3. SUPER LINEAR ALGEBRA

3.1. The category of super vector spaces. Super linear algebra deals with the category of super vector spaces over a field k. We shall fix k and suppose that it is of characteristic 0; in physics k is $\mathbf{R}$ or $\mathbf{C}$. The objects of this category are *super vector spaces* V over k, namely, vector spaces over k which are $\mathbf{Z}_2$-graded, i.e., have decompositions

$$V = V_0 \oplus V_1 \qquad (0, 1 \in \mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}).$$

The elements of V_0 are called *even* and those of V_1 *odd.* If d_i is the dimension of V_i, we say that V has *dimension* $(d_0|d_1)$. For super vector spaces V, W, the morphisms from V to W are linear maps $V \longrightarrow W$ which preserve the gradings. They form a linear space denoted by $\mathrm{Hom}(V, W)$. For any super vector space V the elements in $V_0 \cup V_1$ are called *homogeneous* and if they are nonzero, their *parity* is defined to be 0 or 1 according as they are even or odd. The parity function is denoted by p. In any formula defining a linear or multilinear object in which the parity function appears, it is assumed that the elements involved are homogeneous (so that the formulae make sense) and that the definition is extended to nonhomogeneous elements by linearity. If we take $V = k^{p+q}$ with its standard basis $e_i (1 \le i \le p + q)$ and define e_i to be even (odd) if $i \le p$ $(i > p)$, then V becomes a super vector space with

$$V_0 = \sum_{i=1}^{p} k e_i, \qquad V_1 = \sum_{i=p+1}^{q} k e_i.$$

It is denoted by $k^{p|q}$.

The notion of direct sum for super vector spaces is the obvious one. For super vector spaces V, W, their tensor product is $V \otimes W$ whose homogeneous parts are defined by

$$(V \otimes W)_i = \sum_{j+m=i} V_j \otimes W_m$$

where i, j, m are in $\mathbf{Z}_2$ and $+$ is addition in $\mathbf{Z}_2$. Thus

$$(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1), \qquad (V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0).$$

For super vector spaces V, W, the so-called *internal* Hom, denoted by $\mathbf{Hom}(V, W)$, is the vector space of *all* linear maps from V to W, where the even maps are the ones preserving the grading and the odd maps are those that reverse the grading. In particular,

$$(\mathbf{Hom}(V, W))_0 = \mathrm{Hom}(V, W).$$

If V is a super vector space, we write $\mathbf{End}(V)$ for $\mathbf{Hom}(V, V)$. The dual of a super vector space V is the super vector space V^* where $(V^*)_i$ is the space of linear functions from V to k that vanish on V_{1-i}.

The rule of signs and its consistency. The $\otimes$ in the category of vector spaces is associative and commutative in a natural sense. Thus, for ordinary, i.e., ungraded or purely even vector spaces U, V, W, we have the natural associativity isomorphism

$$(U \otimes V) \otimes W \simeq U \otimes (V \otimes W), \qquad (u \otimes v) \otimes w \longmapsto u \otimes (v \otimes w)$$

and the commutativity isomorphism

$$c_{V,W} : V \otimes W \simeq W \otimes V, \qquad v \otimes w \longmapsto w \otimes v.$$

For the category of super vector spaces the associativity isomorphism remains the same; but the commutativity isomorphism is changed to

$$c_{V,W} : V \otimes W \simeq W \otimes V, \qquad v \otimes w \longmapsto (-1)^{p(v)p(w)} w \otimes v.$$

This is the first example where the defining formula is given only for homogeneous elements and it is assumed to be extended by linearity. Notice that

$$c_{V,W} c_{W,V} = \mathrm{id}.$$

This definition is the source of the *rule of signs* used by physicists which says that whenever two terms are interchanged in a formula, a minus sign will appear if both terms are odd.

The commutativity and associativity isomorphisms are compatible in the following sense. If U, V, W are super vector spaces,

$$c_{U,V\otimes W} = c_{U,W}c_{U,V}, \qquad c_{V,W}c_{U,W}c_{U,V} = c_{U,V}c_{U,W}c_{V,W}$$

as is easily checked. These relations can be extended to products of more than 3 super vector spaces. Suppose that $V_i (1 \leq i \leq n)$ are super vector spaces, and σ is a permutation of $\{1, 2, \ldots, n\}$. Then σ is a product $s_{i_1}s_{i_2}\ldots s_{i_r}$ where s_j is the permutation that just interchanges j and $j+1$. Writing

$$L(s_j) = I \otimes \ldots \otimes c_{V_j,V_{j+1}} \otimes \ldots \otimes V_n$$

and applying these commutativity isomorphisms successively interchanging adjacent terms in $V_1 \otimes \ldots \otimes V_n$ we have an isomorphism

$$L(\sigma) = L(s_{i_1})\ldots L(s_{i_r}) : V_1 \otimes \ldots \otimes V_n \simeq V_{\sigma^{-1}(1)} \otimes \ldots \otimes V_{\sigma^{-1}(n)}.$$

This isomorphism is independent of the way σ is expressed as a composition $s_{i_1}\ldots s_{i_r}$ and is given by

$$L(\sigma) : v_1 \otimes \ldots \otimes v_n \longmapsto (-1)^{p(\sigma)} v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)}$$

where

$$p(\sigma) = \big\{(i,j) \;\big|\; v_i, v_j \text{ odd}, i < j,\ \sigma(i) > \sigma(j)\big\}.$$

Furthermore, we have

$$L(\sigma\tau) = L(\sigma)L(\tau).$$

If all the V_i are the same and equal to V, we have an action of the group S_n in $V \otimes \ldots \otimes V$.

We shall now prove these results. Our convention is that the elements of S_n are mappings of the set $\{1, 2, \ldots, n\}$ onto itself and that the product is composition of mappings. We fix a super vector space V. For $n = 1$ the group S_n is trivial. We begin by discussing the action of S_n on the n-fold tensor product of V with itself. For $n = 2$ the group S_n is $\mathbf{Z}_2$, and we send the nontrivial element to the transformation $c_{V,V}$ on $V \otimes V$ to get the action. Let us assume now that $n \geq 3$. On $V_3 := V \otimes V \otimes V$ we have operators c_{12}, c_{23} defined as follows:

$$\begin{aligned} c_{12} &: v_1 \otimes v_2 \otimes v_3 \longmapsto (-1)^{p(v_1)p(v_2)} v_2 \otimes v_1 \otimes v_3, \\ c_{23} &: v_1 \otimes v_2 \otimes v_3 \longmapsto (-1)^{p(v_2)p(v_3)} v_1 \otimes v_3 \otimes v_2. \end{aligned}$$

We then find by a simple calculation that

$$c_{12}c_{23}c_{12} = c_{23}c_{12}c_{23}.$$

In the group S_3 the interchanges of $1, 2$ and $2, 3$ are represented by involutions s_1, s_2 respectively and S_3 is the group generated by them with the relation

$$s_1 s_2 s_1 = s_2 s_1 s_2.$$

So there is an action of S_3 on V_3 generated by the c_{ij}. This action, denoted by $\sigma \longmapsto L(\sigma)$, can be explicitly calculated for the six elements of S_3 and can be written as follows:

$$L(\sigma) : v_1 \otimes v_2 \otimes v_3 \longmapsto (-1)^{p(\sigma)} v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes v_{\sigma^{-1}(3)}$$

where

$$p(\sigma) = \sum_{(k,\ell)\in N(\sigma)} p(v_k)p(v_\ell), \qquad N(\sigma) = \{(k, \ell) \mid k < \ell, \sigma(k) > \sigma(\ell)\}.$$

This description makes sense for all n and leads to the following formulation.

Proposition 3.1.1. *There is a unique action L of S_n on $V_n := V \otimes \ldots \otimes V$ (n factors) such that for any $i < n$, the element s_i of S_n that sends i to $i + 1$ and vice versa and fixes all the others, goes over to the map*

$$L(s_i) : v_1 \otimes \ldots \otimes v_n \longmapsto (-1)^{p(v_i)p(v_{i+1})} v_1 \otimes \ldots \otimes v_{i+1} \otimes v_i \otimes \ldots \otimes v_n.$$

For arbitrary σ let $N(\sigma), p(\sigma)$ be defined as above. Then

$$L(\sigma) : v_1 \otimes \ldots \otimes v_n \longmapsto (-1)^{p(\sigma)} v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)}.$$

Finally, we can write

$$p(\sigma) = \#\{(k, \ell) \mid k < \ell, v_k, v_\ell \text{ both odd, } \sigma(k) > \sigma(\ell)\}.$$

Proof. The calculation above for $n = 3$ shows that for any $i < n$ we have

$$L(s_i)L(s_{i+1})L(s_i) = L(s_{i+1})L(s_i)L(s_{i+1}).$$

Since S_n is generated by the s_i with the relations

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad (1 \le i \le n-1)$$

it is immediate that there is an action of S_n on V_n that sends s_i to $L(s_i)$ for all i. If we disregard the sign factors this is the action

$$R(\sigma) : v_1 \otimes \ldots \otimes v_n \longmapsto v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)}.$$

Hence except for the sign factor we are done. We shall prove the formula for the sign factor by induction on $\ell(\sigma)$, the *length* of σ, which is by definition the cardinality $\#N(\sigma)$ of the set $N(\sigma)$.

First of all, $\ell(\sigma) = 1$ if and only if $\sigma = s_i$ for some i and the result is then obvious. Suppose $\ell(\sigma) > 1$ and we assume the result for elements of smaller length. We can find i such that $(i, i+1) \in N(\sigma)$; we define $\tau = \sigma s_i$. It is then easily verified that $k < \ell \iff s_i k < s_i \ell$ whenever $k < \ell$ and $(k, \ell) \neq (i, i+1)$, and

$$(k, \ell) \in N(\tau) \iff (s_i k, s_i \ell) \in N(\sigma) \qquad (k < \ell, (k, \ell) \neq (i, i+1))$$

while

$$(i, i+1) \in N(\sigma), \quad (i, i+1) \notin N(\tau).$$

It follows from this that

$$\ell(\tau) = \ell(\sigma) - 1.$$

The result is thus true for τ. Now

$$\begin{aligned} L(\sigma)\,(v_1 \otimes \ldots \otimes v_n) &= (-1)^{p(v_i)p(v_{i+1})} L(\tau)\,(v_{s_i 1} \otimes \ldots \otimes v_{s_i n}) \\ &= (-1)^q R(\sigma)\,(v_1 \otimes \ldots \otimes v_n) \end{aligned}$$

where

$$q = p(v_i)p(v_{i+1}) + \sum_{(k,\ell) \in N(\tau)} p(v_{s_i k}) p(v_{s_i \ell}) = \sum_{(k',\ell') \in N(\sigma)} p(v_{k'}) p(v_{\ell'}) = p(\sigma).$$

This completes the proof.

Corollary 3.1.2. *Let $V_i (i = 1, \ldots, n)$ be super vector spaces. For each $\sigma \in S_n$ let $L(\sigma)$ be the map*

$$L(\sigma) : V_1 \otimes \ldots \otimes V_n \longrightarrow V_{\sigma^{-1}(1)} \otimes \ldots \otimes V_{\sigma^{-1}(n)}$$

be defined by

$$L(\sigma) : v_1 \otimes \ldots \otimes v_n \longmapsto (-1)^{p(\sigma)} v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)}.$$

If $\sigma = s_{i_1} \ldots s_{i_r}$ *then*

$$L(\sigma) = L(s_{i_1}) \ldots L(s_{i_r})$$

where, for all i*,*

$$L(s_i) = I \otimes \ldots \otimes c_{V_i, V_{i+1}} \otimes I \ldots \otimes I.$$

In particular,

$$L(\sigma\tau) = L(\sigma)L(\tau).$$

Proof. Take $V = \oplus V_i$ and apply the proposition. The result is immediate.

Remark. The above corollary shows that the result of applying the exchanges successively at the level of tensors is independent of the way the permutation is expressed as a product of adjacent interchanges. This is the fundamental reason why the rule of signs works in super linear algebra in a consistent manner.

Super algebras. A *super algebra* A is a super vector space which is an associative algebra (always with unit 1 unless otherwise specified) such that multiplication is a morphism of super vector spaces from $A \otimes A$ to A. This is the same as requiring that

$$p(ab) = p(a) + p(b).$$

It is easy to check that 1 is always even, that A_0 is a purely even subalgebra, and that

$$A_0 A_1 \subset A_1, \qquad A_1^2 \subset A_0.$$

If V is a super vector space, $\mathbf{End}(V)$ is a super algebra. For a super algebra its super center is the set of all elements a such that $ab = (-1)^{p(a)p(b)} ba$ for all b; it is often written as $Z(A)$. This definition is an example that illustrates the sign rule. We have

$$Z(\mathbf{End}(V)) = k \cdot 1.$$

It is to be mentioned that the super center is in general different from the center of A viewed as an ungraded algebra. Examples of this will occur in the theory of Clifford algebras that will be treated in Chapter 5. If $V = k^{p|q}$ we write $M(p|q)$ or $M^{p|q}$ for $\mathbf{End}(V)$. Using the standard basis we have the usual matrix representations for elements of $M(p|q)$ in the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where the letters denote matrices with A, B, C, D of orders respectively $p \times p, p \times q, q \times p, q \times q$. The even elements and odd elements are respectively of the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \qquad \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

A super algebra A is said to be *commutative* if

$$ab = (-1)^{p(a)p(b)} ba$$

for all (homogeneous) $a, b \in A$. The terminology can cause some mild confusion because $k[t]$ with t odd and $t^2 = 1$ is a super algebra which is commutative as an algebra but not as a super algebra. Indeed, in a commutative super algebra, we have

$$ab + ba = 0, \qquad a^2 = 0$$

for odd a, b; in particular, odd elements are *nilpotent.* This is false for t in the above example. For this reason, and in order to avoid confusion, commutative super algebras are often called *supercommutative.* The exterior algebra over an even vector space is an example of a supercommutative algebra. If the vector space has finite dimension this super algebra is isomorphic to $k[\theta_1, \ldots, \theta_q]$ where the θ_i are anticommuting, i.e., satisfy the relations $\theta_i \theta_j + \theta_j \theta_i = 0$ for all i, j. If A is supercommutative, A_0 (super)commutes with A. We can formulate supercommutativity of an algebra A by

$$\mu = \mu \circ c_{A,A}, \qquad \mu : A \otimes A \longrightarrow A$$

where μ is multiplication. Formulated in this way there is no difference from the usual definition of commutativity classically. In this definition the sign rule is hidden. In general it is possible to hide all the signs using such devices.

