4. ELEMENTARY THEORY OF SUPERMANIFOLDS

4.1. The category of ringed spaces. The unifying concept that allows us to view differentiable, analytic, or holomorphic manifolds, and also algebraic varieties, is that of a *ringed space*. This is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ (written as $\mathcal{O}$ when there is no doubt as to what X is) is a sheaf of commutative rings (with units) on X. For instance, let X be a Hausdorff second countable space carrying a smooth structure and let $C^\infty(U \longrightarrow C^\infty(U))$ be the sheaf of rings where, for each open set $U \subset X$, $C^\infty(U)$ is the $\mathbf{R}$-algebra of all smooth functions on U. Then (X, C^∞) is a ringed space which is locally isomorphic to the ringed space associated to a ball in $\mathbf{R}^n$ with its smooth structure. To formulate this notion more generally let us start with a topological space X. For each open $U \subset X$ let $R(U)$ be an $\mathbf{R}$-algebra of real *functions* such that the assignment

$$U \longmapsto R(U)$$

is a *sheaf* of algebras of functions. This means that the following conditions are satisfied:

(1) Each $R(U)$ contains the constant and if $V \subset U$ then the restriction map takes $R(U)$ into $R(V)$.

(2) If U is a union of open sets U_i and $f_i \in R(U_i)$, and if the (f_i) are compatible, i.e., given i, j, f_i and f_j have the same restriction to $U_i \cap U_j$, then the function f such that f_i is the restriction of f to U_i belongs to $R(U)$.

We call (X, R) a *ringed space of functions*. If (X, R) and (Y, S) are two such spaces, a *morphism* between (X, R) and (Y, S) is a continuous map $\psi(X \longrightarrow Y)$ such that the pullback map ψ^* takes $S(V)$ into $R(\psi^{-1}(V))$ for each open set $V \subset Y$; here

$$(\psi^*(g))(x) = g(\psi(x)) \qquad (g \in S(V)).$$

We have thus obtained the *category* of ringed spaces of functions. If (X, R) is a ringed space of functions and $Y \subset X$ is an open set, the space (Y, R_Y) is also a ringed space of functions if R_Y is the *restriction of R to Y*, i.e., for any open set $V \subset Y$, $R_Y(U) = R(U)$. We refer to (Y, R_Y) as the *open subspace* of (X, R) defined by Y; the identity map from Y to X is then a morphism.

In order to define specific types of ringed spaces of functions we choose local models and define the corresponding types of ringed spaces of functions as those locally isomorphic to the local models. For example, to define a smooth manifold we start with the ringed spaces $(\mathbf{R}^n, C_n^\infty)$ where

$$C_n^\infty : U \longmapsto C_n^\infty(U),$$

$C_n^\infty(U)$ being the $\mathbf{R}$-algebra of smooth functions on U. Then a differentiable or a smooth manifold can be defined as a ringed space (X, R) of functions such that for each point $x \in X$ there is an open neighborhood U of x and a homeomorphism of h of U with an open set $U^\sim \subset \mathbf{R}^n$ such that h is an isomorphism of (U, R_U) with the ringed space of functions $(U^\sim, C_n^\infty|U^\sim)$, i.e., if $V \subset U$ is open, the algebra $R(V)$ is precisely the algebra of all functions $g \circ h$ where g is a smooth function on $h(V)$. To define an analytic or a complex analytic manifold the procedure is similar; we simply replace $(\mathbf{R}^n, C_n^\infty)$ by $(\mathbf{R}^n, A_n)$ or $(\mathbf{C}^n, H_n)$ where A_n (resp. H_n) is the sheaf of algebras of analytic (resp. complex analytic) functions. It is usual to add additional conditions of separation and globality on X, for instance, that X be Hausdorff and second countable.

In algebraic geometry, Serre pioneered an approach to algebraic varieties by defining them as ringed spaces of functions locally isomorphic to the ringed spaces coming from affine algebraic sets over an algebraically closed field. See Dieudonne[1] for the theory of these varieties which he calls *Serre varieties*. It is possible to go far in the Serre framework; for instance it is possible to give quite a practical and adequate treatment of the theory of affine algebraic groups.

However, as we have mentioned before, Grothendieck realized that ultimately the Serre framework is inadequate and that one has to replace the coordinate rings of affine algebraic sets with *completely arbitrary commutative rings with unit*, i.e., in the structure sheaf the rings of functions are replaced by arbitrary commutative rings with unit. This led to the more general definition of a ringed space leading to the Grothendieck's schemes. It turns out that this more general notion of a ringed space is essential for super geometry.

Definition. A sheaf of rings on a topological space X is an assignment

$$U \longmapsto R(U)$$

where $R(U)$ is a commutative ring with unit, with the following properties:

(1) If $V \subset U$ there is a homomorphism from $R(U)$ to $R(V)$, called *restriction to* V, denoted by r_{VU}; for three open sets $W \subset V \subset U$ we have $r_{WV}r_{VU} = r_{WU}$.

(2) If U is the union of open sets U_i and $f_i \in R(U_i)$ are given, then for the existence of $f \in R(U)$ that restricts on U_i to f_i for each i, it is necessary and sufficient that f_i and f_j have the same restrictions on $U_i \cap U_j$; moreover, f, when it exists, is unique.

A *ringed space* is a pair $(X, \mathcal{O})$ where X is a topological space and $\mathcal{O}$ is a sheaf of rings on X. $\mathcal{O}$ is called the *structure sheaf* of the ringed space. For any open set U the elements of $\mathcal{O}(U)$ are called *sections over* U. If it is necessary to call attention to X we write $\mathcal{O}_X$ for $\mathcal{O}$.

If $x \in X$ and U, V are open sets containing x, we say that two elements $a \in \mathcal{O}(U), b \in \mathcal{O}(V)$ are *equivalent* if there is an open set W with $x \in W \subset U \cap V$ such that a and b have the same restrictions to W. The equivalence classes are as usual called *germs* of sections of $\mathcal{O}$ and form a ring $\mathcal{O}_x$ called the *stalk* of the sheaf at x. The notion of a *space* is then obtained if we make the following definition.

Definition. A ringed space is called a *space* if the stalks are all local rings.

Here we recall that a commutative ring with unit is called *local* if it has a unique maximal ideal. The unique maximal ideal of $\mathcal{O}_x$ is denoted by $\mathfrak{m}_x$. The elements of $\mathcal{O}_x \setminus \mathfrak{m}_x$ are precisely the invertible elements of $\mathcal{O}_x$.

The notion of an open subspace of a ringed space is obtained as before; one just restricts the sheaf to the open set in question. In defining morphisms between ringed spaces one has to be careful because the rings of the sheaf are abstractly attached to the open sets and there is no automatic pullback as in the case when the rings were rings of functions. But the solution to this problem is simple. One gives the pullbacks also in defining morphisms. Thus a morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a continuous map ψ from X to Y together with a sheaf map of $\mathcal{O}_Y$ to $\mathcal{O}_X$ above ψ, i.e., a collection of homomorphisms

$$\psi_V^* : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(\psi^{-1}(V)) \qquad (V \text{ open } \subset Y)$$

which commute with restrictions. The notion of isomorphism of ringed spaces follows at once. We have thus obtained the *category* of ringed spaces. If the objects are spaces we require that the pullback, which induces a map $\mathcal{O}_{Y,\psi(x)} \longrightarrow \mathcal{O}_{X,x}$, is local, i.e., it takes the maximal ideal $\mathfrak{m}_{\psi(x)}$ of $\mathcal{O}_{Y,\psi(x)}$ into the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$.

In the case when the rings $\mathcal{O}(U)$ are actually rings of *functions* with values in a field k, the pullbacks defined earlier are in general the only ones possible. To see this, assume that X and Y are ringed spaces of functions and that the stalks are local rings. For $x \in X$, the elements of $\mathcal{O}_{X,x}$ vanishing at x form an ideal I_x and so is contained in $\mathfrak{m}_x$. Since I_x has codimension 1, being the kernel of the evaluation map $f \longmapsto f(x)$, we must have $I_x = \mathfrak{m}_x$. Then if an element has a nonzero value at a point, its restriction to some open set V containing x is invertible in $\mathcal{O}_X(V)$. Now suppose that we have an arbitrary pullback ψ^* defined as above. Fix $x \in X$ and let $\psi(x) = y$. If $\psi^*(g)(x) \neq g(\psi(x))$ for some $g \in S(V)$, we may, by adding a constant to g assume that $\psi^*(g)(x) = 0, g(\psi(x)) \neq 0$. So g is invertible on some V, hence $\psi^*(g)$ is invertible in an open neighborhood of x, contradicting the assumption that $\psi^*(g)(x) = 0$. This also shows that in this case the locality condition is automatically satisfied.

Using very general results from commutative algebra one can represent any commutative ring with unit as a ring of "functions"on some space, even though the field in which these functions take their values will in general vary from point to point. Indeed, the space is the set of *prime ideals* of the ring, and at any prime ideal we have the field of quotients of the integral domain which is the ring modulo the prime ideal; the value of an element of the ring at this prime ideal is its image in this field. But, as we explained in Chapter 2, this representation need not be faithful; there will be elements which go to the zero function. For instance this is the case for nilpotent elements. This fact makes the discussion of schemes more subtle.

To get super geometric objects we know that we have to replace everywhere the commutative rings by supercommutative rings. Thus a *super ringed space* is a topological space X with a sheaf of supercommuting rings with units, called the *structure sheaf.* The restriction homomorphisms of the sheaf must be morphisms in the super category and so must preserve the gradings. The definition of morphisms of super ringed spaces is exactly the same as for ringed spaces, with the only change that the pullback maps (ψ_V^*) must be morphisms in the category of supercommutative rings, i.e., preserve the gradings. We thus obtain the category of super ringed spaces. For any two objects X, Y in this category, $\mathrm{Hom}(X, Y)$ denotes as usual the set of morphisms $X \longrightarrow Y$. A *superspace* is a super ringed space such that the stalks are local supercommutative rings. A supermanifold is a special type of superspace.

Here we must note that a supercommutative ring is called local if it has a unique maximal homogeneous ideal. Since the odd elements are nilpotent, they are in any homogeneous maximal ideal and so this comes to saying that the even part is a commutative local ring.

4.2. Supermanifolds. To introduce supermanifolds we follow the example of classical manifolds and introduce first the *local models*. A *super domain* $U^{p|q}$ is the super ringed space $(U^p, \mathcal{C}^{\infty\, p|q})$ where U^p is an open set in $\mathbf{R}^p$ and $\mathcal{C}^{\infty\, p|q}$ is the sheaf of supercommuting rings defined by

$$\mathcal{C}^{\infty\, p|q} : V \longmapsto C^\infty(V)[\theta^1, \theta^2, \dots, \theta^q] \qquad (V \subset U \text{ open})$$

where the θ^j are anticommuting variables (indeterminates) satisfying the relations

$${\theta^i}^2 = 0, \qquad \theta^i\theta^j = -\theta^j\theta^i \ (i \neq j) \iff \theta^i\theta^j = -\theta^j\theta^i \ (1 \leq i, j \leq q).$$

Thus each element of $\mathcal{C}^{\infty\, p|q}(V)$ can be written as

$$\sum_{I \subset \{1,2,\dots,q\}} f_I \theta^I$$

where the $f_I \in C^\infty(V)$ and θ^I is given by

$$\theta^I = \theta^{i_1}\theta^{i_2} \dots \theta^{i_r} \qquad (I = \{i_1, \dots, i_r\}, i_1 < \dots < i_r).$$

The dimension of this superdomain is defined to be $p|q$. We omit the reference to the sheaf and call $U^{p|q}$ itself the superdomain. In particular we have the super domains $\mathbf{R}^{p|q}$. A *supermanifold* of dimension $p|q$ is a super ringed space which is locally isomorphic to $\mathbf{R}^{p|q}$. Morphisms between supermanifolds are morphisms between the corresponding super ringed spaces. We add the condition that the underlying topological space of a supermanifold should be Hausdorff and second countable. The superdomains $\mathbf{R}^{p|q}$ and $U^{p|q}$ are special examples of supermanifolds of dimension $p|q$. An open submanifold of a supermanifold is defined in the obvious way. The $U^{p|q}$ are open submanifold of the $\mathbf{R}^{p|q}$.

The definition of supermanifold given is in the smooth category. To yield definitions of real analytic and complex analytic supermanifolds we simply change the local models. Thus a real analytic supermanifold is a super ringed space locally isomorphic to $\mathbf{R}_{\mathrm{an}}^{p|q}$ which is the super ringed space with

$$U \longmapsto \mathcal{A}^{p|q}(U) = \mathcal{A}(U)[\theta^1, \ldots, \theta^q]$$

as its structure sheaf where $\mathcal{A}(U)$ is the algebra of all real analytic functions on U. For a complex analytic supermanifold we take as local models the spaces $\mathbf{C}^{p|q}$ whose structure sheaves are given by

$$\mathbf{C}^{p|q}(U) = H(U)[\theta^1, \ldots, \theta^q],$$

where $H(U)$ is the algebra of holomorphic functions on U. Actually one can even define, as Manin does[2], more general geometric objects, like superanalytic spaces, and even superschemes.

Examples. $\mathbf{R}^{p|q}$: We have already seen $\mathbf{R}^{p|q}$. The coordinates x_i of $\mathbf{R}^p$ are called the *even coordinates* and the θ^j are called the *odd coordinates.*

$GL(1|1)$: Although we shall study this and other super Lie groups in more detail later it is useful to look at them at the very beginning. Let G be the open subset of $\mathbf{R}^2$ with $x_1 > 0, x_2 > 0$. Then $GL(1|1)$ is the open submanifold of the supermanifold $\mathbf{R}^{2|2}$ defined by G. This is an example of a super Lie group and for making this aspect very transparent it is convenient to write the coordinates as a matrix:

$$\begin{pmatrix} x^1 & \theta^1 \\ \theta^2 & x^2 \end{pmatrix}.$$

We shall take up the Lie aspects of this example a little later.

