a \otimes b) \qquad (*)$$

for all $a, b \in A$. Since $t(a \otimes b)(x) = \pm axb$ it follows that $p(t(a \otimes b)) = p(a \otimes b) = p(a) + p(b)$. Moreover as $T1 = \ell$, we have $p(T) = p(\ell)$. Hence applying both sides of $(*)$ to 1 we get

$$(-1)^{p(b)p(\ell)} a\ell b = (-1)^{p(\ell)[p(a)+p(b)]} T(ab).$$

Taking $a = 1$ we see that $Tb = \ell b$ so that the above equation becomes

$$(-1)^{p(b)p(\ell)} a\ell b = (-1)^{p(\ell)[p(a)+p(b)]} \ell ab.$$

Taking $b = 1$ we get $a\ell = (-1)^{p(a)p(\ell)}\ell a$, showing that ℓ lies in the super center of A. So $\ell \in k$. But then $R'' = \mathbf{End}_k(A)$ so that the map $t : A \otimes A^{\text{opp}} \longrightarrow \mathbf{End}_k(A)$ is surjective. By counting dimensions we see that this must be an isomorphism. Thus we have (ii).

(iv) $\Longrightarrow$ (v). It is enough to prove that (v) follows from (ii). But, under (ii), A, as a module for $A \otimes A^{\text{opp}}$, is simple. Since 2-sided homogeneous ideals are stable under t, we get (v).

(v) $\Longrightarrow$ (vi). By (v) we know that any nonzero morphism of A into a super algebra is faithful. Take a simple module M for A. Its super commutant is a super division algebra D and by Lemma 3 we have $A_M = \mathbf{End}_D(M)$. The map $A \longrightarrow A_M$ is faithful and so

$$A \simeq \mathbf{End}_D(M).$$

This implies that

$$A \simeq M^{r|s}(k) \otimes K, \qquad K = D^{\text{opp}}.$$

Since the super center of a product is the product of super centers we see that the super center of K must reduce to k. Thus we have (vi).

(vi) $\Longrightarrow$ (i). It is enough to prove that if K is a super division algebra whose super center is k, then K is CS. Now the left action of K on itself is simple and so semisimple. Thus K is semisimple. We now pass to the algebraic closure $\overline{k}$ of k. Then $\overline{K} = K_{\overline{k}}$ is semisimple and has super center $\overline{k}$. Thus $\overline{K}$ satisfies (iv), and hence (v) so that any nonzero morphism of $\overline{K}$ is faithful. Let M be a simple module for $\overline{K}$ and E the super commutant in M. Then, with $F = E^{\text{opp}}$, $\overline{K} \simeq M^{r|s} \otimes F$. For F there are only two possibilities: $F = \overline{k}, D$. In the first case $\overline{K} \simeq M^{r|s}$ while in the second case $\overline{K} \simeq M^{r+s} \otimes D$ by Lemma 4. Hence $\overline{K}$ is CS.

(iii) $\Longleftrightarrow$ (i). It is enough to prove (i) $\Longrightarrow$ (iii) when k is algebraically closed. It is only a question of the semisimplicity of A as an ungraded algebra. If $A = M^{r|s}$ then the ungraded A is M^{r+s} and so the result is clear. If $A = M^n \otimes D$, then it is a question of proving that the ungraded D is semisimple. But as an ungraded algebra, $D \simeq k[u]$ where $u^2 = 1$ and so $D \simeq k \oplus k$, hence semisimple.

For the converse, let us suppose (iii) is true. Let us write A_u for A regarded as an ungraded algebra. We shall show that A is semisimple as a super algebra. This will give us (iv) and hence (i). We shall assume that k is algebraically closed. We first argue as in the proof of (iv) $\Longrightarrow$ (ii) above that A_u is semisimple as a module for $A_u \otimes A_u^{\text{opp}}$. Take now a filtration of homogeneous left ideals

$$A_0 = A \supset A_1 \supset \ldots \supset A_r \supset A_{r+1} = 0$$

where each $M_i := A_i/A_{i+1}$ is a simple super module. Let R be the set of elements which map to the zero endomorphism in each M_i. Then R is a homogeneous two-sided ideal. If $x \in R$, then $xA_i \subset A_{i+1}$ for all i, and so $x^r = 0$. Now, the ungraded algebra A_u is semisimple by assumption. Hence as R is stable under $A_u \otimes A_u^{\rm opp}$ we can find a two-sided ideal R' such that $A = R \oplus R'$. Since $RR' \subset R \cap R' = 0$ we have $RR' = R'R = 0$. Write $1 = u + u'$ where $u \in R, u' \in R'$. Then $uu' = u'u = 0$ and so $1 = (u + u')^r = u^r + u'^r = u'^r$, showing that $1 \in R'$. Hence $R = R1 = 0$. This means that A_u, and hence A, acts faithfully in $\oplus_i M_i$.

The kernel of A in M_i and M_j are the same if either $M_i \simeq M_j$ or $M_i \simeq \Pi M_j$. Hence, by omitting some of the M_i we may select a subfamily $M_i (1 \le i \le s)$ such that for $i \neq j$ we have $M_i \not\simeq M_j, \Pi M_j$, and that A acts faithfully on $M = \oplus_{1 \le i \le s}$. We may thus suppose that $A = A_M$. Let $P_i(M \longrightarrow M_i)$ be the corresponding projections. If A'_u is the ordinary commutant of A_u it is clear that $P_i \in A'_u$ for all i. We claim that $P_i \in (A'_u)'_u$ for all i. Let $S \in A'_u$ be homogeneous. Then $S[M_i]$ is a super module for A which is a quotient of M_i or ΠM_i and so is either 0 or equivalent to M_i or ΠM_i. Hence it cannot be equivalent to any M_j for $j \neq i$ and so $S[M_i] \subset M_i$ for all i. So S commutes with P_i for all i. Thus $P_i \in (A'_u)'_u$ for all i. But A_u, being semisimple, we have $A_u = (A'_u)'_u$ and so $P_i \in A$ for all i. Hence $P_i \in A \cap A' = \mathrm{sctr}(A) = k$ for all i. Thus there is only one index i and $P_i = 1$ so that M is simple. But then $A = \mathrm{End}_K(M)$ where K is the super commutant of A in M. B is a super division algebra with super center k and so we have (vi). But then as (vi) implies (i) we are done.

(vii) $\Longleftrightarrow$ (i). The argument in the preceding implication actually proves that (vii) implies (i). The reverse is trivial since the left action of A on itself is semisimple and faithful if A is CS.

This completes the proof of the entire proposition.

Proposition 5.4.6. *Let k be arbitrary and A a* CS *super algebra over k. Let M be any module for A and let B be the commutant of A in M. Then the natural map $A \longrightarrow \mathrm{End}_B(M)$ is an isomorphism:*

$$A \simeq \mathrm{End}_B(M).$$

Moreover, the commutants in the simple modules for A are all isomorphic. If B is such a commutant, then B a super division algebra with super center k, and $A \simeq M^{r|s}(k) \otimes B^{\rm opp}$. Finally, if $A = M^{r|s} \otimes K$ where K is a super division algebra with super center k, $K^{\rm opp}$ is the commutant of A in its simple modules.

Proof. The first assertion is immediate from Lemma 3 since A is semisimple by Proposition 5. To prove the second assertion let M, N be two simple modules for

A. Let $\overline{M}, \overline{N}$ be their extensions to $\overline{k}$ as modules for $\overline{A} := A_{\overline{k}}$. We consider two cases.

Case 1: $\overline{A} \simeq \mathbf{End}(V)$ where V is a super vector space over $\overline{k}$. Then $\overline{M} \simeq V \otimes R, \overline{N} \simeq V \otimes S$ where R, S are super vector spaces. Unless one of R, S is purely even and the other purely odd, we have $\mathrm{Hom}_{\overline{A}}(\overline{M}, \overline{N}) \neq 0$. Hence $\mathrm{Hom}_A(M, N) \neq 0$, and so as M and N are simple we must have $M \simeq N$. In the exceptional case we replace N by ΠN to conclude as before that $M \simeq \Pi N$. So to complete the proof we must check that the commutants of A in M and ΠM are isomorphic. But parity reversal does not change the action of A and hence does not change the commutant.

Case 2: $A_{\overline{k}} \simeq \mathrm{End}_{\overline{k}}(V) \otimes D$ where V is a purely even vector space. In this case we have seen that there is a unique simple module and so the result is trivial.

For the last assertion let $A = M^{r|s} \otimes K$ where K is a super division algebra with k as super center. Let $M = k^{r|s} \otimes K$ viewed as a module for A in the obvious manner, K acting on K by left multiplication. It is easy to check that this is a simple module. The commutant is $1 \otimes K^{\mathrm{opp}} \simeq K^{\mathrm{opp}}$ as we wanted to show.

The super Brauer group of a field. Let k be arbitrary. We have seen that if A is a CS super algebra, then A is isomorphic to $M^{r|s}(k) \otimes B$ where B is a CS super division algebra, i.e., a super division algebra with super center k. B is also characterized by the property that B^{opp} is the super commutant of A in its simple modules. Two CS super algebras A_1, A_2 are said to be *similar* if their associated division algebras are isomorphic, i.e., if $A_i \simeq M^{r_i|s_i}(k) \otimes D$ where D is a central super division algebra. Similarity is an equivalence relation which is coarser than isomorphism and the similarity class of A is denoted by $[A]$. We define Brauer multiplication of the similarity classes as before by

$$[A] \cdot [B] = [A \otimes B].$$

It is obvious that this depends only on the classes and not on the representative super algebras in the class. This is a commutative product and has $[k]$ as the unit element. The relation

$$A \otimes A^{\mathrm{opp}} \simeq \mathbf{End}_k(A)$$

shows that $[A^{\mathrm{opp}}]$ is the inverse of $[A]$. Thus the similarity classes from a commutative group. This is the *super Brauer group* of k, denoted by $\mathrm{sBr}(k)$. Our goal is to get information about the structure of $\mathrm{sBr}(k)$ and the subset of classes of the Clifford algebras inside it. We shall in fact show that

$$\mathrm{sBr}(\mathbf{R}) = \mathbf{Z}_8 = \mathbf{Z}/8\mathbf{Z}.$$

This will come out of some general information on $\mathrm{sBr}(k)$ for arbitrary k and special calculations when $k = \mathbf{R}$. We shall also show that the classes of the Clifford algebras exhaust $\mathrm{sBr}(\mathbf{R})$. Finally by looking at the even parts of the Clifford algebras we shall determine the types of the Clifford modules over $\mathbf{R}$.

First of all we have, for k algebraically closed,

$$\mathrm{sBr}(k)(k) = \{[k], [D]\} = \mathbf{Z}_2.$$

In fact this is clear from the fact that

$$(M^n \otimes D) \otimes (M^n \otimes D) \simeq M^{n^2} \otimes (D \otimes D^{\mathrm{opp}}) \simeq M^{n^2} \otimes M^{1|1} \simeq M^{n^2|n^2}$$

so that $[M^n \otimes D]^2 = 1$. For arbitrary k, going from k to $\overline{k}$ gives a homomorphism

$$\mathrm{sBr}(k) \longrightarrow \mathbf{Z}_2.$$

This is surjective because $[D_k]$ goes over to $[D]$. The kernel is the subgroup H of $\mathrm{sBr}(k)$ of those similarity classes of CS super algebras which become isomorphic to $M^{r|s}$ over $\overline{k}$. For example, the Clifford algebras of even dimensional quadratic vector spaces belong to H. In what follows when we write $A \in H$ we really mean $[A] \in H$.

Fix $A \in H$. Then, $\overline{A} = A_{\overline{k}} \simeq \mathbf{End}_{\overline{k}}(S)$ and so, over $\overline{k}$, $\overline{A}$ has two simple super modules, namely S and ΠS. Let $\dim(S) = r|s$ and let

$$I(A) = \{S, \Pi S\}.$$

Changing S to ΠS we may assume that $r > 0$. We may view these as modules for A over $\overline{k}$. Let L denote one of these and let $\sigma \in G_k := \mathrm{Gal}(\overline{k}/k)$. In S we take a homogeneous basis and view L as a morphism of A into $M^{r|s}(\overline{k})$. Then $a \longmapsto L(a)^\sigma$ is again a representation of A in $\overline{k}$, and its equivalence class does not depend on the choice of the basis used to define L^σ. L^σ is clearly simple and so is isomorphic to either S or ΠS. Hence G_k acts on $I(A)$ and so we have a map α_A from G_k to $\mathbf{Z}_2$ identified with the group of permutations of $I(A)$. If A is purely even, i.e., $s = 0$, then it is clear that $S^\sigma \simeq S$ for any $\sigma \in G_k$. So $\alpha_A(\sigma)$ acts as the identity on $I(A)$ for all σ for such A. Suppose now that A is not purely even so that $r > 0, s > 0$. Let Z^+ be the center of A^+ and $\overline{Z}^+$ its extension to $\overline{k}$, the center of $\overline{A}^+$. Then $\overline{Z}^+$ is canonically isomorphic, over $\overline{k}$, to $\overline{k} \oplus \overline{k}$, and has two characters χ_1, χ_2 where the notation is chosen so that $\overline{Z}^+$ acts on S by $\chi_1 \oplus \chi_2$; then it acts on ΠS by $\chi_2 \oplus \chi_1$. So in this case we can identify $I(A)$ with $\{\chi_1, \chi_2\}$ so that $S \mapsto \Pi S$ corresponds to

$(\chi_1, \chi_2) \mapsto (\chi_2, \chi_1)$. Now G_k acts on $\overline{Z}^+$ and hence on $\{\chi_1, \chi_2\}$, and this action corresponds to the action on $I(A)$. In other words, if we write, for any $\overline{k}$–valued character χ of Z^+, χ^σ for the character

$$\chi^\sigma(z) = \chi(z)^\sigma \qquad (z \in Z^+),$$

then σ fixes the elements of $I(A)$ or interchanges them according as

$$(\chi_1^\sigma, \chi_2^\sigma) = (\chi_1, \chi_2) \text{ or } (\chi_1^\sigma, \chi_2^\sigma) = (\chi_2, \chi_1).$$

Proposition 5.4.7. *The map $A \longmapsto \alpha_A$ is a homomorphism of H into the group* $\mathrm{Hom}(G_k, \mathbf{Z}_2)$. *It is surjective and its kernel K is $Br(k)$. In particular we have an exact sequence*

$$1 \longrightarrow \mathrm{Br}(k) \longrightarrow H \longrightarrow k^\times/(k^\times)^2 \longrightarrow 1.$$

Proof. For any simple module S of a super algebra A, the identity map is an odd bijection interchanging S with ΠS, while for arbitrary linear homogeneous maps we have $p(x\otimes y) = p(x)+p(y)$. So, if $A_1, A_2 \in H$ and $\{S_i, \Pi S_i\}$ are the simple modules for A_i, then $A_1 \otimes A_2 \in H$ and its simple modules are $S_1 \otimes S_2 \simeq \Pi S_1 \otimes \Pi S_2, \Pi(S_1 \otimes S_2) \simeq S_1 \otimes \Pi S_2 \simeq \Pi S_1 \otimes S_2$. This shows that $\alpha_{A_1 \otimes A_2}(\sigma) = \alpha_{A_1}(\sigma)\alpha_{A_2}(\sigma)$ for all $\sigma \in G_k$.

To prove the surjectivity of $A \longmapsto \alpha_A$ let $f \in \mathrm{Hom}(G_k, \mathbf{Z}_2)$. We may assume that f is not trivial. The kernel of f is then a subgroup of G_k of index 2 and so determines a quadratic extension $k' = k(\sqrt{a})$ of k for some $a \in k^\times \setminus {k^\times}^2$. We must find $A \in H$ such that the corresponding α_A is just f, i.e., $S^\sigma \simeq S$ if and only if σ fixes $b = \sqrt{a}$. Let $V = k \oplus k$ with the quadratic form $Q = x^2 - ay^2$. If f_1, f_2 is the standard basis for V, then $Q(f_1) = 1, Q(f_2) = -a$ while $\Phi(f_1, f_2) = 0$. Let $e_1 = bf_1 + f_2, e_2 = (1/4a)(bf_1 - f_2)$. Then, writing Q, Φ for the extensions of Q, Φ to $V' = k' \otimes_k V$, and remembering that $Q(x) = \Phi(x, x)$, we have $Q(e_1) = Q(e_2) = 0$ and $\Phi(e_1, e_2) = 1/2$. The simple module S for $C(V')$ has then the basis $\{1, e_2\}$ with

$$e_1 \longmapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e_2 \longmapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Since $2bf_1 = e_1 + 4ae_2, 2f_2 = e_1 - 4ae_2$, we have

$$f_1 \mapsto \begin{pmatrix} 0 & 1/2b \\ 2a/b & 0 \end{pmatrix}, \qquad f_2 \longmapsto \begin{pmatrix} 0 & 1/2 \\ -2a & 0 \end{pmatrix}.$$

The algebra $C(V')^+ = k'[f_1f_2]$ is already abelian and so coincides with $k' \otimes_k Z^+$. In the module S we have

$$f_1f_2 \longmapsto \begin{pmatrix} -a/b & 0 \\ 0 & a/b \end{pmatrix}.$$

If now σ is the nontrivial element of $\mathrm{Gal}(k'/k)$, then σ changes b to $-b$, so that in S^σ we have

$$f_1f_2 \longmapsto \begin{pmatrix} a/b & 0 \\ 0 & -a/b \end{pmatrix}.$$

Thus

$$S^\sigma \simeq \Pi S$$

which is exactly what we wanted to show.

It remains to determine the kernel K of the homomorphism $A \longmapsto \alpha_A$. Certainly A is in K if it is purely even. Suppose that A is not purely even and $\overline{A}$ isomorphic to $M^{r|s}$ with $r > 0, s > 0$. Using the characters of $\overline{Z}^+$ to differentiate between S and ΠS we see that for α_A to be the identity element of $\mathrm{Hom}(G_k, \mathbf{Z}_2)$ it is necessary and sufficient that $\chi_i^\sigma = \chi_i$ on Z^+, i.e., the χ_i take their values in k. So they are k-valued characters of Z^+. It is then obvious that the map $(\chi_1, \chi_2) : Z^+ \longrightarrow k \oplus k$ is an isomorphism. Conversely if $Z^+ \simeq k \oplus k$ it is obvious that α_A is the identity. So we obtain the result that A lies in K if and only if either A is purely even or the center of its even part is isomorphic over k to $k \oplus k$.

We shall now prove that K is isomorphic to $\mathrm{Br}(k)$. For A in K let D be a super division algebra with super center k such that $[A] = [D]$. Then D^+, which is a division algebra over k, cannot contain a subalgebra isomorphic to $k \oplus k$ and so D must be purely even. For any purely even division algebra D with center k, the algebra $A = M^{r|s}(k) \otimes D$ is, for $s = 0$, purely even and is a classical central simple algebra in the similarity class of the central division algebra D, while for $s > 0$,

$$A^+ \simeq (M^{r|s}(k))^+ \otimes D \simeq (M^r(k) \otimes D) \oplus (M^s(k) \otimes D)$$

and so its center is $\simeq k \oplus k$. Thus the elements of K are the precisely the classical similarity classes of purely even division algebras with center k with multiplication as Brauer multiplication. So the kernel is isomorphic to $\mathrm{Br}(k)$.

To complete the proof it only remains to identify $\mathrm{Hom}(G_k, \mathbf{Z}_2)$ with $k^\times/(k^\times)^2$. The nontrivial elements in $\mathrm{Hom}(G_k, \mathbf{Z}_2)$ are in canonical bijection with the subgroups of G_k of index 2, and these in turn are in canonical bijection with the quadratic extensions of k, and so, by standard results in Galois theory, in correspondence with $k^\times/(k^\times)^2$. We need only verify that this correspondence is a group map. Given $a \in k^\times$, make a fixed choice of $\sqrt{a}$ in $\overline{k}$ and write $b = \sqrt{a}$. For $\sigma \in G_k$, b/b^σ is independent of the choice of the square root of a and so it depends only on a. Let $\chi_a(\sigma) = b/b^\sigma$. Then, as $b^\sigma = \pm b$ it follows that χ_a takes values in $\mathbf{Z}_2$. Moreover, the map $a \longmapsto \chi_a$ is a group homomorphism and $\chi_a = 1$ if and only if $a \in (k^\times)^2$. Thus we have the group isomorphism

$$k^\times/(k^\times)^2 \simeq \mathrm{Hom}(G_k, \mathbf{Z}_2).$$

This finishes the proof.

We suggested earlier that when $k = \mathbf{R}$ the type of Clifford modules of a real quadratic vector space depends only on the signature. For arbitrary k there is a similar result that relates the super Brauer group with the Witt group of the field. Recall that $W(k)$, *the Witt group of* k, is the group F/R where F is the free additive abelian group generated by the isomorphism classes of quadratic vector spaces over k and R is the subgroup generated by the relations

$$[V \oplus V_h] - [V] = 0$$

where V_h is hyperbolic, i.e., of the form $(V_1, Q_1) \oplus (V_1, -Q_1)$. If L is an abelian group and $V \longmapsto f(V)$ a map of quadratic spaces into L, it will define a morphism of $W(k)$ into L if and only if

$$f(V \oplus V_h) = f(V).$$

We write $[V]_W$ for the Witt class of V. As an example let us calculate the Witt group of $\mathbf{R}$. Any real quadratic space V of signature (p, q) is isomorphic to $\mathbf{R}^{p,q}$; we write $\mathrm{sign}(V) = p - q$. It is obvious that in $W(\mathbf{R})$,

$$[\mathbf{R}^{0,1}]_W = -[\mathbf{R}^{1,0}]_W, \qquad [\mathbf{R}^{p,q}]_W = (p-q)[\mathbf{R}^{1,0}]_W.$$

Clearly $\mathrm{sign}(V_h) = 0$ and so $\mathrm{sign}(V \oplus V_h) = \mathrm{sign}(V)$. Thus sign induces a morphism s from $W(\mathbf{R})$ into $\mathbf{Z}$. We claim that this is an isomorphism. To see this let t be the morphism from $\mathbf{Z}$ to $W(\mathbf{R})$ that takes 1 to $[\mathbf{R}^{1,0}]_W$. Clearly $st(1) = 1$ and so st is the identity. Also $s([\mathbf{R}^{p,q}]_W) = p - q$ so that

$$ts([\mathbf{R}^{p,q}]_W) = t(p-q) = (p-q)t(1) = (p-q)[\mathbf{R}^{1,0}]_W = [\mathbf{R}^{p,q}]_W$$

by what we saw above. So ts is also the identity. Thus

$$W(\mathbf{R}) \simeq \mathbf{Z}.$$

Now we have a map
$$V \longmapsto [C(V)]$$
from quadratic spaces into the super Brauer group of k and we have already seen that $C(V_h)$ is a full matrix super algebra over k. Hence $[C(V_h)]$ is the identity and so
$$[C(V \oplus V_h)] = [C(V) \otimes C(V_h)] = [C(V)].$$
Thus by what we said above we have a map
$$f : W(k) \longrightarrow \mathrm{sBr}(k)$$
such that for any quadratic vector space V,
$$f([V]_W) = [C(V)].$$

The representation theory of the even parts of CS super algebras. For applications we need the representation theory of the algebra C^+ where C is a Clifford algebra. More generally let us examine the representation theory of algebras A^+ where A is a CS super algebra over k. If A is purely even there is nothing more to do as we are already in the theory of the classical Brauer group. Thus all simple modules of A over k have commutants D^{opp} where D is the (purely even) central division algebra in the similarity class of A. So we may assume that A is not purely even. Then we have the following proposition.