A variant of the definition of supercommutativity leads to the definition of the *opposite* of a super algebra. If A is a super algebra, its opposite A^{opp} has the same super vector space underlying it but

$$a \cdot b = (-1)^{p(a)p(b)} ba$$

where $a \cdot b$ is the product of a and b in A^{opp}. This is the same as requiring that

$$\mu_{\mathrm{opp}} = \mu \circ c_{A,A}.$$

Thus A is supercommutative if and only if $A^{\mathrm{opp}} = A$.

Super Lie algebras. If we remember the sign rule it is easy to define a *Lie super algebra* or *super Lie algebra.* It is a super vector space $\mathfrak{g}$ with a bracket $[\ ,\ ]$ which is a morphism from $\mathfrak{g} \otimes \mathfrak{g}$ to $\mathfrak{g}$ with the following properties:

(a) $[a, b] = -(-1)^{p(a)p(b)}[b, a]$.

(b) The (super) Jacobi identity

$$[a, [b, c]] + (-1)^{p(a)p(b)+p(a)p(c)}[b, [c, a]] + (-1)^{p(a)p(c)+p(b)p(c)}[c, [a, b]] = 0.$$

One can hide the signs above by rewriting these relations as

(a) $[\ ,\ ](1 + c_{\mathfrak{g},\mathfrak{g}}) = 0$.

(b) The (super) Jacobi identity

$$[\ ,[\ ,\ ]](1 + \sigma + \sigma^2) = 0$$

where σ is the automorphism of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ corresponding to the cyclic permutation $(123) \longmapsto (312)$.

Thus, (b) shows that the super Jacobi identity has the same form as the ordinary Jacobi identity for ordinary Lie algebras. Thus the super Lie algebra is defined in exactly the same manner in the category of super vector spaces as an ordinary Lie algebra is in the category of ordinary vector spaces. It thus appears as an entirely natural object. One might therefore say that a super Lie algebra is a *Lie object* in the category of super vector spaces.

There is a second way to comprehend the notion of a super Lie algebra which is more practical. The bracket is skew symmetric if one of the elements is even and symmetric if both are odd. The super Jacobi identity has 8 special cases depending on the parities of the three elements a, b, c. If all three are even the definition is simply the statement that $\mathfrak{g}_0$ is a (ordinary) Lie algebra. The identities with 2 even and 1 odd say that $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module. The identities with 2 odd and 1 even say that the bracket

$$\mathfrak{g}_1 \otimes \mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$$

is a symmetric $\mathfrak{g}_0$-map. Finally, the identities for all three odd elements reduce to

$$[a, [b, c]] + \ldots + \ldots = 0 \qquad (a, b, c \in \mathfrak{g}_1)$$

where $+ \ldots + \ldots$ is cyclic summation in a, b, c. It is not difficult to see that the last requirement is equivalent to

$$[a, [a, a]] = 0 \qquad (a \in \mathfrak{g}_1).$$

Indeed, if this condition is assumed, then replacing a by $xa+yb$ where $a, b \in \mathfrak{g}_1$ and $x, y \in k$ we find that

$$[b,[a,a]] + 2[a,[a,b]] = 0 \qquad (a, b \in \mathfrak{g}_1).$$

But then

$$0 = [a+b+c,[a+b+c,a+b+c]] = 2\left([a,[b,c]] + [b,[c,a]] + [c,[a,b]]\right).$$

Thus a super Lie algebra is a super vector space $\mathfrak{g}$ on which a bilinear bracket $[\ ,\ ]$ is defined such that

(a) $\mathfrak{g}_0$ is an ordinary Lie algebra for $[\ ,\ ]$.

(b) $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module for the action $a \longmapsto \mathrm{ad}(a) : b \longmapsto [a,b]$ $(b \in \mathfrak{g}_1)$.

(c) $a \otimes b \longmapsto [a,b]$ is a symmetric $\mathfrak{g}_0$-module map from $\mathfrak{g}_1 \otimes \mathfrak{g}_1$ to $\mathfrak{g}_0$.

(d) For all $a \in \mathfrak{g}_1$, we have $[a,[a,a]] = 0$.

Except for (d) the other conditions are linear and can be understood within the framework of ordinary Lie algebras and their representations. The condition (d) is nonlinear and is the most difficult to verify in applications when Lie super algebras are constructed by putting together an ordinary Lie algebra and a module for it satisfying (a)-(c).

If A is a super algebra, we define

$$[a,b] = ab - (-1)^{p(a)p(b)}ba \qquad (a, b \in A).$$

It is then an easy verification that $[\ ,\ ]$ converts A into a super Lie algebra. It is denoted by A_L but often we omit the suffix L. If $A = \mathbf{End}(V)$, we often write $\mathfrak{gl}(V)$ for the corresponding Lie algebra; if $V = \mathbf{R}^{p|q}$ we write $\mathfrak{gl}(p|q)$ for $\mathfrak{gl}(V)$.

Let $\mathfrak{g}$ be a super Lie algebra and for $X \in \mathfrak{g}$ let us define

$$\mathrm{ad}\, X : \mathfrak{g} \longrightarrow \mathfrak{g}, \qquad \mathrm{ad}\, X(Y) = [X,Y].$$

Then

$$\mathrm{ad} : X \longmapsto \mathrm{ad}\, X$$

is a morphism of $\mathfrak{g}$ into $\mathfrak{gl}(\mathfrak{g})$. The super Jacobi identity is just the relation

$$[\mathrm{ad}\, X, \mathrm{ad}\, Y] = \mathrm{ad}\,[X,Y] \qquad (X, Y \in \mathfrak{g}).$$

The supertrace. Let $V = V_0 \oplus V_1$ be a finite dimensional super vector space and let $X \in \mathbf{End}(V)$. Then we have

$$X = \begin{pmatrix} X_{00} & X_{01} \\ X_{10} & X_{11} \end{pmatrix}$$

where X_{ij} is the linear map of V_j to V_i such that $X_{ij}v$ is the projection on V_i of Xv for $v \in V_j$. The *super trace* of X is now defined as

$$\mathrm{str}(X) = \mathrm{tr}(X_{00}) - \mathrm{tr}(X_{11}).$$

It is easy to verify that

$$\mathrm{str}(XY) = (-1)^{p(X)p(Y)}\mathrm{str}(YX) \qquad (X, Y \in \underline{\mathrm{End}}(V)).$$

In analogy with the classical situation we write $\mathfrak{sl}(V)$ for the space of elements in $\mathfrak{gl}(V)$ with super trace 0; if $V = \mathbf{R}^{p|q}$, then we write $\mathfrak{sl}(p|q)$ for $\mathfrak{sl}(V)$. Since the odd elements have supertrace 0, $\mathfrak{sl}(V)$ is a sub super vector space of $\mathfrak{gl}(V)$. It is easy to verify that

$$[X, Y] \in \mathfrak{sl}(V) \qquad (X, Y \in \mathfrak{gl}(V)).$$

Thus $\mathfrak{sl}(V)$ is a sub super Lie algebra of $\mathfrak{gl}(V)$. Corresponding to the classical series of Lie algebras $\mathfrak{gl}(n), \mathfrak{sl}(n)$ we thus have the series $\mathfrak{gl}(p|q), \mathfrak{sl}(p|q)$ of super Lie algebras. In Chapter 6 we shall give Kac's classification of simple super Lie algebras over an algebraically closed field of which the $\mathfrak{sl}(p|q)$ are particular examples.

3.2. The super Poincaré algebra of Gol'fand and Likhtman. Although we have given a natural and simple definition of super Lie algebras, historically they emerged first in the works of physicists. Gol'fand and Likhtman constructed the super Poincaré algebra in 1971 and Wess and Zumino constructed the superconformal algebra in 1974. These were ad hoc constructions, and although it was realized that these were new algebraic structures, their systematic theory was not developed till 1975 when Kac[3] introduced Lie super algebras in full generality and classified the simple ones over $\mathbf{C}$ and $\mathbf{R}$. We shall discuss these two examples in some detail because they contain much of the intuition behind the construction of superspacetimes and their symmetries. We first take up the super Poincaré algebra of Gol'fand and Likhtman.

Let $\mathfrak{g}$ be a Lie super algebra. This means that

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

where $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module with appropriate properties. The basic assumptions in all constructions of superspacetimes are the following.

(1) $\mathfrak{g}$ is a *real* Lie super algebra.

(2) $\mathfrak{g}_1$ is a very special type of $\mathfrak{g}_0$-module, namely, it is spinorial.

(3) The spacetime momenta should be captured among the commutators $[A, B]$ where $A, B \in \mathfrak{g}_1$, i.e., $[\mathfrak{g}_1, \mathfrak{g}_1]$ should contain the translation subspace of $\mathfrak{g}_0$.

The condition (2) means that either $\mathfrak{g}_0$ or some quotient of it is an orthogonal Lie algebra, $\mathfrak{g}_0$ acts on $\mathfrak{g}_1$ through this quotient, and the module $\mathfrak{g}_1$ is spinorial, i.e., its complexification is a direct sum of spin modules of this orthogonal Lie algebra. This restriction of $\mathfrak{g}_1$ has its source in the fact that in quantum field theory the objects obeying the anticommutation rules were the spinor fields, and this property was then taken over in the definition of spinor fields at the classical level.

In the example of Gol'fand-Likhtman, $\mathfrak{g}_0$ is the Poincaré Lie algebra, i.e.,

$$\mathfrak{g}_0 = \mathfrak{t} \oplus \mathfrak{l}$$

where $\mathfrak{t} \simeq \mathbf{R}^4$ is the abelian Lie algebra of spacetime translations, $\mathfrak{l}$ is the Lorentz Lie algebra $\mathfrak{so}(1,3)$, namely, the Lie algebra of $\mathrm{SO}(1,3)^0$, and the sum is semidirect with respect to the action of $\mathfrak{l}$ on $\mathfrak{t}$; in particular $\mathfrak{t}$ is an abelian ideal. $\mathfrak{g}_0$ is thus the Lie algebra of the Poincaré group and hence $\mathfrak{g}$ is to be thought of as a *super Poincaré algebra.* We shall also assume that $\mathfrak{g}$ is minimal in the sense that there is no sub super Lie algebra that strictly includes $\mathfrak{g}_0$ and is strictly included in $\mathfrak{g}$.

The Poincaré group P acts on $\mathfrak{g}_1$ and one can analyze its restriction to the translation subgroup in a manner analogous to what was done in Chapter 1 for unitary representations except now the representation is finite dimensional and not unitary. Since $\mathfrak{t}$ is abelian, by Lie's theorem on solvable actions we can find eigenvectors in the complexification $(\mathfrak{g}_1)_{\mathbf{C}}$ of $\mathfrak{g}_1$ for the action of $\mathfrak{t}$. So there is a linear function λ on $\mathfrak{t}$ such that

$$V_\lambda := \left\{ v \in (\mathfrak{g}_1)_{\mathbf{C}}, \mid [X, v] = \lambda(X)v, \ X \in \mathfrak{t} \right\} \neq 0.$$

If L is the Lorentz group $\mathrm{SO}(1,3)^0$ and we write $X \longmapsto X^h$ for the action of $h \in L$ on $\mathfrak{t}$ as well as $\mathfrak{g}_1$, we have

$$\mathrm{ad}\ (X^h) = h \ \mathrm{ad}\ (X)\ h^{-1}.$$

This shows that shows that h takes V_λ to V_{λ^h} where $\lambda^h(X) = \lambda(X^{h^{-1}})$ for $X \in \mathfrak{t}$. But $\mathfrak{g}_{1,\mu}$ can be nonzero only for a finite set of linear functions μ on $\mathfrak{t}$, and hence

$\lambda = 0$. But then $\mathfrak{g}_{1,0}$ is stable under L so that $\mathfrak{g}_0 \oplus \mathfrak{g}_{1,0}$ is a Lie super algebra. By minimality it must be all of $\mathfrak{g}$. Hence in a minimal $\mathfrak{g}$ the action of $\mathfrak{t}$ on $\mathfrak{g}_1$ is 0. This means that $\mathfrak{g}_0$ acts on $\mathfrak{g}_1$ through $\mathfrak{l}$ so that it makes sense to say that $\mathfrak{g}_1$ is spinorial. Furthermore, if $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is any super Lie algebra and $\mathfrak{h}$ is a $\mathfrak{g}_0$-submodule of $\mathfrak{g}_1$, then $\mathfrak{g}_0 \oplus \mathfrak{h}$ is a sub super Lie algebra of $\mathfrak{g}$. Hence if $\mathfrak{g}$ is a minimal extension of $\mathfrak{g}_0$, then $\mathfrak{g}_1$ must be *irreducible.* Since we are working over $\mathbf{R}$, we must remember that $\mathfrak{g}_1$ may not be irreducible after extension of scalars to $\mathbf{C}$.