$GL(p|q)$: We start with $\mathbf{R}^{p^2+q^2|2pq}$ whose coordinates are written as a matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where

$$A = (a_{ij}), \qquad D = (d_{\alpha\beta})$$

are the even coordinates and

$$B = (b_{i\beta}), \qquad C = (c_{\alpha j})$$

are the odd coordinates. If G is the subset where $\det(A)\det(D) \neq 0$, then G is open and the supermanifold $GL(p|q)$ is the open submanifold of $\mathbf{R}^{p^2+q^2|2pq}$ defined by G. Here again the multiplicative aspects will be taken up later.

Exterior bundles of vector bundles on a classical manifold and their relation to supermanifolds. Let M be a classical manifold and let V be a vector bundle on M. Then we have the *exterior bundle* E of V which is also a vector bundle on M. If V_x is the fiber of V at $x \in M$, then the fiber of E at x is $\Lambda(V_x)$, the exterior algebra of V_x. Let $\mathcal{O}$ be the sheaf of sections of E. Then locally on M the sheaf is isomorphic to $U^{p|q}$ where $p = \dim(M)$ and $q =$ rank (V), the rank of V being defined as the dimension of the fibers of V. Indeed, if V is the trivial bundle on N with sections θ_i, then the sections of E are of the form $\sum_I f_I \theta_I$ where $\theta_I = \theta_{i_1} \wedge \ldots \wedge \theta_{i_r}$ so that the sections over N of E can be identified with elements of $C^\infty(N)[\theta_1, \ldots, \theta_q]$. Thus $(M, \mathcal{O})$ is a supermanifold. Let us write $E^\flat$ for this supermanifold. Clearly every supermanifold is locally isomorphic to a supermanifold of the form $E^\flat$; indeed, this is almost the definition of a supermanifold. The extent to which supermanifolds are globally not of the form $E^\flat$ is thus a cohomological question. One can prove (not surprisingly) that any *differentiable* supermanifold is isomorphic to some $E^\flat$, and that this result is no longer true in the analytic category (see Manin's discussion[2]). However, even in the differentiable category we cannot simply replace supermanifolds by the supermanifolds of the form $E^\flat$. The point is that the isomorphism $M \simeq E^\flat$ is not canonical; indeed, as we shall elaborate later on, supermanifolds have *many more* morphisms than the exterior bundles because of the possibility, essential in the applications to physics, that the even and odd coordinates can be mixed under transformations. In other words, between two supermanifolds $E_1^\flat, E_2^\flat$ there are more morphisms in general than the morphisms that one obtains by requiring that they preserve the bundle structure.

The imbedded classical manifold of a supermanifold. If X is a supermanifold, the underlying topological space is often denoted by $|M|$. We shall now show that there is a natural smooth structure on $|M|$ that converts it into a smooth manifold. This gives the intuitive picture of M as essentially this classical manifold surrounded by a cloud of odd stuff. We shall make this more precise through our discussion below.

Let us first observe that if R is a commutative ring, then in the exterior algebra $E = R[\xi_1, \ldots, \xi_r]$, an element

$$s = s_0 + \sum_j s_j \xi_j + \sum_{j<m} s_{jm} \xi_j \xi_m + \ldots,$$

where the coefficients s_0, s_j etc are in R, is invertible in E if and only if s_0 is invertible in R. The map $s \mapsto s_0$ is clearly a homomorphism into R and so if s is invertible, then s_0 is invertible in R. To prove that s is invertible if s_0 is, it is clear that by replacing s with $s_0^{-1}s$ we may assume that $s_0 = 1$; then $s = 1 - n$ where n

is in the ideal generated by the ξ_j and so is nilpotent, so that s is invertible with inverse $1 + \sum_{m \geq 1} n^m$. Taking $R = C^\infty(V)$ where V is an open neighborhood of the origin 0 in $\mathbf{R}^p$, we see that for any section s of E, we can characterize $s_0(0)$ as the unique real number λ such that $s - \lambda$ is not invertible on any neighborhood of 0. We can now transfer this to any point x of a supermanifold M. Then to any section of $\mathcal{O}_M$ on an open set containing x we can associate its *value* at x as the unique real number $s^\sim(x)$ such that $s - s^\sim(x)$ is not invertible in any neighborhood of x. The map

$$s \longmapsto s^\sim(x)$$

is a homomorphism of $\mathcal{O}(U)$ into $\mathbf{R}$. Allowing x to vary in U we see that

$$s \longmapsto s^\sim$$

is a homomorphism of $\mathcal{O}(U)$ onto an algebra $\mathcal{O}'(U)$ of real functions on U. It is clear that the assignment

$$U \longmapsto \mathcal{O}'(U)$$

is a *presheaf* on M. In the case when $(U, \mathcal{O}_U)$ is actually $(V, \mathcal{O}_V)$ where V is an open set in $\mathbf{R}^p$ we see that $\mathcal{O}'_V = C^\infty_V$ and so is actually a sheaf. In other words, for any point of M there is an open neighborhood U of it such that the restriction of $\mathcal{O}'$ to U is a sheaf and defines the structure of a smooth manifold on U. So, if we define $\mathcal{O}^\sim$ to be the sheaf of algebras of functions generated by $\mathcal{O}'$, then $\mathcal{O}^\sim$ defines the structure of a smooth manifold on M. We write $M^\sim$ for this smooth manifold. It is also called the *reduced manifold* and is also written as M_{red}. It is clear that this construction goes through in the real and complex analytic categories also. For $M = U^{p|q}$ we have $M^\sim = U$.

One can also describe the sheaf in another way. If we write

$$\mathcal{J}(U) = \left\{ s \;\middle|\; s^\sim = 0 \text{ on } U \right\}$$

then it is clear that $\mathcal{J}$ is a subsheaf of $\mathcal{O}$. We then have the exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^\sim \longrightarrow 0$$

showing that $\mathcal{O}^\sim$ is the quotient sheaf $\mathcal{O}/\mathcal{J}$. The construction above exhibits $\mathcal{O}^\sim$ explicitly as a sheaf of algebras of *functions* on M.

From our definition of morphisms and the sheaf map $\mathcal{O} \longrightarrow \mathcal{O}^\sim$ it is now clear that the identity map $M^\sim \longrightarrow M$ is a morphism of the classical manifold $M^\sim$ into the supermanifold M. Since the pullback is surjective this is an imbedding and so justifies the intuitive picture that M is essentially the classical manifold $M^\sim$ surrounded by a cloud of odd stuff. Actually we can go further.

We introduce the sheafs $\mathcal{J}^r$ for $r = 1, 2, \ldots$ and define

$$M^r = (M, \mathcal{O}/\mathcal{J}^r)$$

so that $M^1 = M^\sim$. Then one can think of M^r as the r^{th} *infinitesimal neighborhood* of $M^\sim$ in M. The sequence

$$M^1 = M^\sim, M^2, M^3, \ldots$$

actually terminates with

$$M^{q+1} = M.$$

This is the same as saying that

$$\mathcal{J}^{q+1} = 0.$$

To see this we can work locally and take $M = U^{p|q}$. The sections of $\mathcal{J}$ over an open subset V of U are elements of the form

$$\sigma = \sum_j s_j \theta^j$$

where s_j are sections over V; it is obvious that if we take a product $\sigma_1 \ldots \sigma_r$ of such elements, the product is 0 if $r > q$. Notice however that the M^r are *not* supermanifolds; they are in general only superspaces in the sense of Manin.

Suppose now we have a morphism

$$\psi : M \longrightarrow N$$

of supermanifolds. Let ψ^* be the pullback $\mathcal{O}_N \longrightarrow \mathcal{O}_M$. If t is a section of $\mathcal{O}_N$ defined around $y = \psi(x)(x \in M)$ and $s = \psi^*(t)$, then $s - s^\sim(x)$ is not invertible in any neighborhood of x and so $t - s^\sim(x)$ is not invertible in any neighborhood of y, showing that

$$\psi^*(t)^\sim(x) = t^\sim(\psi(x)).$$

In particular

$$\psi^* \mathcal{J}_N \subset \mathcal{J}_M.$$

This shows that we have a morphism

$$\psi^\sim : M^\sim \longrightarrow N^\sim$$

of classical manifolds associated to $\psi(M \longrightarrow N)$. Clearly the assignment $\psi \longrightarrow \psi^\sim$ commutes with composition and so the assignment

$$M \longrightarrow M^\sim$$

is functorial. More generally for any fixed $r \geq 1$, the assignment

$$M \longrightarrow M^r$$

is also functorial, in view of the relation

$$\psi \mathcal{J}_N^r \subset \mathcal{J}_M^r.$$

Remark. If $M = E^\flat$ where E is the exterior bundle of a vector bundle over a classical manifold N, the $\mathcal{O}(U)$ are actually *modules* over $C^\infty(U)$ for U open in N and so we have maps $C^\infty(U) \longrightarrow \mathcal{O}(U)$. This means that we have a map $M \longrightarrow M^\sim$ as well as the imbedding $M^\sim \longrightarrow M$. In other words we have a *projection* $M \longrightarrow M^\sim$. This makes it clear why this is such a special situation.

Construction of supermanifolds by gluing. It is clear from the definition of supermanifolds that general supermanifolds are obtained by gluing superdomains. However the gluing has to be done more carefully than in the classical case because the rings of the sheaf are not function rings and so the gluing data have to be sheaf isomorphisms that have to be specified and do not come automatically.

Let X be a topological space, let $X = \bigcup_i X_i$ where each X_i is open and let $\mathcal{O}_i$ be a sheaf of rings on X_i for each i. Write $X_{ij} = X_i \cap X_j, X_{ijk} = X_i \cap X_j \cap X_k$. Let

$$f_{ij} : (X_{ji}, \mathcal{O}_j | X_{ji}) \longrightarrow (X_{ij}, \mathcal{O}_i | X_{ij})$$

be an isomorphism of sheafs with

$$f_{ij}^\sim = \text{ id }_{X_{ji}} = \text{ the identity map on } X_{ji} = X_{ij}.$$

To say that we *glue the ringed spaces* $(X_i, \mathcal{O}_i)$ *through the* f_{ij} means the construction of a sheaf of rings $\mathcal{O}$ on X and for each i a sheaf isomorphism

$$f_i : (X_i, \mathcal{O}|X_i) \longrightarrow (X_i, \mathcal{O}_i|X_i), \qquad f_i^\sim = \text{ id }_{X_i}$$

such that

$$f_{ij} = f_i f_j^{-1}$$

for all i, j. The conditions, necessary and sufficient, for the existence of $(\mathcal{O}, (f_i))$ are the so-called *gluing conditions*:

(1) $f_{ii} = \text{ id }$ on $\mathcal{O}_i$.

(2) $f_{ij} f_{ji} = \text{ id }$ on $\mathcal{O}_i | X_{ij}$.

(3) $f_{ij} f_{jk} f_{ki} = \text{ id }$ on $\mathcal{O}_i | X_{ijk}$.

The proof of the sufficiency (the necessity is obvious) is straightforward. In fact there is essentially only one way to define the sheaf $\mathcal{O}$ and the f_i. For any open set $U \subset X$ let $\mathcal{O}(U)$ be the set of all (s_i) such that

$$s_i \in \mathcal{O}_i(U \cap X_i), \qquad s_i = f_{ij}(s_j)$$

for all i, j. $\mathcal{O}(U)$ is a subring of the full direct product of the $\mathcal{O}_i(U \cap X_i)$. The f_i are defined by

$$f_i : (s_i) \longmapsto s_i$$

for all i. It is easy but a bit tedious to verify that $(\mathcal{O}, (f_i))$ satisfy the requirements. If $(\mathcal{O}', (f_i'))$ are a second system satisfying the same requirement, and $s_i' = f_i'^{-1}(s_i)$, the s_i' are restrictions of a section $s' \in \mathcal{O}'(U)$ and $(s_i) \longmapsto s'$ is an isomorphism. These isomorphisms give a sheaf isomorphism $\mathcal{O} \longrightarrow \mathcal{O}'$ compatible with te $(f_i), (f_i')$. The details are standard and are omitted. Notice that given the $X_i, \mathcal{O}_i, f_{ij}$, the data $\mathcal{O}, (f_i)$ are unique up to unique isomorphism.

For brevity we shall usually refer to the f_{ij} as isomorphisms of super ringed spaces

$$f_{ij} : X_{ji} \simeq X_{ij}, \qquad X_{ij} = (X_{ij}, \mathcal{O}_i | X_i \cap X_j)$$

above the identity morphisms on $X_i \cap X_j$.