Proposition 5.4.8. *Let A be a* CS *super algebra which is not purely even and write $A = M^{r|s}(k) \otimes B$ where B is the central super division algebra in the similarity class of A. Then we have the following.*

(i) *If B is purely even, $A^+ \simeq M^r(B) \oplus M^s(B)$ where $M^p(B) = M^p \otimes B$.*

(ii) *If B is not purely even, then*
$$A \simeq M^{r+s}(B), \qquad A^+ \simeq M^{r+s}(B^+).$$

In particular, A^+ is always semisimple as a classical algebra, and the types of its simple modules depend only on the class of A in $\mathrm{sBr}(k)$. *In case* (i) *A has two simple modules both with commutants B^{opp} while in case* (ii) *A has a unique simple module with commutant ${B^+}^{\mathrm{opp}}$.*

Proof. Obvious.

Remark. It must be noted that when B is not purely even, B^+ need not be central.

The case of the field of real numbers. Let us now take $k = \mathbf{R}$. Then $\mathbf{R}^\times/\mathbf{R}^{\times 2} = \mathbf{Z}_2$ while $\mathrm{Br}(\mathbf{R}) = \mathbf{Z}_2$ also. Hence, by Proposition 7,

$$|\mathrm{sBr}(\mathbf{R})| = 8.$$

On the other hand, as $W(\mathbf{R}) \simeq \mathbf{R}$, there is a homomorphism f of $\mathbf{Z}$ into $\mathrm{sBr}(\mathbf{R})$ such that if V is a real quadratic space, then

$$[C(V)] = f(\mathrm{sign}(V))$$

where $\mathrm{sign}(V)$ is the signature of V. Since $\mathrm{sBr}(\mathbf{R})$ is of order 8 it follows that the class of $C(V)$ depends only on the signature mod 8.

It remains to determine $\mathrm{sBr}(\mathbf{R})$ and the map $V \longrightarrow [C(V)]$. We shall show that $\mathrm{sBr}(\mathbf{R})$ is actually cyclic, i.e., it is equal to $\mathbf{Z}_8 = \mathbf{Z}/8\mathbf{Z}$, and that f is the natural map $\mathbf{Z} \longrightarrow \mathbf{Z}_8$. We shall show that $\mathbf{R}[\varepsilon]$ has order 8. If V is a real quadratic space of dimension 1 containing a unit vector, $C(V)$ is the algebra $\mathbf{R}[\varepsilon]$ where ε is odd and $\varepsilon^2 = 1$. Its opposite is $\mathbf{R}[\varepsilon^0]$ where ε^0 is odd and ${\varepsilon^0}^2 = -1$:

$$\mathbf{R}[\varepsilon]^{\mathrm{opp}} = \mathbf{R}[\varepsilon^0].$$

Both $\mathbf{R}[\varepsilon]$ and $\mathbf{R}[\varepsilon^0]$ are central super division algebras and so, as the order of $\mathrm{sBr}(\mathbf{R})$ is 8, their orders can only be 2, 4 or 8. We wish to exclude the possibilities that the orders are 2 and 4. We consider only $\mathbf{R}[\varepsilon]$. Write $A = \mathbf{R}[\varepsilon]$.

By direct computation we see that $A \otimes A$ is the algebra $\mathbf{R}[\varepsilon_1, \varepsilon_2]$ where the ε_i are odd, $\varepsilon_i^2 = 1$, and $\varepsilon_1\varepsilon_2 = -\varepsilon_2\varepsilon_1$. We claim that this is a central super division algebra. It is easy to check that the super center of this algebra is just $\mathbf{R}$. We claim that it is a super division algebra. The even part is $\mathbf{R}[\varepsilon_1\varepsilon_2]$, and as $(\varepsilon_1\varepsilon_2)^2 = -1$ it is immediate that it is $\simeq \mathbf{C}$, hence a division algebra. On the other hand $(u\varepsilon_1 + v\varepsilon_2)^2 = u^2 + v^2$ and so $u\varepsilon_1 + v\varepsilon_2$ is invertible as soon as $(u, v) \neq (0, 0)$. Thus $\mathbf{R}[\varepsilon_1, \varepsilon_2]$ is a central super division algebra. We claim that its square, namely the class of $[A]^4$ is nontrivial and in fact is purely even and represented by $\mathbf{H}$, the purely even algebra of quaternions. First of all if $[A]^4$ were trivial we should have $[A]^2 = [A^{\mathrm{opp}}]^2$ which would mean that the corresponding super division algebras must be isomorphic. Thus $\mathbf{R}[\varepsilon_1, \varepsilon_2] \simeq \mathbf{R}[{\varepsilon_1}^0, {\varepsilon_2}^0]$. Then we should be able to find $a, b \in \mathbf{R}$ such that $(a\varepsilon_1 + b\varepsilon_2)^2 = a^2 + b^2 = -1$ which is impossible. So $[A]^4 \neq 1$. Hence $[A]$ must be of order 8, proving that $\mathrm{sBr}(\mathbf{R})$ is cyclic of order 8 and is generated by $\mathbf{R}[\varepsilon]$.

The central super division algebras corresponding to the powers $[\mathbf{R}[\varepsilon]]^m (0 \leq m \leq 7)$ are thus the representative elements of $\mathrm{sBr}(\mathbf{R})$. These can now be written down. For $m = 2$ it is $\mathbf{R}[\varepsilon_1, \varepsilon_2]$. Now, $[\mathbf{R}[\varepsilon]]^2$ becomes isomorphic over $\mathbf{C}$ to $D \otimes D \simeq M^{1|1}$. If we go back to the discussion in Proposition 7 we then see that $[A]^2 \in H$ and $[A]^4 \in \mathrm{Br}(\mathbf{R})$; as $[A]^4$ is a nontrivial element of $\mathrm{Br}(\mathbf{R})$, the corresponding division algebra must be purely even and isomorphic to $\mathbf{H}$. Thus for $m = 4$ it is purely even and $\mathbf{H}$. For $m = 6$ it is the opposite of the case $m = 2$ and so is $\mathbf{R}[\varepsilon_1^0, \varepsilon_2^0]$. We now consider the values $m = 3, 5, 7$. But $[A]^7 = [A^{\mathrm{opp}}]$ and $[A]^5 = [A^{\mathrm{opp}}]^3$. Now $[A]^5 = [A]^4 \cdot [A]$ and so $\mathbf{H} \otimes \mathbf{R}[\varepsilon]$ is in the class $[A]^5$, $\mathbf{H}$ being viewed as purely even. It is immediate that $\mathbf{H} \otimes \mathbf{R}[\varepsilon] = \mathbf{H} \oplus \mathbf{H}\varepsilon$ is already a super division algebra and so is the one defining the class $[A]^5$. Consequently, $[A]^3$ corresponds to the super division algebra $\mathbf{H} \otimes \mathbf{R}[\varepsilon^0]$. We have thus obtained the following result.

Theorem 5.4.9. *The group* $\mathrm{sBr}(\mathbf{R})$ *is cyclic of order* 8 *and is generated by* $[\mathbf{R}[\varepsilon]]$. *If* V *is a real quadratic space then* $[C(V)] = [\mathbf{R}[\varepsilon]]^{\mathrm{sign}(V)}$ *where* $\mathrm{sign}(V)$ *is the signature of* V. *The central super division algebras* $D(m)$ *in the classes* $[\mathbf{R}[\varepsilon]]^m (0 \leq m \leq 7)$ *are given as follows.*

m	$D(m)$
0	$\mathbf{R}$
1	$\mathbf{R}[\varepsilon]$
2	$\mathbf{R}[\varepsilon_1, \varepsilon_2]$
3	$\mathbf{R}[\varepsilon^0]$
4	$\mathbf{H}$
5	$\mathbf{H} \otimes \mathbf{R}[\varepsilon]$
6	$\mathbf{R}[\varepsilon_1^0, \varepsilon_2^0]$
7	$\mathbf{R}[\varepsilon^0]$

In the above $\mathbf{R}[\varepsilon_1, \varepsilon_2]$ *is the* (*super division*) *algebra generated over* $\mathbf{R}$ *by* $\varepsilon_1, \varepsilon_2$ *with* $\varepsilon_j^2 = 1$ $(j = 1, 2)$, $\varepsilon_1\varepsilon_2 = -\varepsilon_2\varepsilon_1$, *while* $\mathbf{R}[\varepsilon_1^0, \varepsilon_2^0]$ *is the* (*super division*) *algebra generated over* $\mathbf{R}$ *by* $\varepsilon_1^0, \varepsilon_2^0$ *with* ${\varepsilon_j^0}^2 = -1$ $(j = 1, 2)$, $\varepsilon_1^0\varepsilon_2^0 = -\varepsilon_2^0\varepsilon_1^0$.

Reality of Clifford modules. We are now in a position to describe the representation theory of the Clifford modules over $\mathbf{R}$, namely the types of the simple modules for $C(V)^+$ where V is a real quadratic vector space. Here we have to go from $C(V)$ to $C(V)^+$ and we use Proposition 8 for this purpose. We must remember during the following discussion that the dimension and signature of a real quadratic vector space are of the same parity. The only purely even central super division algebras over $\mathbf{R}$ are $\mathbf{R}$ and $\mathbf{H}$. If the class of $C(V)$ corresponds to $\mathbf{R}$ (resp. $\mathbf{H}$), then Proposiiton 8 shows that $C(V)^+$ has two simple modules with commutant $\mathbf{R}$ (resp. $\mathbf{H}$). From Theorem 9 we see that this happens if and only if $\mathrm{sign}(V) \equiv 0 \pmod 8$ (resp. $\equiv 4 \pmod 8$) and the corresponding commutant is $\mathbf{R}$ (resp. $\mathbf{H}$). For the remaining values of the signature, the class of $C(V)$ is not purely even. For the values (mod 8) 1, 3, 5, 7 of the signature of V, the commutant of the simple module is respectively $\mathbf{R}, \mathbf{H}, \mathbf{H}, \mathbf{R}$ and for these values C^+ has a unique simple module with commutant respectively $\mathbf{R}, \mathbf{H}, \mathbf{H}, \mathbf{R}$. For the values 2, 6 (mod 8) of the signature of V, $C(V)^+$ has a unique simple module with commutant $\mathbf{C}$. Hence we have proved the following theorem.

Theorem 5.4.10. *Let V be a real quadratic vector space and let $s = \mathrm{sign}(V)$ be its signature. Then $C(V)^+$ is semisimple and the commutants of the simple modules of $C(V)^+$, which are also the commutants of the simple spin modules of* $\mathrm{Spin}(V)$*, are given as follows:*

s mod 8	commutant
0	$\mathbf{R}$, $\mathbf{R}$
1, 7	$\mathbf{R}$
2, 6	$\mathbf{C}$
3, 5	$\mathbf{H}$
4	$\mathbf{H}$, $\mathbf{H}$

Remark. One may ask how much of this theory can be obtained by arguments of a general nature. Let us first consider the case when $\dim(V)$ is odd. Then $C(V)^+_{\mathbf{C}}$ is a full matrix algebra. So we are led to the following general situation. We have a real algebra A with complexification A_c which is a full matrix algebra. So $A_{\mathbf{C}}$ has a unique simple module S and we wish to determine the types of simple modules of A over $\mathbf{R}$. The answer is that A also has a *unique* simple module over $\mathbf{R}$, but this may be either of real type or quaternionic type. To see this we first make the simple remark that if M, N are two real modules for a real algebra and $M_{\mathbf{C}}, N_{\mathbf{C}}$ are their complexifications, then

$$\mathrm{Hom}_{A_{\mathbf{C}}}(M_{\mathbf{C}}, N_{\mathbf{C}}) \neq 0 \Longrightarrow \mathrm{Hom}_A(M, N) \neq 0.$$

Indeed, there is a natural conjugation in the complex Hom space ($\overline{f}(m) = \overline{f(\overline{m})}$) and the real Hom space consists precisely of those elements of the complex Hom space fixed by it, so that the real Hom spans the complex Hom over $\mathbf{C}$. This proves the above implication. This said, let $S_{\mathbf{R}}$ be a real simple module for A and $S_{\mathbf{C}}$ its complexification. If $S_{\mathbf{R}}$ is of type $\mathbf{R}$, then $S_{\mathbf{C}}$ is simple and so $\simeq S$. If S' is another simple real module of type $\mathbf{R}$, its complexification $S'_{\mathbf{C}}$ is also $\simeq S$, and so by the remark above, $\mathrm{Hom}(S_{\mathbf{R}}, S') \neq 0$ showing that $S' \simeq S_{\mathbf{R}}$. If S' were to be of type $\mathbf{H}$, its commutant is of dimension 4 and so $S'_{\mathbf{C}} = 2S$; but then $2S$ has two real forms, namely, $2S_{\mathbf{R}}, S'$, hence $\mathrm{Hom}(S', 2S_{\mathbf{R}}) \neq 0$, a contradiction. If S' is of type $\mathbf{C}$ its commutant is of dimension 2 and so the same is true for $S'_{\mathbf{C}}$; but the commutant in aS is of dimension a^2, so that this case does not arise. Thus A has also a unique simple module but it may be either of type $\mathbf{R}$ or type $\mathbf{H}$. Now, for a Clifford algebra C over $\mathbf{R}$ of odd dimension, $C^+_{\mathbf{C}}$ is a full matrix algebra and so the above situation applies. The conclusion is that there is a unique simple spin module over $\mathbf{R}$ which may be of type $\mathbf{R}$ or $\mathbf{H}$.

In the case when V has even dimension $2m$, the argument is similar but slightly more involved because the even part of the Clifford algebra now has two simple modules over the complexes, say $S^{\pm}$. In fact, if

$$S : C(V)_{\mathbf{C}} \simeq \mathbf{End}\left(\mathbf{C}^{2^{m-1}|2^{m-1}}\right)$$

then

$$S(a) = \begin{pmatrix} S^+(a) & 0 \\ 0 & S^-(a) \end{pmatrix} \qquad (a \in C(V)^+_{\mathbf{C}})$$

and $S^{\pm}$ are the two simple modules for $C(V)^+_{\mathbf{C}}$. However these two are exchanged by inner automorphisms of the Clifford algebra that are induced by real invertible odd elements. Let g be a real invertible odd element of $C(V)$. Then

$$S(g) = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$$

and we find

$$S(gag^{-1}) = \begin{pmatrix} \alpha S^-(a)\alpha^{-1} & 0 \\ 0 & \beta S^+(a)\beta^{-1} \end{pmatrix} \qquad (a \in C(V)^+_{\mathbf{C}})$$

so that

$$S^{+g} \simeq S^-, \qquad S^{-g} \simeq S^+ \qquad (S^{\pm g}(a) = S^{\pm}(gag^{-1}), a \in C(V)^+_{\mathbf{C}}).$$

If now g is real, i.e., $g \in C(V)$, then the inner automorphism by g preserves $C(V)^+$ and exchanges $S^{\pm}$. Such g exist: if $u \in V$ has unit norm, then $u^2 = 1$ so that u is real, odd, and invertible ($u^{-1} = u$). The situation here is therefore of a real algebra A with complexification $A_{\mathbf{C}}$ which is semisimple and has two simple modules $S^{\pm}$ which are exchanged by an automorphism of A. In this case A has either two or one simple modules: if it has two, both are of the same type which is either **R** or **H**. If it has just one, it is of type **C**.

To prove this we remark first that if S' is a simple module for A, $S'_{\mathbf{C}}$ is $S^{\pm}, S^+ \oplus S^-, 2S^{\pm}$ according as S' is of type $\mathbf{R}, \mathbf{C}, \mathbf{H}$. This statement is obvious for the real type. If the type is **C** the commutant has dimension 2; the complexification is $mS^+ \oplus nS^-$, whose commutant has dimension $m^2 + n^2$ and this is 2 only when $m = n = 1$. If S' is of type **H**, the commutant is of dimension 4 and $m^2 + n^2 = 4$ only for $m = 2, n = 0$ or $m = 0, n = 2$. This said, assume first that $S_{\mathbf{R}}$ is a simple module for A of type **R**. Then its complexification is either S^+ or S^-. Using the automorphism g we obtain a second simple module of type **R** whose complexification is the other of $S^{\pm}$. So we have simple modules $S^{\pm}_{\mathbf{R}}$ of type **R** with complexifications $S^{\pm}$. There will be no other simple modules of type **R**, and in fact, no others of other types also. For, if S' is simple of type **C**, its complexification is $S^+ \oplus S^-$ which has 2 real forms, namely $S^+_{\mathbf{R}} \oplus S^-_{\mathbf{R}}$ as well as S' which is impossible by our remark. If S' is quaternionic, the same argument applies to $2S^+ \oplus 2S^-$.

If A has a simple module of complex type, it has to be unique since its complexification is uniquely determined as $S^+ \oplus S^-$, and by the above argument A cannot have any simple module of type **R**. But A cannot have a simple module of type **H** also. For, if S' were to be one such, then the complexification of S' is $2S^{\pm}$, and the argument using the odd automorphism g will imply that A will have two simple modules $S^{\pm}_{\mathbf{H}}$ with complexifications $2S^{\pm}$; but then $2S^+ \oplus 2S^-$ will have two real forms, $S^+_{\mathbf{H}} \oplus S^-_{\mathbf{H}}$ and $2S'$ which is impossible.

Finally, if $S_{\mathbf{R}}$ is of type **H**, then what we have seen above implies that A has two simple modules of type **H** and no others.

However these general arguments cannot decide when the various alternatives occur nor will they show that these possibilities are governed by the value of the signature mod 8. That can be done only by a much closer analysis.

The method of Atiyah–Bott–Shapiro. They worked with the definite case, and among many other things, they determined in[7] the structure of the Clifford algebras and their even parts over the reals. Now all signatures are obtained from the definite quadratic spaces by adding hyperbolic components. In fact,

$$\mathbf{R}^{p,q} = \begin{cases} \mathbf{R}^{p,p} \oplus \mathbf{R}^{0,q-p} \ (0 \le p \le q) \\ \mathbf{R}^{q,q} \oplus \mathbf{R}^{p-q,0} \end{cases} (0 \le q \le p), \qquad [\mathbf{R}^{m,0}] = -[\mathbf{R}^{0,m}].$$

It is therefore enough to determine the types of the Clifford algebras where the quadratic form is *negative definite.* This is what is done in[7]. We shall present a variant of their argument in what follows. The argument is in two steps. We first take care of the definite case, and then reduce the general signature (p, q) to the signature $(0, q')$.

We first consider only negative definite quadratic vector spaces and it is always a question of ungraded algebras and ungraded tensor products. We write C_m for the ungraded Clifford algebra of the real quadratic vector space $\mathbf{R}_{0,m}$. It is thus generated by $(e_j)_{1\le j\le m}$ with relations

$$e_j^2 = -1 \ (1 \le j \le m), \quad e_r e_s + e_s e_r = 0 \ (r \ne s).$$

Let us write M^r for the matrix algebra $M^r(\mathbf{R})$. The algebra generated by e_1, e_2 with the relations

$$e_1^2 = e_2^2 = 1, \quad e_1 e_2 + e_2 e_1 = 0,$$

is clearly isomorphic to M^2 by

$$e_1 \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_2 \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

On the other hand, if $F_\pm = \mathbf{R}[e]$ where $e^2 = \pm 1$, then

$$F+ \simeq \mathbf{R} \oplus \mathbf{R}, \quad a + be \longmapsto (a + b, a - b), \qquad F_- \simeq \mathbf{C}, \quad a + be \longmapsto a + ib.$$

Hence for any algebra A, we have

$$A \otimes F_+ = A[e] = A \oplus A.$$

Finally we have the obvious isomorphisms of Clifford algebras

$$C_1 \simeq \mathbf{C}, \quad C_2 \simeq \mathbf{H}.$$

In what follows we write $\mathbf{C}$ for the complex numbers viewed as an $\mathbf{R}$-algebra.