The irreducible representations of the group $\mathrm{SL}(2, \mathbf{C})$, viewed as a real Lie group, are precisely the representations $\mathbf{k} \otimes \overline{\mathbf{m}}$ where for any integer $r \geq 1$, we write $\mathbf{r}$ for the irreducible *holomorphic* representation of dimension r, $\overline{\mathbf{m}}$ denoting the complex conjugate representation of the representation $\mathbf{m}$. Recall that $\mathbf{1}$ is the trivial representation in dimension 1 and $\mathbf{2}$ is the defining representation in $\mathbf{C}^2$. Of these $\mathbf{2}$ and $\overline{\mathbf{2}}$ are the spin modules. To get a real irreducible spinorial module we take notice that $\mathbf{2} \oplus \overline{\mathbf{2}}$ has a real form. Indeed with $\mathbf{C}^2$ as the space of $\mathbf{2}$, the representation $\mathbf{2} \oplus \overline{\mathbf{2}}$ can be written as

$$\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} g \cdot u \\ \overline{g} \cdot \overline{u} \end{pmatrix} \qquad (g \in \mathrm{SL}(2, \mathbf{C}), u, v \in \mathbf{C}^2)$$

where a bar over a letter denotes complex conjugation. This action commutes with the conjugation

$$\sigma : \begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} \overline{v} \\ \overline{u} \end{pmatrix}.$$

We define $\mathfrak{m}$ to be the real form of $\mathbf{2} \oplus \overline{\mathbf{2}}$ defined by σ. We have

$$\mathfrak{m}_{\mathbf{C}} = \mathbf{2} \oplus \overline{\mathbf{2}}, \qquad \mathfrak{m} = (\mathfrak{m}_{\mathbf{C}})^{\sigma}.$$

Since $\mathbf{2}$ and $\overline{\mathbf{2}}$ are inequivalent, the above is the *only* possible decomposition of $\mathfrak{m}_{\mathbf{C}}$ into irreducible pieces, and so there is no proper submodule of $\mathfrak{m}$ stable under the conjugation σ. Thus $\mathfrak{m}$ is irreducible under $\mathfrak{g}_0$. This is the so-called *Majorana spinor*. Any real irreducible representation of $\mathrm{SL}(2, \mathbf{C})$ is a direct sum of copies of $\mathfrak{m}$ and so minimality forces $\mathfrak{g}_1$ to be $\mathfrak{m}$. Our aim is to show that there is a structure of a super Lie algebra on

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{m}$$

satisfying (3) above. The irreducibility of $\mathfrak{m}$ ensures the minimality of $\mathfrak{g}$ as an extension of $\mathfrak{g}_0$.

To make $\mathfrak{g}$ into a Lie super algebra we must find a symmetric $\mathfrak{g}_0$–map

$$[\ ,\ ] : \mathfrak{m} \otimes \mathfrak{m} \longrightarrow \mathfrak{g}_0$$

such that

$$[a,[a,a]]=0 \qquad (a\in\mathfrak{m}).$$

Now

$$\mathfrak{m}_{\mathbf{C}}\otimes\mathfrak{m}_{\mathbf{C}}=(\mathbf{2}\otimes\mathbf{2})\oplus(\overline{\mathbf{2}}\otimes\overline{\mathbf{2}})\oplus(\mathbf{2}\otimes\overline{\mathbf{2}})\oplus(\overline{\mathbf{2}}\otimes\mathbf{2}).$$

We claim that there is a projectively unique symmetric $\mathfrak{l}$-map

$$L:\mathfrak{m}_{\mathbf{C}}\otimes\mathfrak{m}_{\mathbf{C}}\longrightarrow\mathfrak{t}_{\mathbf{C}}$$

where the right side is the complexification of the 4-dimensional representation of $\mathrm{SO}(1,3)^0$ viewed as a representation of $\mathrm{SL}(2,\mathbf{C})$. To see this we first note that $\mathbf{2}\otimes\overline{\mathbf{2}}$ descends to a representation of $\mathrm{SO}(1,3)^0$ because -1 acts as -1 on both factors and so acts as 1 on their tensor product. Moreover it is the only irreducible representation of dimension 4 of $\mathrm{SO}(1,3)^0$, and we write this as $\mathbf{4_v}$, the vector representation in dimension 4; of course $\mathbf{4_v}\simeq\mathfrak{t}$. Furthermore, using the map $F:u\otimes v\longmapsto v\otimes u$ we have

$$\mathbf{2}\otimes\overline{\mathbf{2}}\simeq\overline{\mathbf{2}}\otimes\mathbf{2}\simeq\mathbf{4_v}.$$

Thus $W=(\mathbf{2}\otimes\overline{\mathbf{2}})\oplus(\overline{\mathbf{2}}\otimes\mathbf{2})\simeq\mathbf{4_v}\oplus\mathbf{4_v}$. On the other hand, W is stable under F and so splits as the direct sum of subspaces symmetric and skew symmetric with respect to F, these being also submodules. Hence each of them is isomorphic to $\mathbf{4_v}$. Now $\mathbf{2}\otimes\mathbf{2}$ is $\mathbf{1}\oplus\mathbf{3}$ where $\mathbf{1}$ occurs in the skew symmetric part and $\mathbf{3}$ occurs in the symmetric part, and a similar result is true for the complex conjugate modules. Hence

$$(\mathfrak{m}_{\mathbf{C}}\otimes\mathfrak{m}_{\mathbf{C}})^{\mathrm{symm}}\simeq\mathbf{3}\oplus\overline{\mathbf{3}}\oplus\mathbf{4_v}$$

showing that there is a projectively unique symmetric $\mathfrak{l}$-map L from $\mathfrak{m}_{\mathbf{C}}\otimes\mathfrak{m}_{\mathbf{C}}$ to $\mathfrak{t}_{\mathbf{C}}$.

We put

$$[a,b]=L(a\otimes b) \qquad (a,b\in\mathfrak{m}_{\mathbf{C}}).$$

Since L goes into the translation part $\mathfrak{t}_{\mathbf{C}}$ which acts as 0 on $\mathfrak{m}_{\mathbf{C}}$, we have automatically

$$[c,[c,c]]=0 \qquad (c\in\mathfrak{m}_{\mathbf{C}}).$$

We have thus obtained a Lie super algebra

$$\mathfrak{s}=(\mathfrak{g}_0)_{\mathbf{C}}\oplus\mathfrak{m}_{\mathbf{C}}.$$

This is the complexified super Poincaré algebra of Gol'fand-Likhtman.

To obtain the appropriate real form of $\mathfrak{s}$ is now easy. Let us denote by φ the conjugation of $\mathfrak{t}_{\mathbf{C}}$ that defines $\mathfrak{t}$. Then, on the one dimensional space of symmetric $\mathfrak{l}$-maps from $\mathfrak{m}_{\mathbf{C}} \otimes \mathfrak{m}_{\mathbf{C}}$ to $\mathfrak{t}_{\mathbf{C}}$ we have the conjugation

$$M \longmapsto \sigma \circ M \circ (\varphi \otimes \varphi),$$

and so there is an element N fixed by this conjugation. If

$$[a, b] = N(a \otimes b) \qquad (a, b \in \mathfrak{m}),$$

then N maps $\mathfrak{m} \otimes \mathfrak{m}$ into $\mathfrak{t}$ and so, as before,

$$[[c, c], c] = 0 \qquad (c \in \mathfrak{m}).$$

Thus we have a Lie super algebra structure on

$$\mathfrak{g} := \mathfrak{g}_0 \oplus \mathfrak{m}.$$

This is the super Poincaré algebra constructed by Gol'fand and Likhtman.

It is to be noted that in constructing this example we have made the following assumptions about the structure of $\mathfrak{g}$:

(1) $\mathfrak{g}_1$ is spinorial,

(2) $[\ ,\ ]$ is not identically zero on $\mathfrak{g}_1 \otimes \mathfrak{g}_1$ and maps it into $\mathfrak{t}$,

(3) $\mathfrak{g}$ is minimal under the conditions (1) and (2).

Indeed, $\mathfrak{g}$ is *uniquely determined by these assumptions.* However, there are other examples if some of these assumptions are dropped. Since $\mathfrak{t}$ is irreducible as a module for $\mathfrak{l}$, it follows that the map $\mathfrak{g}_1 \otimes \mathfrak{g}_1 \longrightarrow \mathfrak{t}$ is surjective and so the spacetime momenta P_μ (which form a basis of $\mathfrak{t}$) can be expressed as anticommutators of the spinorial odd elements (also called spinorial charges), a fact that is important because it often leads to the positivity of the energy operator in susy theories. This aspect of $\mathfrak{g}$ has prompted a heuristic understanding of $\mathfrak{g}$ as the *square root of the Poincaré algebra.* If we choose the usual basis $P_\mu, M_{\mu\nu}$, for $\mathfrak{g}_0$, and use a basis (Q_α) for $\mathfrak{m}_{\mathbf{C}}$, then the commutation rules for $\mathfrak{g}$ can be expressed in terms of the Pauli and Dirac matrices in the following form used by physicists:

$$[P_\mu, Q_\alpha] = 0, \qquad [M^{\mu\nu}, Q_\alpha] = -i\sigma^{\mu\nu}_{\alpha\beta} Q_\beta$$

$$\{Q_\alpha, Q_\beta\} = -2(\gamma^\mu C^{-1})_{\alpha\beta} P_\mu$$

where $\{\ ,\ \}$ is the anticommutator, $\sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$, and a $(+,+,+,-)$ Lorentz metric is used, and C is a so-called charge conjugation matrix.

3.3. Conformal spacetime. The second example we wish to discuss is the super conformal algebra of Wess and Zumino. To this end we need some preliminary discussion of conformality.

The relevance of conformality to the physics of radiation goes back to Weyl. Conformal maps are defined as maps of one Riemannian or pseudo Riemannian manifold into another that take one metric to a multiple of the other, the multiplying constant being a strictly positive function that is allowed to vary from point to point. The simplest example is the dilation $x \mapsto cx$ on the space $\mathbf{R}^{p,q}$, which is Euclidean space $\mathbf{R}^{p+q}$ equipped with a metric of signature (p,q), where $c > 0$ is a constant. Less trivial examples are complex analytic maps f from a domain D in the complex plane to another domain D', df being never 0 on D. Such maps are classically known as conformal maps, which is the reason why the maps in the more general context are also called conformal. Weyl noticed that the Maxwell equations are invariant under all conformal transformations; we have seen this in our discussion of the Maxwell equations. The idea that for radiation problems the symmetry group should be the conformal group on Minkowski spacetime is also natural because the conformal group is the group whose action preserves the forward light cone structure of spacetime. In Euclidean space $\mathbf{R}^n$ (with the usual positive definite metric), the so-called *inversions* are familiar from classical geometry; these are maps $P \mapsto P'$ with the property that P' is on the same ray s OP (O is the origin) and satisfies

$$OP.\,OP' = 1;$$

this determines the map as

$$x \longmapsto x' = \frac{x}{||x||^2}.$$

It is trivial to check that

$$ds'^2 = \frac{1}{r^4} ds^2$$

so that this map is conformal; it is undefined at O, but by going over to the one-point compactification S^n of $\mathbf{R}^n$ via stereographic projection and defining ∞ to be the image of O we get a conformal map of S^n. This is typical of conformal maps in the sense that they are globally defined only after a suitable compactification. The compactification of Minkowski spacetime and the determination of the conformal extension of the Poincaré group go back to the nineteenth century and the work of Felix Klein. It is tied up with some of the most beautiful parts of classical projective geometry. It was resurrected in modern times by the work of Penrose.

In two dimensions the conformal groups are infinite dimensional because we have more or less arbitrary holomorphic maps which act conformally. However this is not true in higher dimensions; for $\mathbf{R}^{p,q}$ with $p+q \geq 3$, the vector fields which are conformal in the sense that the corresponding one parameter (local) groups of diffeomorphisms are conformal, already form a *finite dimensional* Lie algebra, which is in fact isomorphic to $\mathfrak{so}(p+1, q+1)$. Thus $SO(p+1, q+1)$ acts conformally on a compactification of $\mathbf{R}^{p,q}$, and contains the inhomogeneous group $ISO(p, q)$ as the subgroup that leaves invariant $\mathbf{R}^{p,q}$. In particular $SO(1, n+1)$ is the conformal extension of $ISO(n)$, acting on S^n viewed as the one-point compactification of $\mathbf{R}^n$. We shall discuss these examples a little later. For the moment we shall concern ourselves with the case of dimension 4.