We now consider the case when the family (X_i) is closed under intersections. Suppose we have a class $\mathcal{R}$ of open subsets of X closed under intersections such that each $R \in \mathcal{R}$ has a sheaf of rings on it which makes it a ringed space and X is the union of the sets R. Then for these to glue to a ringed space structure on X the conditions are as follows. For each pair $R, R' \in \mathcal{R}$ with $R' \subset R$ there should be an isomorphism of ringed spaces

$$\lambda_{RR'} : R' \simeq R_{R'}$$

where $R_{R'}$ is the ringed space R' viewed as an open subspace of R, and that these $\lambda_{R'R}$ should satisfy

$$\lambda_{RR''} = \lambda_{RR'} \lambda_{R'R''} \qquad (R'' \subset R' \subset R).$$

In this case if Y is a ringed space there is a natural bijection between the morphisms f of X into Y and families (f_R) of morphisms $R \longrightarrow Y$ such that

$$f_{R'} = f_R \lambda_{RR'} \qquad (R' \subset R).$$

The relation between f and the f_R is that f_R is the restriction of f to R. In the other direction, the morphisms from Y to X are described as follows. First of all we must have a map $t(Y^\sim \longrightarrow X^\sim)$; then the morphisms g of S into X above t are in natural bijection with families (g_R) of morphisms from $Y_R := t^{-1}(R)$ into R such that

$$g_{R'} = \lambda_{RR'} g_R.$$

Example 1: Projective superspaces. This can be done over both $\mathbf{R}$ and $\mathbf{C}$. We shall work over $\mathbf{C}$ and let X be the complex projective n-space with homogeneous coordinates $z^i (i = 0, 1, 2, \ldots, n)$. The super projective space $Y = \mathbf{CP}^{n|q}$ can now be defined as follows. Heuristically we can think of it as the set of equivalence classes of systems

$$(z^1, \ldots, z^{n+1}, \theta^1, \ldots, \theta^q)$$

where equivalence is defined by

$$(z^1, \ldots, z^{n+1}, \theta^1, \ldots, \theta^q) \simeq \lambda(z^1, \ldots, z^{n+1}, \theta^1, \ldots, \theta^q)$$

whenever $\lambda \in \mathbf{C}$ is nonzero. For a more precise description we take the reduced manifold to be X. For any open subset $V \subset X$ we look at the preimage V' of V in $\mathbf{C}^{n+1} \setminus \{0\}$ and the algebra $A(V') = H(V')[\theta^1, \ldots, \theta^q]$ where $H(V')$ is the algebra of holomorphic functions on V'. Then $\mathbf{C}^\times$ acts on this super algebra by

$$t : \sum_I f_I(z)\theta^I \longmapsto \sum_I t^{-|I|} f_I(t^{-1}z)\theta^I \qquad (t \in \mathbf{C}^\times).$$

Let

$$\mathcal{O}_Y(V) = A(V')^{\mathbf{C}^\times}$$

be the subalgebra of elements invariant under this action. It is then immediately verified that $\mathcal{O}_Y$ is a sheaf of supercommuting $\mathbf{C}$-algebras on X. Let X^i be the open set where $z^i \neq 0$ and let V above be a subset of X^i. Then V can be identified with an open subset V_1 of the affine subspace of $\mathbf{C}^{n+1}$ where $z^i = 1$. Then

$$A(V') \simeq H(V_1)[\theta^1, \ldots, \theta^q].$$

This shows that Y is a complex analytic supermanifold. This is the projective superspace $\mathbf{CP}^{n|q}$. For a deeper discussion of these and other grassmannians and flag supermanifolds see Manin[2].

Products. The category of supermanifolds admits products. For this purpose we start with the category of ringed spaces and introduce the notion of *categorical products.* Let $X_i (1 \le i \le n)$ be spaces in the category. A ringed space X together with ("projection") maps $P_i : X \longrightarrow X_i$ is called a product of the X_i,

$$X = X_1 \times \ldots \times X_n,$$

if the following is satisfied: for any ringed space Y, the map

$$f \longmapsto (P_1 \circ f, \ldots, P_n \circ f)$$

from $\mathrm{Hom}(S, X)$ to $\prod_i \mathrm{Hom}(S, X_i)$ is a bijection. In other words, the morphisms f from Y to X are identified with n-tuples $(f_1, \ldots, f_n)$ of morphisms $f_i(Y \longrightarrow X_i)$ such that $f_i = P_i \circ f$ for all i. It is easy to see that if a categorical product exists, it is unique up to unique isomorphism. Notice that this is another example of defining an object by giving the set of morphisms of an arbitrary object into it.

We shall now show that in the category of supermanifolds (categorical) products exist. Let $X_i (1 \le i \le n)$ be supermanifolds. Let $X^\sim = X_1^\sim \times \ldots \times X_n^\sim$ be the product of the classical manifolds associated to the X_i. We wish to construct a supermanifold X and morphisms $P_i(X \longrightarrow X_i)$ such that $P_i^\sim$ is the projection $X^\sim \longrightarrow X_i^\sim$ and $(X, (P_i))$ is a product of the X_i. If $X_i = U_i^{p_i|q_i}$ with coordinates $(x_i^1, \ldots, x_i^{p_i}, \theta_i^1, \ldots, \theta_i^{q_i})$ then their product is $U^{p|q}$ where $p = \sum_i p_i, q = \sum_i q_i$, with coordinates (x_i^j, θ_i^m); for the projection P_i we have

$$P_i^* x_i^j = x_i^j, P_i^* \theta_i^m = \theta_i^m.$$

Suppose now the X_i are arbitrary. Let $\mathcal{R}$ be the set of rectangles R in $X^\sim$, $R = U_{1R} \times \ldots \times U_{nR}$, such that the U_{iR} are isomorphic to coordinate superdomains; we choose some isomorphism for each of these. Then each $R \in \mathcal{R}$ can be viewed as a supermanifold with projections P_{iR}. Suppose now that $R' \subset R (R, R' \in \mathcal{R})$ and $P_{iR|R'}$ is the restriction of P_{iR} to R'; then $(R', (P_{iR|R'}))$ is also a product supermanifold structure on R'. Because of the uniquely isomorphic nature of the products, we have a *unique* isomorphism of supermanifolds

$$\lambda_{RR'} : R' \simeq R_{R'}$$

such that

$$P_{iR|R'} = \lambda_{RR'} P_{iR'}.$$

If now $R'' \subset R' \subset R$ we have

$$\lambda_{RR''} P_{iR''} = P_{iR|R''}$$

while

$$\lambda_{R'R} \lambda_{R'R''} P_{iR''} = \lambda_{RR'} P_{iR'|R''} = P_{iR|R''}.$$

Hence by the uniqueness of the λ's we get

$$\lambda_{RR'} \lambda_{R'R''} = \lambda_{RR''}.$$

The discussion above on gluing leads at once to the fact that the rectangles glue together to form a supermanifold X, the projections P_{iR} define projections $P_i(X \longrightarrow X_i)$ and that $(X, (P_i))$ is a product of the X_i. We omit the easy details.

4.3. Morphisms. The fact that the category of supermanifolds is a very viable one depends on the circumstance that morphisms between them can be described (locally) *exactly as in the classical case.* Classically, a map from an open set in $\mathbf{R}^m$ to one in $\mathbf{R}^m$ is of the form

$$(x^1, \ldots, x^m) \longmapsto (y^1, \ldots, y^n)$$

where the y^i are smooth functions of the $x^1, \ldots, x^m$. In the super context the same description prevails. To illustrate what we have in mind we shall begin by discussing an example. This example will also make clear the point we made earlier, namely, that a supermanifold should not be thought of simply as an exterior bundle of some vector bundle on a classical manifold.

A morphism $\mathbf{R}^{1|2} \longrightarrow \mathbf{R}^{1|2}$: What do we do when we describe smooth map between two manifolds? We take local coordinates $(x^i), (y^j)$ and then define the morphism as the map

$$(x^i) \longrightarrow (y^j)$$

where the y^j are smooth functions of the x^i. It is a fundamental fact of the theory of supermanifolds, in fact it is what makes the theory reasonable, that the morphisms in the super category can also be described in the same manner. Before proving this we shall look at an example.

Let $M = \mathbf{R}^{1|2}$. We want to describe a morphism ψ of M into itself such that $\psi^\sim$ is the identity. Let ψ^* be the pullback. We use t, θ^1, θ^2 as the coordinates on M and t as the coordinate on $M^\sim = \mathbf{R}$. Since $\psi^*(t)$ is an even section and $(\psi^\sim)^*(t) = t$, it follows that

$$\psi^*(t) = t + f\theta^1\theta^2$$

where f is a smooth function of t. Similarly

$$\psi^*(\theta^j) = g_j\theta^1 + h_j\theta^2$$

where g_j, h_j are again smooth functions of t. However it is not immediately obvious how ψ^* should be defined for an arbitrary section, although for sections of the form

$$a + b_1\theta^1 + b_2\theta^2$$

where a, b_1, b_2 are *polynomials* in t the prescription is uniquely defined; we simply replace t by $\psi^*(t)$ in a, b_1, b_2. It is already reasonable to expect by Weierstrass's approximation theorem that ψ^* should be uniquely determined. To examine this let us take the case where

$$\psi^*(t) = t + \theta^1\theta^2, \qquad \psi^*(\theta^j) = \theta^j \ (j = 1, 2).$$

If g is a smooth function of t on an open set $U \subset \mathbf{R}$ we want to define $\psi_U^*(g)$. Formally we should define it to be $g(t + \theta^1\theta^2)$ and this definition is even rigorous if g is a polynomial as we observed just now. For arbitrary g let us expand $g(t+\theta^1\theta^2)$ as a formal Taylor series(!) as

$$g(t + \theta^1\theta^2) = g(t) + g'(t)\theta^1\theta^2$$

wherein the series does not continue because $(\theta^1\theta^2)^2 = 0$. We shall now *define* $\psi_U^*(g)$ by the above formula. It is an easy verification that ψ_U^* is then a homomorphism

$$C^\infty(U) \longrightarrow C^\infty(U)[\theta^1, \theta^2].$$

If

$$g = g_0 + g_1\theta^1 + g_2\theta^2 + g_{12}\theta^1\theta^2$$

then we must define

$$\psi_U^*(g) = \psi^*(g_0) + \psi^*(g_1)\theta^1 + \psi^*(g_2)\theta^2 + \psi^*(g_{12})\theta^1\theta^2.$$

It is then clear that ψ_U^* is a homomorphism

$$C^\infty(U)[\theta^1, \theta^2] \longrightarrow C^\infty(U)[\theta^1, \theta^2]$$

with

$$\psi_U^*(t) = t, \quad \psi_U^*(\theta_j) = \theta^j (j = 1, 2).$$

The family (ψ_U^*) then defines a morphism $\mathbf{R}^{1|2} \longrightarrow \mathbf{R}^{1|2}$. It is obvious that this method goes through in the general case also when f, g_1, g_2 are arbitrary instead of 1 as above.

To see that the pullback homomorphism ψ^* is uniquely defined we must prove that $\psi_U^*(g) = g + g'\theta^1\theta^2$ for $g \in C^\infty(U)$. Now $\psi_U^*(g)$ must be even and so we can write

$$\psi^*(g) = g + D(g)\theta^1\theta^2.$$

Clearly D is an endomorphism of $C^\infty(U)$. The fact that ψ^* is a homomorphism now implies that D is a derivation. But $D(t) = 1$ and so D and d/dt are two derivations of $C^\infty(U)$ that coincide for t. They are therefore identical. So $D = d/dt$, showing that $\psi_U^*(g) = g + g'\theta^1\theta^2$.

This example also shows that the supermanifold $\mathbf{R}^{1|2}$ admits more self morphisms than the exterior bundle of rank 2 over $\mathbf{R}$. Thus the category of exterior bundles is not equivalent to the category of supermanifolds even in the differentiable case, as we have already observed. Automorphisms such as the one discussed above are the geometric versions of true Fermi-Bose symmetries characteristic of supersymmetry where the even and odd coordinates are thoroughly mixed.

The main result on morphisms can now be formulated.

Theorem 4.3.1. *Let $U^{p|q}$ be an open submanifold of $\mathbf{R}^{p|q}$. Suppose M is a supermanifold and ψ is a morphism of M into $U^{p|q}$. If*

$$f_i = \psi^*(t^i),\ g_j = \psi^*(\theta^j) \qquad (1 \le i \le p, 1 \le j \le q)$$

then the f_i (g_j) are even (odd) elements of $\mathcal{O}_M(M)$. Conversely, if $f_i, g_j \in \mathcal{O}_M(M)$ are given with f_i even and g_j odd, there is a unique morphism $\psi(M \longrightarrow U^{p|q})$ such that

$$f_i = \psi^*(t^i),\ g_j = \psi^*(\theta^j) \qquad (1 \le i \le p, 1 \le j \le q).$$

Only for the converse does one need a proof. In some sense at the heuristic level the uniqueness part of this theorem is not a surprise because if a morphism is given on the coordinates x^i, θ^j, then it is determined on all sections of the form $\sum_I p_I\theta^I$ where the p_I are *polynomials* in the x^i, and clearly some sort of continuity argument should imply that it is determined uniquely for the sections where the p_I are merely smooth. In fact (as Berezin did in his original memoirs) this argument can be made rigorous by introducing a topology - the usual one on smooth functions - on the sections and showing first that morphisms are continuous. But we shall avoid the topological arguments in order to bring more sharply into focus the analogy with schemes by remaining in the algebraic framework throughout as we shall do (see the paper of Leites[3]). In this approach the polynomial approximation is carried out using the Taylor series only up to terms of order q, and use is made of the principle that if two sections have the same Taylor series up to and including terms of order q at all points of an open set, the two sections are identical. So, before giving the formal proof of the theorem we shall formulate and prove this principle.

Let M be a supermanifold and let $\mathcal{O} = \mathcal{O}_M$ be its structure sheaf. Let $m \in M^\sim$ be a fixed point. We can speak of germs of sections defined in a neighborhood of m. The germs form a supercommutative $\mathbf{R}$-algebra $\mathcal{O}_m = \mathcal{O}_{M,m}$. For any section f defined around m let $[f]_m = [f]$ denote the corresponding germ. We have previously considered the ideal $\mathcal{J}_m = \mathcal{J}_{M,m}$ of germs $[f]$ such that $[f^\sim] = 0$. We now introduce the larger ideal $\mathcal{I}_m = \mathcal{I}_{M,m}$ of germs for which $f^\sim(m) = 0$, i.e.,

$$\mathcal{I}_m = \mathcal{I}_{M,m} = \Big\{ [f]_m \;\Big|\; f^\sim(m) = 0 \Big\}.$$

By the definition of a supermanifold there is an isomorphism of an open neighborhood of m with $U^{p|q}$. Let x^i, θ^j denote the pullbacks of the coordinate functions of $U^{p|q}$. We may assume that ${x^i}^\sim(m) = 0 (1 \le i \le p)$.