We consider first the case when $m = 2n$ is even. Then we know that the center of C_{2n} is $\mathbf{R}$. Let

$$f_1 = e_1 \ldots e_{2n-2} e_{2n-1}, \quad f_2 = e_1 \ldots e_{2n-2} e_{2n}.$$

It is then immediate that the f_i commute with the $e_j (1 \le j \le 2n - 2)$, while

$$f_1^2 = f_2^2 = (-1)^n, \quad f_1 f_2 + f_2 f_1 = 0.$$

Hence the algebra A_n generated by f_1, f_2 is isomorphic to C_2 if n is odd and to M^2 if n is even. Moreover
$$C_{2n} = C_{2n-2} \otimes A_n.$$
We therefore have
$$C_{4n+2} = C_{4n} \otimes \mathbf{H}, \qquad C_{4n} = C_{4n-2} \otimes M^2.$$
Using the fact that $\mathbf{H} \otimes \mathbf{H} = \mathbf{H} \otimes \mathbf{H}^0 = M^4$ we obtain
$$C_{4n} = C_{4n-2} \otimes M^2 = C_{4n-4} \otimes M^2 \otimes \mathbf{H} = C_{4n-6} \otimes M^4 \otimes \mathbf{H} = C_{4n-8} \otimes M^{16}$$
$$C_{4n+2} = C_{4n} \otimes \mathbf{H} = C_{4n-8} \otimes M^{16} \otimes \mathbf{H} = C_{4n-6} \otimes M^{16}.$$
Thus we have the periodicity
$$C_{2n+8} = C_{2n} \otimes M^{16}.$$
Moreover,
$$C_2 = \mathbf{H},\ C_4 = \mathbf{H} \otimes M^2,\ C_6 = \mathbf{H} \otimes M^2 \otimes \mathbf{H} = M^8,\ C_8 = M^8 \otimes M^2 = M^{16}.$$
We thus obtain the following table:
$$\begin{array}{lll} C_2 = \mathbf{H} & C_4 = M^2 \otimes \mathbf{H} & \\ C_6 = M^8 & C_8 = M^{16} & C_{2n+8} = C_{2n}. \end{array}$$

We take up next the C_m with odd $m = 2n+1$. Take the basis as $e_j (0 \leq j \leq 2n+1)$ and let
$$\gamma = e_0 e_1 \ldots e_{2n}.$$
Then by Proposition 2.1. γ commutes with all the $e_j (1 \leq j \leq 2n)$ and
$$\gamma^2 = (-1)^{n+1}.$$
Moreover,
$$C_{2n+1} \simeq C_{2n}, \qquad C_{2n+1} \simeq C_{2n} \otimes \mathbf{R}[\gamma]$$
by Proposition 2.2. Hence we have
$$C_{4n+1} = C_{4n} \otimes \mathbf{C}, \qquad C_{4n+3} = C_{4n+2} \otimes F = C_{4n+2} \oplus C_{4n+2}.$$

Thus, writing A for $\mathbf{C}$ or F, we have

$$C_{2n+9} = C_{2n+8} \otimes A = C_{2n} \otimes A \otimes M^{16} = C_{2n+1} \otimes M^{16}$$

since $2n+9$ and $2n+1$ have the same residue mod 4. So there is again periodicity mod 8. Now $C_5 = \mathbf{H} \otimes \mathbf{C} \otimes M^2$ while $\mathbf{H} \otimes \mathbf{C}$, viewed as a complex algebra, is just $M^2(\mathbf{C}) = M^2 \otimes \mathbf{C}$, so that $C_5 = M^4 \otimes \mathbf{C}$. Hence we have the following table:

$$\begin{array}{lll} C_1 = \mathbf{C} & C_3 = \mathbf{H} \oplus \mathbf{H} & \\ C_5 = M^4 \otimes \mathbf{C} & C_7 = M^8 \oplus M^8 & \mathbf{C}_{2n+9} = C_{2n+1} \otimes M^{16}. \end{array}$$

Combining the odd and even cases we thus have finally the table:

$$\begin{array}{lll} C_1 = \mathbf{C} & C_2 = \mathbf{H} & \\ C_3 = \mathbf{H} \oplus \mathbf{H} & C_4 = M^2 \otimes \mathbf{H} & C_{m+8} = C_m \\ C_5 = M^4 \otimes \mathbf{C} & C_6 = M^8 & \\ C_7 = M^8 \oplus M^8 & C_8 = M^{16}. & \end{array}$$

It only remains to determine the structure of the even parts. We have

$$C^+_{n+1} = C_n$$

since the $e_0 e_j (1 \le j \le n)$ generate C^+_{n+1} and they generate also C_n. Also

$$C^+_1 = \mathbf{R}.$$

Hence we have the table:

$$\begin{array}{lll} C^+_1 = \mathbf{R} & C^+_2 = \mathbf{C} & \\ C^+_3 = \mathbf{H} & C^+_4 = \mathbf{H} \oplus \mathbf{H} & C^+_{m+8} = C^+_m \otimes M^{16} \\ C^+_5 = M^2 \otimes \mathbf{H} & C^+_6 = M^4 \otimes \mathbf{C} & \\ C^+_7 = M^8 & C^+_8 = M^8 \oplus M^8. & \end{array}$$

We now take up the case of the general signature (p, q). Once again it is a matter of ungraded algebras and tensor products. We write $C_{p,q}$ for the ungraded Clifford algebra of $\mathbf{R}^{p,q}$, namely, the algebra with generators $e_i (1 \le i \le D = p+q)$ and relations $e_i^2 = \varepsilon_i, e_i e_j + e_j e_i = 0 (i \ne j)$; here the ε_i are all ± 1 and exactly q of them are equal to -1. We also write, for typographical reasons, $M(r)$ for $M^r(\mathbf{R})$, and $2A$ for $A \oplus A$. By convention $C_{0,0} = C^+_{0,0} = \mathbf{R}$.

We first note that $C^+_{p,q}$ is generated by the $g_r = e_1 e_r (2 \le r \le D)$, with the relations

$$g_r^2 = -\varepsilon_1 \varepsilon_r, \qquad g_r g_s + g_s g_r = 0 \; (r \neq s).$$

If both p and q are ≥ 1 we can renumber the basis so that ε_1 takes both values ± 1, and in case one of them is 0, we have no choice about the sign of ε_1. Hence we have

$$C^+_{p,q} = C^+_{q,p} = C_{p,q-1} = C_{q,p-1} \qquad (p, q \ge 0, p+q \ge 1)$$

with the convention that when p or q is 0 we omit the relation involving $C_{a,b}$ where one of a, b is < 0.

First assume that D is even and ≥ 2. Then $C_{p,q}$ is a central simple algebra. As in the definite case we write

$$f_1 = e_1 \ldots e_{D-2} e_{D-1}, \qquad f_2 = e_1 \ldots e_{D-2} e_D.$$

Then it is immediate that the f_j commute with all the $e_i (1 \le i \le D-2)$, while $f_1 f_2 + f_2 f_1 = 0$ and

$$f_1^2 = (-1)^{\frac{D}{2}-1} \varepsilon_1 \ldots \varepsilon_{D-2} \varepsilon_{D-1}, \qquad f_2^2 = (-1)^{\frac{D}{2}-1} \varepsilon_1 \ldots \varepsilon_{D-2} \varepsilon_D.$$

If $\varepsilon_j (j = D-1, D)$ are of opposite signs, the algebra generated by f_1, f_2 is $C_{1,1}$ while the algebra generated by the $e_i (1 \le i \le D-2)$ is $C_{p-1,q-1}$. Hence we get

$$C_{p,q} = C_{p-1,q-1} \otimes M(2).$$

Repeating this process we get

$$C_{p,q} = \begin{cases} C_{0,q-p} \otimes M(2^p) \; (1 \le p \le q, \; D = p+q \; \text{ is even}) \\ C_{p-q,0} \otimes M(2^q) \; (1 \le q \le p, \; D = p+q \; \text{ is even}). \end{cases}$$

Let us now take up the case when D is odd. Let $\gamma = e_1 e_2 \ldots e_D$. By Propositions 2.1 and 2.2, $\gamma^2 = (-1)^{\frac{p-q-1}{2}}$ and $C_{p,q} = C^+_{p,q} \otimes \mathrm{ctr}(C_{p,q})$ while $\mathrm{ctr}(C_{p,q}) = \mathbf{R}[\gamma]$. We have already seen that $C^+_{p,q} = C_{p,q-1}$ while $\mathbf{R}[\gamma] = \mathbf{R} \oplus \mathbf{R}$ or $\mathbf{C}$ according as $q - p$ is of the form $4\ell + 3$ or $4\ell + 1$. Hence

$$C_{p,q} = \begin{cases} 2C^+_{p,q} & \text{if } p - q = 4\ell + 1 \\ C^+_{p,q} \otimes \mathbf{C} & \text{if } p - q = 4\ell + 3. \end{cases}$$

From this discussion it is clear that the structure of $C_{p,q}$ can be determined for all p, q.

We are now in a position to determine the types of the simple modules of $C^+_{p,q}$ using our results for the algebras $C_{0,n}$ and $C^+_{0,n}$, especially the periodicity mod 8 established for them. It is enough to consider the case $p \le q$.

D odd: If $p < q$, then $C^+_{p,q} = C_{p,q-1}$ is a central simple algebra and so has a unique simple module. Since $C_{p,q-1} = C_{0,q-p-1} \otimes M(2^p)$, it is immediate that the type of the simple modules of $C^+_{p,q}$ is determined by $q-p$ mod 8; it is $\mathbf{R}$ or $\mathbf{H}$ according as $q-p \equiv 1, 7$ mod 8 or $q-p \equiv 3, 5$ mod 8.

D even: We first assume that $0 < p < q$ so that $q \ge p+2$. Then

$$C^+_{p,q} = C_{p,q-1} = \begin{cases} 2C_{p,q-2} & \text{if } q-p = 4\ell \\ C_{p,q-2} \otimes \mathbf{C} & \text{if } q-p = 4\ell+2. \end{cases}$$

Since $C_{p,q-2} = C_{0,q-p-2} \otimes M(2^p)$ it is now clear that $C^+_{p,q}$ has two simple modules, both with the same commutant, when $q-p \equiv 0,\ 4$ mod 8, the commutant being $\mathbf{R}$ when $q-p \equiv 0$ mod 8, and $\mathbf{H}$ when $q-p \equiv 4$ mod 8. If $q-p \equiv 2,\ 6$ mod 8, there is a unique simple module with commutant $\mathbf{C}$.

There remains the case $p = q$. In this case $C^+_{p,p}$ is a direct sum of two copies of $M(2^{p-1})$ and so there are two simple modules of type $\mathbf{R}$.

Theorem 10 is now an immediate consequence. The following table summarizes the discussion.

$q-p$ mod 8	$C_{p,q}$	$C^+_{p,q}$
0	$M(2^{D/2})$	$2M(2^{(D-2)/2})$
1	$M(2^{(D-1)/2}) \otimes \mathbf{C}$	$M(2^{(D-1)/2})$
2	$M(2^{(D-2)/2}) \otimes \mathbf{H}$	$M(2^{(D-2)/2}) \otimes \mathbf{C}$
3	$2M(2^{(D-3)/2}) \otimes \mathbf{H}$	$M(2^{(D-3)/2}) \otimes \mathbf{H}$
4	$M(2^{(D-2)/2}) \otimes \mathbf{H}$	$2M(2^{(D-4)/2}) \otimes \mathbf{H}$
5	$M(2^{(D-1)/2}) \otimes \mathbf{C}$	$M(2^{(D-3)/2}) \otimes \mathbf{H}$
6	$M(2^{D/2})$	$M(2^{(D-2)/2}) \otimes \mathbf{C}$
7	$2M(2^{(D-1)/2})$	$M(2^{(D-1)/2)})$.

5.5. Pairings and morphisms. For various purposes in physics one needs to know the existence and properties of morphisms

$$S_1 \otimes S_2 \longrightarrow \Lambda^r(V) \qquad (r \geq 0)$$

where S_1, S_2 are irreducible spin modules for a quadratic vector space V and $\Lambda^r(V)$ is the r^{th} exterior power with $\Lambda^0(V) = k$, k being the ground field. For applications to physics the results are needed over $k = \mathbf{R}$, but to do that we shall again find it convenient to work over $\mathbf{C}$ and then use descent arguments to come down to $\mathbf{R}$. Examples of questions we study are the existence of $\mathrm{Spin}(V)$-invariant forms on $S_1 \times S_2$ and whether they are symmetric or skewsymmetric, needed for writing the mass terms in the Lagrangian; the existence of symmetric morphisms $S \otimes S \longrightarrow V$ as well as $S \otimes S \longrightarrow \Lambda^r(V)$ needed for the construction of super Poincaré and super conformal algebras we need; and the existence of morphisms $V \otimes S_1 \longrightarrow S_2$ needed for defining the Dirac operators and writing down kinetic terms in the Lagrangians we need. Our treatment follows closely that of Deligne[8].

We begin by studying the case $r = 0$, i.e., forms invariant under the spin groups (over $\mathbf{C}$). Right at the outset we remark that if S is an irreducible spin module, the forms on S, by which we always mean nondegenerate bilinear forms on $S \times S$, define isomorphisms of S with its dual and so, by irreducibility, are unique up to scalar factors (whenever they exist). The basic lemma is the following.

Lemma 5.5.1. *Let V be a complex quadratic vector space and S a spinorial module, i.e., a $C(V)^+$-module. Then a form $(\cdot,\cdot)$ is invariant under* $\mathrm{Spin}(V)$ *if and only if*

$$(as, t) = (s, \beta(a)t) \qquad (s, t \in S, a \in C(V)^+) \tag{$*$}$$

where β is the principal antiautomorphism of $C(V)$.

Proof. We recall that β is the unique antiautomorphism of $C(V)$ which is the identity on V. If the above relation is true, then taking $a = g \in \mathrm{Spin}(V) \subset C(V)^+$ shows that $(gs, t) = (s, g^{-1}t)$ since $\beta(g) = g^{-1}$. In the other direction, if $(\cdot,\cdot)$ is invariant under $\mathrm{Spin}(V)$, we must have $(as, t) + (s, at) = 0$ for $a \in C^2 \simeq \mathrm{Lie}(\mathfrak{so}(V))$. But, for $a = uv - vu$ where $u, v \in V$, we have $\beta(a) = -a$ so that $(as, t) = (s, \beta(a)t)$ for $a \in C^2$. Since C^2 generates $C(V)^+$ as an associative algebra we have $(*)$.

It is not surprising that information about invariant forms is controlled by antiautomorphisms. For instance, suppose that U is a purely even vector space and $A = \mathrm{End}(U)$; then there is a bijection between antiautomorphisms β of A and forms $(\cdot,\cdot)$ on U defined up to a scalar multiple such that

$$(as, t) = (s, \beta(a)t) \qquad (s, t \in U, a \in A).$$

In fact, if $(\cdot, \cdot)$ is given, then for each $a \in A$ we can define $\beta(a)$ by the above equation and then verify that β is an antiautomorphism of A. The form can be changed to a multiple of it without changing β. In the reverse direction, suppose that β is an antiautomorphism of A. Then we can make the dual space U^* a module for A by writing

$$(as^*)[t] = s^*[\beta(a)t]$$

and so there is an isomorphism $B_\beta : U \simeq U^*$ of A-modules. The form

$$(s, t) := B_\beta(s)[t]$$

then has the required relationship with β. Since B_β is determined up to a scalar, the form determined by β is unique up to a scalar multiple. If $(s, t)' := (t, s)$, it is immediate that $(as, t)' = (s, \beta^{-1}(a)t)'$ and so $(\cdot, \cdot)'$ is the form corresponding to β^{-1}. In particular, β is involutive if and only if $(\cdot, \cdot)'$ and $(\cdot, \cdot)$ are proportional, i.e., $(\cdot, \cdot)$ is either symmetric or skewsymmetric. Now suppose that U is a *super* vector space and $A = \mathbf{End}(U)$; then for *even* β U^* is a super module for A and so is either isomorphic to U or its parity reversed module ΠU, so that B_β above is even or odd. Hence the corresponding form is even or odd accordingly. Recall that for an even (odd) form we have $(s, t) = 0$ for unlike(like) pairs s, t. Thus we see that if $A \simeq M^{r|s}$ and β is an involutive even antiautomorphism of A, we can associate to (A, β) two invariants coming from the form associated to β, namely, the parity $\pi(A, \beta)$ of the form which is a number 0 or 1, and the symmetry $\sigma(A, \beta)$ which is a sign $\pm$, $+$ for symmetric and $-$ for skewsymmetric forms.

In view of these remarks and the basic lemma above we shall base our study of invariant forms for spin modules on the study of pairs $(C(V), \beta)$ where $C(V)$ is the Clifford algebra of a complex quadratic vector space and β is its principal antiautomorphism, namely the one which is the identity on V. Inspired by the work in §4 we shall take a more general point of view and study pairs (A, β) where A is a CS super algebra over $\mathbf{C}$ and β is an even involutive antiautomorphism of A. If $A = C(V)$ then the symbol β will be exclusively used for its principal antiautomorphism. The idea is to define the notion of a tensor product and a similarity relation for such pairs and obtain a group, in analogy with the super Brauer group, a group which we shall denote by $B(\mathbf{C})$. It will be proved that $B(\mathbf{C}) \simeq \mathbf{Z}_8$, showing that the theory of forms for spin modules is governed again by a periodicity mod 8; however this time it is the *dimension* of the quadratic vector space mod 8 that will tell the story. The same periodicity will be shown to persist for the theory of morphisms.

If $(A_i, \beta_i)(i = 1, 2)$ are two pairs, then

$$(A, \beta) = (A_1, \beta_1) \otimes (A_2, \beta_2)$$

is defined by

$$A = A_1 \otimes A_2, \quad \beta = \beta_1 \otimes \beta_2, \qquad \beta(a_1 \otimes a_2) = (-1)^{p(a_1)p(a_2)}\beta_1(a_1) \otimes \beta_2(a_2).$$

The definition of β is made so that it correctly reproduces what happens for Clifford algebras. In fact we have the following.

Lemma 5.5.2. *If $V_i (i = 1, 2)$ are quadratic vector spaces and $V = V_1 \oplus V_2$, then*

$$(C(V), \beta) = (C(V_1), \beta) \otimes (C(V_2), \beta).$$

Proof. If $u_i (1 \le i \le p) \in V_1, v_j (1 \le j \le q) \in V_2$, then

$$\beta(u_1 \ldots u_p \ldots v_1 \ldots v_q) = v_q \ldots v_1 u_p \ldots u_1 = (-1)^{pq}\beta(u_1 \ldots u_p)\beta(v_1 \ldots v_q)$$

which proves the lemma.

We need to make a remark here. The definition of the tensor product of two β's violates the sign rule. One can avoid this by redefining it without altering the theory in any essential manner (see Deligne[8]), but this definition is more convenient for us. As a result, in a few places we shall see that the sign rule gets appropriately modified. The reader will notice these aberrations without any prompting.

For the pairs (A, β) the tensor product is associative and commutative as is easy to check. We now define the pair (A, β) to be *neutral* if $A \simeq M^{r|s}$ and the form corresponding to β which is defined over $\mathbf{C}^{r|s}$ is *even* and *symmetric.* We shall say that $(A, \beta), (A', \beta')$ are *similar* if we can find neutral $(B_1, \beta_1), (B_2, \beta_2)$ such that

$$(A, \beta) \otimes (B_1, \beta_1) \simeq (A', \beta') \otimes (B_2, \beta_2).$$

If (A, β) is a pair where $A \simeq M^{r|s}$, we write $\pi(A, \beta), \sigma(A, \beta)$ for the parity and symmetry of the associated form on $\mathbf{C}^{r|s}$. When we speak of the parity and sign of a pair (A, β) it is implicit that A is a full matrix super algebra. Notice that on a full matrix super algebra we can have forms of arbitrary parity and symmetry. Indeed, forms are defined by invertible matrices x, symmetric or skewsymmetric, in the usual manner, namely $\varphi_x(s, t) = s^T xt$. The involution β_x corresponding to x is

$$\beta_x(a) = x^{-1}a^T x \qquad (a \in M^{r|s}).$$

Note that β_x is even for x homogeneous and involutive if x is symmetric or skewsymmetric. We have the following where in all cases β_x is even and involutive:

$$A = M^{r|r}, \quad x = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \varphi_x = \text{even and symmetric}$$
$$A = M^{r|r}, \quad x = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \varphi_x = \text{odd and symmetric}$$
$$A = M^{2r|2r}, \quad x = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}, \; J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \varphi_x = \text{even and skewsymmetric}$$
$$A = M^{2r|2r}, \quad x = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \varphi_x = \text{odd and skewsymmetric.}$$

Lemma 5.5.3. *Let π, σ_i be the parity and symmetry of $(A_i, \beta_i)(i = 1, 2)$. Then for the parity π and symmetry σ of the tensor product $(A_1 \otimes A_2, \beta_1 \otimes \beta_2)$ we have*

$$\pi = \pi_1 + \pi_2, \qquad \sigma = (-1)^{\pi_1\pi_2}\sigma_1\sigma_2.$$

Proof. It is natural to expect that the form corresponding to $\beta_1 \otimes \beta_2$ is the tensor product of the corresponding forms for the β_i. But because the definition of the tensor product has violated the sign rule one should define the tensor product of forms with suitable sign factors so that this can be established. Let $A_i = \mathbf{End}(S_i)$. Let us define

$$(s_1 \otimes s_2, t_1 \otimes t_2) = C(s_1, s_2, t_1, t_2)(s_1, t_1)(s_2, t_2)$$

where C is a sign factor depending on the parities of the s_i, t_j. The requirement that this corresponds to $\beta_1 \otimes \beta_2$ now leads to the equations

$$C(s_1, s_2, \beta_1(a_1)t_1, \beta_2(a_2)t_2) = (-1)^{p(a_2)[p(s_1)+p(t_1)+p(a_1)]}C(a_1s_1, a_2s_2, t_1, t_2)$$

which is satisfied if we take

$$C(s_1, s_2, t_1, t_2) = (-1)^{p(s_2)[p(s_1)+p(t_1)]}.$$

Thus the correct definition of the tensor product of two forms is

$$(s_1 \otimes s_2, t_1 \otimes t_2) = (-1)^{p(s_2)[p(s_1)+p(t_1)]}(s_1, t_1)(s_2, t_2).$$

If $(s,t) \neq 0$ then $\pi = p(s) + p(t)$ and so choosing $(s_i, t_i) \neq 0$ we have $\pi = p(s_1) + p(s_2) + p(t_1) + p(t_2) = \pi_1 + \pi_2$. For σ we get

$$\sigma = (-1)^{[p(s_1)+p(t_1)][p(s_2)+p(t_2)]}\sigma_1\sigma_2 = (-1)^{\pi_1\pi_2}\sigma_1\sigma_2.$$

It follows from this that if the (A_i, β_i) are neutral so is their tensor product. From this we see that similarity is an equivalence relation, obviously coarser than isomorphism, and that similarity is preserved under tensoring. In particular we can speak of similarity classes and their tensor products. The similarity classes form a commutative semigroup and the neutral elements form the identity element of this semigroup. We denote it by $B(\mathbf{C})$. The parity and symmetry invariants do not change when tensored by a neutral pair so that they are really invariants of similarity classes.

We wish to prove that $B(\mathbf{C})$ is a group and indeed that it is the cyclic group $\mathbf{Z}_8$ of order 8. Before doing this we define, for the parity group $P = \{0,1\}$ and sign group $\Sigma = \{\pm 1\}$, their product $P \times \Sigma$ with the product operation defined by the lemma above:

$$(\pi_1, \sigma_1)(\pi_2, \sigma_2) = (\pi_1 + \pi_2, (-1)^{\pi_1\pi_2}\sigma_1\sigma_2).$$

It is a trivial calculation that $P \times \Sigma$ is a group isomorphic to $\mathbf{Z}_4$ and is generated by $(1,+)$. Let $B_0(\mathbf{C})$ be the semigroup of classes of pairs (A, β) where $A \simeq M^{r|s}$. The map

$$\varphi : (A, \beta) \longmapsto (\pi, \sigma)$$

is then a homomorphism of $B_0(\mathbf{C})$ into $P \times \Sigma$. We assert that φ is surjective. It is enough to check that $(1,+)$ occurs in its image. Let V_2 be a two-dimensional quadratic vector space with basis $\{u, v\}$ where $\Phi(u,u) = \Phi(v,v) = 0$ and $\Phi(u,v) = 1/2$, so that $u^2 = v^2 = 0$ and $uv + vu = 1$. Then $C(V_2) \simeq M^{1|1}$ via the standard representation that acts on $\mathbf{C} \oplus \mathbf{C}v$ as follows:

$$v \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} : 1 \mapsto v, \quad v \mapsto 0, \qquad u \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} : 1 \mapsto 0, \quad v \mapsto 1.$$

The principal involution β is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} d & b \\ c & a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The form corresponding to β is then defined by the invertible symmetric matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and so is odd and symmetric. Thus $(C(V_2), \beta)$ gets mapped to $(1,+)$ by

φ. Thus φ is surjective. Moreover the kernel of φ is just the neutral class. Hence $B_0(\mathbf{C})$ is already a group isomorphic to $\mathbf{Z}_4$ and is generated by the class of the Clifford algebra in dimension 2. In particular the parity and symmetry of the forms determine the class of any element of $B_0(\mathbf{C})$.