The variety of lines in projective space: the Klein quadric. We now treat the case of dimension 4 in greater detail. We start with a complex vector space T of dimension 4 and the corresponding projective space $\mathbf{P} \simeq \mathbf{CP}^3$ of lines (= one dimensional linear subspaces) in T. We denote by $\mathbf{G}$ the Grassmannian of all 2-dimensional subspaces of T which can be thought of as the set of all lines in $\mathbf{P}$. The group $\mathrm{GL}(T) \simeq \mathrm{GL}(4, \mathbf{C})$ acts transitively on $\mathbf{G}$ and so we can identify $\mathbf{G}$ with the homogeneous space $\mathrm{GL}(4, \mathbf{C})/P_0$ where P_0 is the subgroup of elements leaving invariant the plane π_0 spanned by e_1, e_2. Thus P_0 consists of matrices of the form

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where A, B, D are 2×2 matrices, so that $\mathbf{G}$ becomes a complex manifold of dimension $16 - 12 = 4$. The group $\mathrm{SL}(T) \simeq \mathrm{SL}(4, \mathbf{C})$ already acts transitively on $\mathbf{G}$. We omit the reference to $\mathbf{C}$ hereafter and write $\mathrm{GL}(T), \mathrm{SL}(T)$ etc for the above groups. Associated to T we also have its second exterior power $E = \Lambda^2(T)$. The action of $\mathrm{GL}(T)$ on T lifts to a natural action on E: for $g \in \mathrm{GL}(T)$, $g(u \wedge v) = gu \wedge gv$. It is well-known that this action gives an irreducible representation of $\mathrm{SL}(T)$ on E.

We shall now exhibit a $\mathrm{SL}(T)$-equivariant imbedding of $\mathbf{G}$ in the projective space $\mathbf{P}(E)$ of E. If π is a plane in T and a, b is a basis for it, we have the element $a \wedge b \in E$; if we change the basis to another $(a', b') = (a, b)u$ where u is an invertible 2×2 matrix, then $a' \wedge b' = \det(u) a \wedge b$ and so the image $[a \wedge b]$ of $a \wedge b$ in the projective space $\mathbf{P}(E)$ of E is uniquely determined. This gives the *Plücker map* $P\ell$:

$$P\ell : \pi \longmapsto [a \wedge b] \qquad (a, b \text{ a basis of } \pi).$$

The Plücker map is an *imbedding*. To see this, recall first that if a, b are 2 linearly independent vectors in T, then, for any vector c the condition $c \wedge a \wedge b = 0$ is necessary and sufficient for c to lie in the plane spanned by a, b; this is obvious if we take $a = e_1, b = e_2$ where $(e_i)_{1 \le i \le 4}$ is a basis for T. So, if $a \wedge b = a' \wedge b'$ where a', b' are also linearly independent, then $c \wedge a' \wedge b' = 0$ when $c = a, b$, and hence a, b lie on the plane spanned by a', b' and so the planes spanned by a, b and a', b' are the same. Finally it is obvious that $P\ell$ is equivariant under $\mathrm{GL}(T)$.

If we choose a basis (e_i) for T and define $e_{ij} = e_i \wedge e_j$, then $(e_{ij})_{i<j}$ is a basis for E, and one can compute for any plane π of T the homogeneous coordinates of $P\ell(\pi)$. Let π be a plane in T with a basis (a, b) where

$$a = \sum_i a_i e_i, \qquad b = \sum_i b_i e_i.$$

Let $y_{ij} = -y_{ji}$ be the minor defined by rows i, j in the 4×2 matrix

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \end{pmatrix}$$

so that

$$a \wedge b = \sum_{i<j} y_{ij} e_i \wedge e_j.$$

The $(y_{ij})_{i<j}$ are by definition the (homogeneous) *Plücker coordinates* of π. These of course depend on the choice of a basis for T.

The image of $\mathbf{G}$ under the Plücker map can now be determined completely. In fact, if $p = a \wedge b$, then $p \wedge p = 0$; conversely, if $p \in E$, say $p = (y_{12}, y_{23}, y_{31}, y_{14}, y_{24}, y_{34})$, and $p \wedge p = 0$, we claim that there is a $\pi \in \mathbf{G}$ such that $[p]$ is the image of π under $P\ell$. The condition $p \wedge p = 0$ becomes

$$y_{12}y_{34} + y_{23}y_{14} + y_{31}y_{24} = 0. \tag{\textbf{K}}$$

To prove our claim, we may assume, by permuting the ordering of the basis vectors e_i of T if necessary, that $y_{12} \neq 0$, and hence that $y_{12} = 1$. Then

$$y_{34} = -y_{31}y_{24} - y_{23}y_{14}$$

so that we can take

$$p = a \wedge b, \qquad a = e_1 - y_{23}e_3 - y_{24}e_4, \quad b = e_2 - y_{31}e_3 + y_{14}e_4$$

which proves the claim.

Actually the quadratic function in ($\mathbf{K}$) depends only on the choice of a volume element on T, i.e., a basis for $\Lambda^4(T)$. Let $0 \neq \mu \in \Lambda^4(T)$. Then

$$p \wedge p = Q_\mu(p) p \wedge p.$$

If $\mu = e_1 \wedge e_2 \wedge e_3 \wedge e_4$ then $Q_\mu(p)$ is given by the left side of the equation ($\mathbf{K}$). The equation

$$Q_\mu(p) = 0 \iff p \wedge p = 0,$$

which is the equation ($\mathbf{K}$) above, in the Plücker coordinates with respect to the basis (e_{ij}), defines a quadric in the projective space $\mathbf{P}(E)$. It is called the *Klein quadric* and is denoted by $\mathbf{K}$. Klein discovered it and used it extensively in the study of the geometry of lines in projective space. The Plücker map is then a bijection of $\mathbf{G}$ with $\mathbf{K}$. The variety $\mathbf{K}$ is nonsingular because the gradient of the function Q never vanishes at any point of $\mathbf{K}$.

By the definition of Q_μ we have, for any $y \in E$,

$$y \wedge y = Q_\mu(y)\mu$$

and so it follows at once that

$$Q_\mu(g \cdot y) = Q_\mu(y) \qquad (g \in \mathrm{SL}(T)).$$

Thus the action of $\mathrm{SL}(T)$ in E maps $\mathrm{SL}(T)$ into the complex orthogonal group $O(E) \simeq O(6)$; it is actually into $\mathrm{SO}(E) \simeq \mathrm{SO}(6)$ because the image has to be connected. It is easy to check that the kernel of this map is ± 1. In fact, if $g(u \wedge v) = u \wedge v$ for all u, v, then g leaves all 2-planes stable, hence all lines stable, and so is a scalar c with $c^4 = 1$; then $u \wedge v = c^2 u \wedge v$ so that $c^2 = 1$. Since both $\mathrm{SL}(T)$ and $\mathrm{SO}(E)$ have dimension 15, we then have the exact sequence

$$1 \longrightarrow (\pm 1) \longrightarrow \mathrm{SL}(T) \longrightarrow \mathrm{SO}(E) \longrightarrow 1.$$

We may therefore view $\mathrm{SL}(T)$as the spin group of $\mathrm{SO}(E)$. Let $\mathbf{4}$ be the defining 4-dimensional representation of $\mathrm{SL}(T)$ (in T) and $\mathbf{4}^*$ its dual representation (in T^*). Then $\mathbf{4}$ and $\mathbf{4}^*$ are the two spin representations of $\mathrm{SO}(E)$. This follows from the fact (see Chapter 5, Lemma 5.6.1) that all other nontrivial representations of $\mathrm{SL}(4)$ have dimension > 4.

Let $(e_i)_{1 \leq i \leq 4}$ be a basis for T. Let π_0 be the plane spanned by e_1, e_2 and π_∞ the plane spanned by e_3, e_4. We say that a plane π is *finite* if its intersection with π_∞ is 0. This is equivalent to saying that the projection $T \longrightarrow \pi_0$ corresponding to the direct sum $T = \pi_0 \oplus \pi_\infty$ is an isomorphism of π with π_0. In this case we have a uniquely determined basis

$$a = e_1 + \alpha e_3 + \gamma e_4, \quad b = e_2 + \beta e_3 + \delta e_4$$

for π, and conversely, any π with such a basis is finite. It is also the same as saying that $y_{12} \neq 0$ as we have seen above. Indeed, if $y_{12} \neq 0$ and $(a_i), (b_i)$ are the coordinate vectors of a basis for π, the 12-minor of the matrix with columns as these two vectors is nonzero and so by right multiplying by the inverse of the 12-submatrix we have a new basis for π of the above form. Let $\mathbf{K}^\times$ be the set of all finite planes. As it is defined by the condition $y_{12} \neq 0$, we see that $\mathbf{K}^\times$ is an open subset of $\mathbf{K}$ which is easily seen to be dense. Thus, the assignment

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \pi(A)$$

where $\pi(A)$ is the plane spanned by a, b above, gives a parametrization of the open dense set of finite planes in the Klein quadric. Since $\mathbf{K}$ also has dimension 4 we see that the Plücker map allows us to view the Klein quadric as the *compactification* of *complex spacetime* with coordinates $\alpha, \beta, \gamma, \delta$, identified with the space $M_2(\mathbf{C})$ of complex 2×2 matrices A. If $g \in \mathrm{GL}(T)$ and π is a finite plane parametrized by the matrix A, then for

$$g = \begin{pmatrix} L & M \\ N & R \end{pmatrix}$$

the plane $\pi' = g \cdot \pi$ has the basis

$$\begin{pmatrix} L + MA \\ N + RA \end{pmatrix}$$

so that π' is parametrized by

$$(N + RA)(L + MA)^{-1}$$

provided it is also finite; the condition for π' to be finite is that $L+MA$ be invertible. This g acts on $\mathbf{K}$ generically as the *fractional linear map*

$$g : A \longmapsto (N + RA)(L + MA)^{-1}.$$

The situation is reminiscent of the action of $\mathrm{SL}(2)$ on the Riemann sphere by fractional linear transformations, except that in the present context the complement of the set of finite planes is not a single point but a variety which is actually a cone, the cone at infinity. It consists of the planes which have nonzero intersection with π_∞. In the interpretation in $\mathbf{CP}^3$ these are lines that meet the line π_∞.

We shall now show that the subgroup P of $\mathrm{SL}(T)$ that leaves $\mathbf{K}^\times$ invariant is precisely the subgroup P_∞ that fixes π_∞. The representation of π_∞ is the matrix

$$\begin{pmatrix} 0 \\ I \end{pmatrix}$$

and so if

$$g\begin{pmatrix} I \\ A \end{pmatrix} = \begin{pmatrix} L + MA \\ N + RA \end{pmatrix},$$

then g fixes π_∞ if and only if $M = 0$. On the other hand, the condition that g leaves $\mathbf{K}^\times$ stable is also the condition that it leaves its complement invariant. Now from elementary projective geometry we know that if ℓ is a line in $\mathbf{CP}^3$ and $[\ell]$ is the set of all lines that meet ℓ, then ℓ is the only line that meets all the lines in $[\ell]$. Hence any g that leaves $[\ell]$ stable must necessarily fix ℓ. Thus g preserves $\mathbf{K}^\times$ if and only if g fixes π_∞, i.e., $M = 0$. We need a variant of this result where only lines which are *real* with respect to some conjugation are involved. We shall prove this variant algebraically. The condition that g preserves $\mathbf{K}^\times$ is that $L+MA$ be invertible for all A. We shall prove that $M = 0$ assuming only that g maps all π with A *Hermitian* into finite planes. Taking $A = 0$ we see that L should be invertible, and then the condition becomes that $I + L^{-1}MA$ should be invertible for all Hermitian A; we must show then that $M = 0$. Replacing M by $X = L^{-1}M$ we must show that if $I + XA$ is invertible for all Hermitian A, then $X = 0$. If $X \neq 0$, take an ON basis (f_i) such that $(Xf_1, f_1) = c \neq 0$. If $A = uP$ where P is the orthogonal projection on the one dimensional span of f_1, then computing determinants in the basis (f_i) we find that

$$\det(I + XA) = 1 + uc = 0 \text{ for } u = -c^{-1}.$$

We have thus proved that $P = P_\infty$ and is the subgroup of all g of the form

$$\begin{pmatrix} L & 0 \\ NL & R \end{pmatrix} =: (N, L, R) \qquad (L, R \text{ invertible}).$$

The action of P on $\mathbf{K}^\times$ is given by

$$A \longmapsto N + RAL^{-1}.$$

Using the correspondence $g \longmapsto (N, L, R)$ we may therefore identify P with the semidirect product

$$P = M_2 \times' H, \quad H = \mathrm{SL}(2\times 2) := \left\{ \begin{pmatrix} L & 0 \\ 0 & R \end{pmatrix} \,\middle|\, L, R \in GL(2), \det(L)\det(R) = 1 \right\}$$

with H and P acting on M_2 respectively by

$$A \longmapsto RAL^{-1}, \qquad A \longmapsto N + RAL^{-1}.$$

The group $\mathrm{SL}(2) \times \mathrm{SL}(2)$ is a subgroup of H and as such its action is just the action $A \longmapsto g_2 A g_1^{-1}$. H itself is the product of this subgroup and the group of dilations consisting of elements (c, c^{-1}) which act by $A \longmapsto c^{-2}A$. We have thus imbedded the complex spacetime inside its compactification $\mathbf{K}$ and the complex Poincaré group (plus the dilations) inside $\mathrm{SL}(T)$ as P, in such a way that the Poincaré action goes over to the action by its image in P.