Lemma 4.3.2. *We have the following.*

(1) *$\mathcal{I}_{M,m}$ is generated by $[x^i]_m, [\theta^j]_m$. Moreover if $\psi(M \longrightarrow N)$ is a morphism, then for any $n \in N$ and any $k \ge 0$,*

$$\psi^*(\mathcal{I}^k_{N,n}) \subset \mathcal{I}^k_{M,m} \qquad (m \in M, \psi(m) = n).$$

(2) *If $k > q$ and f is a section defined around m such that $[f]_{m'} \in \mathcal{I}^k_{m'}$ for all m' in some neighborhood of m, then $[f]_m = 0$.*

(3) *For any k and any section f defined around m there is a polynomial $P = P_{k,f,m}$ in the $[x^i], [\theta^j]$ such that*

$$f - P \in \mathcal{I}^k_m.$$

Proof. All the assertions are local and so we may assume that $M = U^{p|q}$ where U is a *convex* open set in $\mathbf{R}^p$ and $m = 0$. By Taylor series where the remainder is given as an integral, we know that if g is a smooth function of the x^i defined on *any* convex open set V containing 0, then, for any $k \geq 0$,

$$g(x) = g(0) + \sum_j x^j (\partial_j g)(0) + \ldots + \frac{1}{k!} \sum_{j_1, \ldots, j_k} x^{j_1} \ldots x^{j_k} (\partial_{j_1} \ldots \partial_{j_k} g)(0) + R_k(x)$$

where

$$R_k(x) = \frac{1}{k!} \sum_{j_1, \ldots, j_{k+1}} x^{j_1} \ldots x^{j_{k+1}} g_{j_1 j_2 \ldots j_{k+1}},$$

the $g_{j_1 j_2 \ldots j_{k+1}}$ being smooth functions on V defined by

$$g_{j_1 j_2 \ldots j_{k+1}}(x) = \int_0^1 (1-t)^k (\partial_{j_1} \ldots \partial_{j_{k+1}} g)(tx) dt.$$

Take first $k = 0$ and let $g(0) = 0$. Then

$$g = \sum_j x^j g_j.$$

If now $f = f_0 + \sum_I f_I \theta^I$ is in $\mathcal{O}(V)$ and $f_0(0) = 0$, then taking $g = f_0$ we obtain the first assertion in (1). For the assertion about ψ we have already seen that it is true for $k = 1$ (the locality of morphisms). Hence it is true for all k.

Let us first remark that because any section h can be written as $\sum_I h_I \theta^I$, it makes sense to speak of the evaluation $h(n) = \sum_I h_I(n) \theta^I$ at any point n; this is not to be confused with $h^\sim(n)$ which is invariantly defined and lies in $\mathbf{R}$ while $h(n)$ depends on the coordinate system and lies in $\mathbf{R}[\theta^1, \ldots, \theta^q]$. To prove (2) let $k > q$ and let us consider $\mathcal{I}_0^k$. Any product of k elements chosen from $x^1, \ldots, x^p, \theta^1, \ldots, \theta^q$ is zero, unless there is at least one x^j. So

$$\mathcal{I}_0^k \subset \sum_j [x^j] \mathcal{O}_0.$$

Therefore, if $[f] \in \mathcal{I}_0^k$ then

$$f(0) = 0 \qquad (*)$$

where $f(0)$ is the evaluation of the section at 0. Suppose now f is in $\mathcal{O}$ and lies in $\mathcal{I}_n^k$ for all n in some open neighborhood N of 0. Then $(*)$ is applicable with 0 replaced by n. Hence

$$f(n) = 0 \qquad (n \in N).$$

This proves that the germ $[f]$ is 0.

To prove (3) we take any section defined around 0, say $f = \sum_I f_I \theta^I$. Fix $k \geq 1$. Writing $f_I = g_I + R_I$ where g_I is the Taylor expansion of f_I at 0 and R_I is in the ideal generated by the monomials in the x^j of degree k, it follows at once that for $P = \sum g_I \theta^I$ and $R = \sum_I R_I \theta^I$ we have $f = P + R$. Going over to germs at 0 we see that $[P]$ is a polynomial in the $[x]$'s and $[\theta]$'s while $[R]$ is in the ideal $\mathcal{I}_0^k$. This proves (3).

Proof of Theorem 4.3.1. We are now in a position to prove (converse part of) Theorem 4.3.1.

Uniqueness. Let $\psi_i (i = 1, 2)$ be two pullbacks such that $\psi_1^*(u) = \psi_2^*(u)$ for $u = x^i, \theta^j$. This means that $\psi_1^\sim = \psi_2^\sim$. We must prove that $\psi_1^*(u) = \psi_2^*(u)$ for all $u \in C^\infty(U)[\theta^1, \ldots, \theta^q]$. This equality is true for all polynomials in x^i, θ^j. Let $u \in \mathcal{O}(V)$ where V is an open set contained in U. Write $g = \psi_1^*(u) - \psi_2^*(u)$. Let $k > n$ where M has dimension $m|n$. Let $x \in M$ and let $y = \psi_1^\sim(x) = \psi_2^\sim(x) \in U$. By (3) of the Lemma we can find a polynomial P in the x^i, θ^j such that $[u]_y = [P]_y + [R]_y$ where $[R]_y$ is in $\mathcal{I}_y^k$. Applying ψ_i^* to this relation, noting that $\psi_1^*([P]_y) = \psi_2^*([P]_y)$, we obtain, in view of (1) of the Lemma, that $[g]_x \in \mathcal{I}_{M,x}^k$. But $x \in M$ is arbitrary except for the requirement that it goes to $y \in U$ under $\psi_1^\sim = \psi_2^\sim$. Hence $g = 0$ by (2) of the Lemma.

Existence. We write M as a union of open sets W on each of which we have coordinate systems. In view of the uniqueness it is enough to construct the morphism $W \longrightarrow U$ and so we can take $M = W$. We follow the method used in the example of the morphism $\mathbf{R}^{1|2} \longrightarrow \mathbf{R}^{1|2}$ discussed earlier. It is further enough, as in the example above, to construct a homomorphism $C^\infty(U) \longrightarrow \mathcal{O}(W)_0$ taking x^i to f_i; such a homomorphism extends at once to a homomorphism of $C^\infty(U)[\theta^1, \ldots, \theta^q]$ into $\mathcal{O}(W)$ which takes θ^j to g_j. Write $f_i = r_i + n_i$ where $r_i \in C^\infty(W)$ and $n_i = \sum_{|I| \geq 1} n_{iI} \varphi^I$ (here y^r, φ^s are the coordinates on W). If $g \in C^\infty(U)$ we define $\psi^*(g)$ by the *formal* Taylor expansion

$$\psi^*(g) = g(r_1 + n_1, \ldots r_p + n_p) := \sum_\gamma \frac{1}{\gamma!} (\partial^\gamma g)(r_1, \ldots, r_p) n^\gamma$$

the series being finite because of the nilpotency of the n_i. To verify that $g \longmapsto \psi^*(g)$ is a homomorphism we think of this map as a composition of the map

$$A : g \longmapsto \sum_\gamma \frac{1}{\gamma!} (\partial^\gamma g) T^\gamma$$

from $C^\infty(U)$ to $C^\infty(U)[T^1, \ldots, T^q]$, the T^r being indeterminates, followed by the substitution $x^i \longmapsto r_i$, followed by the specialization $T^i \longmapsto n_i$. Since all these are homomorphisms we are done.

The theorem is fully proved.

Remark. This theorem shows that morphisms between supermanifolds can be written in local coordinates in the form

$$x^1, \ldots, x^m, \theta^1, \ldots, \theta^n \longmapsto y^1, \ldots, y^p, \varphi^1, \ldots, \varphi^q$$

where y^i, φ^j are even and odd sections respectively. The theory of supermanifolds thus becomes very close to the theory of classical manifolds and hence very reasonable. Also, the fact that Taylor series of arbitrary order were used in the proof suggests that it is not possible to define supermanifolds in the C^k category for finite k unless one does artificial things like coupling the number of odd coordinates to the degree of smoothness.

The symbolic way of calculation. This theorem on the determination of morphisms is the basis of what one may call the *symbolic way of calculation.* Thus, if M, N are supermanifolds where (x^i, θ^j) are coordinates on M and (y^r, φ^s) are coordinates on N, we can think of a morphism $\psi(M \longrightarrow N)$ symbolically as

$$(x, \theta) \longrightarrow (y, \varphi), \qquad y = y(x, theta), \quad \varphi = \varphi(x, \theta)$$

which is an abuse of notation for the map ψ^* such that

$$\psi^*(y^r) = y^r(x, \theta) \in \mathcal{O}_M(M)_0, \quad \psi^*(\varphi^s) = \varphi^s(x, \theta) \in \mathcal{O}_M(M)_1.$$

We shall see later how useful this symbolic point of view is in making calculations free of pedantic notation.

4.4. Differential calculus. The fundamental result is the differential criterion for a system of functions to form a coordinate system at a point. This leads as usual to results on the local structure of isomorphisms, immersions, and submersions.

Derivations and vector fields. Let us first look at derivations. Recall that a derivation of a k-superalgebra B (k a field of characteristic 0) is a k-linear map $D : B \longrightarrow B$ such that

$$D(ab) = (Da)b + (-1)^{p(D)p(a)}a(Db) \qquad (a, b \in B).$$

Let R be a commutative k-algebra with unit element and let $A = R[\theta^1, \ldots, \theta^q]$ as usual. Then R is a supercommutative k-algebra and one has the space of k-derivations of R. If ∂ is a derivation of R it extends uniquely as an even derivation of A which vanishes for all the θ^i. We denote this by ∂ again. On the other hand if we fix i there is a unique odd derivation of A which is 0 on A and takes θ^j to $\delta_{ij}\theta^i$. We denote this by $\partial/\partial\theta^i$. Thus

$$\partial \sum f_I \theta^I = \sum_I (\partial f)\theta^I, \qquad \frac{\partial}{\partial\theta^j}\left(\sum_{j\notin I} f_I\theta^I + \sum_{j\notin I} f_{j,I}\theta^j\theta^I\right) = \sum_{j\notin I} f_{j,I}\theta^I.$$

If M is a supermanifold one can then define *vector fields* on M as derivations of the sheaf $\mathcal{O}_M$. More precisely they are families of derivations $(D_U) : \mathcal{O}(U) \longrightarrow \mathcal{O}(U)$ which are compatible with restrictions. The derivations form a sheaf of modules over the structure sheaf $\mathcal{O}$. It is called the *tangent sheaf* of M is analogy with what happens in the classical case. Let us denote it by $\mathcal{T}M$. To see what its local structure is let us now consider the case $M = U^{p|q}$. If $R = C^\infty(U)$ we thus have derivations

$$\frac{\partial}{\partial x^i}, \frac{\partial}{\partial\theta^j}$$

on $\mathcal{O}(U)$. We shall now show by the technique of polynomial approximation used earlier that the derivations of $\mathcal{O}(U)$ form a module isomorphic to the free module $A^{p|q}$ where $A = \mathcal{O}(U)$, with basis as the partials listed above. Indeed, let D be any derivation (even or odd) of $\mathcal{O}(U)$ and let us write $y^1, \ldots, y^m$ for the entire set of coordinates $x^1, \ldots, \theta^q (m = p + q)$. Let $a_j = Dy^j$; we wish to show that $D = \sum_j a_j\partial/\partial y^j$ (the freeness is clear since this derivation must take y^j to a_j and so is 0 only if the a_j are all 0). Let D' be the derivation $D - \sum_j a_j\partial/\partial y^j$. Then $D'y^j = 0$ for all j and so, by the derivation property $D'P = 0$ for all polynomials in the y^j. Suppose now that $f \in \mathcal{O}(U)$ and $u \in U$. Then there is a polynomial P_k in the y^j such that for $g = f - P_k$, $[g]_u \in \mathcal{I}_u^k$. Hence $[D'f]_u = [D'g]_u$. But $[D'g]_u \in \mathcal{I}_u^{k-1}$ and so, if $k > q+1$, we can conclude that $[D'f]_u \in \mathcal{I}_k^q + 1$. Since $u \in U$ is arbitrary, we have $D'f = 0$.

Thus the tangent sheaf $\mathcal{T}M$ on M is locally isomorphic to the free module $\mathcal{O}(U)^{p|q}$. It is thus an example of a *vector bundle* on the supermanifold on M, i.e., a sheaf of $\mathcal{O}$-modules on M which is locally isomorphic to $\mathcal{O}^{r|s}$ for suitable r, s.

Once the partial derivatives with respect to the coordinate variables are defined, the differential calculus on supermanifolds takes almost the same form as in the classical case, except for the slight but essential differences originating from the presence odd derivations. For explicit formulas we have

$$\frac{\partial}{\partial x^i}\sum f_I\theta^I = \sum_I \frac{\partial f}{\partial x^i}\theta^I, \qquad \frac{\partial}{\partial \theta^j}\left(\sum_{j\notin I} f_I\theta^I + \sum_{j\notin I} f_{j,I}\theta^j\theta^I\right) = \sum_{j\notin I} f_{j,I}\theta^I.$$

Tangent space. The tangent map of a morphism. Let M be a supermanifold and let $m \in M$. Then as in the classical case we define a *tangent vector* to M at m as a derivation of the stalk $\mathcal{O}_m$ into $\mathbf{R}$. More precisely a tangent vector ξ at m is a linear map

$$\xi : \mathcal{O}_m \longrightarrow \mathbf{R}$$

such that

$$\xi(fg) = \xi(f)g(m) + (-1)^{p(\xi)p(f)} f(m)\xi(g) \qquad (f, g \in \mathcal{O}_m).$$

If x^i, θ^j are local coordinates for M at some point, the tangent space has

$$\left(\frac{\partial}{\partial x^i}\right)_m, \left(\frac{\partial}{\partial \theta^j}\right)_m$$

as a basis and so is a super vector space of dimension $p|q$; this is done in the same way as we did the case of vector fields by polynomial approximation. This is thus true in general. We denote by $T_m(M)$ the tangent space of M at m. If $\psi(M \longrightarrow N)$ is a morphism of supermanifolds and , $m \in M, n = \psi(m) \in N$, then

$$\xi \longmapsto \xi \circ \psi^*$$

is a morphism of super vector spaces from $T_m(M)$ to $T_n(N)$, denoted by $d\psi_m$:

$$d\psi_m : T_m(M) \longrightarrow T_n(N).$$

This is called the *tangent map* of ψ at m. It is obvious that the assignment

$$\psi \longmapsto d\psi_m$$

preserves composition in the obvious sense. In local coordinates this is a consequence of the chain rule which we shall derive presently in the super context.