Proposition 5.5.4. *$B(\mathbf{C})$ is a group isomorphic to the cyclic group $\mathbf{Z}_8$ of order 8 and is generated by the class of the Clifford algebra in dimension 1, namely $(\mathbf{C}[\varepsilon], \beta)$, where ε is odd and $\varepsilon^2 = 1$, and $\beta(\varepsilon) = \varepsilon$. The subgroup $B_0(\mathbf{C}) \simeq \mathbf{Z}_4$ is generated by the class of the Clifford algebra in dimension 2.*

Proof. If A is a complex CS super algebra which is not a full matrix algebra, then it is of the form $M^n \otimes \mathbf{C}[\varepsilon]$ and so $A \otimes A \simeq M^{n^2|n^2}$. Thus the square of any element x in $B(\mathbf{C})$ is in $B_0(\mathbf{C})$ and hence $x^8 = 1$. This proves that $B(\mathbf{C})$ is a group and $B(\mathbf{C})/B_0(\mathbf{C}) \simeq \mathbf{Z}_2$. The square of the class of the Clifford algebra in dimension 1 is the Clifford algebra in dimension 2 which has been shown to be a generator of $B_0(\mathbf{C})$. Thus $(\mathbf{C}[\varepsilon], \beta)$ generates $B(\mathbf{C})$ and has order 8.

Corollary 5.5.5. *The inverse of the class of $(\mathbf{C}[\varepsilon], \beta)$ is the class of $(\mathbf{C}[\varepsilon], \beta^0)$ where $\beta^0(\varepsilon) = -\varepsilon$.*

Proof. Since $\mathbf{C}[\varepsilon]$ is its own inverse in the super Brauer group $\mathrm{sBr}(\mathbf{C})$, the inverse in question has to be $(\mathbf{C}[\varepsilon], \beta')$ where $\beta' = \beta$ or β^0. The first alternative is impossible since $(\mathbf{C}[\varepsilon], \beta)$ has order 8, not 2.

There is clearly a unique isomorphism of $B(\mathbf{C})$ with $\mathbf{Z}_8$ such that the class of $(\mathbf{C}[\varepsilon], \beta)$ corresponds to the residue class of 1. We shall identify $B(\mathbf{C})$ with $\mathbf{Z}_8$ through this isomorphism. We shall refer to the elements of $B_0(\mathbf{C})$ as the even classes and the elements of $B(\mathbf{C}) \setminus B_0(\mathbf{C})$ as the odd classes. For D-dimensional V_D the class of $(C(V_D), \beta)$ is in $B_0(\mathbf{C})$ if and only if D is even. Since the class of $(C(V_D), \beta)$ is the D^{th} power of the class of $(C(V_1), \beta) = (\mathbf{C}[\varepsilon], \beta)$, it follows that the class of $(C(V_8), \beta)$ is 1 and hence that $(C(V_D), \beta)$ and $(C(V_{D+8}), \beta)$ are in the same class, giving us the periodicity mod 8. The structure of invariant forms for the Clifford algebras is thus governed by the dimension mod 8. The following table gives for the even dimensional cases the classes of the Clifford algebras in terms of the parity and symmetry invariants. Let $D = \dim(V)$ and let $\overline{D}$ be its residue class mod 8.

Table 1

$\overline{D}$	π	σ
0	0	$+$
2	1	$+$
4	0	$-$
6	1	$-$

However for determining the nature of forms invariant under $\mathrm{Spin}(V)$ we must go from the Clifford algebra to its even part. We have the isomorphism $C(V_D) \simeq \mathbf{End}(S)$ where S is an irreducible super module for $C(V_D)$. The above table tells us that for $\overline{D} = 0, 4$ there is an even invariant form for S, respectively symmetric and skewsymmetric. Now under the action of $C(V_{2m})^+$ we have $S = S^+ \oplus S^-$ where $S^\pm$ are the semispin representations. So both of these have invariant forms which are symmetric for $\overline{D} = 0$ and skewsymmetric for $\overline{D} = 4$. For $\overline{D} = 2, 6$ the invariant form for S is odd and so what we get is that $S^\pm$ are dual to each other. In this case there will be no invariant forms for $S^\pm$ individually; for, if for example S^+ has an invariant form, then S^+ is isomorphic to its dual and so is isomorphic to S^- which is impossible. When the form is symmetric the spin group is embedded inside the orthogonal group of the spin module, while in the skew case it is embedded inside the symplectic group. Later on we shall determine the imbeddings much more precisely when the ground field is $\mathbf{R}$. Thus we have the table

Table 2

$\overline{D}$	forms on $S^\pm$
0	symmetric on $S^\pm$
2	$S^\pm$ dual to each other
4	skewsymmetric on $S^\pm$
6	$S^\pm$ dual to each other

We now examine the odd classes in $B(\mathbf{C})$. Here the underlying algebras A are of the form $M \otimes Z$ where M is a purely even full matrix algebra and Z is the center (not super center) of the algebra, with $Z \simeq \mathbf{C}[\varepsilon]$:

$$A \simeq A^+ \otimes Z, \qquad Z = \mathbf{C}[\varepsilon],\ \varepsilon \text{ odd},\ \varepsilon^2 = 1.$$

Note that Z is a super algebra. If β is an even involutive antiautomorphism of A then β leaves Z invariant and hence also $Z^{\pm}$. It acts trivially on $Z^+ = \mathbf{C}$ and as a sign $s(\beta)$ on Z^-. We now have the following key lemma.

Lemma 5.5.6. *We have the following.*

(i) *Let $(A, \beta), (A', \beta')$ be pairs representing an odd and even class respectively in $B(\mathbf{C})$. Then*

$$s(\beta \otimes \beta') = (-1)^{\pi} s(\beta)$$

where π' is the parity of the form corresponding to β'. In particular the sign $s(\beta)$ depends only on the similarity class of (A, β).

(ii) *With the identification $B(\mathbf{C}) \simeq \mathbf{Z}_8$ (written additively), the elements x^+, x of $B(\mathbf{C})$ corresponding to (A^+, β) and (A, β) respectively are related by*

$$x^+ = x - s(\beta)1.$$

In particular the similarity class of (A^+, β) depends only on that of (A, β).

Proof. Let $(A'', \beta'') = (A, \beta) \otimes (A', \beta')$. The center of A'' is again of dimension $1|1$. If A' is purely even, then Z is contained in the center of A'' and so has to be its center and the actions of β, β'' are then the same. Suppose that $A' = M^{r|s}$ where $r, s > 0$. Let

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in A'.$$

It is trivial to check that η commutes with A'^+ and anticommutes with A'^-, and that it is characterized by this property up to a scalar multiple. We claim that $\varepsilon \otimes \eta$ lies in the odd part of the center of A''. This follows from the fact that ε and η commute with $A \otimes 1$ and $1 \otimes A'^+$, while they anticommute with $1 \otimes A'^-$. Hence $\varepsilon \otimes \eta$ *spans* the odd part of the center of A''. Now

$$\beta''(\varepsilon \otimes \eta) = \beta(\varepsilon) \otimes \beta'(\eta).$$

The first factor on the right side is $s(\beta)\varepsilon$. On the other hand, by the characterization of η mentioned above, we must have $\beta'(\eta) = c\eta$ for some constant c, and so to prove (i) we must show that $c = (-1)^{\pi'}$. If the form corresponding to β' is even, there are even s, t such that $(s, t) \neq 0$; then $(s, t) = (\eta s, t) = (s, c\eta t) = c(s, t)$, so that $c = 1$. If the form is odd, then we can find even s and odd t such that $(s, t) \neq 0$; then $(s, t) = (\eta s, t) = (s, c\eta t) = -c(s, t)$ so that $c = -1$. This finishes the proof of (i).

For proving (ii) let x, x^+, z be the elements of $B(\mathbf{C})$ corresponding to $(A, \beta), (A^+, \beta), (Z, \beta)$ respectively. Clearly $x = x^+ + z$. If $s(\beta) = 1$, (Z, β) is the class of the Clifford algebra in dimension 1 and so is given by the residue class of 1. Thus $x^+ = x - 1$. If $s(\beta) = -1$, then (Z, β) is the inverse of the class of the Clifford algebra in dimension 1 by Corollary 5 and hence $x^+ = x + 1$.

For the odd classes of pairs (A, β) in $B(\mathbf{C})$ we thus have two invariants: the sign $s(\beta)$ and the symmetry $s(A^+)$ of the form associated to the similarity class of (A^+, β). We then have the following table:

Table 3

Residue class	$s(A^+)$	$s(\beta)$
1	$+$	$+$
3	$-$	$-$
5	$-$	$+$
7	$+$	$-$

To get this table we start with $(\mathbf{C}[\varepsilon], \beta)$ with $\beta(\varepsilon) = \varepsilon$ for which the entries are $+, +$. For 7 the algebra remains the same but the involution is β^0 which takes ε to $-\varepsilon$, so that the entries are $+, -$. From Table 1 we see that the residue class 4 in $B_0(\mathbf{C})$ is represented by any full matrix super algebra with an even invariant skewsymmetric form; we can take it to be the purely even matrix algebra $M = M^2$ in dimension 2 with the invariant form defined by the skewsymmetric matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let β_M be the corresponding involution. Then 5 is represented by $(M, \beta_M) \otimes (\mathbf{C}[\varepsilon], \beta)$. Using Lemma 3 we see that the signs of the form and the involution are $-, +$. To get the invariants of the residue class 3 we remark that as $3 = 4 - 1$ it is represented by $(M, \beta_M) \otimes (\mathbf{C}[\varepsilon], \beta^0)$ and so its invariants are $-, -$.

If $D = 2m + 1$ is odd, $(C(V_D), \beta)$ is similar to the D^{th} power of the class of $(\mathbf{C}[\varepsilon], \beta)$. Hence there is periodicity in dimension mod 8 and the invariants for the residue classes of D mod 8 are the same as in the above table. For the symmetry of the Spin(V)-invariant forms we simply read the first column of the above table. We have thus the following theorem.

Theorem 5.5.7. *The properties of forms on the spin modules associated to complex quadratic vector spaces V depend only on the residue class $\overline{D}$ of $D = \dim(V)$ mod 8. The forms, when they exist, are unique up to scalar factors and are given by the following table.*

$\overline{D}$	forms on S, $S^{\pm}$
0	symmetric on $S^{\pm}$
1, 7	symmetric on S
2, 6	$S^{\pm}$ dual to each other
3, 5	skewsymmetric on S
4	skewsymmetric on $S^{\pm}$

When $S^{\pm}$ are dual to each other, there are no forms on $S^{\pm}$ individually.

Forms in the real case. We shall now extend the above results to the case of *real* spin modules. The results are now governed by *both* the dimension and the signature mod 8.

We are dealing with the following situation. $S_{\mathbf{R}}$ is a real irreducible module for $C(V)^+$ where $C(V)$ is the Clifford algebra of a real quadratic vector space V; equivalently $S_{\mathbf{R}}$ is an irreducible module for Spin(V). The integers p, q are such that $V \simeq \mathbf{R}^{p,q}$, $D = p + q, \Sigma = p - q$, D and Σ having the same parity. $\overline{D}, \overline{\Sigma}$ are the residue classes of D, Σ mod 8. We write σ for the conjugation of $S_{\mathbf{C}}$ that defines $S_{\mathbf{R}}$, $S_{\mathbf{C}}$ being the complexification of $S_{\mathbf{R}}$. If $S_{\mathbf{R}}$ is of type $\mathbf{R}$ then $S_{\mathbf{C}}$ is the irreducible spin module S or $S^{\pm}$; if $S_{\mathbf{R}}$ is of type $\mathbf{H}$ then $S_{\mathbf{C}} = S \otimes W$ where S is an irreducible complex spin module and $\dim(W) = 2$, $C(V)^+$ acting on $S_{\mathbf{C}}$ only through the first factor. If $S_{\mathbf{R}}$ is of type $\mathbf{C}$, this case occuring only when D is even, then $S_{\mathbf{C}} = S^+ \oplus S^-$.

Let $\mathbf{A}$ be the commutant of the image of $C(V)^+$ in End$(S_{\mathbf{R}})$. Then $\mathbf{A} \simeq \mathbf{R}, \mathbf{H}$, or $\mathbf{C}$. We write $\mathbf{A}_1$ for the group of elements of norm 1 in $\mathbf{A}$. Notice that this is defined independently of the choice of the isomorphism of $\mathbf{A}$ with these algebras, and

$$\mathbf{A}_1 \simeq \{\pm 1\}, \mathrm{SU}(2), T$$

in the three cases, T being the multiplicative group of complex numbers of absolute value 1. If β is any invariant form for $S_{\mathbf{R}}$, and $a \in \mathbf{A}_1$,

$$a{\cdot}\beta : (u,v) \longmapsto \beta(a^{-1}u, a^{-1}v)$$

is also an invariant form for $S_{\mathbf{R}}$. Thus we have an action of $\mathbf{A}_1$ on the space of invariant forms for $S_{\mathbf{R}}$. We shall determine this action also below. Actually, when $\mathbf{A} = \mathbf{R}$, we have $\mathbf{A}_1 = \{\pm 1\}$ and the action is trivial, so that only the cases $\mathbf{A} = \mathbf{H}, \mathbf{C}$ need to be considered.

The simplest case is when $S_{\mathbf{R}}$ is of type $\mathbf{R}$. This occurs when $\overline{\Sigma} = 0, 1, 7$. If $\overline{\Sigma} = 1, 7$ then D is odd and the space of invariant bilinear forms for S is of dimension 1. It has a conjugation $B \mapsto B^{\sigma}$ defined by $B^{\sigma}(s,t) = B(s^{\sigma}, t^{\sigma})^{\mathrm{conj}}$ and if B is a real element, then B spans this space and is an invariant form for $S_{\mathbf{R}}$. The symmetry of the form does not change and the conclusions are given by the first column of the first table of Theorem 10 below. If $\overline{\Sigma} = 0$ the conclusions are again the same for the spin modules $S^{\pm}_{\mathbf{R}}$ for $\overline{D} = 0, 4$. When $\overline{D} = 2, 6$, $S^{\pm}$ are in duality which implies that $S^{\pm}_{\mathbf{R}}$ are also in duality. We have thus verified the first column of the second table of Theorem 10 below.

To analyze the remaining cases we need some preparation. For any complex vector space U we define a *pseudo conjugation* to be an antilinear map τ of U such that $\tau^2 = -1$. For example, if $U = \mathbf{C}^2$ with standard basis $\{e_1, e_2\}$, then

$$\tau : e_1 \longmapsto e_2, \qquad e_2 \longmapsto -e_1$$

defines a pseudo conjugation. For an arbitrary U, if τ is a pseudo conjugation or an ordinary conjugation, we have an induced conjugation on $\mathrm{End}(U)$ defined by $a \longmapsto \tau a \tau^{-1}$ (conjugations of $\mathrm{End}(U)$ have to preserve the product by definition). If we take τ to be the conjugation of $\mathbf{C}^2$ that fixes the e_i, then the induced conjugation on $\mathrm{End}(\mathbf{C}^2) = M^2(\mathbf{C})$ is just $a \longmapsto a^{\mathrm{conj}}$ with the fixed point algebra $M^2(\mathbf{R})$, while for the pseudo conjugation τ defined above, the induced conjugation is given by

$$a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \begin{pmatrix} \overline{\delta} & -\overline{\gamma} \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

so that its fixed points form the algebra of matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \qquad (\alpha, \beta \in \mathbf{C}).$$

If $\alpha = a_0 + ia_1, \beta = a_2 + ia_3, (a_j \in \mathbf{R})$, then

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \longmapsto a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

is an isomorphism of the above algebra with $\mathbf{H}$ and the elements of $\mathbf{H}_1$ correspond to SU(2). If $U = \mathbf{C}^{2m}$, then, with (e_i) as the standard basis,

$$\tau : e_j \mapsto e_{m+j}, \quad e_{m+j} \mapsto -e_j \qquad (1 \leq j \leq m)$$

is a pseudo conjugation. Pseudo conjugations cannot exist if the vector space is odd dimensional. Indeed, on $\mathbf{C}$, any antilinear transformation is of the form $z \mapsto cz^{\mathrm{conj}}$, and its square is $z \mapsto |c|^2 z$, showing that it can never be -1. The same argument shows that no pseudo conjugation of an arbitrary vector space U can fix a line. If T is a pseudo conjugation on U, then for any nonzero u, the span U' of u, Tu is of dimension 2 and stable under T, and so T induces a pseudo conjugation on U/U'; an induction on dimension shows that pseudo conjugations do not exist when U is odd dimensional. Any two conjugations of U are equivalent; if U_i are the real forms defined by two conjugations $\sigma_i (i = 1, 2)$, then for any element $g \in \mathrm{GL}(U)$ that takes U_1 to U_2 we have $g\sigma_1 g^{-1} = \sigma_2$. In particular any conjugation is equivalent to the standard conjugation on $\mathbf{C}^m$. The same is true for pseudo conjugations also. Indeed if $\dim(U) = 2m$ and τ is a pseudo conjugation, let W be a maximal subspace of U such that $W \cap \tau(W) = 0$; we claim that $U = W \oplus \tau(W)$. Otherwise, if $u \notin W' := W \oplus \tau(W)$, and L is the span of u and τu, then $L \cap W'$ is τ-stable and so has to have dimension 0 or 2, hence has dimension 0 as otherwise we will have $L \subset W'$. The span W_1 of W and u then has the property that $W_1 \cap \tau(W_1) = 0$, a contradiction. So $U = W \oplus \tau(W)$. It is then clear that τ is isomorphic to the pseudo conjugation of $\mathbf{C}^{2m}$ defined earlier.

Lemma 5.5.8. *Any conjugation of* $\mathrm{End}(U)$ *is induced by a conjugation or a pseudo conjugation of* U *which is unique up to a scalar factor of absolute value* 1.

Proof. Choose some conjugation θ of U and let $a \mapsto a^\theta$ be the induced conjugation of $\mathrm{End}(U)$:

$$a^\theta = \theta a \theta, \qquad a^\theta u = (au^\theta)^\theta \qquad (u \in U, a \in \mathrm{End}(U)).$$

Let $a \mapsto a^*$ be the given conjugation of $\mathrm{End}(U)$. Then $a \mapsto (a^\theta)^*$ is an automorphism and so we can find an $x \in \mathrm{GL}(U)$ such that $(a^\theta)^* = xax^{-1}$. Replacing a by a^θ this gives $a^* = xa^\theta x^{-1}$. So $a = (a^*)^* = xx^\theta a(xx^\theta)^{-1}$ showing that $xx^\theta = c1$ for a constant c, and hence that $xx^\theta = x^\theta x = c1$. Thus c is real, and replacing x

by $|c|^{-1/2}x$ we may assume that $xx^\theta = \pm 1$. Let τ be defined by $u^\tau = xu^\theta (u \in U)$. Then τ is antilinear and $\tau^2 = \pm 1$. Clearly $*$ is induced by τ. If τ' is another such, then $\tau'^{-1}\tau$ induces the identity automorphism on $\mathrm{End}(U)$ and so $\tau = c\tau'$ where c is a scalar. Since $\tau^2 = |c|^2\tau'^2$ we must have $|c| = 1$.

For any conjugation or pseudo conjugation α of U we write $\widehat{\alpha}$ for the induced conjugation $a \mapsto \alpha a \alpha^{-1}$ of $\mathrm{End}(U)$.

Lemma 5.5.9. *Let $S_{\mathbf{R}}$ be of type $\mathbf{H}$ and let $S_{\mathbf{C}}, S, W, \sigma$ be as above. Then $\sigma = \tau \otimes \tau_1$ where τ (resp. τ_1) is a pseudo conjugation of S (resp. W). τ and τ_1 are unique up to scalar factors of absolute value 1 and τ commutes with the action of $C(V)^+$. Conversely if S is an irreducible spin module for $\mathrm{Spin}(V_{\mathbf{C}})$ and $\mathrm{Spin}(V)$ commutes with a pseudo conjugation, the real irreducible spin module(s) is of type $\mathbf{H}$.*

Proof. The complexifications of the image of $C(V)^+$ in $\mathrm{End}(S_{\mathbf{R}})$ and its commutant are $\mathrm{End}(S) \simeq \mathrm{End}(S) \otimes 1$ and $\mathrm{End}(W) \simeq 1 \otimes \mathrm{End}(W)$ respectively. Hence the conjugation $\widehat{\sigma}$ of $\mathrm{End}(S_{\mathbf{C}})$ induced by σ leaves both $\mathrm{End}(S)$ and $\mathrm{End}(W)$ invariant. So, by the above lemma there are conjugations or pseudo conjugations τ, τ_1 on S, W inducing the restrictions of $\widehat{\sigma}$ on $\mathrm{End}(S)$ and $\mathrm{End}(W)$ respectively. Since $\mathrm{End}(S)$ and $\mathrm{End}(W)$ generate $\mathrm{End}(S_{\mathbf{C}})$ we have $\widehat{\sigma} = \widehat{\tau} \otimes \widehat{\tau}_1 = \widehat{(\tau \otimes \tau_1)}$. It follows that for some $c \in \mathbf{C}$ with $|c| = 1$ we must have $\sigma = c(\tau \otimes \tau_1)$. Replacing τ_1 by $c\tau_1$ we may therefore assume that $\sigma = \tau \otimes \tau_1$. Since σ commutes with the action of $C(V)^+$ and $C(V)^+$ acts on $S \otimes W$ only through the first factor, it follows easily that τ commutes with the action of $C(V)^+$. Now the subalgebra of $\mathrm{End}(W)$ fixed by $\widehat{\tau}_1$ is $\mathbf{H}$ and so τ_1 must be a pseudo conjugation. Hence, as σ is a conjugation, τ must also a pseudo conjugation.