We shall now show that the action of $\mathrm{SL}(T)$ on $\mathbf{K}$ is conformal. To this end we should first define a conformal metric on $\mathbf{K}$. A conformal metric on a complex or real manifold X is an assignment that associates to each point x of X a set of nonsingular quadratic forms on the tangent space at x, any two of which are nonzero scalar multiples of each other, such that on a neighborhood of each point we can choose a holomorphic (resp. smooth, real analytic) metric whose quadratic forms belong to this assignment; we then say that the metric defines the conformal structure on that neighborhood. The invariance of a conformal metric under an automorphism α of X has an obvious definition, namely that if α takes x to y, the set of metrics at x goes over to the set at y under $d\alpha$; if this is the case, we say that α is *conformal.* We shall now show that on the tangent space at each point π of $\mathbf{K}$ there is a set F_π of metrics uniquely defined by the requirement that they are changed into multiples of themselves under the action of the stabilizer of π and further that any two members of F_π are proportional. Moreover we shall show that on a suitable neighborhood of any π we can choose a metric whose multiples define this structure. This will show that $\pi \mapsto F_\pi$ is the unique conformal metric on $\mathbf{K}$ invariant for the action of $\mathrm{SL}(T)$. To verify the existence of F_π we can, in view of the transitivity of the action of $\mathrm{SL}(T)$, take $\pi = \pi_0$. Then the stabilizer P_{π_0} consists of the matrices

$$\begin{pmatrix} L & M \\ 0 & R \end{pmatrix} \qquad (L, R \text{ invertible}).$$

Now $\pi_0 \in \mathbf{K}^\times \simeq M_2$, where the identification is

$$A \longmapsto \begin{pmatrix} I \\ A \end{pmatrix}$$

with the action

$$A \longmapsto RA(L + MA)^{-1}.$$

We identify the tangent space at π_0 with M_2; then the tangent action of the element of the stabilizer above is then

$$A \longmapsto \left(\frac{d}{dt}\right)_{t=0} tRA(L+tMA)^{-1} = RAL^{-1}.$$

The map

$$q : A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \det(A) = \alpha\delta - \beta\gamma$$

is a nondegenerate quadratic form on M_2 which changes into cq where $c = \det(R)\det(L)^{-1}$ under the above tangent action. Moreover, as the subgroup of the stabilizer defined by $M = 0, R, L \in \mathrm{SL}(2)$ has no characters, any quadratic form that is changed into a multiple of itself by elements of this subgroup will have to be invariant under it, and as the action of this subgroup is already irreducible, such a form has to be a multiple of q. We may then take F_π to be the set of nonzero multiples of q. It is easy to construct a *holomorphic* metric on $\mathbf{K}^\times$ that defines the conformal structure. The *flat* metric

$$\mu = d\alpha d\delta - d\beta d\gamma$$

on $\mathbf{K}^\times$ is invariant under the translations, and, as the translations are already transitive on $\mathbf{K}^\times$, μ *has to define* the conformal structure on $\mathbf{K}^\times$. The form of the metric on $\mathbf{K}^\times$ is not the usual flat one; but if we write

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}$$

then

$$d\alpha d\delta - d\beta d\gamma = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

the right side of which is the usual form of the metric.

We now turn to what happens over the reals where the story gets more interesting. Any conjugation of T defines a real form of T and hence defines a real form of E. The corresponding real form of $\mathbf{K}$ is simply the Klein quadric of the real form of T. For our purposes we need a conjugation of E that does not arise in this manner. We have already seen that real Minkowski space can be identified with the space of 2×2 *Hermitian* matrices in such a way that SL(2) acts through $A \mapsto gAg^*$. So it is appropriate to start with the conjugation

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \begin{pmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\beta} & \overline{\delta} \end{pmatrix}$$

on $\mathbf{K}^\times$. Since the Plücker coordinates of the corresponding plane are

$$(1, -\alpha, -\beta, \delta, -\gamma, \alpha\delta - \beta\gamma)$$

it is immediate that the conjugation θ on E defined by

$$\theta : (y_{12}, y_{23}, y_{31}, y_{14}, y_{24}, y_{34}) \longmapsto (\overline{y_{12}}, \overline{y_{23}}, \overline{y_{24}}, \overline{y_{14}}, \overline{y_{31}}, \overline{y_{34}})$$

determines a conjugation on $\mathbf{P}(E)$ preserving $\mathbf{K}$ that extends the conjugation defined above on $\mathbf{K}^\times$. This is clearly unique and we shall write θ again for it. Let

$$E_{\mathbf{R}} = \{e \in E \mid e^\theta = e\}.$$

Then $E_{\mathbf{R}}$ is a real form of E; $y \in E_{\mathbf{R}}$ if and only if $y_{12}, y_{23}, y_{34}, y_{14}$ are real and $y_{31} = \overline{y_{24}}$. The restriction $Q_{\mathbf{R}}$ of Q to $E_{\mathbf{R}}$ is then real and is given by

$$Q_{\mathbf{R}}(y) = y_{12}y_{34} + y_{23}y_{14} + y_{31}\overline{y_{31}} \qquad (y \in E_{\mathbf{R}})$$

which is real and has signature $(4, 2)$. Let $\mathbf{K}_{\mathbf{R}}$ be the fixed point set of θ on $\mathbf{K}$. Then $\mathbf{K}_{\mathbf{R}}$ is the image of the set of zeros of Q on $E_{\mathbf{R}}$. In fact, let $u \in E$ be such that its image lies in $\mathbf{K}_{\mathbf{R}}$, then $u^\theta = cu$ for some $c \neq 0$; since θ is involutive, we must have $|c| = 1$, and so we can write $c = \overline{d}/d$ for some d with $|d| = 1$. Then for $v = d^{-1}u$ we have $v^\theta = v$. Thus

$$\mathbf{K}_{\mathbf{R}} = \{[y] \mid y \in E_{\mathbf{R}}, Q_{\mathbf{R}}(y) = 0\}.$$

We also note at this time that $\mathrm{SO}(E_{\mathbf{R}})^0$ is transitive on $K_{\mathbf{R}}$. In fact, in suitable real coordinates (u, v) with $u \in \mathbf{R}^4, v \in \mathbf{R}^2$, the equation to $K_{\mathbf{R}}$ is $u{\cdot}u - v{\cdot}v = 0$; given a nonzero point (u, v) on this cone we must have both u and v nonzero and so without changing the corresponding point in projective space we may assume that $u{\cdot}u = v{\cdot}v = 1$. Then we can use $\mathrm{SO}(4, \mathbf{R}) \times \mathrm{SO}(2, \mathbf{R})$ to move (u, v) to $((1, 0, 0, 0), (1, 0))$.

Now θ induces an involution $Q' \longmapsto {Q'}^\theta$ on the space of quadratic forms on E: ${Q'}^\theta(u) = Q(u^\theta)^{\mathrm{conj}}$. Since Q and Q^θ coincide on $E_{\mathbf{R}}$ they must be equal, i.e., $Q = Q^\theta$. Hence $g^\theta := \theta g \theta$ lies in $\mathrm{SO}(Q)$ if and only if $g \in \mathrm{SO}(Q)$. So we have a conjugation $g \mapsto g^\theta$ on $\mathrm{SO}(E)$. It is easy to check that the subgroup of fixed points for this involution is $\mathrm{SO}(E_{\mathbf{R}})$, the subgroup of $\mathrm{SO}(E)$ that leaves $E_{\mathbf{R}}$ invariant. Since $\mathrm{SL}(T)$ is simply connected, θ lifts to a unique conjugation of $\mathrm{SL}(T)$, which we shall also denote by θ. Let

$$G = \mathrm{SL}(T)^\theta.$$

We wish to show the following:

(1) G is connected and is the full preimage of $\mathrm{SO}(E_{\mathbf{R}})^0$ in $\mathrm{SL}(T)$ under the (spin) map $\mathrm{SL}(T) \longrightarrow \mathrm{SO}(E_{\mathbf{R}})^0$.

(2) There is a Hermitian form $\langle\ ,\ \rangle$ of signature $(2,2)$ on T such that G is the subgroup of $\mathrm{SL}(T)$ preserving it, so that $G \simeq \mathrm{SU}(2,2)$.

(3) A plane in T defines a point of $\mathbf{K_R}$ if and only if it is a null plane with respect to $\langle\ ,\ \rangle$ and that G acts transitively on the set of null planes.

The differential of the spin map is the identity on the Lie algebra and so, whether G is connected or not, the image of G^0 under the spin map is all of $\mathrm{SO}(E_{\mathbf{R}})^0$. We shall first prove that G^0 is the full preimage of $\mathrm{SO}(E_{\mathbf{R}})^0$, and for this it is enough to show that $-1 \in G^0$. Consider, for $z \in \mathbf{C}$ with $|z| = 1$,

$$\delta(z) = \text{ diagonal } (z, \overline{z}, z, \overline{z}).$$

Its action on E is by the matrix

$$\gamma(z) = \text{diagonal } (1, 1, z^2, 1, \overline{z}^2, 1).$$

Then $\gamma(z)$ leaves $E_{\mathbf{R}}$ invariant and so lies in $\mathrm{SO}(E_{\mathbf{R}})^0$ for all z. If h is the map $\mathrm{SL}(T) \longrightarrow \mathrm{SO}(E)$ and $h(g)^\theta = h(g)$, then $g^\theta = \pm g$. Hence $\delta(z)^\theta = \pm\delta(z)$ for all z. By continuity we must have the $+$ sign for all z and so $\delta(z) \in G$ for all z, hence $\delta(z) \in G^0$ for all z. But $\delta(1) = 1, \delta(-1) = -1$, proving that $-1 \in G^0$.

Now it is known that any real form of $\mathfrak{sl}(4)$ is conjugate to one of $\mathfrak{sl}(4, \mathbf{R}), \mathfrak{su}(p, q) (0 \le p \le q, p + q = 4)$ and hence any conjugation of $\mathfrak{sl}(4)$ is conjugate to either $X \longmapsto X^{\mathrm{conj}}$ or to $X \longmapsto -FX^*F$ where F is the diagonal matrix with p entries equal to 1 and q entries equal to -1. The corresponding conjugations of $\mathrm{SL}(4)$ are $g \longmapsto g^{\mathrm{conj}}$ and $g \longmapsto Fg^{*-1}F$ respectively. The fixed point groups of conjugations of $\mathrm{SL}(4)$ are thus conjugate to $\mathrm{SL}(4, \mathbf{R})$ and $\mathrm{SU}(p, q)$. But these are all connected[4]. So G is connected. Furthermore if K is a maximal compact subgroup of G, then G goes *onto* a maximal compact subgroup of $\mathrm{SO}(4, 2)$ with kernel $\{\pm\}$ and so, as the dimension of the maximal compacts of $\mathrm{SO}(4, 2)$, which are all conjugate to $\mathrm{SO}(4) \times \mathrm{SO}(2)$, is 7, the dimension of the maximal compacts of G is also 7. But the maximal compacts of $\mathrm{SL}(4, \mathbf{R}), \mathrm{SU}(4), \mathrm{SU}(1, 3), \mathrm{SU}(2, 2)$ are respectively $\mathrm{SO}(4), \mathrm{SU}(4), (\mathrm{U}(1) \times \mathrm{U}(3))_1, (\mathrm{U}(2) \times \mathrm{U}(2))_1$ where the suffix 1 means that the determinant has to be 1, and these are of dimension $6, 15, 9, 7$ respectively. Hence $G \simeq \mathrm{SU}(2, 2)$. However a calculation is needed to determine the Hermitian form left invariant by G and to verify that the planes that are fixed by θ are precisely the null planes for this Hermitian form. It is interesting to notice that the images of the real forms $\mathrm{SL}(4, \mathbf{R}), \mathrm{SU}(4), \mathrm{SU}(1,3), \mathrm{SU}(2,2)$ are respectively $\mathrm{SO}(3,3), \mathrm{SO}(6), \mathrm{SO}^*(6)$, and $\mathrm{SO}(4,2)$. In particular the real form $\mathrm{SO}(5,1)$ is *not* obtained this way.