Let us now derive the chain rule. Let

$$\psi : U^{p|q} \longrightarrow V^{m|n}$$

be a morphism and let (y^j) (z^k) be the coordinates on $U^{p|q}$ $(V^{m|n})$ where we are including both the even and odd coordinates in this notation. Then for any $f \in \mathcal{O}(V)$ we have

$$\frac{\partial \psi^*(f)}{\partial y^i} = \sum_k \frac{\partial \psi^*(z^k)}{\partial y^i} \psi^* \left(\frac{\partial f}{\partial z^k} \right). \qquad \textbf{(chain rule)}$$

If we omit reference to ψ^* as we usually do in classical analysis, this becomes the familiar

$$\frac{\partial}{\partial y^i} = \sum_k \frac{\partial z^k}{\partial y^i} \frac{\partial}{\partial z^k}.$$

This is proved as before. Let D be the difference between the two sides. Then D is a derivation from $\mathcal{O}(V)$ to $\mathcal{O}(U)$ in the sense that

$$D(fg) = (Df)g + (-1)^{p(D)p(f)} d(Dg),$$

where $p(D)$ is just the parity of y^i, and it is trivial that $Dz^k = 0$ for all k. Hence $D = 0$. The argument is again by polynomial approximation.

In the above formula the coefficients have been placed to the left of the derivations. This will of course have sign consequences when we compose two morphisms. Let

$$\psi : U \longrightarrow V, \qquad \varphi : V \longrightarrow W, \qquad \tau = \varphi\psi.$$

Let $(y^k), (z^r), (t^m)$ be the coordinates on U, V, W respectively. If we write $p(y^k)$ for the parity of y^k and so on, then the parity of $\partial z^r / \partial y^k$ is $p(z^r) + p(y^k)$. The chain rule gives

$$\frac{\partial t^m}{\partial y^k} = \sum_r (-1)^{p(z^r)(p(y^k)+p(t^m)+1)+p(y^k)p(t^m)} \frac{\partial t^m}{\partial z^r} \frac{\partial z^r}{\partial y^k}$$

if we remember that $p(z^r)^2 = 1$. Hence if we define

$$z^r_{,k} = (-1)^{(p(z^r)+1)p(y^k)} \frac{\partial z^r}{\partial y^k},$$

and also

$$t^m_{,r} = (-1)^{(p(t^m)+1)p(z^r)} \frac{\partial t^m}{\partial z^r}, \quad t^m_{,k} = (-1)^{(p(t^m)+1)p(y^k)} \frac{\partial t^m}{\partial y^k},$$

then we have

$$t^m_{,k} = \sum_r t^m_{,r} z^r_{,k}.$$

So if we write

$$J\psi = (z^r_{,k}),$$

then composition corresponds to matrix multiplication. In terms of even and odd coordinates x^i, θ^j for U and y^s, φ^n for V with

$$\psi^*(y^s) = f_s, \qquad \psi^*(\varphi^n) = g_n$$

we obtain

$$J\psi = \begin{pmatrix} \frac{\partial f}{\partial x} & -\frac{\partial f}{\partial \theta} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial \theta} \end{pmatrix}.$$

For computations involving the tangent map this refinement has no effect. In fact, with respect to the bases

$$\left(\frac{\partial}{\partial x^i}\right)_m, \left(\frac{\partial}{\partial \theta^j}\right)_m, \left(\frac{\partial}{\partial y^r}\right)_n, \left(\frac{\partial}{\partial \varphi^k}\right)_n$$

the matrix of $d\psi_m$ is

$$\begin{pmatrix} \frac{\partial f}{\partial x}^{\sim}(m) & 0 \\ 0 & \frac{\partial g}{\partial \theta}^{\sim}(m) \end{pmatrix}$$

as it should be, since $d\psi_m$ is an even map.

Differential criteria. Let us work with a supermanifold M and let $m \in M$. Let

$$f_1, \ldots, f_p, g_1, \ldots, g_q$$

be sections of $\mathcal{O}$ with f_i even and g_j odd, defined around m. Then there is an open neighborhood V of m and a unique morphism ψ of the supermanifold V into $\mathbf{R}^{p|q}$ such that

$$\psi^*(x^i) = f_i, \qquad \psi^*(\theta^j) = g_j \qquad (1 \le i \le p, 1 \le j \le q).$$

We say that the (f_i, g_j) *form a coordinate system for M at m* if ψ is an isomorphism of a neighborhood of m with an open submanifold of $\mathbf{R}^{p|q}$.

Theorem 4.4.1. *The following are equivalent.*

(1) *The (f_i, g_j) form a coordinate system for M at m.*

(2) *ψ is an isomorphism of supermanifolds from a neighborhood of m in M to a neighborhood of $\psi(m)$ in $\mathbf{R}^{p|q}$.*

(3) *$d\psi_m$ is a linear isomorphism of $T_m(M)$ with $T_{\psi(m)}(N)$.*

(4) *We have*

$$\det\left(\frac{\partial f}{\partial x}\right)^{\sim}(m)\ \det\left(\frac{\partial g}{\partial \theta}\right)^{\sim}(m) \neq 0.$$

Proof. The equivalence (1) $\Longleftrightarrow$ (2) is just the definition. Also it is obvious that (3) $\Longleftrightarrow$ (4). The implication (2)$\Longrightarrow$(3) is also easy; if $n = \psi(m)$ and φ is the inverse of ψ, then $d\varphi_n d\psi_m = d\psi_m d\varphi_n = 1$ showing that $d\psi_m$ is a linear isomorphism. So it remains to prove that (3)$\Longrightarrow$(2). In this proof we shall use either (3) or (4) interchangeably. We also suppose that $M = U^{p|q}$.

Since $\det(\partial f^{\sim}/\partial x)^{\sim}(m) \neq 0$, we know that $(f_1^{\sim}, \ldots, f_p^{\sim})$ form a system of coordinates for the classical manifold $M^{\sim}$ near m and so $f_1^{\sim}, \ldots, f_p^{\sim}, \theta^1, \ldots, \theta^q$ is a system of coordinates for $U^{p|q}$ at m. So we may assume (after shrinking U) that

$$f_i \equiv x_i(\mathcal{J})$$

where $\mathcal{J}$ is the ideal generated by the θ^j. Now

$$g_j = \sum_k f_{jk}\theta^k + \sum_{km} f_{jkm}\theta^k\theta^m + \ldots$$

where f_{jk}, f_{jkm} etc are smooth functions defined near m. By assumption the matrix $(f_{jk}^{\sim})$ is invertible at m and hence near m. So

$$x^1, \ldots, x^p, \varphi^1, \ldots, \varphi^q \qquad \varphi^j = \sum_k f_{jk}\theta^k$$

is again a system of coordinates. So we may assume that

$$g_j \equiv \theta^j(\mathcal{J}^2).$$

So we have a morphism

$$\psi : U \longrightarrow V$$

such that

$$\psi^*(y^i) = f_i \equiv x^i(\mathcal{J}), \qquad \psi^*(\varphi^j) \equiv \theta^j(\mathcal{J}^2)$$

and we wish to prove that ψ^* is an isomorphism on a suitably small neighborhood of m. Note that the reduced morphism is the identity so that $U = V$. Let μ be the morphism $V \longrightarrow U$ such that $\mu^*(x^i) = y^i, \mu^*(\theta^j) = \varphi^j$. The morphism μ is not the inverse of ψ that we are after but is like a *parametrix*, i.e., $\mu\psi$ is close to the identity in some sense. Actually

$$\psi^*\mu^* = 1 + N$$

where N is *nilpotent*. We shall in fact show that $N^{q+1} = 0$. Let τ be the morphism $\mu\psi$ from U to U so that $\tau^* = \psi^*\mu^* = 1 + N$. Clearly $N1 = 0$ while

$$Nx^i \equiv 0(\mathcal{J}) \qquad N\theta^j \equiv 0(\mathcal{J}^2).$$

Since $\tau^*(\theta^j) \equiv \theta^j(\mathcal{J}^2)$ it follows that $\tau^*(\theta^j) \in \mathcal{J}$ and hence, as τ^* is a homomorphism, $\tau^*(\mathcal{J}) \subset \mathcal{J}$. Thus $\tau^*(\mathcal{J}^k) \subset \mathcal{J}^k$ for all $k \geq 1$. By definition $\tau^\sim = \mu^\sim\psi^\sim$ is the identity and so it is clear that $\tau^*(g) \equiv g(\mathcal{J})$ for all $g \in \mathcal{O}(V)$. We now claim that N maps $\mathcal{J}^k$ into $\mathcal{J}^{k+1}$ for all $k \geq 1$. Since N is not a homomorphism we have to do this for each k. This means showing that $\tau^*(g) \equiv g(\mathcal{J}^{k+1})$ if $g \in \mathcal{J}^k$. Take $g = h\theta^J$ where $|J| \geq k$. Then $\tau^*(g) = \tau^*(h)\tau^*(\theta^J)$. Now

$$\tau^*(\theta^{j_1} \ldots \theta^{j_r}) = (\theta^{j_1} + \beta_1) \ldots (\theta^{j_r} + \beta_r)$$

where the $\beta_j \in \mathcal{J}^2$ and so

$$\tau^*(\theta^J) \equiv \theta^J(\mathcal{J}^{r+1}).$$

Hence, if $|J| = r \geq k$,

$$\tau^*(g) = \tau^*(h)\tau^*(\theta^J) = (h + w)(\theta^J + \xi)$$

where $w \in \mathcal{J}$ and $\xi \in \mathcal{J}^{k+1}$ so that

$$\tau^*(g) \equiv g(\mathcal{J}^{k+1}) \qquad (g \in \mathcal{J}^k).$$

Thus N maps $\mathcal{J}^k$ into $\mathcal{J}^{k+1}$ for all k, hence $N^{q+1} = 0$.

The fact that $N^r = 0$ for $r > q$ implies that $1 + N$ is invertible, indeed, $(1+N)^{-1} = \sum_{s\geq 0}(-1)^s N^s$. Let ν^* be the inverse of τ^*. Thus $\psi^*\mu^*\nu^* = 1$ showing that ψ^* has a right inverse. So there is a morphism φ from V to U such that $\varphi\psi = 1_V$. On the other hand, as the invertibility of $d\varphi$ follows from the above relation, we can apply the preceding result to φ to fine a morphism ψ' such that $\psi'\varphi = 1_U$. So $\psi' = \psi'\varphi\psi = \psi$. Thus ψ is an isomorphism.

Corollary 4.4.2 *If $\psi(M \longrightarrow N)$ is a morphism such that $d\psi_m$ is bijective everywhere, then $\psi^\sim$ maps M onto an open subspace N' of N; if $\psi^\sim$ is also one-one, then ψ is an isomorphism of M with N' as supermanifolds.*

Local structure of morphisms. The above criterion makes it possible, exactly as in classical geometry, to determine the canonical forms of immersions and submersions. The general problem is as follows. Let M, N be supermanifolds and let $m \in M, n \in N$ be fixed. Let ψ be a morphism from M to N with $\psi(m) = n$. If γ (γ') is a local automorphism of $M(N)$ fixing $m(n)$ then $\psi' = \gamma' \circ \psi \circ \gamma$ is also a morphism from M to N taking m to n. We then say that $\psi \simeq \psi'$. The problem is to describe the equivalence classes. The representatives of the equivalence classes are called *local models.* Clearly the linear maps $d\psi_m$ and $d\psi'_m$ are equivalent in the sense that $d\psi'_m = g' d\psi_m g$ where g (g') is an automorphism of $T_m(M)$ $(T_n(N))$. The even and odd ranks and nullities involved are thus invariants. The morphism ψ is called an *immersion at* m if $d\psi_m$ is injective, and a *submersion at* m if $d\psi_m$ is surjective. We shall show now that the local models for an immersion are

$$M = U^{p|q}, \qquad (x^i, \theta^j), \qquad N = M \times V^{r|s}(0 \in V), \qquad (x^i, y^s, \theta^j, \varphi^k)$$

with

$$\begin{aligned} &\psi^\sim : m \longmapsto (m, 0) \\ &\psi^* : x^i \mapsto x^i, \theta^j \mapsto \theta^j;\ y^r, \varphi^k \mapsto 0. \end{aligned}$$

We shall also show that for the submersions the local models are *projections*, namely

$$N = U^{p|q}, \qquad (x^i, \theta^j), \qquad M = N \times V^{r|s}, \qquad (x^i, y^s, \theta^j, \varphi^k)$$

with

$$\begin{aligned} &\psi^\sim : (m, v) \longmapsto m \\ &\psi^* : x^i \mapsto x^i, \theta^j \mapsto \theta^j. \end{aligned}$$

Theorem 4.4.3. *The above are local models for immersions and submersions.*

Proof. *Immersions.* Let $\psi(U^{p+r|q+s} \longrightarrow V^{p|q})$ be an immersion at $0 \in U$, with (x^i, θ^j) as coordinates for U and (u^a, ξ^b) as coordinates for V. Write $\psi^*(g) = g^*$. Since $d\psi_m$ is separately injective on $T_m(M)_0$ and $T_m(M)_1$ we see that the matrices

$$\left(\frac{\partial u^{a*}}{\partial x^i}\right)_{1 \le i \le p, 1 \le a \le p+r}, \qquad \left(\frac{\partial \xi^{s*}}{\partial \theta^j}\right)_{1 \le j \le q, 1 \le s \le q+s}$$

have ranks respectively p and q at m. By permuting the u^r and ξ^s we may therefore assume that the matrices

$$\left(\frac{\partial u^{r*}}{\partial x^i}\right)_{1 \le i \le p, 1 \le r \le p}, \qquad \left(\frac{\partial \xi^{s*}}{\partial \theta^j}\right)_{1 \le j \le q, 1 \le s \le q}$$

which are composed of the first p columns of the first matrix and the first q columns of the second matrix are invertible at m. This means that

$$u^{1^*}, \dots, u^{p*}, \xi^{1^*}, \dots, \xi^{q*}$$

form a coordinate system for $U^{p|q}$ at m. We may therefore assume that

$$u^{r*} = x^r (1 \le r \le p), \qquad \xi^{s*} = \theta^q.$$

However $u^{r*}(r > p), \xi^{s*}(s > q)$ may not map to 0 as in the local model. Let

$$u^{r*} = \sum_I g_{rI}\theta^I, \qquad \xi^{s*} = \sum_J h_{sJ}\theta^J$$

where g_{rI}, h_{sJ} are C^∞-functions of $x^1, \dots, x^p$. Let

$$w^r = \sum_I g_{rI}(u^1, \dots, u^p)\xi^I (r > p), \qquad \eta^s = \sum_J h_{rJ}(u^1, \dots, u^p)\xi^J (s > q).$$

Then ψ^* maps $u'^r = u^r - w^r(r > p)$ and $\xi'^s = \xi^s - \eta^s(s > q)$ to 0. It is obvious that

$$u^1, \dots, u^p, u'^{p+1}, \dots, u'^m, \xi^1, \dots, \xi^q, \xi'^{q+1}, \dots, \xi'^n$$

is a coordinate system at $\psi(m)$. With this coordinate system the morphism ψ is in the form of the local model.