For the converse choose a W of dimension 2 with a pseudo conjugation $\tau - 1$. Let τ be the pseudo conjugation on S commuting with $\mathrm{Spin}(V)$. Then $\sigma = \tau \otimes \tau - 1$ is a conjugation on $S \otimes W$ commuting with $\mathrm{Spin}(V)$ and so $2S$ has a real form $S'_{\mathbf{R}}$. This real form must be irreducible; for otherwise, if $S'_{\mathbf{R}}$ is a proper irreducible constituent, then $S"_{\mathbf{C}} \simeq S$ which will imply that S has a real form. So $\mathrm{Spin}(V)$ must commute with a conjugation also, an impossibility. This proves the entire lemma.

Suppose now that S has an invariant form. The space of these invariant forms is of dimension 1, and τ, since it commutes with $C(V)^+$, induces a conjugation $B \mapsto B^\tau$ on this space where $B^\tau(s,t) = B(s^\tau, t^\tau)^{\mathrm{conj}}$. Hence we may assume that S has an invariant form $B = B^\tau$. The space of invariant forms for $S \otimes W$ is now $B \otimes J$ where J is the space of bilinear forms for W which is a natural module for $\mathbf{A}_1$ and which carries a conjugation, namely the one induced by τ_1. We select a basis e_1, e_2 for W so that $\tau_1(e_1) = e_2, \tau_1(e_2) = -e_1$. Then $\mathbf{A}_1 = \mathrm{SU}(2)$ and its action on

W commutes with τ_1. Clearly $J = J_1 \oplus J_3$ where J_k carries the representation $\mathbf{k}$ of dimension k, where J_1 is spanned by skewsymmetric forms while J_3 is spanned by symmetric forms, and both are stable under τ_1. Hence

$$\mathrm{Hom}(S_{\mathbf{R}} \otimes S_{\mathbf{R}}, \mathbf{C}) = B \otimes J^{\tau_1}$$

where $B = B^\tau$ is an invariant form for S and J^{τ_1} is the subspace of J fixed by the conjugation induced by τ_1. For a basis of J_1 we can take the symplectic form $b_0 = b_0^{\tau_1}$ given by $b_0(e_1, e_2) = 1$. Then $B_{\mathbf{R},0} = B \otimes b_0$ is invariant under σ and defines an invariant form for $S_{\mathbf{R}}$, fixed by the action of $\mathbf{A}_1$, and is unique up to a scalar factor. If $b_j, j = 1, 2, 3$ are a basis for $J_3^{\tau_1}$, then $B_{\mathbf{R},j} = B \otimes b_j$ are symmetric invariant forms for $S_{\mathbf{R}}$, defined up to a transformation of SO(3). The symmetry of $B_{\mathbf{R},0}$ is the *reverse* of that of B while those of the $B_{\mathbf{R},j}$ are the *same* as that of B. This takes care of the cases $\overline{\Sigma} = 3, 5$, $\overline{D}$ arbitrary, and $\overline{\Sigma} = 4, \overline{D} = 0,\ 4$. In the latter case the above argument applies to $S^{\pm}_{\mathbf{R}}$.

Suppose that $\overline{\Sigma} = 4$, $\overline{D} = 2,\ 6$. Then S^+ and S^- are dual to each other. We have the irreducible spin modules $S^{\pm}_{\mathbf{R}}$ with complexifications $S^{\pm}_{\mathbf{C}} = S^{\pm} \otimes W^{\pm}$ and conjugations $\sigma^{\pm} = \tau^{\pm} \otimes \tau_1^{\pm}$ (with the obvious notations). The invariant form

$$B : S^+_{\mathbf{C}} \times S^-_{\mathbf{C}} \longrightarrow \mathbf{C}$$

is unique up to a scalar factor and so, as before, we may assume that $B = B^{\mathrm{conj}}$ where

$$B^{\mathrm{conj}}(s^+, s^-) = B((s^+)^{\tau^+}, (s^-)^{\tau^-})^{\mathrm{conj}} \qquad (s^{\pm} \in S^{\pm}_{\mathbf{C}}).$$

For any form $b(W^+ \times W^- \longrightarrow \mathbf{C})$ such that $b^{\mathrm{conj}} = b$ where the conjugation is with respect to $\tau_1^{\pm}$,

$$B \otimes b : S^+_{\mathbf{C}} \otimes W^+ \times S^-_{\mathbf{C}} \otimes W^- \longrightarrow \mathbf{C}$$

is an invariant form fixed by σ and so restricts to an invariant form

$$S^+_{\mathbf{R}} \otimes S^-_{\mathbf{R}} \longrightarrow \mathbf{R}.$$

Thus $S^+_{\mathbf{R}}$ and $S^-_{\mathbf{R}}$ are in duality. As before there are no invariant forms on $S^{\pm}_{\mathbf{R}} \times S^{\pm}_{\mathbf{R}}$ separately.

In this case, although there is no question of symmetry for the forms, we can say a little more. We may clearly take $W^+ = W^- = W, \tau_1^+ = \tau_1^- = \tau_1$. Then we can identify the $\mathbf{H}_1$–actions on $W^{\pm}$ with the standard action of SU(2) on $W = \mathbf{C}e_1 + \mathbf{C}e_2$ where $\tau(e_1) = e_2, \tau(e_2) = -e_1$. The space of forms on $S^+_{\mathbf{R}} \times S^-_{\mathbf{R}}$ is then $B^{\pm} \otimes J^{\tau_1}$

where $B^\pm = B^{\pm\tau}$ is a basis for the space of forms on $S^+ \times S^-$. We then have a decomposition

$$\mathrm{Hom}(S^+_{\mathbf{R}} \otimes S^-_{\mathbf{R}}, \mathbf{R}) = (B^\pm \otimes J_1^{\tau_1}) \oplus (B^\pm \otimes J_3^{\tau_1}).$$

We have thus verified the second column in the two tables of Theorem 10 below.

The case $\overline{\Sigma} = 2, 6$ when the real spin modules $S_{\mathbf{R}}$ are of type $\mathbf{C}$ remains. In this case $S_{\mathbf{C}} = S^+ \oplus S^-$ and is self dual. We have a conjugate linear isomorphism $u \longmapsto u^*$ of S^+ with S^- and

$$\sigma : (u, v^*) \longmapsto (v, u^*)$$

is the conjugation of $S_{\mathbf{C}}$ that defines $S_{\mathbf{R}}$. The space of maps from $S_{\mathbf{C}}$ to its dual is of dimension 2 and so the space spanned by the invariant forms for $S_{\mathbf{C}}$ is of dimension 2. This space as well as its subspaces of symmetric and skewsymmetric elements are stable under the conjugation induced by σ. Hence the space of invariant forms for $S_{\mathbf{R}}$ is also of dimension 2 and spanned by its subspaces of symmetric and skewsymmetric forms. If $\overline{D} = 0$ (resp. 4), $S^\pm$ admit symmetric (resp. skewsymmetric) forms, and so we have two linearly independent symmetric (resp. skewsymmetric) forms for $S_{\mathbf{R}}$. If $\overline{D} = 2, 6$, $S^\pm$ are dual to each other. The pairing between $S^\pm$ then defines two invariant forms on $S^+ \oplus S^-$, one symmetric and the other skewsymmetric. Hence both the symmetric and skewsymmetric subspaces of invariant forms for $S_{\mathbf{C}}$ have dimension 1. So $S_{\mathbf{R}}$ has both symmetric and skewsymmetric forms.

It remains to determine the action of $\mathbf{A}_1 = T$ on the space of invariant forms for $S_{\mathbf{R}}$. For $b \in T$ its action on $S_{\mathbf{R}}$ is given by

$$(u, u^*) \longmapsto (bu, b^{\mathrm{conj}} u^*).$$

$\overline{D} = 0, 4$. In this case the space of invariant forms for S^+ is nonzero and has a basis β. The form

$$\beta^* : (u^*, v^*) \longmapsto \beta(u, v)^{\mathrm{conj}}$$

is then a basis for the space of invariant forms for S^-. The space of invariant forms for $S_{\mathbf{C}}$ is spanned by β, β^* and the invariant forms for $S_{\mathbf{R}}$ are those of the form

$$\beta_c = c\beta + c^{\mathrm{conj}} \beta^* (c \in \mathbf{C}).$$

The induced action of T is then given by

$$\beta_c \longmapsto \beta_{b^{-2}c}.$$

Thus the space of invariant forms for $S_{\mathbf{R}}$ is the module $\mathbf{2}$ for T given by

$$\mathbf{2} : e^{i\theta} \longmapsto \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos\theta \end{pmatrix}$$

with respect to the basis β_1, β_i. In particular there are no forms fixed by T.

$\overline{D} = 2, 6$. In this case we have a bilinear duality of $S^{\pm}$ given by

$$(u, v^*) \longmapsto \langle u, v^* \rangle.$$

The space of invariant forms for $S_{\mathbf{C}}$ is spanned by

$$((u, v^*), (u', {v'}^*)) \longmapsto \langle u, {v'}^* \rangle \pm \langle u', v^* \rangle.$$

The space of invariant forms for $S_{\mathbf{R}}$ is then spanned by

$$((u, u^*), (u', {u'}^*)) \longmapsto \langle u, {u'}^* \rangle \pm \langle u', u^* \rangle.$$

Clearly these are both invariant for the action $(u, u^*) \longmapsto (bu, b^{\mathrm{conj}} u^*)$.

We now have the following Theorem. Note that when $S^{\pm}_{\mathbf{R}}$ are dual, there are no forms on $S^{\pm}_{\mathbf{R}}$ individually. For the second columns in the two tables $\mathbf{k}$ denotes the representation of dimension k for $\mathrm{SU}(2)$ while for the third column in the second table the number $\mathbf{k}$ denotes the representation of T in which $e^{i\theta}$ goes over to the rotation by $2k\theta$. The notation $\pm[\mathbf{k}]$ means that the space of forms with symmetry $\pm$ carries the representation $[\mathbf{k}]$ of $\mathbf{A}_1$. When there is no number attached to a symmetry it means that the form is unique up to a real scalar factor.

Theorem 5.5.10 *The forms for the real irreducible spin modules are given by the following tables. Here $\overline{D}, \overline{\Sigma}$ denote the residue class of D, Σ mod 8, and dual pair means that $S^{\pm}_{\mathbf{R}}$ are dual to each other. Also $+$ and $-$ denote symmetric and skewsymmetric forms, and* d.p. *means dual pair.*

$\overline{D}\backslash\overline{\Sigma}$	$1, 7(\mathbf{R})$	$3, 5(\mathbf{H})$
$1,\ 7$	$+$	$-[\mathbf{1}], +[\mathbf{3}]$
$3,\ 5$	$-$	$+[\mathbf{1}], -[\mathbf{3}]$

$\overline{D}\backslash\Sigma$	$0(\mathbf{R},\mathbf{R})$	$4(\mathbf{H},\mathbf{H})$	$2,6(\mathbf{C})$
0	$+$	$-[\mathbf{1}],+[\mathbf{3}]$	$+[\mathbf{2}]$
4	$-$	$+[\mathbf{1}],-[\mathbf{3}]$	$-[\mathbf{2}]$
2, 6	d.p.	d.p.$[\mathbf{1}]\oplus[\mathbf{3}]$	$+[\mathbf{0}],-[\mathbf{0}]$

Morphisms from spinors, spinors to vectors and exterior tensors. As mentioned earlier we need to know, for real irreducible spin modules S_1, S_2, the existence of symmetric morphisms $S_1 \otimes S_2 \longrightarrow V$ in the construction of super spacetimes, and more generally morphisms $S_1 \otimes S_2 \longrightarrow \Lambda^r(V)$ in the construction of super Poincaré and super conformal algebras. We shall now study this question which is also essentially the same as the study of morphisms $\Lambda^r(V) \otimes S_1 \longrightarrow S_2$ (for $r = 1$ these morphisms allow one to define the Dirac or Weyl operators). Here again we first work over $\mathbf{C}$ and then come down to the real case.

Let $D = \dim(V)$. We shall first assume that D is even. In this case we have the two semispin modules $S^{\pm}$ and their direct sum S_0 which is a simple super module for the full Clifford algebra. Write ρ for the isomorphism

$$\rho : C(V) \simeq \mathbf{End}(S_0) \qquad \dim(S_0) = 2^{(D/2)-1}|2^{(D/2)-1}.$$

Since $(S^{\pm})^* = S^{\pm}$ or $S^{\mp}$ where $*$ denotes duals, it is clear that S_0 is self dual. Since $-1 \in \mathrm{Spin}(V)$ goes to -1 in S_0 it follows that -1 goes to 1 in $S_0 \otimes S_0$ and so $S_0 \otimes S_0$ is a $\mathrm{SO}(V)$-module. We have, as $\mathrm{Spin}(V)$-modules,

$$S_0 \otimes S_0 \simeq S_0 \otimes S_0^* \simeq \mathrm{End}(S_0)$$

where $\mathrm{End}(S_0)$ is viewed as an ungraded algebra on which $g \in \mathrm{Spin}(V)$ acts by $t \longmapsto \rho(g)t\rho(g)^{-1} = \rho(gtg^{-1})$. Since ρ is an isomorphism of $C(V)$ with $\mathrm{End}(S_0)$ (ungraded), it follows that the action of $\mathrm{Spin}(V)$ on $C(V)$ is by inner automorphisms and so is the one coming from the action of the image of $\mathrm{Spin}(V)$ in $\mathrm{SO}(V)$. Thus

$$S_0 \otimes S_0 \simeq C(V).$$

Lemma 5.5.11. *If D is even then S_0 is self dual and*

$$S_0 \otimes S_0 \simeq 2\left(\oplus_{0 \le r \le D/2-1} \Lambda^r(V)\right) \oplus \Lambda^{D/2}(V).$$

In particular, as the $\Lambda^r(V)$ are irreducible for $r \le D/2-1$, we have,

$$\dim\left(\mathrm{Hom}(S_0 \otimes S_0, \Lambda^r(V))\right) = 2 \quad (0 \le r \le D/2-1).$$

Proof. In view of the last relation above it is a question of determining the $\mathrm{SO}(V)$-module structure of $C(V)$. This follows from the results of Section 2. The Clifford algebra $C = C(V)$ is filtered and the associated graded algebra is isomorphic to $\Lambda = \Lambda(V)$. The skewsymmetrizer map (see Section 2)

$$\lambda : \Lambda \longrightarrow C$$

is manifestly equivariant with respect to $\mathrm{Spin}(V)$ and so we have, with $\Lambda^r = \Lambda^r(V)$,

$$\lambda : \Lambda = \oplus_{0 \le r \le D} \Lambda^r \simeq C \tag{1}$$

is an isomorphism of $\mathrm{SO}(V)$-modules. If we now observe that $\Lambda^r \simeq \Lambda^{D-r}$ and that the Λ^r are irreducible for $0 \le r \le D/2-1$ the lemma follows immediately.

Suppose now A, B, L are three modules for a group G. Then $\mathrm{Hom}(A, B) \simeq \mathrm{Hom}(A \otimes B^*, \mathbf{C})$, where $\alpha(A \longrightarrow B)$ corresponds to the map (also denoted by α) of $A \otimes B^* \longrightarrow \mathbf{C}$ given by

$$\alpha(a \otimes b^*) = b^*(\alpha(a)).$$

So

$$\mathrm{Hom}(A \otimes B, L) \simeq \mathrm{Hom}(A \otimes B \otimes L^*, \mathbf{C}) \simeq \mathrm{Hom}(B \otimes L^*, A^*).$$

If A and L have invariant forms we can use these to identify them with their duals, and obtain a correspondence

$$\mathrm{Hom}(A \otimes B, L) \simeq \mathrm{Hom}(L \otimes B, A) \qquad \gamma' \leftrightarrow \gamma$$

where the corresponding elements γ', γ of the two Hom spaces are related by

$$(\gamma(\ell \otimes b), a) = (\gamma'(a \otimes b), \ell) \qquad (a \in A, b \in B, \ell \in L).$$

We remark that the correspondence $\gamma' \leftrightarrow \gamma$ depends on the choices of invariant forms on A and L. We now apply these considerations to the case when $G = \mathrm{Spin}(V)$ and $A = B = S_0, L = \Lambda^r$. The invariant form on V lifts to one on Λ^r. Now the Clifford algebra $C = C(V)$ is isomorphic to $\mathbf{End}(S_0)$ and so, the theory of the B-group discussed earlier associates to (C, β) the invariant form $(\cdot, \cdot)$ on $S_0 \times S_0$ for which we have $(as, t) = (s, \beta(a)t)(a \in C)$. We then have a correspondence

$$\gamma' \leftrightarrow \gamma, \qquad \gamma' \in \mathrm{Hom}(S_0 \otimes S_0, \Lambda^r), \gamma \in \mathrm{Hom}(\Lambda^r \otimes S_0, S_0)$$

such that

$$(\gamma'(s \otimes t), v) = (\gamma(v \otimes s), t) \qquad (s, t \in S, v \in \Lambda^r).$$

Let (see Section 2) λ be the skewsymmetrizer map (which is $\mathrm{Spin}(V)$-equivariant) of Λ onto C. The action of C on S_0 then gives a $\mathrm{Spin}(V)$-morphism

$$\gamma_0 : v \otimes s \longmapsto \lambda(v)s.$$

Let Γ_0 be the element of $\mathrm{Hom}(S_0 \otimes S_0, \Lambda^r)$ that corresponds to γ_0. We then have, with respect to the above choices of invariant forms,

$$(\Gamma_0(s \otimes t), v) = (\lambda(v)s, t) = (s, \beta(\lambda(v))t) \qquad (s, t \in S, v \in \Lambda^r). \qquad (*)$$

Note that Γ_0, γ_0 are both nonzero since $\lambda(v) \neq 0$ for $v \neq 0$. To the form on S_0 we can associate its parity π and the symmetry σ. Since $\lambda(v)$ has parity $p(r)$, it follows that $(\lambda(v)s, t) = 0$ when $p(r) + p(s) + p(t) + \pi = 1$. Thus $\Gamma_0(s \otimes t) = 0$ under the same condition. In other words, Γ_0 is even or odd, and

$$\text{parity } (\Gamma_0) = p(r) + \pi.$$

Since

$$\beta(\lambda(v)) = (-1)^{r(r-1)/2}\lambda(v) \qquad (v \in \Lambda^r)$$

it follows that Γ_0 is symmetric or skewsymmetric and

$$\text{symmetry } (\Gamma) = (-1)^{r(r-1)/2}\sigma.$$

The parity and symmetry of Γ_0 are thus dependent only on $\overline{D}$.

In case Γ_0 is even, i.e., when $\pi = p(r)$, Γ_0 restricts to nonzero maps

$$\Gamma^{\pm} : S^{\pm} \times S^{\pm} \longrightarrow \Lambda^r.$$

To see why these are nonzero, suppose for definiteness that $\Gamma^+ = 0$. Then $\Gamma_0(s \otimes t) = 0$ for $s \in S^+, t \in S^{\pm}$ and so $(\lambda(v)s, t) = 0$ for $s \in S^+, t \in S_0, v \in \Lambda^r$. Then $\lambda(v) = 0$ on S^+ for all $v \in \Lambda^r$ which is manifestly impossible because if (e_i) is an ON basis for V and $v = e_{i_1} \wedge \ldots \wedge e_{i_r}$, then $\lambda(v) = e_{i_1} \ldots e_{i_r}$ is invertible and so cannot vanish on S^+. The maps $\Gamma^\pm$ may be viewed as linearly independent elements of $\mathrm{Hom}(S_0 \otimes S_0, \Lambda^r)$. Since this Hom space has dimension 2 it follows that Γ^+, Γ^- form a basis of this Hom space. It follows that

$$\mathrm{Hom}(S^\pm \otimes S^\pm, \Lambda^r) = \mathbf{C}\Gamma^\pm, \qquad \mathrm{Hom}(S^\pm \otimes S^\mp, \Lambda^r) = 0.$$

If $\pi = p(r) = 0$ then $S^\pm$ are self dual. Let $\gamma^\pm$ be the restrictions of γ_0 to $S^\pm$ and

$$\mathrm{Hom}(\Lambda^r \otimes S^\pm, S^\pm) = \mathbf{C}\gamma^\pm, \qquad \mathrm{Hom}(\Lambda^r \otimes S^\pm, S^\mp) = 0.$$

From Table 1 we see that $\pi = 0$ when $\overline{D} = 0, 4$, and then $\sigma = +, -$ respectively. Thus $\Gamma^\pm$ have the symmetry $(-1)^{r(r-1)/2}$ and $-(-1)^{r(r-1)/2}$ respectively in the two cases.

If $\pi = p(r) = 1$ then $S^\pm$ are dual to each other, $\gamma^\pm$ map $\Lambda^r \otimes S^\pm$ to $S^\mp$, and we argue similarly that

$$\mathrm{Hom}(\Lambda^r \otimes S^\pm, S^\mp) = \mathbf{C}\gamma^\pm, \qquad \mathrm{Hom}(\Lambda^r \otimes S^\pm, S^\pm) = 0.$$

We see from Table 1 that $\pi = 1$ when $\overline{D} = 2, 6$ with $\sigma = +, -$ respectively. Thus $\Gamma^\pm$ have the symmetry $(-1)^{r(r-1)/2}$ and $-(-1)^{r(r-1)/2}$ respectively in the 2 cases.

If Γ is odd, i.e., when $\pi = p(r) + 1$, the discussion is entirely similar. Then Γ_0 is 0 on $S^\pm \otimes S^\pm$ and it is natural to define $\Gamma^\pm$ as the restrictions of Γ_0 to $S^\pm \otimes S^\mp$. Thus

$$\Gamma^\pm : S^\pm \times S^\mp \longrightarrow \Lambda^r$$

and these are again seen to be nonzero. We thus obtain as before

$$\mathrm{Hom}(S^\pm \otimes S^\mp, \Lambda^r) = \mathbf{C}\Gamma^\pm, \qquad \mathrm{Hom}(S^\pm \otimes S^\pm, \Lambda^r) = 0.$$

If $\pi = 1, p(r) = 0$ then $S^\pm$ are dual to each other, and

$$\mathrm{Hom}(\Lambda^r \otimes S^\pm, S^\pm) = \mathbf{C}\gamma^\pm, \qquad \mathrm{Hom}(\Lambda^r \otimes S^\pm, S^\mp) = 0.$$

This happens when $\overline{D} = 2, 6$ and there is no symmetry.