Let $\mathfrak{g}$ be the Lie algebra of G. If Z is an endomorphism of T its action $\rho(Z)$ on E is given by $e_i \wedge e_j \mapsto Ze_i \wedge e_j + e_i \wedge Ze_j$. If $Z = (z_{ij})_{1 \le i,j \le 4}$, the matrix of $\rho(Z)$ in the basis $e_{12}, e_{23}, e_{14}, e_{34}, e_{31}, e_{24}$ is

$$\begin{pmatrix} z_{11}+z_{22} & -z_{13} & z_{24} & 0 & -z_{23} & -z_{14} \\ -z_{31} & z_{22}+z_{33} & 0 & -z_{24} & -z_{21} & z_{34} \\ z_{42} & 0 & z_{11}+z_{44} & z_{13} & -z_{43} & z_{12} \\ 0 & -z_{42} & z_{31} & z_{33}+z_{44} & z_{41} & z_{32} \\ -z_{32} & -z_{12} & -z_{34} & z_{14} & z_{33}+z_{11} & 0 \\ -z_{41} & z_{43} & z_{21} & z_{23} & 0 & z_{22}+z_{44} \end{pmatrix}.$$

The condition that $Z \in \mathfrak{g}$ is that the action of this matrix commutes with θ. If Θ is the matrix of θ, this is the condition

$$\rho(Z)\Theta = \Theta\overline{\rho(Z)}.$$

Now

$$\Theta = \begin{pmatrix} I_4 & 0 \\ 0 & J_2 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and so, writing

$$\rho(Z) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

we get

$$A = \overline{A}, \qquad DJ_2 = J_2\overline{D}, \qquad BJ_2 = \overline{B}, \qquad J_2\overline{C} = C.$$

Remembering that $\sum z_{jj} = 0$ these reduce to the conditions

$$\begin{aligned} z_{11}+z_{22}, z_{22}+z_{33} &\in \mathbf{R} \\ z_{13}, z_{24}, z_{31}, z_{42} &\in \mathbf{R} \\ z_{22}+z_{44} &\in (-1)^{1/2}\mathbf{R} \end{aligned}$$

and

$$z_{14} = \overline{z_{23}}, \quad z_{34} = -\overline{z_{21}}, \quad z_{12} = -\overline{z_{43}}, \quad z_{32} = \overline{z_{41}}.$$

It is not difficult to check that these are equivalent to saying that Z must be of the form

$$\begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} \qquad B, C \text{ Hermitian}, A \text{ arbitrary}$$

where * denotes adjoints and all letters denote 2×2 matrices. If

$$F = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}$$

then this amounts to

$$FZ^* + ZF = 0.$$

In other words, the Lie algebra $\mathfrak{g}$ of G is the same as the fixed point set of the conjugation

$$Z \longmapsto -FZ^*F$$

and so

$$Z^\theta = -FZ^*F \qquad (Z \in \mathfrak{sl}(T)).$$

This means that

$$g^\theta = Fg^{*-1}F \qquad (g \in \mathrm{SL}(T)).$$

Let us write (,) for the usual positive definite Hermitian form on $\mathbf{C}^4$ and let

$$F(u, v) = (Fu, v) \qquad (u, v \in \mathbf{C}^4).$$

Then F is a Hermitian form and G is the group that leaves it invariant. This Hermitian form has signature $(2, 2)$; indeed, if $T^\pm$ are the 2-dimensional eigenspaces of F for the eigenvalues ± 1, then $T = T^+ \oplus T^-$ and for $u = u^+ + u^-$ with $u^\pm \in T^\pm$, we have

$$F(u, u) = (||u^+||^2 - ||u^-||^2)$$

where $|| \cdot ||$ is the usual norm in $\mathbf{C}^4$. This finishes the proof that $G \simeq \mathrm{SU}(2, 2)$.

The plane π_0 is certainly a null plane for F. As G is transitive on $\mathbf{K_R}$ (because $\mathrm{SO}(E_{\mathbf{R}})$ is transitive), it follows that all the planes that are fixed by θ are null planes for F. There are no other null planes. To see this, we note that $F\pi$ is orthogonal to π and is also a null plane. Define $f_3 = -iFf_1, f_4 = -iFf_2$. Then a simple calculation shows that (f_i) is an ON basis for T such that f_1, f_2 span π and $(f_1, f_3) = (f_2, f_4) = i$ while all other scalar products between the f_j vanish. If $g \in \mathrm{U}(4)$ takes the e_i to f_i, we have $g{\cdot}\pi_0 = \pi$. But then $g \in \mathrm{U}(2, 2)$ also, and if its determinant is not 1, we change g to $h = cg$ for some scalar so that $\det(h) = 1$; then $h \in \mathrm{SU}(2, 2)$ and h takes π_0 to π. Thus π is fixed by θ. All of our claims are thus proved.

Recall that $\mathbf{K_R}$ is the fixed point set of the Klein quadric $\mathbf{K}$ with respect to the conjugation θ. Then real Minkowski space (corresponding to Hermitian A in M_2) is imbedded as a dense open set $\mathbf{K}_{\mathbf{R}}^{\times}$ of $\mathbf{K_R}$. If

$$g = \begin{pmatrix} L & M \\ N & R \end{pmatrix} \in \mathrm{SL}(T)$$

is to leave $\mathbf{K}_{\mathbf{R}}^{\times}$ invariant the condition is that $L + MA$ is invertible for all Hermitian A, and we have seen that this is equivalent to requiring that $M = 0$ and L be invertible. If moreover $g \in G$ the conditions reduce to saying that the subgroup $P_{\mathbf{R}}$ leaving $\mathbf{K}_{\mathbf{R}}^{\times}$ invariant is the set of all matrices of the form

$$\begin{pmatrix} L & 0 \\ NL & L^{*-1} \end{pmatrix} \qquad (N \text{ Hermitian}, \det(L) \in \mathbf{R})$$

with the action on $\mathbf{K}_{\mathbf{R}}^{\times}$ given by

$$A \longmapsto N + L^{*-1} A L^{-1}.$$

This is just the real Poincaré action. The conformal metric on $\mathbf{K}$ becomes $(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$ which is real and of signature $(1, 3)$ on $\mathbf{K}_{\mathbf{R}}^{\times}$. Since $\det(A)$ goes over to $\det(A)\det(L)^{-2}$ it follows that we have a real conformal structure which is invariant under G. The group G thus acts conformally and transitively on $\mathbf{K}_{\mathbf{R}}$, and the subgroup that leaves real Minkowski invariant the inhomogeneous Lorentz group (plus the dilations). The Lie algebra of $\mathrm{SU}(2, 2)$ is isomorphic to $\mathfrak{so}(2, 4)$ which is thus viewed as the conformal extension of the Poincaré Lie algebra.

Conformality in higher dimensions. The above considerations can be generalized considerably. In fact it is true that $\mathbf{R}^{m,n}$, affine Minkowski space of signature (m, n), can be imbedded as a dense open subset of a compact manifold which has a conformal structure and on which the group $\mathrm{SO}(m + 1, n + 1)$ acts transitively and conformally, and further that the Poincaré group (= the inhomogeneous group $\mathrm{ISO}(m, n)^0 = \mathbf{R}^{m,n} \times' \mathrm{SO}(m, n)^0$) can be imbedded inside $\mathrm{SO}(m+1, n+1)$ in such a fashion that the action of $\mathrm{ISO}(m, n)^0$ goes over to the action of $\mathrm{SO}(m+1, n+1)^0$. For $m = 1, n = 3$ we obtain the imbedding of the usual Poincaré group inside $\mathrm{SO}(2, 4)^0$ treated above. Throughout we assume that $0 \le m \le n$ and $n \ge 1$.

We start with the imbedding of $\mathrm{ISO}(m, n)^0$ in $\mathrm{SO}(m + 1, n + 1)^0$. Let V be a real vector space of dimension $m + n + 2$ with a nondegenerate symmetric bilinear form $(\ ,\ )$ of signature $(m + 1, n + 1)$ given on it. Let Ω be the "light cone"of V, namely the variety of nonzero elements p of V such that $(p, p) = 0$;

$$\Omega = \{p \in V \mid p \ne 0, (p, p) = 0\}.$$

We write $H = \mathrm{SO}(V)^0$. In a linear basis for V in which the quadratic form of V becomes

$$Q = x_0^2 + \ldots + x_m^2 - y_0^2 - \ldots - y_n^2$$

the equation defining Ω is a homogeneous quadratic polynomial and so defines a quadric cone in the projective space $\mathbf{P}(V) \simeq \mathbf{RP}^{m+n+1}$ of V, stable under the action of $\mathrm{SO}(V)$. We write $[\Omega]$ for this cone and in general $[p]$ for the image in projective space of any nonzero $p \in V$. Since the gradient of Q is never zero at any point of Ω it follows that Ω and $[\Omega]$ are both smooth manifolds and that the map $\Omega \longrightarrow [\Omega]$ is submersive everywhere. Clearly $\dim([\Omega]) = m + n$. The action of H on Ω gives rise to an action of H on $[\Omega]$. Let $p \in \Omega$ and let Ω_p (resp. $[\Omega]_p$) be the tangent space to Ω (resp. $[\Omega]$) at p (resp. $[p]$). Finally, let H_p be the stabilizer of p in H. In what follows we shall fix p and choose $q \in \Omega$ such that $(q, p) = 1$. This is always possible and we write W_p for the orthogonal complement of the span of the linear span of q, p. It is easy to see that $V = \mathbf{R}p \oplus \mathbf{R}q \oplus W_p$.

Notice first that the tangent map $\Omega_p \longrightarrow [\Omega]_p$ has kernel $\mathbf{R}p$ and so $(\ ,\ )$ induces a quadratic form on $[\Omega]_p$. It is immediate from the above decomposition of V that $W_p \simeq [\Omega]_p$ and so $[\Omega]_p$ has signature (m, n) with respect to this quadratic form. If p' is any other point of Ω above $[p]$, then $p' = \lambda p (\lambda \neq 0)$ and the quadratic form induced on $[\Omega]_p$ gets multiplied by λ^2 if we use p' in place of p. Moreover if we change p to $h{\cdot}p$ for some $h \in H$ the quadratic forms at $h{\cdot}[p]$ are the ones induced from the quadratic forms at $[p]$ by the tangent map of the action of h. It follows that we have a *conformal structure* on $[\Omega]$ and that the action of H is conformal.

We shall first verify that H^0 acts transitively on $[\Omega]$. We use coordinates and write the equation of Ω as

$$x^2 = y^2 \qquad (x^2 = x_0^2 + \ldots + x_m^2, y^2 = y_0^2 + \ldots + y_n^2).$$

Clearly $\mathbf{x} := (x_0, \ldots, x_m), \mathbf{y} := (y_0, \ldots, y_n)$ are both nonzero for any point of Ω. So without changing the image in projective space we nay assume that $x^2 = y^2 = 1$. Then we can use the actions of $\mathrm{SO}(m+1, \mathbf{R})$ and $\mathrm{SO}(n+1, \mathbf{R})$ to assume that $\mathbf{x} = (1, 0, \ldots, 0), \mathbf{y} = (1, 0, \ldots, 0)$; in case $m = 0$ we have to take $\mathbf{x}$ as $(\pm 1, 0, \ldots, 0)$. So the transitivity is proved when $m > 0$. If $m = 0$ we change $\mathbf{y}$ to $(\pm 1, 0, \ldots, 0)$ so that in all cases any point of $[\Omega]$ can be moved to the image of $(1, 0 \ldots, 0, 1, 0, \ldots, 0)$. This proves transitivity and hence also the connectedness of $[\Omega]$.

We shall now show that $H_p^0 \simeq \mathrm{ISO}(m, n)^0$ giving us an imbedding of the latter in $\mathrm{SO}(m+1, n+1)^0$. We proceed as in Chapter 1, Section 5. Let $h \in H_p$ and write V_p be the tangent space to Ω at p so that V_p is the orthogonal complement to p. Then h leaves V_p stable and so we have a flag $\mathbf{R}p \subset V_p \subset V$ stable under h. We claim that $h{\cdot}q - q \in V_p$. Certainly $h{\cdot}q = bq + v$ for some $v \in V_p$. But then $1 = (q, p) = (h{\cdot}q, p) = b$ showing that $b = 1$. It is then immediate that $h{\cdot}r - r \in V_p$ for any r. We write $t(h)$ for the image of $h{\cdot}q - q$ in $W_p' := V_p/\mathbf{R}p \simeq W_p$. On the other hand h induces an action on $V_p/\mathbf{R}p$ which preserves the induced quadratic form there and so we have a map $H_p^0 \longrightarrow \mathrm{SO}(W_p')$ which must go into $\mathrm{SO}(W_p')^0$. So we have the image $r(h) \in \mathrm{SO}(W_p')^0$ of h. We thus have a map

$$J : h \longmapsto (t(h), r(h)) \in \mathrm{ISO}(W_p')^0.$$

We claim that J is an isomorphism. First of all J is a homomorphism. Indeed, let $h, h' \in H_p^0$. Then $h' \cdot q = q + t(h') + c(h')p$ where $t(h') \in W_p$ so that $hh' \cdot p - q \equiv t(h) + r(h) \cdot t(h') \bmod \mathbf{R}p$, showing that $J(hh') = J(h)J(h')$. It is obvious that J is a morphism of Lie groups. Now the dimension of the Lie algebra of H_p, which is the dimension subspace $\{L \in \mathrm{Lie}(H) \mid Lp = 0\}$, is easily computed to be $\dim \mathrm{ISO}(m,n)$; for this one can go to the complexes and work in the complex orthogonal groups and take p to be the point $(1, i, 0, \ldots, 0)$, and then it is a trivial computation. Hence if we prove that J is injective we can conclude that it is an isomorphism. Suppose then that $J(h)$ is the identity. This means that $h \cdot q = q + a(h)p$, and further, that for any $v \in V_p$, $h \cdot v = v + b(h)p$. Taking the scalar product of the first relation with $h \cdot q$ we find that $0 = (q,q) = (h \cdot q, h \cdot q) = 2a(h)$, giving $h \cdot q = q$. Taking the scalar product of the second relation with $h \cdot q$ we find $(v,q) = (h \cdot v, h \cdot q) = (v + b(h)p, q) = (v,q) + b(h)$, giving $b(h) = 0$. So $h \cdot v = v$, proving that $h = 1$. We thus have

$$\mathrm{ISO}(m,n)^0 \simeq H_p^0 \hookrightarrow \mathrm{SO}(m+1, n+1)^0.$$

For $h \in H_p^0$ we also write $t(h)$ and $r(h)$ for the elements of $\mathrm{ISO}(W_p')$ that are respectively the translation by $t(h)$ and action by $r(h)$ in W_p'.