Submersions. Let ψ be a submersion of $V^{p+r|q+s}$ on $M = U^{p|q}$ with $m = 0, \psi(m) = n = 0$. Let (x^i, θ^j) be the coordinates for $U^{p|q}$ and (y^a, φ^b) coordinates for $V^{p+r|q+s}$. The map $d\psi_0$ is surjective separately on $T_0(M)_0$ and $T_0(M)_1$. So the matrices

$$\left(\frac{\partial x^{i^*}}{\partial y^a}\right)_{1\le a\le p+r, 1\le i\le p}, \qquad \left(\frac{\partial \theta^{j^*}}{\partial \varphi^b}\right)_{1\le b\le q+s, 1\le j\le q}$$

have ranks respectively p and q at 0. We may therefore assume that the submatrices composed of the first p rows of the first matrix and of the first q rows of the second matrix are invertible at 0. This means that

$$x^{1^*}, \dots, u^{p*}, y^{p+1}, \dots, y^{p+r}, \theta^{1^*}, \dots, \varphi^{q*}, \varphi^{q+1}, \dots, \varphi^{q+s}$$

form a coordinate system for $V^{p+r,q+s}$ at 0. The morphism is in the form of the local model in these coordinates. This proves the theorem.

4.5. Functor of points. In algebraic geometry as well as in supergeometry, the basic objects are somewhat strange and the points of their underlying topological space do not have geometric significance. There is a second notion of points which is geometric and corresponds to our geometric intuition. Moreover in the supergeometric context this notion of points is essentially the one that the physicists use in their calculations. The mathematical basis for this notion is the so-called *functor of points.*

Let us first consider affine algebraic geometry. The basic objects are algebraic varieties defined by polynomial equations

$$p_r(z^1, \ldots, z^n) = 0 \qquad (r \in I) \tag{$*$}$$

where the polynomials p_r have coefficients in $\mathbf{C}$ and the z^i are complex variables. It is implicit that the solutions are from $\mathbf{C}^n$ and the set of solutions forms a variety with its Zariski topology and structure sheaf. However Grothendieck focussed attention on the fact that one can look for solutions in A^n where A is any commutative $\mathbf{C}$-algebra with unit. Let $V(A)$ be the set of these solutions; $V(\mathbf{C})$ is the underlying set for the classical complex variety defined by these equations. The elements of $V(A)$ are called the *A-points* of the variety $(*)$. We now have an assignment

$$\mathbf{V} : A \longmapsto V(A)$$

from the category of commutative $\mathbf{C}$-algebras with units into the category of sets. This is the *functor of points* of the variety $(*)$. That the above assignment is functorial is clear: if B is a $\mathbf{C}$-algebra with a map $A \longrightarrow B$, then the map $A^n \longrightarrow B^n$ maps $V(A)$ into $V(B)$. It turns out that *the functor V contains the same information* as the classical complex variety, and the set of morphisms between two affine varieties is bijective with the set of natural maps between their functors of points. The set $V(A)$ itself can also be described as $\mathrm{Hom}(\mathbf{C}[V], A)$. Obviously an arbitrary functor from $\mathbf{C}$-algebras to sets will not rise as the functor points of an affine variety or the algebra of polynomial functions on such a variety (by Hilbert's zeros theorem these are the algebras which are finitely generated over $\mathbf{C}$ and reduced in the sense that they have no nonzero nilpotents). If a functor has this property it is called *representable.* Thus affine algebraic geometry is the same as the theory of representable functors. Notice that the sets $V(A)$ *have no structure*; it is their *functorial property* that contains the information residing in the classical variety.

Now the varieties one encounters in algebraic geometry are not always affine; the projective ones are obtained by gluing affine ones. In the general case they are schemes. The duality between varieties and algebras makes it clear that for a given scheme X one has to understand by its points any morphism from an *arbitrary scheme* into X. In other words, given a scheme X, the functor

$$S \longmapsto \mathrm{Hom}(S, X) \qquad (S \text{ an arbitrary scheme})$$

is called the *functor of points of* X; it is denoted by $X(S)$. Heuristically we may think of $X(S)$ as points of X *parametrized by* S. This notion of points is much closer to the geometric intuition than the points of the underlying space of a scheme. For example, the underlying topological space of the product of two schemes X, Y is *not* the product of X and Y; however, this is true for S-points: $(X \times Y)(S) \simeq X(S) \times Y(S)$ canonically. A functor from schemes to sets is called *representable* if it is naturally isomorphic to the functor of points of a scheme; the scheme is then uniquely determined up to isomorphism and is said to *represent* the functor. In many problems, especially in the theory of moduli spaces, it is most convenient to define first the appropriate functor of points and then prove its representability.

We take over this point of view in supergeometry. The role of schemes is played by supermanifolds and the role of affine schemes or their coordinate rings is played by supercommutative algebras. If X is a supermanifold, its functor points is

$$S \longmapsto X(S) \qquad (S \text{ a supermanifold})$$

where

$$X(S) = \mathrm{Hom}(S, X) = \text{ set of morphisms } S \longrightarrow X.$$

If X, Y are supermanifolds then $(X \times Y)(S) \simeq X(S) \times Y(S)$ canonically. A morphism ψ from $\mathbf{R}^{0|0}$ into X is really a point of $X^\sim$ is the classical sense; indeed, if U is open in $X^\sim$, the odd elements of $\mathcal{O}(U)$ must map to 0 under ψ^* and so ψ^* factors through to a homomorphism of $\mathcal{O}^\sim$ into $\mathbf{R}$. To define morphisms that see the odd structure of X we must use supermanifolds themselves as domains for the morphisms. Later on, when we treat super Lie groups we shall see the usefulness of this point of view.

Consider the simplest example, namely $\mathbf{R}^{p|q}$. If S is a supermanifold, the S-points of $\mathbf{R}^{p|q}$ are systems $(x^1, \ldots, x^p, \theta^1, \ldots, \theta^q)$ where $x^i \in \mathcal{O}_S(S)_0, \theta^j \in \mathcal{O}_S(S)_1$. This is not any different from the heuristic way of thinking of $\mathbf{R}^{p|q}$ as the set of all systems $(x^1, \ldots, x^p, \theta^1, \ldots, \theta^q)$ where the x^i are even variables and the θ^j are odd variables. One can think of $\mathbf{R}^{p|q}$ as a "group"with the group law

$$(x, \theta) + (x', \theta') \longrightarrow (x + x', \theta + \theta').$$

At the level of S-points this is *exactly* a group law; the symbols denote elements of $\mathcal{O}_S(S)$ of the appropriate parity. Thus the informal or symbolic way of thinking and writing about supermanifolds is essentially the same as the mode of operation furnished by the language of the functor of points.

4.6. Integration on supermanifolds. Integration on supermanifolds consists of integrating with respect to both the even and odd variables. For the even variables it is the standard classical theory but integration in anticommuting variables is new and was discovered by Berezin who also discovered the change of variables formula.

The integral on an exterior algebra

$$A = \mathbf{R}[\theta^1, \ldots, \theta^q]$$

is a linear function

$$A \longrightarrow \mathbf{R}, \qquad a \longmapsto \int a = \int a \; d^q\theta,$$

uniquely determined by the following properties:

$$\int \theta^I = 0 \quad (|I| < q), \qquad \int \theta^Q = 1, \quad (Q = \{1, 2, \ldots, q\}).$$

We use the notation

$$Q = \{1, 2, \ldots, q\}$$

throughout this section. Thus integration is also differentiation, and

$$\int = (\partial/\partial\theta^q)(\partial/\partial\theta^{q-1})\ldots(\partial/\partial\theta^1).$$

For a superdomain $U^{p|q}$ the integral is a linear form

$$\mathcal{O}_c(U) \longrightarrow \mathbf{R}, \qquad s \longmapsto \int s = \int s \; d^p x d^q \theta$$

where the suffix c means that the sections are *compactly supported*; the integral is evaluated by repeated integration. Thus

$$\int \sum_I s_I \theta^I = \int s_Q \; d^p x.$$

Sometimes we write

$$\int s = \int_U s$$

to emphasize that the integration is over U. Thus *the integral picks out just the coefficient of* θ^Q *and integrates it in the usual way with respect to the even variables.* This might seem very peculiar till one realizes that any definition should be made in such a way that one has a nice formula for changing variables in the integral. Now the Berezinian is the replacement of the determinant in the super context and we shall see that this definition of the integral is precisely the one for which one can prove a change of variables formula exactly analogous to the classical one, with Ber replacing det.

Statement of the change of variables formula. Let

$$\psi : U^{p|q} \longrightarrow V^{p|q}$$

be an isomorphism of supermanifolds. In symbolic notation we write this transformation as

$$(x, \theta) \longmapsto (y, \varphi);$$

if (x, θ) are coordinates for U and (u, ξ) are coordinates for V, this means that

$$\psi^*(u^i) = y^i(x, \theta), \quad \psi^*(\xi^j) = \varphi^j(x, \theta).$$

We then have the modified tangent map with matrix

$$J\psi = \begin{pmatrix} \partial y/\partial x & -\partial y/\partial \theta \\ \partial \varphi/\partial x & \partial \varphi/\partial \theta \end{pmatrix}.$$

Notice that y is even and φ is odd so that this matrix even, i.e., has even elements in the diagonal blocks and odd elements in the off diagonal blocks. It is also invertible because ψ is a super diffeomorphism. Hence its Berezinian makes sense. We then have the following theorem.

Theorem 4.6.1. *For all compactly supported sections* $s \in \mathcal{O}_V(V)$*, we have*

$$\int_V s = \int_U \psi^*(s) \mathrm{Ber}(J\psi).$$

The proof of this remarkable formula is a little involved. It is mostly a question of accommodating the odd variables in the classical formula for change of variables. The method of proving this is to exhibit the diffeomorphism ψ as a composition of simpler diffeomorphisms and then use the multiplicative property of both $J\psi$ and Ber to reduce the proof to the case of the simpler diffeomorphisms.

We can already make a simplification. Since $\psi^\sim$ is a diffeomorphism of the reduced manifolds associated to $U^{p|q}$ and $V^{p|q}$, we can introduce the diffeomorphism τ from $U^{p|q}$ to $V^{p|q}$ which is defined by

$$\tau : (x, \theta) \longrightarrow (y^\sim, \theta).$$

For this change of variables the theorem is just the classical change of variables formula; and as $\tau^{-1}\psi$ is an isomorphism of $U^{p|q}$ with itself we may replace ψ by $\tau^{-1}\psi$. Thus we may assume that

$$U = V, \quad y(x, \theta) \equiv x \ (\mathrm{mod}\ \mathcal{J}).$$

Here we recall that $\mathcal{J}$ is the ideal in $\mathcal{O}_U(U)$ generated by the θ^j.

The purely odd case. We first deal with the case $p = 0$. Thus we are dealing with isomorphisms of $\mathbf{R}^{0|q}$ with itself, i.e., automorphisms of the exterior algebra $A = \mathbf{R}[\theta^1, \ldots, \theta^q]$. In the general case of such a transformation $\theta \longrightarrow \varphi$ we have

$$\varphi^i \equiv \sum_j c_{ij}\theta^j \ (\mathrm{mod}\ \mathcal{J}^3)$$

where the matrix (c_{ij}) is invertible. By a linear transformation we can make it the identity and so we may assume that

$$\varphi \equiv \theta^i (\mathrm{mod}\ \mathcal{J}^3) \quad (1 \leq i \leq q).$$

Consider first the case in which ψ changes just one of the coordinates, say θ^1. Thus we have

$$\psi : \theta \longrightarrow \varphi, \qquad \varphi^1 = \theta^1 + \alpha, \quad \varphi^j = \theta^j (j > 1).$$

Then $\partial\alpha/\partial\theta^1$ is even and lies in $\mathcal{J}^2$. Write

$$\alpha = \theta^1\beta + \gamma, \quad \beta, \gamma \in \mathbf{R}[\theta^2, \ldots, \theta^q].$$

Then

$$\alpha_{,\,1} := \partial\alpha/\partial\theta^1 = \beta$$

and

$$\mathrm{Ber}(J\psi) = (1 + \alpha_{,\,1})^{-1} = (1 + \beta)^{-1}.$$

Notice the inverse here; the formula for the Berezinian involves the *inverse* of the matrix corresponding to the odd-odd part. Thus we have to prove that

$$\int u = \int \psi^*(u)(1 + \alpha_{,1})^{-1}.$$

This comes to showing that

$$\int \varphi^I (1 + \beta)^{-1} = \begin{cases} 0 & \text{if } |I| < q \\ 1 & \text{if } I = Q. \end{cases}$$

We must remember that Ber is even and so commutes with everything and φ^I is the expression obtained by making the substitution $\theta^j \longmapsto \varphi^j$. If $r < q$ we have

$$\int \theta^2 \dots \theta^r (1 + \beta)^{-1} = 0$$

because the integrand does not involve θ^1. Suppose we consider θ^I where $|I| < q$ and contains the index 1, say, $I = \{1, 2, \dots, r\}$ with $r < q$. Then, with $\gamma_1 = \gamma(1 + \beta)^{-1}$, we have

$$\begin{aligned} \int (\theta^1(1 + \beta) + \gamma)\theta^2 \dots \theta^r (1 + \beta)^{-1} &= \int (\theta^1 + \gamma_1)\theta^2 \dots \theta^r \\ &= \int \theta^1 \dots \theta^r + \int \gamma_1 \theta^2 \dots \theta^r \\ &= 0, \end{aligned}$$

the last equality following from the fact that the first term involves only $r < q$ odd variables and the second does not involve θ^1. For the case θ^Q the calculation is essentially the same. We have

$$\int (\theta^1(1 + \beta) + \gamma)\theta^2 \dots \theta^q (1 + \beta)^{-1} = \int (\theta^1 + \gamma_1)\theta^2 \dots \theta^q = 1.$$