If $\pi = 0, p(r) = 1$ then $S^\pm$ are self dual, γ maps $\Lambda^r \otimes S^\pm$ to $S^\mp$, and

$$\mathrm{Hom}(\Lambda^r \otimes S^\pm, S^\mp) = \mathbf{C}\gamma^\pm, \qquad \mathrm{Hom}(\Lambda^r \otimes S^\pm, S^\pm) = 0.$$

This happens when $\overline{D} = 0, 4$ and there is no symmetry.

This completes the treatment of the case when D, the dimension of V, is even.

We turn to the case when D is odd. As usual the center Z of $C(V)$ now enters the picture. We have $Z = \mathbf{C}[\varepsilon]$ where ε is odd, $\varepsilon^2 = 1$, and $C(V) = C = C^+ \otimes Z$. The even algebra C^+ is isomorphic to $\mathrm{End}(S)$ where S is the spin module and $S \oplus S$ is the simple super module for C in which C^+ acts diagonally and ε acts as the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The basic lemma here is the following.

Lemma 5.5.12. *If D is odd then S is self dual and*

$$S \otimes S \simeq \oplus_{0 \le r \le (D-1)/2} \Lambda^r$$

is the decomposition of $S \otimes S$ into irreducible components under $\mathrm{SO}(V)$*. In particular the maps*

$$S \otimes S \longrightarrow \Lambda^r, \quad \Lambda^r \otimes S \longrightarrow S$$

are unique up to a scalar factor.

Proof. The skewsymmetrizer isomorphism λ of $\Lambda(V)$ with C takes $\Lambda^{\mathrm{even}} := \oplus_{0 \le k \le (D-1)/2} \Lambda^{2k}$ onto C^+. We have

$$S \otimes S \simeq \mathrm{End}(S) \simeq C^+ \simeq \Lambda^{\mathrm{even}}.$$

But now r and $D - r$ have opposite parity and so exactly one of them is even. Hence

$$\Lambda^{\mathrm{even}} \simeq \oplus_{0 \le r \le (D-1)/2} \Lambda^r.$$

This proves the decomposition formula for $S \otimes S$ and gives

$$\dim(\mathrm{Hom}(S \otimes S, \Lambda^r)) = \dim(\mathrm{Hom}(\Lambda^r \otimes S, S)) = 1.$$

The rest of the discussion is essentially the same as in the case of even D. The form $(\cdot, \cdot)$ on S is such that $(as, t) = (s, \beta(a)t)$ for all $a \in C^+$, $s, t \in S$.

If r is even, we have $\lambda(v) \in C^+$ for all $v \in \Lambda^r$, and so the map $\gamma : v \otimes s \longmapsto \lambda(v)s$ is a nonzero element of $\mathrm{Hom}(\Lambda^r \otimes S, S)$. We then obtain $\Gamma \in \mathrm{Hom}(S \otimes S, \Lambda^r)$ defined by

$$(\Gamma(s \otimes t), v) = (\lambda(v)s, t) \qquad (s, t \in S, v \in \Lambda^r, r \text{ even }).$$

There is no question of parity as S is purely even and

$$\text{symmetry } (\Gamma) = (-1)^{r(r-1)/2} \sigma$$

where σ is the symmetry of $(\cdot,\cdot)$. We use Table 3 for the values of σ which depend only on $\overline{D}$. Since $\mathrm{Hom}(S\otimes S,\Lambda^r)$ has dimension 1 by Lemma 12, we must have

$$\mathrm{Hom}(S\otimes S,\Lambda^r)=\mathbf{C}\Gamma,\qquad \mathrm{Hom}(\Lambda^r\otimes S,S)=\mathbf{C}\gamma.$$

The symmetry of Γ is $(-1)^{r(r-1)/2}$ or $-(-1)^{r(r-1)/2}$ according as $\overline{D}=1,7$ or $3,5$.

If r is odd, $\varepsilon\lambda(v)\in C^+$ for all $v\in\Lambda^r$ and so, if we define

$$\gamma_\varepsilon : v\otimes s\longmapsto \varepsilon\lambda(v)s,$$

then

$$0\neq\gamma_\varepsilon\in\mathrm{Hom}(\Lambda^r\otimes S,S).$$

We now define Γ_ε by

$$(\Gamma_\varepsilon(s\otimes t),v)=(\varepsilon\lambda(v)s,t)\qquad (s,t\in S,v\in\Lambda^r,r\text{ odd})$$

and obtain as before

$$\mathrm{Hom}(S\otimes S,\Lambda^r)=\mathbf{C}\Gamma_\varepsilon,\mathrm{Hom}(\Lambda^r\otimes S,S)=\mathbf{C}\gamma_\varepsilon.$$

To calculate the symmetry of Γ_ε we must note that β acts on ε by $\beta(\varepsilon)=s(\beta)\varepsilon$ and so

$$(\varepsilon\lambda(v)s,t)=s(\beta)(-1)^{r(r-1)/2}(s,\varepsilon\lambda(v)t).$$

Hence

$$\text{symmetry }(\Gamma)=(-1)^{r(r-1)/2}s(\beta)\sigma.$$

We now use Table 3 for the values of σ and $s(\beta)$. The symmetry of γ_ε is $(-1)^{r(r-1)/2}$ or $-(-1)^{r(r-1)/2}$ according as $\overline{D}=1,3$ or $5,7$.

We can summarize our results in the following theorem. Here S_1,S_2 denote the irreducible spin modules $S^\pm$ when D is even and S when D is odd. Also $r\leq D/2-1$ or $r\leq(D-1)/2$ according as D is even or odd. Let

$$\sigma_r=(-1)^{r(r-1)/2}.$$

Theorem 5.5.13. *For complex quadratic vector spaces V the existence and symmetry properties of maps*

$$\Gamma : S_1\otimes S_2\longrightarrow\Lambda^r(V)\qquad \gamma:\Lambda^r\otimes S_1\longrightarrow S_2$$

depend only on the residue class $\overline{D}$ *of* $D = \dim(V) \bmod 8$. *The maps, when they exist, are unique up to scalar factors and are related by*

$$(\Gamma(s_1 \otimes s_2), v) = (\gamma(v \otimes s_1), s_2).$$

The maps γ *exist only when* $S_1 = S_2 = S$ *(*D *odd),* $S_1 = S_2 = S^{\pm}$ *(*D, r *both even),* $S_1 = S^{\pm}, S_2 = S^{\mp}$ *(*D *even and* r *odd). In all cases the* γ *are given up to a scalar factor by the following table.*

$r \ \backslash \ D$	even	odd
even	$\gamma(v \otimes s^{\pm}) = \lambda(v)s^{\pm}$	$\gamma(v \otimes s) = \lambda(v)s$
odd	$\gamma(v \otimes s^{\pm}) = \lambda(v)s^{\pm}$	$\gamma_{\varepsilon}(v \otimes s) = \varepsilon\lambda(v)s$

Here ε *is a nonzero odd element in the center of* $C(V)$ *with* $\varepsilon^2 = 1$. *The maps* Γ *do not exist except in the cases described in the tables below which also give their symmetry properties.*

	$\overline{D}$	maps	symmetry
r even	0	$S^{\pm} \otimes S^{\pm} \to \Lambda^r$	σ_r
	1, 7	$S \otimes S \to \Lambda^r$	σ_r
	2, 6	$S^{\pm} \otimes S^{\mp} \to \Lambda^r$	
	3, 5	$S \otimes S \to \Lambda^r$	$-\sigma_r$
	4	$S^{\pm} \otimes S^{\pm} \to \Lambda^r$	$-\sigma_r$

	$\overline{D}$	maps	symmetry
r odd	0, 4	$S^{\pm} \otimes S^{\mp} \to \Lambda^r$	
	1, 3	$S \otimes S \to \Lambda^r$	σ_r
	2	$S^{\pm} \otimes S^{\pm} \to \Lambda^r$	σ_r
	5, 7	$S \otimes S \to \Lambda^r$	$-\sigma_r$
	6	$S^{\pm} \otimes S^{\pm} \to \Lambda^r$	$-\sigma_r$

Morphisms over the reals. The story goes along the same lines as it did for the forms. V is now a real quadratic vector space and the modules Λ^r are real and define conjugations on their complexifications. For a real irreducible spin module $S_{\mathbf{R}}$ the space of morphisms $S_{\mathbf{R}} \otimes S_{\mathbf{R}} \longrightarrow \Lambda^r$ carries, as in the case of forms, an action by $\mathbf{A}_1$. In this case the space of morphisms $\Lambda^r \otimes S_{\mathbf{R}} \longrightarrow S_{\mathbf{R}}$ also carries an action of $\mathbf{A}_1$ and the identification of these two Hom spaces respects this action.

Let $S_{\mathbf{R}}$ be of type $\mathbf{R}$, i.e., $\overline{\Sigma} = 1, 7, 0$. The morphisms from $S \otimes S, S^{\pm} \otimes S^{\pm}, S^{\pm} \otimes S^{\mp}$ to Λ^r over $\mathbf{C}$ span one-dimensional spaces stable under conjugation. Hence we can choose basis elements for them which are real. The morphisms $\Lambda^r \otimes S_{\mathbf{R}} \longrightarrow S_{\mathbf{R}}$ defined in Theorem 13 make sense over $\mathbf{R}$ (we must take ε to be real) and span the corresponding Hom space over $\mathbf{R}$. The results are then the same as in the complex case. The symmetries remain unchanged.

Let $S_{\mathbf{R}}$ be of type $\mathbf{H}$, i.e., $\overline{\Sigma} = 3, 5, 4$. Let $B(A_1, A_2 : R)$ be the space of morphisms $A_1 \otimes A_2 \longrightarrow R$. The relevant observation is that if S_1, S_2 are complex irreducible spin modules and U is a $\mathrm{Spin}(V_{\mathbf{C}})$-module such that $\dim(B(S_1, S_2 : U)) = 0$ or 1, then the space of morphisms $(S_1 \otimes W_1) \otimes (S_2 \otimes W_2) \longrightarrow U$ is just $B(S_1, S_2 : U) \otimes B(W_1, W_2 : \mathbf{C})$. The arguments are now the same as in the case of scalar forms; all one has to do is to replace the complex scalar forms by the complex maps into $V_{\mathbf{C}}$. The symmetries follow the same pattern as in the case of $r = 0$.

The last case is when $S_{\mathbf{R}}$ is of type $\mathbf{C}$, i.e., $\overline{\Sigma} = 2, 6$. The morphisms $S_{\mathbf{C}} \otimes S_{\mathbf{C}} \longrightarrow \Lambda^r(V_{\mathbf{C}})$ form a space of dimension 2, and this space, as well as its subspaces of symmetric and skewsymmetric elements, are stable under the conjugation on the Hom space. From this point on the argument is the same as in the case $r = 0$.

Theorem 5.5.14 (odd dimension). *For a real quadratic vector space V of odd dimension D the symmetry properties of morphisms $S_{\mathbf{R}} \otimes S_{\mathbf{R}} \longrightarrow \Lambda^r$ are governed by the residue classes $\overline{D}, \overline{\Sigma}$ as in the following table. If no number is attached to a symmetry sign, then the morphism is determined uniquely up to a real scalar factor.*

	$\overline{D} \backslash \overline{\Sigma}$	$1, 7(\mathbf{R})$	$3, 5(\mathbf{H})$
r even	1, 7	σ_r	$-\sigma_r[\mathbf{1}], \sigma_r[\mathbf{3}]$
	3, 5	$-\sigma_r$	$\sigma_r[\mathbf{1}], -\sigma_r[\mathbf{3}]$
r odd	1, 3	σ_r	$-\sigma_r[\mathbf{1}], \sigma_r[\mathbf{3}]$
	5, 7	$-\sigma_r$	$\sigma_r[\mathbf{1}], -\sigma_r[\mathbf{3}]$.

Theorem 5.5.15 (even dimension). *For real quadratic vector spaces V of even dimension D the symmetry properties of the maps $S^{\pm}_{\mathbf{R}} \otimes S^{\pm}_{\mathbf{R}}, S_{\mathbf{R}} \otimes S_{\mathbf{R}} \longrightarrow \Lambda^r$ are described in the following table. The notation* d.p. *means that the morphism goes from $S^{\pm}_{\mathbf{R}} \otimes S^{\mp}_{\mathbf{R}}$ to Λ^r. If no number is attached to a symmetry sign, then the morphism is determined uniquely up to a real scalar factor.*

	$\overline{D} \backslash \overline{\Sigma}$	$0(\mathbf{R}, \mathbf{R})$	$4(\mathbf{H}, \mathbf{H})$	$2, 6(\mathbf{C})$
r even	0	σ_r	$-\sigma_r[\mathbf{1}], \sigma_r[\mathbf{3}]$	$\sigma_r[\mathbf{2}]$
	2, 6	d.p.	d.p.$[\mathbf{1}] \oplus [\mathbf{3}]$	$+[\mathbf{0}], -[\mathbf{0}]$
	4	$-\sigma_r$	$\sigma_r[\mathbf{1}], -\sigma_r[\mathbf{3}]$	$-\sigma_r[\mathbf{2}]$
r odd	0, 4	d.p.	d.p.$[\mathbf{1}|] \oplus [\mathbf{3}]$	$+[\mathbf{0}], -[\mathbf{0}]$
	2	σ_r	$-\sigma_r[\mathbf{1}], \sigma_r[\mathbf{3}]$	$\sigma_r[\mathbf{2}]$
	6	$-\sigma_r$	$\sigma_r[\mathbf{1}], -\sigma_r[\mathbf{3}]$	$-\sigma_r[\mathbf{2}]$.

Symmetric morphisms from spinor, spinor to vector in the Minkowski case. An examination of the tables in Theorems 14 and 15 reveals that when V has signature $(1, D-1)$ and $S_{\mathbf{R}}$ is a real spin module irreducible over $\mathbf{R}$, there is *always* a unique (up to a real scalar factor) non-trivial *symmetric* morphism $\Gamma : S_{\mathbf{R}} \otimes S_{\mathbf{R}} \longrightarrow V$ invariant with respect to the action of $\mathbf{A}_1$. Indeed, the cases where there is a unique $\mathbf{A}_1$-invariant symmetric morphism $S_{\mathbf{R}} \otimes S_{\mathbf{R}} \longrightarrow V$ are given by

$$\overline{\Sigma} = 1, 7, \overline{D} = 1, 3; \overline{\Sigma} = 3, 5, \overline{D} = 5, 7; \overline{\Sigma} = 2, 6, \overline{D} = 0, 4; \overline{\Sigma} = 0, \overline{D} = 2; \overline{\Sigma} = 4, \overline{D} = 6$$

which include all the cases when the signature is Minkowski since this case corresponds to the relations $\overline{D} \pm \overline{\Sigma} = 2$. It turns out (see Deligne[8]) that this morphism is positive definite in a natural sense. Let $V^{\pm}$ be the sets in V where the quadratic form Q of V is > 0 or < 0, and let $(\cdot, \cdot)$ be the bilinear form associated to Q.

Theorem 5.5.16. *Let V be a real quadratic vector space of dimension D and signature $(1, D-1)$, and let $S_{\mathbf{R}}$ be a real spin module irreducible over $\mathbf{R}$. Then there is a non-trivial $\mathbf{A}_1$-invariant symmetric morphism*

$$\Gamma : S_{\mathbf{R}} \otimes S_{\mathbf{R}} \longrightarrow V$$

which is unique up to a real scalar factor. Moreover we can normalize the sign of the scalar factor so that for $0 \neq s \in S_{\mathbf{R}}$ we have

$$(v, \Gamma(s,s)) > 0 \qquad (v \in V^+).$$

Finally, whether $S_{\mathbf{R}}$ is irreducible or not, there is always a non-trivial symmetric morphism $S_{\mathbf{R}} \otimes S_{\mathbf{R}} \longrightarrow V$.

Proof. We have already remarked that the existence and (projective) uniqueness of Γ follows from the tables of Theorems 14 and 15. It is thus a question of of proving the positivity. Write $S = S_{\mathbf{R}}$ for brevity.

For this we give the argument of Deligne[8]. First of all we claim that the form $b_v(s,t) = (\Gamma(s,t), v)$ cannot be identically 0 for any $v \neq 0$; for if this is true for some v, it is true for all $g.v (g \in \mathrm{Spin}(V))$ and so, by irreducibility of S, for all elements of V. This is a contradiction. Fix now a $v \in V$ such that $Q(v) > 0$. Then b_v is invariant with respect to the stabilizer K of v in $\mathrm{Spin}(V)$. Because V is of Minkowski signature, it follows that $K \simeq \mathrm{Spin}(D-1)$ and is a maximal compact of $\mathrm{Spin}(V)$. If $\overline{D} = 2$ so that $V \simeq \mathbf{R}^{1,8k+1}$, $\overline{\Sigma} = 0$ so that we have two simple spin modules for $\mathrm{Spin}(V)$, $S^{\pm}_{\mathbf{R}}$, of type $\mathbf{R}$. The dimensions of $S^{\pm}_{\mathbf{R}}$ are equal to 2^{4k} which is also the dimension of the spin module of $\mathrm{Spin}(8k+1)$. Since spin modules restrict on quadratic subspaces to spinorial modules, the restrictions to K of $S^{\pm}_{\mathbf{R}}$ are irreducible. But K is compact and so leaves a unique (up to a scalar) definite form invariant, and hence b_v is definite. We are thus done when $\overline{D} = 2$. In the general case we consider $V_0 = V \oplus V_1$ where V_1 is a negative definite quadratic space so that $\dim(V_0) \equiv 2\ (8)$. By the above result there are positive $\mathbf{A}_1$-invariant symmetric morphisms $\Gamma^{\pm}_0 : S^{\pm}_{0,\mathbf{R}} \longrightarrow V_0$. Let P be the projection $V_0 \longrightarrow V$. Now the representation $S^{+}_{0,\mathbf{R}} \oplus S^{-}_{0,\mathbf{R}}$ is faithful on $C(V_0)^+$, hence on $C(V)^+$. We claim that $S_{\mathbf{R}}$ is contained in $2S^{+}_{0,\mathbf{R}} \oplus 2S^{-}_{0,\mathbf{R}}$. Indeed, let U be $S^{+}_{0,\mathbf{R}} \oplus S^{-}_{0,\mathbf{R}}$ viewed as a $C(V)^+$-module. Then $U_{\mathbf{C}}$, being faithful on $C(V_{\mathbf{C}})^+$, contains all the complex irreducibles of $C(V_{\mathbf{C}})^+$. If $S_{\mathbf{R}}$ is of type $\mathbf{R}$ or $\mathbf{C}$, we have $S_{\mathbf{C}} = S, S^{\pm}$ and so $\mathrm{Hom}(S_{\mathbf{C}}, U_{\mathbf{C}}) \neq 0$, showing that $\mathrm{Hom}(S_{\mathbf{R}}, U) \neq 0$. If $S_{\mathbf{R}}$ is of type $\mathbf{H}$, then $S_{\mathbf{C}} = 2S^{\pm}$ and so $\mathrm{Hom}(S_c, 2U_{\mathbf{C}}) \neq 0$. Thus we have $S \hookrightarrow S^{+}_{0,\mathbf{R}}$ or $S \hookrightarrow S^{-}_{0,\mathbf{R}}$. Then we can define

$$\Gamma(s,t) = P\Gamma^{\pm}_0(s,t) \qquad (s,t \in S \hookrightarrow S^{\pm}_{0,\mathbf{R}}).$$

It is obvious that Γ is positive. This finishes the proof in the general case.

5.6. Image of the real spin group in the complex spin module. From Theorem 5.10 we find that when $\overline{D} = 1, \overline{\Sigma} = 1$ the spin module $S_{\mathbf{R}}$ is of type $\mathbf{R}$

and has a symmetric invariant form and so the spin group $\mathrm{Spin}(V)$ is embedded in a real orthogonal group. The question naturally arises as to what the *signature* of this orthogonal group is. More generally, it is a natural question to ask what can be said about the image of the real spin group in the spinor space. This question makes sense even when the complex spin representation does not have a real form. In this section we shall try to answer this question. The results discussed here can be found in[10]. They are based on E. Cartan's classification of the real forms of complex simple Lie algebras[11] and a couple of simple lemmas.

Let V be a real quadratic vector space. A complex irreducible representation of $\mathrm{Spin}(V)$ is said to be *strict* if it does not factor through to a representation of $\mathrm{SO}(V)^0$. The spin and semispin representations are strict but so are many others. Indeed, the strict representations are precisely those that send the nontrivial central element of the kernel of $\mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V)^0$ to -1 in the representation space. If $D = \dim(V) = 1$, $\mathrm{Spin}(V)$ is $\{\pm 1\}$ and the only strict representation is the spin representation which is the nontrivial character. In dimension 2, if V is definite, we have $\mathrm{Spin}(V) = \mathrm{U}(1)$ with $\mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V)^0 \simeq \mathrm{U}(1)$ as the map $z \longmapsto z^2$, and the strict representations are the characters $z \longrightarrow z^n$ where n is an odd integer; the spin representations correspond to $n = \pm 1$. If V is indefinite, $\mathrm{Spin}(V) = \mathbf{R}^\times, \mathrm{SO}(V)^0 = \mathbf{R}^\times_+$, and the covering map is $t \longmapsto t^2$; the strict representations are the characters $t \longrightarrow \mathrm{sgn}\,(t)|t|^z$ where $z \in \mathbf{C}$, and the spin representations correspond to $z = \pm 1$. In dimension 3 when $\mathrm{Spin}(V) = \mathrm{SL}(2, \mathbf{C})$, the strict representations are the nontrivial representations of even dimension; the spin representation is the one with dimension 2.

Lemma 5.6.1. *If $D > 2$, the spin representations are precisely the strict representations of minimal dimension, i.e., if a representation is strict and different from the spin representation, its dimension is strictly greater than the dimension of the spin representation.*

Proof. We go back to the discussion of the basic structure of the orthogonal Lie algebras in Section 3. Let $\mathfrak{g} = \mathfrak{so}(V)$.