The tangent space V_p of Ω at p intersects Ω in a cone; we write C_p for it and $C_{[p]}$ for its image in $[\Omega]$. Clearly H_p fixes $C_{[p]}$. Let $A_{[p]} = [\Omega] \setminus C_{[p]}$. Then $A_{[p]}$ is an open dense subset of $[\Omega]$, stable under H_p^0; the density is an easy verification. We wish to show that there is an isomorphism of $A_{[p]}$ with W_p' in such a manner that the action of H_p^0 goes over to the action of $\mathrm{ISO}(W_p')$.

Let T and M be the preimages under J in H_p^0 of the translation and linear subgroups of $\mathrm{ISO}(W_p')$. Now $[q] \in A_{[p]}$ and we shall first prove that for any $[r] \in A_{[p]}$ there is a unique $h \in T$ such that the translation $t(h)$ takes $[q]$ to $[r]$. Since $[r] \in A_{[p]}$, we have $(r,p) \neq 0$ and so we may assume that $(r,p) = 1$. Hence $t' = r - q \in V_p$ and hence defines an element $t \in W_p'$. There is then a unique $h \in T$ such that $t(h) = t$, i.e., $J(h)$ is translation by t. We claim that $J(h)$ takes $[q]$ to $[r]$. By definition of h we have $h \cdot q - q$ has t as its image in W_p'. Then $r - q$ and $h \cdot q - q$ have the same image in W_p' and so $h \cdot q - r \in V_p$ and has image 0 in W_p'. So $h \cdot q = r + bp$. But then $0 = (q,q) = (h \cdot q, h \cdot q) = (r,r) + 2b(r,p) = 2b$ showing that $b = 0$. In other words, the translation group T acts simply transitively on $A_{[p]}$. We thus have a bijection $h \cdot [q] \longrightarrow t(h)$ from $A_{[p]}$ to W_p'. It is an easy check that the action of H_p^0 on $A_{[p]}$ goes over to the action of $\mathrm{ISO}(W_p')$.

The metric on W_p' induced from V has signature (m,n) as we saw earlier. We can regard it as a flat metric on W_p' and so transport it to $A_{[p]}$ and it becomes invariant under the action of H_p^0. Clearly it belongs to the conformal structure on $[\Omega]$. So all of our assertions are proved.

3.4. The superconformal algebra of Wess and Zumino. In 1974 Wess and Zumino[2] constructed a real super Lie algebra whose even part *contains* the *conformal extension* $\mathfrak{so}(4,2) \simeq \mathfrak{su}(2,2)$ of the Poincaré Lie algebra considered above. The (complexified) Wess-Zumino algebra was the first example constructed of a *simple* super Lie algebra.

A word of explanation is in order here about the word *contains* above. Ideally one would like to require that the super extension have the property that its even part is *exactly* $\mathfrak{so}(4,2)$. This turns out to be impossible and for a super extension of minimal dimension (over $\mathbf{C}$) the even part of the super extension becomes $\mathfrak{so}(4,2) \oplus \mathbf{R}$ where the action of the elements of $\mathbf{R}$ on the odd part of the super extension generates a *rotation group* ("compact R-symmetry").

Let us operate first over $\mathbf{C}$. The problem is then the construction of super Lie algebras whose even parts are isomorphic to $\mathfrak{sl}(4)$, at least up to a central direct factor. We have already come across the series $\mathfrak{sl}(p|q)$ of super Lie algebras. The even part of $\mathfrak{g} = \mathfrak{sl}(p|q)$ consists of complex matrices

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$$

where X, Y are $p \times p, q \times q$ and

$$\mathrm{tr}(X) = \mathrm{tr}(Y).$$

Thus the even part is isomorphic to $\mathfrak{sl}(p) \oplus \mathfrak{sl}(q) \oplus \mathbf{C}$. In particular, the even part of $\mathfrak{sl}(4,1)$ is $\mathfrak{sl}(4) \oplus \mathbf{C}$. The elements of the odd part of $\mathfrak{sl}(4|1)$ are matrices of the form

$$\begin{pmatrix} 0 & a \\ b^t & 0 \end{pmatrix} \qquad (a, b \text{ column vectors in } \in \mathbf{C}^4)$$

so that the odd part is the module $\mathbf{4} \oplus \mathbf{4}^*$. Now $[\mathfrak{g}_1, \mathfrak{g}_1]$ is stable under the adjoint action of $\mathfrak{sl}(4)$ and has nonzero intersection with both $\mathfrak{sl}(4)$ and the one-dimensional center of $\mathfrak{g}_0$. It is then immediate that $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_0$. At this time it is noteworthy that the even part is not precisely $\mathfrak{sl}(4)$ but has a one dimensional central component with basis element R given by

$$R = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{3} \end{pmatrix}.$$

We note the following formulae. If we write

$$(X,x)=\begin{pmatrix}X & 0\\ 0 & x\end{pmatrix},\ (\mathrm{tr}(X)=x)\qquad (a,b)=\begin{pmatrix}0 & a\\ b^T & 0\end{pmatrix}$$

then

$$[(X,x),(a,b)]=((X-x)a,-(X-x)^Tb),\ [(a,b),(a',b')]=(ab'^T+a'b^T,b^Ta'+b'^Ta).$$

In particular, the projections $X(a,b),R(a,b)$ of $[(a,b),(a,b)]$ on $\mathfrak{sl}(4),\mathbf{C}R$ respectively are given by

$$X(a,b)=2ab^T-(1/2)(b^Ta)I,\qquad R(a,b)=(3/2)(b^Ta)R;$$

note that $X(a,b)$ has trace 0 as it should have. R acts nontrivially on the odd part; indeed, ad R is -1 on $\mathbf{4}$ and $+1$ on $\mathbf{4}^*$, as is seen easily from the above formulae.

We shall first show that there does not exist a super Lie algebra $\mathfrak{h}$ with the properties: (1) $\mathfrak{h}_0=\mathfrak{sl}(4)$ and $\mathfrak{h}_1$ spinorial, and (2) $[\mathfrak{h}_1,\mathfrak{h}_1]$ has a nonzero intersection with $\mathfrak{h}_0$ (then we must have that this commutator is all of $\mathfrak{h}_0$). The spinorial condition means that $\mathfrak{h}_1$ should be the sum of copies of $\mathbf{4}$ and $\mathbf{4}^*$. It is not possible that $\mathfrak{h}_1$ contains only copies of $\mathbf{4}$. To see this, note that $\mathbf{4}\otimes\mathbf{4}$ cannot contain the trivial representation as $\mathbf{4}$ and $\mathbf{4}^*$ are not equivalent, and so, as its dimension is 16, it cannot contain the adjoint representation either which has dimension 15. We thus see that both $\mathbf{4}$ and $\mathbf{4}^*$ must occur in the odd part. So, for a particular choice of subspaces of type $\mathbf{4}$ and $\mathbf{4}^*$, the space $\mathfrak{h}_1=\mathfrak{sl}(4)\oplus\mathbf{4}\oplus\mathbf{4}^*$ is a super Lie algebra with the same properties as $\mathfrak{h}$. Since the even part is exactly $\mathfrak{sl}(4)$ we must have a $\mathfrak{sl}(4)$-map from $\mathbf{4}\otimes\mathbf{4}^*$ into $\mathfrak{sl}(4)$ satisfying the cubic condition for super Lie algebras. We claim that this is impossible. To see this notice that such a map is projectively unique and so has to be a multiple of the map obtained from the map $[\ ,\ ]$ of the super Lie algebra $\mathfrak{sl}(4|1)$ by following it with the projection on the $\mathfrak{sl}(4)$ factor. From the formula above we find that

$$[(a,b),X(a,b)]=(-(3/2)(b^Ta)a,+(3/2)(b^Ta)b)$$

which is obviously not identically zero. So there is no super Lie algebra with properties (1) and (2). The dimension of any super Lie algebra with properties (1) and (2) above with the modification in (1) that the even part *contains* $\mathfrak{sl}(4)$ must then be at least 24; if it is to be 24 then the even part has to be the direct sum of $\mathfrak{sl}(4)$ and an one-dimensional central factor. We shall now show that up to isomorphism, $\mathfrak{g}$ is the only super Lie algebra in dimension 24 of the type we want. Let $[\ ,\ ]'$ be another super bracket structure on $\mathfrak{g}$ such that $[\ ,\ ]'$ coincides with $[\ ,\ ]$ on $\mathfrak{sl}(4)\times\mathfrak{g}$. Then $\mathrm{ad}'(R)$ will act like a scalar $-\alpha$ on the $\mathbf{4}$ part and a scalar β on the $\mathbf{4}^*$ part. Moreover, if Z_{00}, Z_{01} denote projections of an element $Z\in\mathfrak{g}_0$ into $\mathfrak{sl}(4)$ and $\mathbf{C}R$ respectively, then there must be nonzero constants γ,δ such that

$$[Y_1,Y_2]' = \gamma[Y_1,Y_2]_{00} + \delta[Y_1,Y_2]_{01} \qquad (Y_1,Y_2\in\mathfrak{g}_1).$$

Thus

$$[(a,b),(a,b)]' = \gamma X(a,b) + \delta R(a,b).$$

The cubic condition then gives the relations

$$\alpha=\beta, \gamma = \alpha\delta.$$

Thus there are nonzero α,δ such that

$$\begin{aligned}[R,Y]' &= \alpha[R,Y]\\ [Y_1,Y_2]' &= \alpha\delta[Y_1,Y_2]_{00} + \delta[Y_1,Y_2]_{01}\end{aligned}.$$

If τ is the linear automorphism of $\mathfrak{g}$ such that

$$\tau(Z) = Z (Z\in\mathfrak{sl}(4)), \quad \tau(R)=\alpha R, \quad \tau(Y) = (\alpha\delta)^{1/2}Y (Y\in\mathfrak{g}_1),$$

then

$$\tau([X_1,X_2]') = [\tau(X_1),\tau(X_2)] \qquad (X_1,X_2\in\mathfrak{g}).$$

We thus have

Theorem 3.4.1. *There is no super Lie algebra whose even part is* $\mathfrak{sl}(4)$ *and is spanned by the commutators of odd elements. Moreover* $\mathfrak{sl}(4|1)$ *is the unique* (*up to isomorphism*) *super Lie algebra of minimum dimension such that* $\mathfrak{sl}(4)$ *is contained in the even part and is spanned by the commutators of odd elements.*

The real form. We now examine the real forms of $\mathfrak{g}$. We are only interested in those real forms whose even parts have their semisimple components isomorphic to $\mathfrak{su}(2,2)$. We shall show that up to an automorphism of $\mathfrak{g}$ there are only two such and that the central factors of their even parts act on the odd part with respective eigenvalues $\mp i, \pm i$ on the $\mathbf{4}$ and $\mathbf{4}^*$ components. The two have the same underlying super vector space and the only difference in their bracket structures is that the commutator of two odd elements in one is the negative of the corresponding commutator in the other. They are however not isomorphic over $\mathbf{R}$. One may call them *isomers.*

The unitary super Lie algebras. We begin with a description of the general unitary series of super Lie algebras. Let V be a complex super vector space. A *super hermitian form* is a morphism

$$f : V \otimes_{\mathbf{R}} V \longrightarrow \mathbf{C}$$

of super vector spaces over $\mathbf{R}$ which is linear in the first variable and conjugate-linear in the second variable such that

$$f \circ c_{V,V} = f^{\mathrm{conj}}.$$

This means that the complex-valued map $(u, v) \longmapsto f(u, v)$ is linear in u, conjugate-linear in v, has the symmetry property

$$f(v, u) = (-1)^{p(u)p(v)}\overline{f(u, v)},$$

and the consistency property

$$f(u, v) = 0 \qquad (u, v \text{ are of opposite parity}).$$

Thus f (resp. if) is an ordinary Hermitian form on $V_0 \times V_0$ (resp. $V_1 \times V_1$). Suppose that f is a nondegenerate super Hermitian form, i.e., its restrictions to the even and odd parts are nondegenerate. We define the super Lie algebra $\mathfrak{su}(V; f)$ to be the super vector space spanned by the set of all homogeneous $Z \in \mathfrak{sl}(V)$ such that

$$f(Zu, v) = -(-1)^{p(Z)p(u)} f(u, Zv).$$

It is not difficult to check that the above formula defines a real super Lie algebra. Let $V = \mathbf{C}^{p|q}$ with $V_0 = \mathbf{C}^p, V_1 = \mathbf{C}^q$ and let $f_\pm$ be given by

$$f_\pm((u_0, u_1), (v_0, v_1)) = (Fu_0, v_0) + \pm i(Gu_1, v_1)$$

with

$$F = \begin{pmatrix} I_r & 0 \\ 0 & -I_{p-r} \end{pmatrix}, \qquad G = \begin{pmatrix} I_s & 0 \\ 0 & -I_{q-s} \end{pmatrix}.$$

Here I_t is the unit $t \times t$ matrix. We denote the corresponding super Lie algebra by $\mathfrak{su}(r, p-r|s, q-s)$. To see that this is a real form of $\mathfrak{sl}(p|q)$ we shall construct a conjugation of $\mathfrak{sl}(p|q)$ whose fixed points form $\mathfrak{su}(r, p-r|s, q-s)$. Let

$$\sigma_\pm : \begin{pmatrix} X & A \\ B^T & Y \end{pmatrix} \longmapsto \begin{pmatrix} -F\overline{X}^T F & \pm iF\overline{B}G \\ \pm i(F\overline{A}G)^T & -G\overline{Y}^T G \end{pmatrix}.$$

It is a simple calculation to verify that $\sigma_\pm$ are conjugate-linear and preserve the super bracket. Hence they are conjugations of the super Lie algebra structure and their fixed points constitute real super Lie algebras. We denote them by $\mathfrak{su}(r, p-r|s, q-s)_\pm = \mathfrak{su}(\mathbf{C}^{p|q} : f_\pm)$ where $f_\pm$ is defined as above. In particular

$$\mathfrak{su}(r, p-r|s, q-s)_\pm = \left\{ \begin{pmatrix} X & A \\ B^T & Y \end{pmatrix} \middle| X = -F\overline{X}^T X, Y = -G\overline{Y}^T G, A = \pm iF\overline{B}G \right\}.$$

Notice that we can take B to be a completely arbitrary *complex* matrix of order $q \times p$, and then A is determined by the above equations. In particular, changing B to iB we see that the underlying super vector spaces of the two real forms are the same and that they are in fact isomers in the sense we defined earlier. It is not difficult to show that they are *not* isomorphic over $\mathbf{R}$.