Clearly this calculation remains valid if the transformation changes just one odd variable, not necessarily the first. Let us say that such transformations are of level 1. A transformation of level r then changes exactly r odd variables. We shall establish the result for transformations of level r by induction on r, starting from the case $r = 1$ proved just now. The induction step is carried out by exhibiting any transformation of level $r + 1$ as a composition of a transformation of level 1 and one of level r. Suppose that we have a transformation of level $r + 1$ of the form

$$\theta \longrightarrow \varphi, \quad \varphi^i = \theta^i + \gamma^i$$

where $\gamma^i \in \mathcal{J}^3$ and is 0 for $i > r+1$. We write this as a composition

$$\theta \longrightarrow \tau \longrightarrow \varphi$$

where

$$\tau^i = \begin{cases} \theta^i + \gamma^i & \text{if } i \le r \\ \theta^i & \text{if } i > r \end{cases} \qquad \varphi^i = \begin{cases} \tau^i & \text{if } i \ne r+1 \\ \tau^{r+1} + \gamma' & \text{if } i = r+1 \end{cases}$$

with a suitable choice of γ'. The composition is then the map

$$\theta \longrightarrow \varphi$$

where

$$\varphi^i = \begin{cases} \theta^i + \gamma^i & \text{if } i \le r \\ \theta^{r+1} + \gamma'(\tau) & \text{if } i = r+1 \\ \theta^i & \text{if } i > r+1. \end{cases}$$

Since $\theta \longrightarrow \tau$ is an even automorphism of the exterior algebra it preserves $\mathcal{J}^3$ and is an automorphism on it, and so we can choose γ' such that $\gamma'(\tau) = \gamma^{r+1}$. The induction step argument is thus complete and the result established in the purely odd case, i.e., when $p = 0$.

The general case. We consider the transformation

$$(x, \theta) \longrightarrow (y, \varphi), \quad y \equiv x \ (\mathrm{mod}\ \mathcal{J}^2).$$

This can be regarded as the composition

$$(x, \theta) \longrightarrow (z, \tau) \longrightarrow (y, \varphi)$$

where

$$z = x, \quad \tau = \varphi, \qquad \text{and} \qquad y = y(z, \varphi), \quad \varphi = \tau.$$

So it is enough to treat these two cases separately.

Case 1*:* $(x, \theta) \longrightarrow (x, \varphi)$. If σ denotes this map, then we can think of σ as a *family* (σ_x) of x-dependent automorphisms of $\mathbf{R}[\theta^1, \ldots, \theta^q]$. Clearly

$$\mathrm{Ber}(J\sigma)(x) = \mathrm{Ber}(J\sigma_x)$$

and so the result is immediate from the result for the purely odd case proved above.

Case 2*:* $(x, \theta) \longrightarrow (y, \theta)$ with $y \equiv x \ (\mathrm{mod}\ \mathcal{J}^2)$. Exactly as in the purely odd case we introduce the *level* of the transformation and show that any transformation of this type of level $r+1$ is the composition of a transformation of level 1 with one of level r. Indeed, the key step is the observation that if τ is a transformation of level r, it induces an automorphism of $\mathcal{J}^2$ and so, given any $\gamma \in \mathcal{J}^2$ we can find a $\gamma' \in \mathcal{J}^2$ such that $\gamma = \gamma'(\tau)$. We are thus reduced to the case of level 1. So we may assume that

$$y^1 = x^1 + \gamma(x, \theta), \quad y^i = x^i (i > 1), \quad \varphi^j = \theta^j.$$

In this case the argument is a little more subtle. Let ψ denote this transformation. Then

$$\mathrm{Ber}(J\psi) = 1 + \partial\gamma/\partial x^1 =: 1 + \gamma_{,1}.$$

Note that there is no inverse here unlike the purely odd case. We want to prove that for a compactly supported smooth function f one has the formula

$$\int f(x^1 + \gamma, x^2, \ldots, x^p)\theta^I (1 + \gamma_{,1}) d^p x d^q \theta = \int f(x)\theta^I d^p x d^q \theta.$$

Clearly it is enough to prove that

$$\int f(x^1 + \gamma, x^2, \ldots, x^p)(1 + \partial\gamma/\partial x^1) d^p x = \int f(x) d^p x. \qquad (*)$$

The variables other than x^1 play no role in $(*)$ and so we need to prove it only for $p = 1$. Write $x = x^1$. Thus we have to prove that

$$\int f(x + \gamma)(1 + \gamma') dx = \int f(x) dx \quad (\gamma' = d\gamma/dx).$$

We expand $f(x + \gamma)$ as a Taylor series which terminates because $\gamma \in \mathcal{J}^2$. Then,

$$\begin{aligned}
\int f(x+\gamma)(1+\gamma')dx &= \sum_{r \geq 0} \frac{1}{r!} \int f^{(r)} \gamma^r (1 + \gamma') dx \\
&= \int f dx + \sum_{r\geq 0} \frac{1}{(r+1)!} \int f^{(r+1)} \gamma^{r+1} dx + \sum_{r \geq 0} \frac{1}{r!} \int f^{(r)} \gamma^r \gamma' dx \\
&= \int f dx + \sum_{r \geq 0} \frac{1}{(r+1)!} \int \left(f^{(r+1)} \gamma^{r+1} + (r+1) f^{(r)} \gamma^r \gamma' \right) dx \\
&= 0
\end{aligned}$$

because

$$\int \left(f^{(r+1)} \gamma^{r+1} + (r+1) f^{(r)} \gamma^r \gamma' \right) dx = 0$$

as we can see by integrating by parts.

This completes the proof of Theorem 4.6.1.

There is no essential difficulty in now carrying over the theory of integration to an arbitrary supermanifold M whose reduced part is orientable. One can introduce the so-called *Berezinian bundle* which is a line bundle on M such that its sections are *densities* which are the objects to be integrated over M. Concretely, one can define a density given an atlas of coordinate charts (x, θ) covering M as a choice of a density

$$\delta(x, \theta) d^p x d^q \theta$$

for each chart, so that on the overlaps they are related by

$$\delta(y(x, \theta), \varphi(x, \theta)) \mathrm{Ber}(J\psi) = \delta(y, \varphi)$$

where ψ denotes the transformation

$$\psi : (x, \theta) \longrightarrow (y, \varphi).$$

We do not go into this in more detail. For a more fundamental way of proving the change of variable formula see[4]. See also [3,5].

4.7. Submanifolds. Theorem of Frobenius. Let M be a supermanifold. Then a *submanifold* of M (sub supermanifold) is a pair (N, j) where N is a supermanifold, $j(N \longrightarrow M)$ is a morphism such that $j^\sim$ is an imbedding of $N^\sim$ onto a closed or locally closed submanifold of $M^\sim$, and j itself is an immersion of supermanifolds. From the local description of immersions it follows that if $n \in N$ it follows that the morphisms from a given supermanifold S into N are precisely the morphisms f from S to M with the property that $f^\sim(S^\sim) \subset j^\sim(N^\sim)$. Let $M = U^{p|q}$ with $0 \in U$, and let

$$f_1, \ldots, f_r, g_1, \ldots, g_s$$

be sections on U such that

(1) the f_i are even and the g_j are odd
(2) the matrices

$$\left(\frac{\partial f_a}{\partial x^i}\right), \qquad \left(\frac{\partial g_b}{\partial \theta^j}\right)$$

have ranks r and s respectively at 0.

This is the same as requiring that there are even $f_{r+1}, \ldots, f_p$ and odd $g_{s+1}, \ldots, g_q$ such that

$$f_1, \ldots, f_p, g_1, \ldots, g_q$$

form a coordinate system at 0. Then

$$f_1 = \ldots = f_r = g_1 = \ldots = g_s = 0$$

defines a submanifold of $U^{p|q}$.

We do not go into this in more detail. The local picture of immersions makes it clear what submanifolds are like locally.

The theorem of Frobenius. We shall now discuss the super version of the classical *local* Frobenius theorem. Let M be a supermanifold and let $\mathcal{T}$ be the tangent sheaf. We start with the following definition.

Definition. A *distribution* over M is a graded subsheaf $\mathcal{D}$ of $\mathcal{T}$ which is locally a direct factor.

There are some important consequences of this definition. To obtain these we first note that in the super context *Nakayama's lemma* remains valid. Let A be a supercommutative ring which is local. Consider an arbitrary but finitely generated A-module E. Then $V = E/\mathfrak{m}E$ is a finite dimensional vector space over the field $A/\mathfrak{m}$. If (v_i) is a basis for V and $e_i \in E$ is homogeneous and lies above v_i, then the (e_i) form a basis for E. This is proved exactly as in the classical case[6]. In our case we apply this to the modules $\mathcal{D}_m, \mathcal{T}_m$ of germs of elements $\mathcal{D}$ and $\mathcal{T}$ at a point m of M. We can then find germs of homogeneous vector fields $X_m^1, \ldots, X_m^a$ and $Y_m^1, \ldots, Y_m^b$ such that $\mathcal{D}_m$ is spanned by the X's and $\mathcal{T}$ is spanned by the X's and Y's. If r, s are the numbers of even and odd vector fields among the X's, and p, q the corresponding numbers for the Y's, then we refer to $r|s$ as the *rank* of $\mathcal{D}$ at m; of course $p + r|q + s$ is the dimension $c|d$ of M. If we assume M is connected, then the numbers $r|s$ are the same at all points. We say then that $\mathcal{D}$ is of rank $r|s$.

Definition. A distribution $\mathcal{D}$ is *involutive* if $\mathcal{D}_m$ is a (super) Lie algebra for each point $m \in M$.

Theorem 4.7.1. *A distribution is involutive if and only if at each point there is a coordinate system* (x, θ) *such that* $\mathcal{D}_m$ *is spanned by* $\partial/\partial x^i, \partial/\partial\theta^j$ $(1 \le i \le r, 1 \le j \le s)$.

The "if"part is trivial. So we need to prove that if $\mathcal{D}$ is involutive, it has the local structure described in the theorem.

Some lemmas on the local structure of an involutive distribution. We need some lemmas of a local nature before we can prove the theorem. We assume that $M = U^{p|q}$ with coordinates (z, η) and $m = 0$.

Lemma 4.7.2. *Let X be an even vector field whose value is a nonzero tangent vector at the point m. Then there is a coordinate system (y, η) at m in which $X = \partial/\partial z^1$.*

Proof. Assume that $M = U^{p|q}$ with $m = 0$, the coordinates being (x, ξ). If there are no odd variables the result is classical and so going over to the reduced manifold we may assume that

$$X = \frac{\partial}{\partial x^1} + \sum_j a_j \frac{\partial}{\partial x^j} + \sum_\rho \beta_\rho \frac{\partial}{\partial \xi^\rho}$$

where a_j are even, β_ρ are odd, and they are all in $\mathcal{J}$. Here and in the rest of the section we use the same symbol $\mathcal{J}$ to denote the ideal sheaf generated by the odd elements of $\mathcal{O}_U$ in any coordinate system. The evenness of a_j then implies that $a_j \in \mathcal{J}^2$. Moreover we can find an even matrix $b = (b_{\rho\tau})$ such that $\beta_\rho \equiv \sum_\tau b_{\rho\tau}\xi^\tau$ mod $\mathcal{J}^2$. Thus mod $\mathcal{J}^2$ we have

$$X \equiv \frac{\partial}{\partial x^1} + \sum_{\rho\tau} b_{\rho\tau}\xi^\tau \frac{\partial}{\partial \xi^\rho}.$$

We now make a transformation $U^{p|q} \longrightarrow U^{p|q}$ given by

$$(x, \xi) \longrightarrow (y, \eta)$$

where

$$y = x, \quad \eta = g(x)\xi, \qquad g(x) = (g_{\rho\tau}(x))$$

and g is an invertible matrix of smooth functions to be chosen suitably. Then we have a diffeomorphism and a simple calculation shows that

$$X \equiv \frac{\partial}{\partial x^1} + \sum_\rho \gamma_\rho \frac{\partial}{\partial \eta^\rho} (\text{ mod } \mathcal{J}^2)$$

and

$$\gamma_\rho = \frac{\partial g_{\rho\tau}}{\partial x^1} + g_{\rho\sigma} b_{\sigma\tau}.$$

We choose g so that it satisfies the matrix differential equations

$$\frac{\partial g}{\partial x^1} = -gb, \quad g(0) = I.$$

It is known that this is possible and that g is invertible. Hence

$$X \equiv \frac{\partial}{\partial y^1} \quad (\bmod \mathcal{J}^2).$$

We now show that one can choose in succession coordinate systems such that X becomes $\equiv \partial/\partial x^1 \bmod \mathcal{J}^k$ for $k = 3, 4, \ldots$. This is done by induction on k. Assume that $X \equiv \partial/\partial x^1 \bmod \mathcal{J}^k$ in a coordinate system (x, ξ). We shall then show that if we choose a suitable coordinate system (y, η) defined by

$$(x, \xi) \longrightarrow (y, \eta), y^i = x^i + a_i, \quad \eta^\rho = \xi^\rho + \beta_\rho$$

where $a_i, \beta_\rho \in \mathcal{J}^k$ are suitably chosen, then $X \equiv \partial/\partial y^1 \bmod \mathcal{J}^{k+1}$. Let

$$X = \frac{\partial}{\partial x^1} + \sum_j g_j \frac{\partial}{\partial x^j} + \sum_\rho \gamma_\rho \frac{\partial}{\partial \xi^\rho}$$

where the $g_j, \gamma_\rho \in \mathcal{J}^k$. Then in the new coordinate system

$$\frac{\partial}{\partial x^j} = \frac{\partial}{\partial y^j} + \sum_k (\partial a_k/\partial x^j) \frac{\partial}{\partial y^k} + \sum_\tau (\partial \beta_\tau/\partial x^j) \frac{\partial}{\partial \eta^\tau} = \frac{\partial}{\partial y^j} + V_j$$

where $V_j \equiv 0 \bmod \mathcal{J}^k$. Similarly,

$$\frac{\partial}{\partial \xi^\rho} = \frac{\partial}{\partial \eta^\rho} + \sum_k (\partial a_k/\partial \xi^\rho) \frac{\partial}{\partial y^k} + \sum_\tau (\partial \beta_\tau/\partial/\partial \xi^\rho) \frac{\partial}{\partial \eta^\tau} = \frac{\partial}{\partial \eta^\rho} + W_\rho$$

where $W_\rho \equiv 0 \bmod \mathcal{J}^{k-1}$. Hence, as $2k \geq k+1, 2k-1 \geq k+1$, we have

$$X = \frac{\partial}{\partial y^1} + \sum_j \left(g_j + \partial a_j/\partial x^1\right) \frac{\partial}{\partial y^j} + \sum_\tau \left(\gamma_\tau + \partial \beta_\tau/\partial x^1\right) \frac{\partial}{\partial \eta^\tau} + Z$$

where $Z \equiv 0 \bmod \mathcal{J}^{k+1}$. If we now choose, as is clearly possible, the a_j, β_τ such that

$$\partial a_j/\partial x^1 = -g_j, \quad \partial \beta_\tau/\partial x^1 = -\gamma_\tau,$$

we see that $X \equiv 0 \bmod \mathcal{J}^{k+1}$. This finishes the proof.