$\mathfrak{g} = D_\ell$: The positive roots are

$$a_i - a_j \ (1 \le i < j \le \ell), \quad a_p + a_q \ (1 \le p < q \le \ell).$$

If $b_1, \ldots, b_\ell$ are the fundamental weights then we have

$$b_i = a_1 + \ldots + a_i \ (1 \le i \le \ell - 2)$$

while

$$b_{\ell-1} = \frac{1}{2}(a_1 + \ldots + a_{\ell-1} - a_\ell) \quad b_\ell = \frac{1}{2}(a_1 + \ldots + a_{\ell-1} + a_\ell).$$

For any dominant integral linear form λ we write π_λ for the irreducible representation with highest weight λ. The weights of V are $(\pm a_i)$ and it is not difficult to verify (see[12], Chapter 4) that

$$\Lambda^r \simeq \pi_{b_r}\ (1 \le r \le \ell - 2), \quad \Lambda^{\ell-1} \simeq \pi_{b_{\ell-1}+b_\ell}, \quad \Lambda^\ell \simeq \pi_{2b_{\ell-1}} \oplus \pi_{2b_\ell}.$$

The most general highest weight is $\lambda = m_1 b_1 + \ldots + m_\ell b_\ell$ where the m_i are integers ≥ 0. Expressing it in terms of the a_i we see that it is an integral linear combination of the a_i if and only if $m_{\ell-1}$ and m_ℓ have the same parity, and this is the condition that the representation π_λ occurs among the tensor spaces over V. So the strictness condition is that $m_{\ell-1}$ and m_ℓ have opposite parities. The semispin representations correspond to the choices where $m_i = 0$ for $1 \le i \le \ell - 2$ and $(m_{\ell-1}, m_\ell) = (1, 0)$ or $(0, 1)$. If $m_{\ell-1}$ and m_ℓ have opposite parities, then one of $m_{\ell-1}, m_\ell$ is odd and so ≥ 1. Hence

$$(m_1, \ldots, m_\ell) = \begin{cases} (m_1, \ldots, m_{\ell-2}, m_{\ell-1} - 1, m_\ell) + (0, \ldots, 0, 1, 0)\ (m_{\ell-1} \ge 1) \\ (m_1, \ldots, m_{\ell-1}, m_\ell - 1) + (0, \ldots, 0, 1)\ (m_\ell \ge 1). \end{cases}$$

The result follows if we remark that the Weyl dimension formula for π_μ implies that

$$\dim(\pi_{\mu+\nu}) > \dim(\pi_\mu) \qquad (\nu \ne 0)$$

where μ, ν are dominant integral.

$\mathfrak{g} = B_\ell$: The positive roots are

$$a_i - a_j\ (1 \le i < j \le \ell), \quad a_p + a_q\ (1 \le p < q \le \ell), \quad a_i\ (1 \le i \le \ell).$$

If $b_1, \ldots, b_\ell$ are the fundamental weights then we have

$$b_i = a_1 + \ldots + a_i\ (1 \le i \le \ell - 1), \quad b_\ell = \frac{1}{2}(a_1 + \ldots + a_\ell).$$

For a dominant integral $\lambda = m_1 b_1 + \ldots + m_\ell b_\ell$ we find that it is an integral linear combination of the a_i's if and only if m_ℓ is even. So the strictness condition is that m_ℓ should be odd. If m_ℓ is odd we can write

$$(m_1, \ldots, m_\ell) = (m_1, \ldots, m_{\ell-1}, m_\ell - 1) + (0, \ldots, 0, 1)$$

from which the lemma follows again by using Weyl's dimension formula.

Let $d_0 = 1$ and let d_p $(p \ge 1)$ be the dimension of the spin module(s) of $\mathrm{Spin}(p)$. Recall from Section 3 that

$$d_p = 2^{[\frac{p+1}{2}]-1} \qquad (p \ge 1).$$

Lemma 5.6.2. *Let π be a representation of* $\mathrm{Spin}(p,q)$ *in a vector space* U *with the property that* $\pi(\varepsilon) = -1$ *where* ε *is the nontrivial element in the kernel of* $\mathrm{Spin}(p,q) \longrightarrow \mathrm{SO}(p,q)^0$, *and let* $K_{p,q}$ *be the maximal compact of* $\mathrm{Spin}(p,q)$ *lying above* $K_0 = \mathrm{SO}(p) \times \mathrm{SO}(q)$. *If* W *is any nonzero subspace of* U *invariant under* $\pi(K_{p,q})$ *then*

$$\dim(W) \geq d_p d_q.$$

In particular, if H *is a real connected semisimple Lie subgroup of* $\mathrm{GL}(U)$ *such that* $\pi(K_{p,q}) \subset H$, *and* L *a maximal compact subgroup of* H, *then for any nonzero subspace of* U *invariant under* L, *we have*

$$\dim(W) \geq d_p d_q.$$

Proof. The cases $p = 0, 1, q = 0, 1, 2$ are trivial since the right side of the inequality to be established is 1. We separate the remaining cases into $p = 0$ and $p > 0$.

a) $p = 0, 1, q \geq 3$: Then $K_{p,q} = \mathrm{Spin}(q)$. We may obviously assume that W is irreducible. Then we have a strict irreducible representation of $\mathrm{Spin}(q)$ in W and hence, by Lemma 1, we have the desired inequality.

b) $2 \leq p \leq q$: In this case we use the description of $K_{p,q}$ given in the remark following Theorem 3.7 so that ε_r maps on ε for $r = p, q$. We can view the restriction of π to $K_{p,q}$ as a representation ρ of $\mathrm{Spin}(p) \times \mathrm{Spin}(q)$ acting irreducibly on W. Then $\rho \simeq \rho_p \times \rho_q$ where ρ_r is an irreducible representation of $\mathrm{Spin}(r), (r = p, q)$. Since $\rho(\varepsilon_p) = \rho(\varepsilon_q) = -1$ by our hypothesis it follows that $\rho_p(\varepsilon_p) = -1, \rho(\varepsilon_q) = -1$. Hence ρ_r is a strict irreducible representation of $\mathrm{Spin}(r), (r = p, q)$ so that $\dim(\rho_r) \geq d_r, (r = p, q)$. But then

$$\dim(W) = \dim(\rho_p)\dim(\rho_q) \geq d_p d_q.$$

This proves the first statement.

Choose a maximal compact M of H containing $\pi(K_{p,q})$; this is always possible because $\pi(K_{p,q})$ is a compact connected subgroup of H. There is an element $h \in H$ such that $hLh^{-1} = M$. Since W is invariant under L if and only if $h[W]$ is invariant under hLh^{-1}, and $\dim(W) = \dim(h[W])$, it is clear that we may replace L by M. But then W is invariant under $\pi(K_{p,q})$ and the result follows from the first assertion. This finishes the proof of the lemma.

Corollary 5.6.3. *Suppose π is the irreducible complex spin representation. Let* $N = \dim(\pi)$ *and let* H, L *be as in the lemma. Then, for any nonzero subspace* W

of U invariant under L we have

$$\dim(W) \geq \begin{cases} \frac{N}{2} & \text{if one of } p, q \text{ is even} \\ N & \text{if both } p, q \text{ are odd.} \end{cases}$$

In particular, when both p and q are odd, the spin module of $\mathrm{Spin}(p,q)$ *is already irreducible when restricted to its maximal compact subgroup.*

Proof. We can assume p is even for the first case as everything is symmetric between p and q. Let $p = 2k, q = 2\ell$ or $2\ell + 1$, we have $d_p = 2^{k-1}, d_q = 2^{\ell-1}$ or 2^ℓ while $N = 2^{k+\ell-1}$ or $2^{k+\ell}$ and we are done. If $p = 2k+1, q = 2\ell+1$ then $d_p = 2^k, d_q = 2^\ell, N = 2^{k+\ell}$ and hence $d_p d_q = N$. This implies at once that U is already irreducible under $K_{p,q}$.

Real forms. If $\mathfrak{g}$ is a complex Lie algebra, by a *real form* of $\mathfrak{g}$ we mean a real Lie algebra $\mathfrak{g}_0 \subset \mathfrak{g}$ such that $\mathfrak{g} \simeq \mathbf{C} \otimes_{\mathbf{R}} \mathfrak{g}_0$. This comes to requiring that there is a basis of $\mathfrak{g}_0$ over $\mathbf{R}$ which is a basis of $\mathfrak{g}$ over $\mathbf{C}$. Then the map $X + iY \longmapsto X - iY$ $(X, Y \in \mathfrak{g})$ is a *conjugation* of $\mathfrak{g}$, i.e., a conjugate linear map of $\mathfrak{g}$ onto itself preserving brackets, such that $\mathfrak{g}_0$ is the set of fixed points of this conjugation. If G is a connected complex Lie group, a connected real Lie subgroup $G_0 \subset G$ is called a *real form* of G if $\mathrm{Lie}(G_0)$ is a real form of $\mathrm{Lie}(G)$. E. Cartan determined all real forms of complex simple Lie algebras $\mathfrak{g}$ up to conjugacy by the adjoint group of $\mathfrak{g}$, leading to a classification of real forms of the complex classical Lie groups. We begin with a summary of Cartan's results[11]. Note that if ρ is any conjugate linear transformation of $\mathbf{C}^n$, we can write $\rho(z) = Rz^\sigma$ where R is a linear transformation and $\sigma : z \mapsto z^{\mathrm{conj}}$ is the standard conjugation of $\mathbf{C}^n$; if $R = (r_{ij})$, then the r_{ij} are defined by $\rho e_j = \sum_i r_{ij} e_i$. We have $R\overline{R} = \pm 1$ according as ρ is a conjugation or a pseudo conjugation. We say ρ corresponds to R; the standard conjugation corresponds to $R = I_n$. If we take $R = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ we get the standard pseudo conjugation τ of $\mathbf{C}^{2n}$ given by $\tau : (z, w) \longmapsto (-\overline{w}, \overline{z})$. If L is an endomorphism of $\mathbf{C}^n$, then L commutes with the antilinear transformation defined by R if and only if $LR = R\overline{L}$.

$G = \mathrm{SL}(n, \mathbf{C})$.

The real forms are

$$(\sigma)\ \mathrm{SL}(n, \mathbf{R}), \quad \mathrm{SU}(a,b) (a \leq b, a+b = n), \quad (\tau)\ \mathrm{SU}^*(2m) \simeq \mathrm{SL}(m, \mathbf{H})\ (n = 2m)$$

where the notation is the usual one and the symbol placed before the real form means that it is the subgroup commuting with the conjugation or pseudo conjugation described. We write $\mathrm{SU}(n)$ for $\mathrm{SU}(0,n)$. It is the unique (up to conjugacy) *compact* real form. The isomorphism

$$\mathrm{SU}^*(2m) \simeq \mathrm{SL}(m, \mathbf{H})$$

needs some explanation. If we identify $\mathbf{C}^2$ with the quaternions $\mathbf{H}$ by $(z, w) \longmapsto z + \mathbf{j}w$ then the action of $\mathbf{j}$ from the right on $\mathbf{H}$ corresponds to the pseudo conjugation $(z, w) \longmapsto (-\overline{w}, \overline{z})$. If we make the identification of $\mathbf{C}^{2m}$ with $\mathbf{H}^m$ by

$$(z_1, \ldots, z_m, w_1, \ldots, w_m) \longmapsto (z_1 + \mathbf{j}w_1, \ldots, z_m + \mathbf{j}w_m)$$

then we have an isomorphism between $\mathrm{GL}(m, \mathbf{H})$ and the subgroup G of $\mathrm{GL}(2m, \mathbf{C})$ commuting with the pseudo conjugation τ. It is natural to call the subgroup of $\mathrm{GL}(m, \mathbf{H})$ that corresponds to $G \cap \mathrm{SL}(2m, \mathbf{C})$ under this isomorphism as $\mathrm{SL}(m, \mathbf{H})$. The group G is a direct product of $H = G \cap \mathrm{U}(2m)$ and a vector group. If J is as above, then H is easily seen to be the subgroup of $\mathrm{U}(2m)$ preserving the symplectic form with matrix J and so is $\mathrm{Sp}(2m)$, hence connected. So G is connected. On the other hand, the condition $gJ = J\overline{g}$ implies that $\det(g)$ is real for all elements of G. Hence the determinant is > 0 for all elements of G. It is clear then that G is the direct product of $G \cap \mathrm{SL}(2m, \mathbf{C})$ and the positive homotheties, i.e., $G \simeq G \cap \mathrm{SL}(2m, \mathbf{C}) \times \mathbf{R}_+^\times$. Thus $\mathrm{GL}(m, \mathbf{H}) \simeq \mathrm{SL}(m, \mathbf{H}) \times \mathbf{R}_+^\times$.

$G = \mathrm{SO}(n, \mathbf{C})$.

The real forms are

$$(\sigma_a)\ \mathrm{SO}(a, b)(a \le b, a + b = n), \quad (\tau)\ \mathrm{SO}^*(2m)\ (n = 2m).$$

σ_a is the conjugation corresponding to $R_a = \begin{pmatrix} I_a & 0 \\ 0 & -I_b \end{pmatrix}$; if $x = \begin{pmatrix} I_a & 0 \\ 0 & iI_b \end{pmatrix}$ then it is easily verified that $x\mathrm{SO}(a, b)x^{-1}$ is the subgroup of $\mathrm{SO}(n, \mathbf{C})$ fixed by σ_a. It is also immediate that

$$g^T g = I_{2m}, \quad gJ_{2m} = J_{2m}\overline{g} \iff g^T g = I_{2m}, \quad \overline{g}^T J_{2m} g = J_{2m}$$

so that $\mathrm{SO}^*(2m)$ is also the group of all elements of $\mathrm{SO}(2m, \mathbf{C})$ that leave invariant the skew hermitian form

$$-z_1\overline{z}_{m+1} + z_{m+1}\overline{z}_1 - z_2\overline{z}_{m+2} + z_{m+2}\overline{z}_2 - \ldots - z_m\overline{z}_{2m} + z_{2m}\overline{z}_m.$$

We write $\mathrm{SO}(n)$ for $\mathrm{SO}(0, n)$; it is the compact form.

$\mathrm{Sp}(2n, \mathbf{C})$.

We remind the reader that this is the group of all elements g in $\mathrm{GL}(2n, \mathbf{C})$ such that $g^T J_{2n} g = J_{2n}$ where J_{2n} is as above. It is known that $\mathrm{Sp}(2n, \mathbf{C}) \subset \mathrm{SL}(2n, \mathbf{C})$. Its real forms are

$$(\sigma)\ \mathrm{Sp}(2n, \mathbf{R}), \quad (\tau_a)\ \mathrm{Sp}(2a, 2b)(a \leq b, a + b = n)$$

where τ_a is the pseudo conjugation

$$\tau_a : z \longmapsto J_a \overline{z}, \quad J_a = \begin{pmatrix} 0 & 0 & I_a & 0 \\ 0 & 0 & 0 & -I_b \\ -I_a & 0 & 0 & 0 \\ 0 & I_b & 0 & 0 \end{pmatrix}$$

and it can be shown as in the previous case that the subgroup in question is also the subgroup of $\mathrm{Sp}(2n, \mathbf{C})$ preserving the invariant Hermitian form $\overline{z}^T B_{a,b} z$ where

$$B_{a,b} = \begin{pmatrix} I_a & 0 & 0 & 0 \\ 0 & -I_b & 0 & 0 \\ 0 & 0 & I_a & 0 \\ 0 & 0 & 0 & -I_b \end{pmatrix}.$$

We write $\mathrm{Sp}(2n)$ for $\mathrm{Sp}(0, 2n)$. It is the compact form.

The groups listed above are all connected and the fact that they are real forms is verified at the Lie algebra level. Cartan's theory shows that there are no others.

Lemma 5.6.4. *Let G be a connected real Lie group and let $G \subset M$ where M is a complex connected Lie group. If $M = \mathrm{SO}(n, \mathbf{C})$ (resp. $\mathrm{Sp}(2n, \mathbf{C})$), then for G to be contained in a real form of M it is necessary that G commute with either a conjugation or a pseudo conjugation of $\mathbf{C}^n$ (resp. $\mathbf{C}^{2n}$); if G acts irreducibly on $\mathbf{C}^n$ (resp. $\mathbf{C}^{2n}$), this condition is also sufficient and then the real form containing G is unique and is isomorphic to $\mathrm{SO}(a, b)$ (resp. $\mathrm{Sp}(a, b)$). If $M = \mathrm{SL}(n, \mathbf{C})$, then for G to be contained in a real form of M it is necessary that G commute with either a conjugation or a pseudo conjugation of $\mathbf{C}^n$ or G leave invariant a nondegenerate Hermitian form on $\mathbf{C}^n$. If G acts irreducibly on $\mathbf{C}^n$ and does not leave a nondegenerate Hermitian form invariant, then the above condition is also sufficient and the real form, which is isomorphic to either $\mathrm{SL}(n, \mathbf{R})$ or $\mathrm{SU}^*(n)(n = 2m)$, is then unique.*

Proof. The first assertion is clear since the real forms of $\mathrm{SO}(n, \mathbf{C})$ and $\mathrm{Sp}(2n, \mathbf{C})$ are those that commute with either a conjugation or a pseudo conjugation of the underlying vector space. Let $M = \mathrm{SO}(n, \mathbf{C})$ or $\mathrm{Sp}(2n, \mathbf{C})$ and suppose that G acts irreducibly. If G commutes with a conjugation σ, then the space of invariant forms for G is one dimensional, and so this space is spanned by the given form on $\mathbf{C}^n$ or $\mathbf{C}^{2n}$ in the two cases. This means that the given form transforms into a multiple of itself under σ and hence M is fixed by σ. But then $G \subset M^\sigma$ showing that G is contained in a real form of M. If there is another real form containing G, let λ be the conjugation or pseudo conjugation commuting with G. Then $\sigma^{-1}\lambda$ is an automorphism of $\mathbf{C}^n$ or $\mathbf{C}^{2n}$ commuting with G and so must be a scalar c as G acts irreducibly. Thus $\lambda = c\sigma$, showing that $M^\sigma = M^\lambda$. Let $M = \mathrm{SL}(n, \mathbf{C})$. The necessity and sufficiency are proved as before, and the uniqueness also follows as before since we exclude the real forms $\mathrm{SU}(a, b)$.

Theorem 5.6.5. *Let V be a real quadratic space of dimension D. When $D = 1$ the spin group is $\{\pm 1\}$ and its image is $\mathrm{O}(1)$. If $D = 2$ we have $\mathrm{Spin}(2) \simeq \mathrm{U}(1)$ and the spin representations are the characters $z \longmapsto z, z^{-1}$, while $\mathrm{Spin}(1,1) \simeq \mathrm{GL}(1, \mathbf{R}) \simeq \mathbf{R}^\times$ and the spin representations are the characters $a \longmapsto a, a^{-1}$. In all other cases the restriction of the complex spin representation(s) to $\mathrm{Spin}(V)$ is contained in a unique real form of the appropriate classical group of the spinor space according to the following tables.*

$$N = \text{ dimension of the complex spin module(s) .}$$

Spin(V) noncompact

	real	quaternionic	complex
orthogonal	$\mathrm{SO}(\frac{N}{2}, \frac{N}{2})$	$\mathrm{SO}^*(N)$	$\mathrm{SO}(N, \mathbf{C})_\mathbf{R}$
symplectic	$\mathrm{Sp}(N, \mathbf{R})$	$\mathrm{Sp}(\frac{N}{2}, \frac{N}{2})$	$\mathrm{Sp}(N, \mathbf{C})_\mathbf{R}$
dual pair	$\mathrm{SL}(N, \mathbf{R})$	$\mathrm{SU}^*(N)$	$\mathrm{SU}(\frac{N}{2}, \frac{N}{2})$

Spin(V) compact

real	quaternionic	complex
$\mathrm{SO}(N)$	$\mathrm{Sp}(N)$	$\mathrm{SU}(N)$

Proof. The arguments are based on the lemmas and corollary above. Let us consider first the case when the Spin group is noncompact so that $V \simeq \mathbf{R}^{p,q}$ with $1 \le p \le q$. Let Γ be the image of $\mathrm{Spin}(V)$ in the spinor space.

Spin representation(s) orthogonal (orthogonal spinors). This means $\overline{D} = 0, 1, 7$. Then Γ is inside the complex orthogonal group and commutes with either a conjugation or a pseudo conjugation according as $\overline{\Sigma} = 0, 1, 7$ or $\overline{\Sigma} = 3, 4, 5$. In the second case $\Gamma \subset \mathrm{SO}^*(N)$ where N is the dimension of the spin representation(s). In the first case $\Gamma \subset \mathrm{SO}(a,b)^0$ and we claim that $a = b = N/2$. Indeed, we first note that p and q cannot both be odd; for, if $\overline{D} = 1, 7$, $p - q$ is odd, while for $\overline{D} = 0$, both $p + q$ and $p - q$ have to be divisible by 8 which means that p and q are both divisible by 4. For $\mathrm{SO}(a,b)^0$ a maximal compact is $\mathrm{SO}(a) \times \mathrm{SO}(b)$ which has invariant subspaces of dimension a and b, and so, by Corollary 3 above we must have $a, b \ge N/2$. Since $a + b = N$ we see that $a = b = N/2$. There still remains the case $\overline{\Sigma} = 2, 6$, i.e., when the real spin module is of the complex type. But the real forms of the complex orthogonal group commute either with a conjugation or a pseudo conjugation and this cannot happen by Lemma 5.9. So there is no real form of the complex orthogonal group containing Γ. The best we can apparently do is to say that the image is contained in $\mathrm{SO}(N, \mathbf{C})_{\mathbf{R}}$ where the suffix $\mathbf{R}$ means that it is the real Lie group underlying the complex Lie group.

Spin representation(s) symplectic (symplectic spinors). This means that $\overline{D} = 3, 4, 5$. Here Γ is inside the complex symplectic group of spinor space. Then Γ commutes with either a conjugation or a pseudo conjugation according as $\overline{\Sigma} = 0, 1, 7$ or $\overline{\Sigma} = 3, 4, 5$. In the first case $\Gamma \subset \mathrm{Sp}(N, \mathbf{R})$. In the second case we have $\Gamma \subset \mathrm{Sp}(2a, 2b)$ with $2a + 2b = N$. The group $\mathrm{S}(\mathrm{U(a)} \times \mathrm{U(b)})$ is a maximal compact of $\mathrm{Sp}(2a, 2b)$ and leaves invariant subspaces of dimension $2a$ and $2b$. Moreover in this case both of p, q cannot be odd; for, if $\overline{D} = 3, 5$, $p - q$ is odd, while, for $\overline{D} = 4$, both $p - q$ and $p + q$ are divisible by 4 so that p and q will have to be even. By Corollary 3 above we have $2a, 2b \ge N/2$ so that $2a = 2b = N/2$. Once again in the complex case $\Gamma \subset \mathrm{Sp}(N, \mathbf{C})_{\mathbf{R}}$. We shall see below that there is equality for $\mathrm{Spin}(1, 3)$.

Dimension is even and the spin representations are dual to each other (linear spinors). Here $\overline{D} = 2, 6$. If the spin representations are real, then they admit no invariant bilinear forms and the only inclusion we have is that they are inside the special linear group of the spinor space. Hence, as they commute with a conjugation, we have, by the lemma above, $\Gamma \subset \mathrm{SL}(N, \mathbf{R})$. If the spin representations are quaternionic, Γ commutes with a pseudo conjugation τ while admitting no invariant bilinear form. We claim that Γ does not admit an invariant Hermitian form either. In fact, if h is an invariant Hermitian form, then $s, t \longmapsto h(s, \tau(t))$ is an invariant bilinear form which is impossible. So we must have $\Gamma \subset \mathrm{SU}^*(N)$. If the real spin representation is of the complex type the argument is more interesting. Let S be the real irreducible spin module so that $S_{\mathbf{C}} = S^+ \oplus S^-$. Let J be the conjugation in $S_{\mathbf{C}}$ that defines S. Then $JS^{\pm} = S^{\mp}$. There exists a pairing $(\cdot, \cdot)$ between $S^{\pm}$. Define $b(s^+, t^+) = (s^+, Jt^+), (s^+, t^+ \in S^+)$. Then b is a $\mathrm{Spin}^+(V)$-invariant sesquilinear form; as S^+ is irreducible, the space of invariant sesquilinear forms is of dimension 1 and so b is a basis for this space. Since this space is stable under adjoints, b is either Hermitian or skew Hermitian, and replacing b by ib if necessary we may assume that S^+ admits a Hermitian invariant form. Hence $\Gamma \subset \mathrm{SU}(a, b)$. The maximal compact argument using Corollary 3 above implies as before that $a, b \geq N/2$. Hence $\Gamma \subset \mathrm{SU}(\frac{N}{2}, \frac{N}{2})$. This finishes the proof of the theorem when $\mathrm{Spin}(V)$ is noncompact.

Spin group compact. This means that $p = 0$ so that $\overline{D} = -\overline{\Sigma}$. So we consider the three cases when the real spin module is of the real, quaternionic or complex types. If the type is real, the spin representation is orthogonal and so $\Gamma \subset \mathrm{SO}(N)$. If the type is quaternionic, Γ is contained in a compact form of the complex symplectic group and so $\Gamma \subset \mathrm{Sp}(N)$. Finally if the real spin module is of the complex type, the previous discussion tells us that Γ admits a Hermitian invariant form, and so as the action of γ is irreducible, this form has to be definite (since the compactness of γ implies that it admits an invariant definite hermitian form anyway). Hence $\gamma \subset \mathrm{SU}(N)$. This finishes the proof of the theorem.

Low dimensional isomorphisms.

In dimensions $D = 3, 4, 5, 6, 8$ the dimension of the spin group is the same as the dimension of the real group containing its image in spinor space and so the spin representation(s) defines a covering map. We need the following lemma.

Lemma 5.6.6. *Let V be a real quadratic space of dimension $D \neq 4$. Then the spin representation(s) is faithful except when $D = 4k$ and $V \simeq \mathbf{R}^{a,b}$ where both a and b are even. In this case the center of $\mathrm{Spin}(V) \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2$ in such a way that the diagonal subgroup is the kernel of the covering map $\mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V)^0$, and the two semispin representations have kernels as the two subgroups of order 2 in the center which are different from the diagonal subgroup.*

Proof. If D is odd, $\mathrm{Spin}(V_{\mathbf{C}})$ has center $C \simeq \mathbf{Z}_2$. Since $\mathfrak{so}(V_{\mathbf{C}})$ is simple, the kernel of the spin representation is contained in C. It cannot be C as then the spin represents would descend to the orthogonal group. So the spin representation is faithful.

For D even the situation is more delicate. Let C be the center of $\mathrm{Spin}(V_{\mathbf{C}})$ (see end of Section 3). If $D = 4k + 2$, we have $C \simeq \mathbf{Z}_4$ and the nontrivial element of the kernel of $\mathrm{Spin}(V_{\mathbf{C}}) \longrightarrow \mathrm{SO}(V_{\mathbf{C}})$ is the unique element of order 2 in C, and this goes to -1 under the (semi)spin representations. It is then clear that they are faithful on C, and the simplicity argument above (which implies that their kernels are contained in C) shows that they are faithful on the whole group.

If $D = 4k$, then $C \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2$. From our description of the center of $\mathrm{Spin}(V_{\mathbf{C}})$ in Section 3 we see that after identifying $\mathbf{Z}_2$ with $\{0, 1\}$, the nontrivial element z of the kernel of the covering map $\mathrm{Spin}(V_{\mathbf{C}}) \longrightarrow \mathrm{SO}(V_{\mathbf{C}})$ is $(1, 1)$. Let $z_1 = (1, 0), z_2 = (0, 1)$. Since $z = z_1 z_2$ goes to -1 under the semispin representations $S^{\pm}$, each of $S^{\pm}$ must map exactly one of z_1, z_2 to 1. They cannot both map the same z_i to 1 because the representation $S^+ \oplus S^-$ of $C(V_{\mathbf{C}})^+$ is faithful. Hence the kernels of $S^{\pm}$ are the two subgroups of order 2 inside C other than the diagonal subgroup. We now consider the restriction to $\mathrm{Spin}(V)$ of $S^{\pm}$. Let $V = \mathbf{R}^{a,b}$ with $a + b = D$. If a, b are both odd, and I is the identity endomorphism of V, $-I \notin \mathrm{SO}(a) \times \mathrm{SO}(b)$ and so the center of $\mathrm{SO}(V)^0$ is trivial. This means that the center of $\mathrm{Spin}(V)$ is $\mathbf{Z}_2$ and is $\{1, z\}$. So the semispin representations are again faithful on $\mathrm{Spin}(V)$. Finally suppose that both a and b are even. Then $-I \in \mathrm{SO}(a) \times \mathrm{SO}(b)$ and so the center of $\mathrm{Spin}(V)^0$ consists of $\pm I$. Hence the center of $\mathrm{Spin}(V)$ has 4 elements and so coincides with C, the center of $\mathrm{Spin}(V_{\mathbf{C}})$. Thus the earlier discussion for complex quadratic spaces applies without change and the two spin representations have as kernels the two $\mathbf{Z}_2$ subgroups of C that do not contain z. This finishes the proof of the lemma.

The case $D = 4$ is a little different because the orthogonal Lie algebra in dimension 4 is not simple but splits into two simple algebras. Nevertheless the table remains valid and we have

$$\mathrm{Spin}(0, 4) \longrightarrow \mathrm{SU}(2), \qquad \mathrm{Spin}(2, 2) \longrightarrow \mathrm{SL}(2, \mathbf{R}).$$

The groups on the left have dimension 6 while those on the left are of dimension 3, and so the maps are not covering maps. The case of $\mathrm{Spin}(1, 3)$ is more interesting. We can identify it with $\mathrm{SL}(2, \mathbf{C})_{\mathbf{R}}$ where the suffix $\mathbf{R}$ means that the group is the underlying real Lie group of the complex group. Let $\mathcal{H}$ be the space of 2×2 hermitian matrices viewed as a quadratic vector space with the metric $h \longmapsto \det(h) (h \in \mathcal{H})$. If we write

$$h = \begin{pmatrix} x_0 + x_3 & x_1 + i x_2 \\ x_1 - i x_2 & x_0 - x_3 \end{pmatrix} \qquad (x_\mu \in \mathbf{R})$$

then

$$\det(h) = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

so that $\mathcal{H} \simeq \mathbf{R}^{1,3}$. The action of $\mathrm{SL}(2, \mathbf{C})$ on $\mathcal{H}$ is given by

$$g, h \longmapsto g h \overline{g}^T$$

which defines the covering map

$$\mathrm{SL}(2,\mathbf{C})_{\mathbf{R}} \longrightarrow \mathrm{SO}(1,3)^0.$$

The spin representations are

$$\mathbf{2} : g \longmapsto g, \qquad \overline{\mathbf{2}} : g \longmapsto \overline{g}$$

and their images are exactly $\mathrm{SL}(2,\mathbf{C})_{\mathbf{R}}$.

The following special isomorphisms follow from the lemma above. The symbol $A \xrightarrow{2} B$ means that A is a double cover of B.

$D = 3$

$$\begin{aligned} \mathrm{Spin}(1,2) &\simeq \mathrm{SL}(2,\mathbf{R}) \\ \mathrm{Spin}(3) &\simeq \mathrm{SU}(2) \end{aligned}$$

$D = 4$

$$\mathrm{Spin}(1,3) \simeq \mathrm{SL}(2,\mathbf{C})_{\mathbf{R}}$$

$D = 5$

$$\begin{aligned} \mathrm{Spin}(2,3) &\simeq \mathrm{Sp}(4,\mathbf{R}) \\ \mathrm{Spin}(1,4) &\simeq \mathrm{Sp}(2,2) \\ \mathrm{Spin}(5) &\simeq \mathrm{Sp}(4) \end{aligned}$$

$D = 6$

$$\begin{aligned} \mathrm{Spin}(3,3) &\simeq \mathrm{SL}(4,\mathbf{R}) \\ \mathrm{Spin}(2,4) &\simeq \mathrm{SU}(2,2) \\ \mathrm{Spin}(1,5) &\simeq \mathrm{SU}^*(4) \simeq \mathrm{SL}(2,\mathbf{H}) \\ \mathrm{Spin}(6) &\simeq \mathrm{SU}(4) \end{aligned}$$

$D = 8$

$$\begin{aligned} \mathrm{Spin}(4,4) &\xrightarrow{2} \mathrm{SO}(4,4) \\ \mathrm{Spin}(2,6) &\xrightarrow{2} \mathrm{SO}^*(8) \\ \mathrm{Spin}(8) &\xrightarrow{2} \mathrm{SO}(8) \end{aligned}$$

Finally, the case $D = 8$ deserves special attention. In this case the Dynkin diagram has three extreme nodes and so there are 3 fundamental representations of $\mathrm{Spin}(V)$ where V is a complex quadratic vector space of dimension 8. They are the vector representation and the two spin representations. They are *all* of dimension 8 and their kernels are the three subgroups of order 2 inside the center C of $\mathrm{Spin}(V)$. In this case the group of automorphisms of the Dynkin diagram is $\mathfrak{S}_3$, the group of permutations of $\{1, 2, 3\}$. This is the group of automorphisms of $\mathfrak{g}$ modulo the group of inner automorphisms and so is also the group of automorphisms of $\mathrm{Spin}(V)$ modulo the inner automorphisms. Thus $\mathfrak{S}_3$ itself operates on the set of equivalence classes of irreducible representations. Since it acts transitively on the extreme nodes it permutes transitively the three fundamental representations. Thus the three fundamental representations are all on the same footing. This is the famous *principle of triality*, first discovered by E. Cartan[13]. Actually, $\mathfrak{S}_3$ *itself* acts on $\mathrm{Spin}(V)$.

5.7. Appendix: Some properties of the orthogonal groups. We would like to sketch a proof of Cartan's theorem on reflections and some consequences of it. We work over $k = \mathbf{R}$ or $\mathbf{C}$ and V a quadratic vector space over k. The notations are as in §5.3. We write however $\Phi(u, v) = (u, v)$ for simplicity.

For any nonisotropic vector $v \in V$ the *reflection* R_v is the orthogonal transformation that reflects every vector in the hyperplane orthogonal to v. It can be computed to be given by

$$R_v x = x - 2\frac{(x, v)}{(v, v)} v \qquad (x \in V).$$

By a reflection we simply mean an element of the form R_v for some nonisotropic v. Cartan's theorem says that any element of $\mathrm{O}(V)$ is a product of at most n reflections where $n = \dim(V)$. The simplest example is when $V = \mathbf{K}^2$ with the metric such that $(e_1, e_1) = (e_2, e_2) = 0, (e_1, e_2) = 1$. Then, for $v = e_1 + av_2$ where $a \neq 0$ the reflection R_v is given by

$$R_v = \begin{pmatrix} 0 & -a^{-1} \\ -a & 0 \end{pmatrix}$$

so that

$$T_c = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} = R_v R_{v'}, \quad v = e_1 + ae_2, v' = e_1 + ace_2.$$

However in the general case T can be more complicated, for instance can be unipotent, and so it is a more delicate argument. For the proof the following special case is essential. Here $V = k^4$ with basis e_1, e_2, f_1, f_2 where

$$(e_i, e_j) = (f_i, f_j) = 0, \qquad (e_i, f_j) = \delta_{ij}$$

and

$$T = \begin{pmatrix} I_2 & B \\ 0 & I_2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with I_2 as the unit 2×2 matrix. In this case let

$$R_1 = R_{e_2+f_2}, \qquad R_2 = R_{e_2+cf_2}, \quad (c \neq 0, \neq 1).$$

Then a simple calculation shows that

$$S := R_2 R_1 T : e_1 \longmapsto e_1, \quad f_1 \longmapsto f_1, \quad e_2 \longmapsto c^{-1} e_2 \quad f_2 \longmapsto cf_2.$$

Hence S is the direct sum of I and T_c and so is a product of 2 reflections, showing that T is a product of 4 reflections.

We can now give Cartan's proof which uses induction on $n = \dim(V)$. If $T \in \mathrm{O}(V)$ fixes a nonisotropic vector it leaves the orthogonal complement invariant and the result follows by induction; T is then a product of at most $n-1$ reflections. Suppose that $x \in V$ is not isotropic and the vector $Tx - x$ is also not isotropic. Then the reflection R in the hyperplane orthogonal to $Tx - x$ will also send x to Tx. So $RTx = x$ and as x is not isotropic the argument just given applies. However it may happen that for all nonisotropic x, $Tx - x$ is isotropic. Then by continuity $Tx - x$ will be isotropic for all $x \in V$. We may also assume that T fixes no nonisotropic x. We shall now show that in this case $n = 4q$ and T is a direct sum of p transformations of the example in dimension 4 discussed above.

Let L be the image of V under $T - I$. Then L is an isotropic subspace of V and so $L \subset L^\perp$. We claim that $L = L^\perp$. If $x \in L^\perp$ and $y \in V$, then $Tx = x + \ell$ and $Ty = y + \ell'$ where $\ell, \ell' \in L$. Since $(Tx, Ty) = (x, y)$ we have $(x, \ell') + (y, \ell) + (\ell, \ell') = 0$. But $(x, \ell) = (\ell, \ell') = 0$ and so $(y, \ell) = 0$. Thus $\ell = 0$, showing that T is the identity on $L^\perp$. Since T cannot fix any nonisotropic vector this means that $L^\perp$ is isotropic and so $L^\perp \subset L$, proving that $L = L^\perp$. Thus $n = 2p$ where p is the dimension of $L = L^\perp$. In this case it is a standard result that we can find another isotropic subspace M such that $V = L \oplus M$ and $(\cdot, \cdot)$ is nonsingular on $L \times M$. Hence with respect to the direct sum $V = L \oplus M$, T has the matrix

$$\begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \qquad B \in \mathrm{Hom}(M, L),$$

and the condition that $(Tx, Ty) = (x, y)$ for all $x, y \in V$ gives

$$(Bm, m') + (Bm', m) = 0 \qquad (m, m' \in M).$$

We now claim that B is an isomorphism of M with L. Suppose that $Bm = 0$ for some nonzero $m \in M$. We choose $\ell \in L$ such that $(m, \ell) \neq 0$ and then a constant a such that $m + a\ell$ is not isotropic. Since $Bm = 0$ we have $T(m + a\ell) = m + a\ell$, a contradiction as T cannot fix any nonisotropic vector.

Thus B is an isomorphism of M with L. The nonsingularity of B implies that the skewsymmetric bilinear form

$$m, m' \longmapsto (Bm, m')$$

is nondegenerate and so we must have $p = 2q$ and there is a basis (m_i) of M such that $(Bm_i, m_j) = \delta_{j,q+i}$ $(1 \leq i \leq q)$. If (ℓ_i) is the dual basis in L then the matrix of T in the basis (ℓ_i, m_j) is

$$\begin{pmatrix} I_{2q} & J_{2q} \\ 0 & I_{2q} \end{pmatrix} \qquad J_{2q} = \begin{pmatrix} 0 & I_q \\ -I_q & o \end{pmatrix}$$

where I_r is the unit $r \times r$ matrix. Then $\dim(V) = 4q$ and T is a direct sum of q copies of the 4×4 matrix treated earlier as an example and the result for T follows immediately. This finishes the proof. We have thus proved the following.

Theorem 5.7.1. *Let V be a quadratic vector space over $k = \mathbf{R}$ or $\mathbf{C}$ of dimension n. Then any element of $\mathrm{O}(V)$ is a product of at most n reflections. An element of $\mathrm{O}(V)$ lies in $\mathrm{SO}(V)$ if and only if it is a product of an even number $2r \leq n$ of reflections.*

Connected components. We shall now determine the identity component of $\mathrm{O}(V)$. Since the determinant is ± 1 for elements of $\mathrm{O}(V)$ it is clear that the identity component is contained in $\mathrm{SO}(V)$. But $\mathrm{SO}(V)$ is not always connected. In the complex case it is standard that $\mathrm{SO}(V)$ is connected[10] and so we need to consider only the real case. We want to obtain the result as a consequence of the above theorem of Cartan, as Cartan himself did in[5]. Let $V = \mathbf{R}_{p,q}$. We may assume that

$0 \leq p \leq q$. If $p = 0$, we are in the case of the real orthogonal groups and the group $\mathrm{SO}(V)$ is again connected.

First assume that $p \geq 2$. The quadratic form is

$$x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2$$

and let $(e_i)_{1 \leq i \leq p+q}$ be the standard basis for V. Let us call a nonisotropic vector u *timelike* if $(u, u) > 0$ and *spacelike* if $(u, u) < 0$. Let $V^{\pm}$ be the subspaces spanned by $(e_i)_{1 \leq i \leq p}$ and $(e_i)_{p+1 \leq i \leq p+q}$. The matrix of an element T of $\mathrm{SO}(V)$ is of the form

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$$

corresponding to the direct sum $V = V^+ \oplus V^-$. We claim that $\det(A) \neq 0$. If not, there is a nonzero timelike vector u^+ such that Tu^+ is spacelike, a contradiction. So on any component of $\mathrm{SO}(V)$ the sign of $\det(T)$ is constant and so we already have the parts $\mathrm{SO}(V)^{\pm}$ where this sign is > 0 or < 0. Any element T of $\mathrm{SO}(V)$ can be written as $R_{v_1} \ldots R_{v_{2r}}$ where each v_j is either timelike or spacelike. But $R_v R_w = R_{R_v w} R_v$, and $R_v w$ is like w, and so we can arrange that in the product representation of T we have all the timelike and spacelike reflections together.

Any vector x with $(x, x) = 1$ can be written as $\cosh t\, u^+ + \sinh t\, u^-$ where $t \geq 0$, and $u^{\pm} \in V^{\pm}$ with $(u^{\pm}, u^{\pm}) = \pm 1$. It is clear that u^+ can be continuously joined to e_1, u^- similarly to e_{p+1}, and, then changing t continuously to 0 we see that x can be continuously joined to e_1. Thus the timelike vectors form a connected domain. A similar argument shows that the spacelike vectors also form a connected domain. Since the map that takes a vector to the reflection in the orthogonal hyperplane is continuous, it follows that any element $T \in \mathrm{SO}(V)$ can be continuously joined to an element of the form $R_{e_1}^r R_{e_{p+1}}^r$ where r is 0 or 1. Clearly $r = 0$ or 1 according as $T \in \mathrm{SO}(V)^{\pm}$ and the cases are distinguished by whether T is the product of an even or odd number each of timelike and spacelike reflections. So we see that $\mathrm{SO}(V)^{\pm}$ are themselves connected and the identity component is $\mathrm{SO}(V)^+$ which is characterized as the set of T expressible as a product of an even number each of timelike and spacelike reflections.

It remains to discuss the case when $p = 1$. Assume that $q \geq 2$. The argument for the connectedness of the set of spacelike vectors remains valid, but for the timelike vectors there are two connected components, depending on whether they can be connected to $\pm e_1$. For any timelike vector $x = \sum_i x_i e_i$ we have $x_1^2 - x_2^2 - \ldots - x_{q+1}^2 > 0$ and so $x_1^2 > 0$, so that the sign of x_1 is constant on any connected component. But $\pm e_1$ define the same reflection and so the argument to determine the identity component of $\mathrm{SO}(V)$ remains valid. The case $p = q = 1$ is trivial. We have thus proved the following.

Theorem 5.7.2. *The group* $\mathrm{SO}(p, q)$ *is connected if and only if either* p *or* q *is* 0*. Otherwise it has* 2 *connected components and the identity component consists of those elements which can be expressed as a product of an even number each of the timelike and spacelike reflections.*

The case $p = 1$ deserves some additional remarks since it is the Minkowski signature and so plays an important role in physics. To avoid trivialities let us assume that $q \geq 2$. Number the standard basis vectors as $e_0, e_1, \ldots, e_q$ where $(e_0, e_0) = 1$ and $(e_j, e_j) = -1$ for $j = 1, 2, \ldots, q$. In this case the timelike vectors $x = x_0e_0 + \sum_j x_je_j$ are such that $x_0^2 > \sum_j x_j^2$ and hence the two components are those where $x_0 >$ or < 0. These are the forward and backward light cones. If x is a unit vector in the forward cone we can use a rotation in the space V^- to move x to a vector of the form $x_0e_0 + x_1e_1$; and then using hyperbolic rotations in the span of e_0, e_1 we can move it to e_0. Suppose now that x, x' are two unit vectors in the forward cone. We claim that $(x, x') > 1$ unless $x = x'$ (in which case $(x, x') = 1$). For this we may assume that $x = e_0$. Then $(x, x') = x'_0 \geq 1$; if this is equal to 1, then $x'_j = 0$ for $j \geq 1$ and so $x' = e_0$. Thus

$$(x, x') > 1,\ = 1 \iff x = x' \qquad ((x, x) = (x', x') = 1, x_0, x'_0 > 0). \qquad (*)$$

We can now modify the argument of Theorem 1 to show that any $T \in \mathrm{O}(1, q)$ is a product of at most $n = q + 1$ spacelike reflections. This is by induction on q. Let $T \in \mathrm{O}(1, q)$ and suppose that x is a timelike unit vector. If $Tx = x$, then the orthogonal complement of x is a definite space of dimension $n - 1$ and since there are only spacelike reflections we are through by induction. Otherwise $Tx = x'$ is a timelike vector distinct from x. Then

$$(x - x', x - x') = 2 - 2(x, x') < 0$$

by $(*)$ so that $x - x'$ is a spacelike vector. The reflection $R = R_{x-x'}$ is the spacelike and takes x to x' also. Hence $T' = RT$ fixes x and induction applies once again. Thus we have proved the following.

Theorem 5.7.3. *If* $p = 1 \leq q$*, all elements of* $\mathrm{O}(1, q)$ *are products of at most* $n = q + 1$ *spacelike reflections, and they belong to* $\mathrm{SO}(1, q)^0$ *if and only if the number of reflections is even.*

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