Lemma 4.7.3. *Let Y be an odd vector field such that $Y^2 = 0$ and Y spans a distribution. Then in some coordinate system we have $Y = \partial/\partial\theta^1$.*

Proof. The proof is patterned after the classical proof where a single vector field is considered. There the corresponding differential equations are written down and solved for arbitrary initial conditions in the "time"variable t, the initial conditions corresponding to $t = 0$. Here we do the same thing, with an odd variable θ^1 in place of t and with initial conditions at $\theta^1 = 0$. If we write

$$Y = \sum_i \alpha_i(z,\eta)\frac{\partial}{\partial z^i} + \sum_\rho a_\rho(z,\eta)\frac{\partial}{\partial \eta^\rho},$$

then the condition for Y to generate a distribution is that

$$a_1(0,0) \neq 0.$$

We now consider a map

$$\mathbf{R}^{0|1} \times U^{p|q-1} \longrightarrow U^{p|q}$$

where we use θ^1 as coordinate for $\mathbf{R}^{0|1}$, $(x, \theta^2, \ldots, \theta^q)$ for coordinates on $U^{p|q-1}$. The map is given by

$$z^i = x^i + \theta^1\alpha_i(x,0,\eta'), \quad \eta^1 = \theta^1 a_1(x,0,\eta'), \quad \eta^\rho = \theta^\rho + \theta^1 a_\rho(x,0,\eta')\ (\rho \geq 2).$$

Here $\eta' = (\eta^2, \ldots, \eta^q)$. At $x = 0$, the tangent map of this map has the matrix

$$\begin{pmatrix} I_p & * & 0 \\ 0 & a_1(0,0) & 0 \\ 0 & * & I_{q-1} \end{pmatrix}$$

which has nonzero determinant because $a_1(0,0) \neq 0$. So we have a local isomorphism which we assume is defined on U by shrinking U. Under this isomorphism the vector field $\partial/\partial\theta^1$ goes over to the vector field

$$\sum_i \alpha'_i \frac{\partial}{\partial z^i} + \sum_\rho a'_\rho \frac{\partial}{\partial \eta^\rho}$$

where

$$\alpha'_i = \alpha_i(x,0,\eta'), \quad a'_\rho = a_\rho(x,0,\eta').$$

But

$$\alpha_i(z,\eta) = \alpha_i(\ldots, x^i + \theta^1\alpha'_i, \ldots, \theta^1 a'_1, \theta^\rho + \theta^1 a'_\rho).$$

Hence by Taylor expansion (terminating) we get

$$\alpha_i = \alpha'_i + \theta^1\beta_i.$$

Similarly we have

$$a_\rho = a'_\rho + \theta^1 b_\rho.$$

Hence $\partial/\partial\theta^1$ goes over to a vector field of the form $Y - \theta^1 Z$ where Z is an even vector field and we have to substitute for θ^1 its expression in the coordinates (z, η). Let V be the vector field in the (x, θ)-coordinates that corresponds to Z. Then

$$\frac{\partial}{\partial\theta^1} + \theta^1 V \longrightarrow Y$$

where $\longrightarrow$ means that the vector fields correspond under the isomorphism being considered. Since $Y^2 = 0$ we must have $(\partial/\partial\theta^1 + \theta^1 V)^2 = 0$. But a simple computation shows that

$$\left(\frac{\partial}{\partial\theta^1} + \theta^1 V\right)^2 = V - \theta^1 W = 0$$

where W is an odd vector field. Hence $V = \theta^1 W$. But then

$$\frac{\partial}{\partial\theta^1} + \theta^1 V = \frac{\partial}{\partial\theta^1} \longrightarrow Y$$

as we wanted to show.

Lemma 4.7.4. *The even part of $\mathcal{D}_m$ has a basis consisting of commuting* (*even*) *vector field germs.*

Proof. Choose a coordinate system (z^i, η^ρ) around m. Let $X^i (1 \le i \le r)$ be even vector fields whose germs at m form a basis for the even part $\mathcal{D}_m$. Then the matrix of coefficients of these vector field has the form

$$T = (\,a \quad \alpha\,)$$

where a is an even $r \times c$ matrix of rank r, while α is odd. Multiplying from the left by invertible matrices of function germs changes the given basis into another and so we may assume, after a suitable reordering of the even coordinates z, that

$$a = (\,I_r \quad a' \quad \beta\,).$$

So we have a new basis for the even part of $\mathcal{D}_m$ (denoted again by X^i) consisting of vector fields of the following form:

$$X^i = \frac{\partial}{\partial z^i} + \sum_{k>r} a'_{ik}\frac{\partial}{\partial z^k} + \sum_\rho \beta_{i\rho}\frac{\partial}{\partial\eta^\rho} \qquad (1 \le i \le r).$$

The commutator $[X^i, X^j]$ must be a combination $\sum_{t\le r} f_t X^t$ and so f_t is the coefficient of $\partial/\partial z^t$ in the commutator. But it is clear from the above formulae that the commutator in question is a linear combination of $\partial/\partial z^k (k > r)$ and the $\partial/\partial\eta^\rho$. Hence all the f_t are 0 and so the X^i commute with each other.

Lemma 4.7.5. *There is a coordinate system (z, θ) such that the even part of $\mathcal{D}_m$ is spanned by the $\partial/\partial z^i (1 \le i \le r)$.*

Proof. Let $(X^i)_{1\le i\le r}$ be commuting vector fields spanning the even part of $\mathcal{D}_m$. We shall prove first that there is a coordinate system (z, η) in which the X^i have the triangular form, i.e.,

$$X^i = \frac{\partial}{\partial z^i} + \sum_{j<i} a_{ij} \frac{\partial}{\partial z^j}.$$

We use induction on r. The case $r = 1$ is just Lemma 4.7.2. Let $r > 1$ and assume the result for $r - 1$ commuting even vector fields. Then for suitable coordinates we may assume that

$$X^i = \frac{\partial}{\partial z^i} + \sum_{j<i} a_{ij} \frac{\partial}{\partial z^j} \quad (i < r).$$

Write

$$X^r = \sum_t f_t \frac{\partial}{\partial z^t} + \sum_\rho g_\rho \frac{\partial}{\partial \eta^\rho}.$$

Then, for $j < r$,

$$[X^j, X^r] = \sum_t (X^j f_t) \frac{\partial}{\partial z^t} + \sum_\rho (X^j g_\rho) \frac{\partial}{\partial \eta^\rho} = 0.$$

Hence

$$X^j f_t = 0, \quad X^j g_\rho = 0.$$

The triangular form of the X^j now implies that these equations are valid with $\partial/\partial z^j$ replacing X^j for $j \le r-1$. Hence the f_t and g_ρ *depend only on the variables* $z^k (k \ge r, \eta^\tau)$. So we can write

$$X^r = \sum_{t\le r-1} h_t \frac{\partial}{\partial z^t} + X'$$

where X' is an even vector field whose coefficients depend only on $z^k (k \ge r), \eta^\sigma$. By Lemma 4.7.2 we can change $z^k (k \ge r), \eta^\sigma$ to another coordinate system $(w^k (k \ge r), \zeta^\sigma)$ such that X' becomes $\partial/\partial w^r$. If we make the change of coordinates

$$z, \eta \longrightarrow z^1, \ldots, z^{r-1}, w^r, \zeta^\sigma$$

it is clear that the $\partial/\partial z^i$ for $i \le r-1$ remain unchanged while X^r goes over to

$$\frac{\partial}{\partial z^r} + \sum_{t<r} k_t \frac{\partial}{\partial z^t}$$

which proves what we claimed. The triangular form of the X^i now shows that they span the same distribution as the $\partial/\partial z^i$. This proves the lemma.

Lemma 4.7.6. *In a suitable coordinate system at m, there is a basis for $\mathcal{D}_m$ of the form*

$$\frac{\partial}{\partial z^i}(1 \le i \le r), \quad Y^\rho$$

where the vector fields supercommute.

Proof. Take a coordinate system (z, η) in which

$$\frac{\partial}{\partial z^i}(1 \le i \le r), \quad Y^\rho(1 \le \rho \le s)$$

span $\mathcal{D}_m$ where the Y^ρ are odd vector fields. The matrix of coefficients has the form

$$\begin{pmatrix} I_r & a & \alpha \\ \beta_1 & \beta_2 & b \end{pmatrix}$$

where b is an even $s \times q$ matrix of rank s. Multiplying from left and reordering the odd variables if necessary we may assume that

$$b = (I_s, b').$$

Thus

$$Y^\rho = \frac{\partial}{\partial \eta^\rho} + \sum \gamma_{\rho j} \frac{\partial}{\partial z^j} + \sum_{\tau > s} c_{\rho\tau} \frac{\partial}{\partial \eta^\tau}.$$

Since the $\partial/\partial z^j$ for $j \le r$ are already in $\mathcal{D}_m$, we may remove the corresponding terms and so we may assume that

$$Y^\rho = \frac{\partial}{\partial \eta^\rho} + \sum_{j>r} \gamma_{\rho j} \frac{\partial}{\partial z^j} + \sum_{\tau > s} c_{\rho\tau} \frac{\partial}{\partial \eta^\tau}. \qquad (*)$$

The commutators

$$[\partial/\partial z^i, Y^\rho] \ (i \le r), \ [Y^\sigma, Y^\tau]$$

must be of the form

$$\sum_{t\leq r} f_t \frac{\partial}{\partial z^t} + \sum_{\rho\leq s} g_\rho Y^\rho$$

and so the f_t, g_ρ are the coefficients of the associated vector fields in the coomutators. But these coefficients are 0 and so the commutators must vanish. This finishes the proof. The argument is similar to Lemma 4.7.3.

Remark. It should be noted that the supercommutativity of the basis follows as soon as the vector fields Y^ρ are in the form $(*)$. We shall use this in the proof of Theorem 4.7.1.

Proof of Theorem 4.7.1. For $s = 0$, namely a purely even distribution, we are already done by Lemma 4.7.4. So let $s > 1$ and let the result be assumed for distributions of rank $r|s-1$. Let us work in a coordinate system with the property of the preceding lemma. The span of

$$\frac{\partial}{\partial z^i}(1 \leq i \leq r), \quad Y^\rho(1 \leq \rho \leq s-1)$$

is also a distribution, say $\mathcal{D}'$, because of the supercommutativity of these vector fields (the local splitting is true because $\mathcal{D} = \mathcal{D}' \oplus \mathcal{E}$ where $\mathcal{E}$ is the span of Y^s). We may therefore assume that $Y^\rho = \partial/\partial\eta^\rho(1 \leq \rho \leq s-1)$. Then we have

$$Y^s = b\frac{\partial}{\partial \eta^s} + \sum_j \alpha_j \frac{\partial}{\partial z^j} + \sum_{\tau\neq s} a_\tau \frac{\partial}{\partial \eta^\tau}.$$

Since $\partial/\partial z^j(1 \leq j \leq r)$ and $\partial/\partial\eta^\rho(1 \leq \rho \leq s-1)$ are in $\mathcal{D}_m$ we may assume that in the above formula the index j is $> r$ and the index $\tau > s$. We may assume that $b(m) \neq 0$, reordering the odd variables $\eta^\sigma(\sigma \geq s)$ if needed. Thus we may assume that $b = 1$. Hence we may suppose that

$$Y^s = \frac{\partial}{\partial \eta^s} + \sum_{j>r} \alpha_j \frac{\partial}{\partial z^j} + \sum_{\tau>s} a_\tau \frac{\partial}{\partial \eta^\tau}.$$

By the remark following Lemma 4.7.5 we then have

$$[\partial/\partial z^i, Y^s] = 0, \quad [\partial/\partial\eta^\sigma, Y^s] = 0(i \leq r-1, \sigma \leq s-1), \quad (Y^s)^2 = 0.$$

These conditions imply in the usual manner that the α_j, a_τ depend only on $z^k(k > r), \eta^\sigma(\sigma \geq s)$. Lemma 4.7.2 now shows that we can change $z^k(k > r), \eta^\tau(\tau \geq s)$

into a new coordinate system $w^k(k > r), \zeta^\tau(\tau \geq s)$ such that in this system Y^s has the form $\partial/\partial\zeta^s$. hence in the coordinate system

$$z^1, \ldots, z^r, w^k(k > r), \eta^1, \ldots, \eta^{r-1}, \zeta^s, \ldots,$$

the vector fields

$$\frac{\partial}{\partial z^i}(i \leq r), \frac{\partial}{\partial \eta^\tau}(\tau \leq r-1), \frac{\partial}{\partial \zeta^s}$$

span $\mathcal{D}_m$. This finishes the proof.

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