We return to the case of $\mathfrak{g} = \mathfrak{sl}(4|1)$. In this case let

$$\mathfrak{g}_{\mathbf{R},\pm} = \mathfrak{su}(2,2|1,0)_\pm.$$

This is the precise definition of the super Lie algebras discovered by Wess and Zumino. They are the real forms defined by the conjugations

$$\sigma_\pm : \begin{pmatrix} X & a \\ b^T & x \end{pmatrix} \longmapsto \begin{pmatrix} -F\overline{X}^T F & \pm iF\overline{b} \\ \pm i(F\overline{a})^T & -\overline{x} \end{pmatrix}, \qquad F = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

We shall now show that up to an automorphism of $\mathfrak{g}$ these are the only real forms whose even parts have their simple components $\simeq \mathfrak{su}(2,2)$.

In the first place suppose $\mathfrak{h}$ is such a real form. Write V for the super vector space with $V_0 = \mathbf{C}^4, V_1 = \mathbf{C}$, with the standard unitary structure. The restriction of $\mathfrak{h}_0$ to V_0 is the Lie algebra of elements X such that $X = -H\overline{X}^T H$ where H is a Hermitian matrix of signature $(2,2)$. By a unitary isomorphism of V_0 we can change H to F where

$$F = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

Let τ be the conjugation of $\mathfrak{sl}(4|1)$ that defines $\mathfrak{h}$. Then τ and σ coincide on $\mathfrak{su}(2,2)$.

Now, on $\mathfrak{g}_1$, we have $\mathrm{ad}(X) = \lambda \mathrm{ad}(X)\lambda^{-1}$ for $\lambda = \tau, \sigma$ and $X \in \mathfrak{su}(2,2)$. So $\tau = \rho\sigma$ where ρ is a linear automorphism of $\mathfrak{g}_1$ that commutes with the action of $\mathfrak{su}(2,2)$ on $\mathfrak{g}_1$. Thus ρ must be of the form

$$(a, b) \longmapsto (k_1 a, k_2 b)$$

for nonzero scalars k_1, k_2. In other words τ is given on $\mathfrak{g}_1$ by

$$(a, b) \longmapsto (k_1 F\overline{b}, k_2 F\overline{a}) \qquad (k_1\overline{k_2} = 1);$$

the condition on the k's is a consequence of the fact that τ is an involution. The condition

$$[(a, b)^\tau, (a', b')^\tau] = [(a, b), (a', b')]^\tau$$

plus the fact that the commutators span $\mathfrak{g}_0$ shows that on $\mathfrak{g}_0$ one must have

$$\tau : \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix} \longmapsto \begin{pmatrix} k_1 k_2 F\overline{X}^T F & 0 \\ 0 & k_1 k_2 \overline{x} \end{pmatrix}.$$

Taking $x = 0$ we find that $k_1 k_2 = -1$ since $\tau = \sigma$ on $\mathfrak{su}(2, 2)$. Thus $k_1 = ir, k_2 = ir^{-1}$ where r is a nonzero real number, and

$$\tau : \begin{pmatrix} X & a \\ b^T & x \end{pmatrix} \longmapsto \begin{pmatrix} -F\overline{X}^T F & irF\overline{b} \\ ir^{-1}F\overline{a} & -\overline{x} \end{pmatrix} \qquad (0 \neq r \in \mathbf{R}).$$

Let θ be the linear automorphism of $V = \mathbf{C}^{4|1}$ which is $|r|^{1/2}I$ on V_0 and I on V_1. We write θ also for the corresponding automorphism of $\mathfrak{g}$. It is a simple calculation that

$$\tau = \theta \sigma_{\mathrm{sgn}(r)} \theta^{-1}.$$

Thus all real forms of $\mathfrak{g}$ of the type we are interested are conjugate to $\mathfrak{g}_{\mathbf{R},\pm}$ by an automorphism of $\mathfrak{g}$ coming from an automorphism of $\mathbf{C}^{4|1}$.

Theorem 3.4.2. *Any real form of* $\mathfrak{sl}(4|1)$ *whose even part has simple component* $\simeq \mathfrak{su}(2, 2)$*, is conjugate to one of* $\mathfrak{su}(2, 2|1, 0)_\pm$.

3.5. Modules over a supercommutative super algebra. In the theory of manifolds, what happens at one point is entirely linear algebra, mostly of the tangent space and the space of tensors and spinors at that point. However if one wants an algebraic framework for what happens on even a small open set one needs the theory of modules over the smooth functions on that open set. For example, the space of vector fields and exterior differential forms on an open set are modules over the algebra of smooth functions on that open set. The situation is the same in the theory of supermanifolds also. We shall therefore discuss some basic aspects of the theory of modules over supercommutative algebras.

Let A be a supercommutative super algebra over the field k (of characteristic 0 as always). Modules are vector spaces over k on which A acts from the left; the action

$$a \otimes m \longmapsto a \cdot m \qquad (a \in A, m \in M)$$

is assumed to be a morphism of super vector spaces, so that

$$p(a \cdot m) = p(a) + p(m).$$

We often write am instead of $a \cdot m$. As in the classical theory left modules may be viewed as right modules and vice versa, but in the super case this involves sign factors; thus M is viewed as a right module for A under the action

$$m \cdot a = (-1)^{p(a)p(m)} a \cdot m \qquad (a \in A, m \in M).$$

A morphism $M \longrightarrow N$ of A-modules is an even k–linear map T such that $T(am) = aT(m)$. For modules M, N one has $M \otimes N$ defined in the usual manner by dividing $M \otimes_k N$ by the k-linear subspace spanned by the relations

$$ma \otimes n = m \otimes an.$$

The internal Hom $\mathbf{Hom}(M, N)$ is defined to be the space of k-linear maps T from M to N such that $T(am) = (-1)^{p(T)p(a)} aT(m)$. It is easily checked that $\mathbf{Hom}(M, N)$ is the space of k-linear maps T from M to N such that $T(ma) = T(m)a$. Thus

$$\begin{aligned} T \in (\mathbf{Hom}(M, N))_0 &\iff T(am) = aT(m) \\ T \in (\mathbf{Hom}(M, N))_1 &\iff T(am) = (-1)^{p(a)} aT(m). \end{aligned}$$

$\mathbf{Hom}(M, N)$ is again a A-module if we define

$$(aT)(m) = aT(m).$$

If we take $N = A$ we obtain the dual module to M, namely M',

$$M' = \mathbf{Hom}(M, A).$$

In all of these definitions it is noteworthy how the rule of signs is used in carrying over to the super case the familiar concepts of the commutative theory.

A *free module* is an A-module which has a free *homogeneous* basis. If $(e_i)_{1 \leq i \leq p+q}$ is a basis with e_i even or odd according as $i \leq p$ or $p + 1 \leq i \leq p + q$, we denote it by $A^{p|q}$, and define its *rank* as $p|q$. Thus

$$A^{p|q} = Ae_1 \oplus \ldots \oplus Ae_{p+q} \qquad (e_i \text{ even or odd as } i \leq \text{ or } > p).$$

To see that p, q are uniquely determined, we use the argument of "taking all the odd variables to 0". More precisely, let

$$J = \text{ the ideal in } A \text{ generated by } A_1.$$

Then

$$J = A_1 \oplus A_1^2, \qquad A_1^2 \subset A_0, \qquad A/J \simeq A_0/A_1^2.$$

All elements of J are nilpotent and so $1 \notin J$, i.e., J is a proper ideal. Now A/J is a commutative ring with unit and so we can find a field F and a homomorphism of A/J into F. Then $A^{p|q} \otimes_F F$ is $F^{p|q}$, a super vector space of dimension $p|q$. Hence p and q are uniquely determined.

Morphisms between different $A^{p|q}$ can as usual be described through matrices, but a little more care than in the commutative case is necessary. We write elements of $A^{p|q}$ as $m = \sum_i e_i x^i$ so that

$$m \longleftrightarrow \begin{pmatrix} x^1 \\ \vdots \\ x^{p+q} \end{pmatrix}.$$

This means that m is even (resp. odd) if and only if the x^i are even (resp. odd) for $i \leq p$ and odd (resp. even) for $i > p$, while for m to be odd, the conditions are reversed. If

$$T : A^{p|q} \longrightarrow A^{r|s}, \qquad T \in \mathbf{Hom}(M, N)$$

then

$$Te_j = \sum_j e_i t^i_j$$

so that T may be identified with the matrix (t^i_j); composition then corresponds to matrix multiplication. The matrix for T is then of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and T is even or odd according as the matrix is of the form

$$\begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{pmatrix} \text{ or } \begin{pmatrix} \text{odd} & \text{even} \\ \text{even} & \text{odd} \end{pmatrix}$$

where "even" etc refer to matrices whose elements are all even etc. If $A = k$, there are no odd elements of A and so we recover the description given earlier. In the general case $\mathbf{Hom}(A^{p|q}, A^{p|q})$ is a super algebra and the associated super Lie algebra is denoted by $\mathfrak{gl}_A(p|q)$. Because there are in general nonzero odd elements in A the definition of the supertrace has to be slightly modified. We put

$$\mathrm{str}(T) = \mathrm{tr}(A) - (-1)^{p(T)}\mathrm{tr}(D), \qquad T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

It can be checked that

$$\mathrm{str}(TU) = (-1)^{p(T)p(U)}\mathrm{str}(UT).$$

Let M be a free module over A of rank $p|q$ with basis e_i where the e_i are even for $i \leq p$ and odd for $i > p$. Let $M' = \mathbf{Hom}(M, A)$. Let $e'_i \in M'$ be defined by

$$e'_i(e_j) = \delta^i_j.$$

Then

$$p(e'_i) = 0 \ (1 \leq i \leq p), \qquad p(e'_i) = 1 \ (p+1 \leq i \leq p+q)$$

and (e'_i) is a free homogeneous basis for M' so that $M' \simeq A^{p|q}$ also. For $m' \in M', m \in M$ we write

$$m'(m) = \langle m', m \rangle.$$

If $T \in \mathbf{Hom}(M, N)$ we define $T' \in \mathbf{Hom}(N', M')$ by

$$\langle T'n', m \rangle = (-1)^{p(T)p(n')}\langle n', Tm \rangle.$$

If

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

then

$$T' = \begin{pmatrix} A^t & C^t \\ -B^t & D^t \end{pmatrix} \qquad (T \text{ even})$$

$$T' = \begin{pmatrix} A^t & -C^t \\ B^t & D^t \end{pmatrix} \qquad (T \text{ odd})$$

as can be easily checked. Unlike the commutative case, $T \longmapsto T'$ is not of period 2 but of period 4. We have

$$p(S') = p(S), \qquad (ST)' = (-1)^{p(S)p(T)}T'S'.$$

Derivations of super algebras. Let A be a super algebra which need not be associative. A *derivation* of A is a k-linear map $D(A \longrightarrow A)$ such that

$$D(ab) = (Da)b + (-1)^{p(D)p(a)}a(Db).$$

Notice the use of the sign rule. If D is even this reduces to the usual definition but for odd D this gives the definition of the *odd derivations.* Let $\mathcal{D} :=: \mathrm{Der}(A)$ be the super vector space of derivations. Then $\mathcal{D}$ becomes a super Lie algebra if we define

$$[D_1, D_2] = D_1D_2 - (-1)^{p(D_1)p(D_2)}D_2D_1.$$

3.6. The Berezinian (superdeterminant). One of the most striking discoveries in super linear algebra is the notion of *superdeterminant.* It was due to Berezin who was a pioneer in super geometry and super analysis, and who stressed the fact that this subject is a vast generalization of classical geometry and analysis. After his untimely and unfortunate death the superdeterminant is now called the *Berezinian.* Unlike the classical determinant, the Berezinian *is defined only for invertible linear transformations*; this is already an indication that it is more subtle than its counterpart in the classical theory. It plays the role of the classical Jacobian in problems where we have to integrate over supermanifolds and have to change coordinates. At this time we shall be concerned only with the linear algebraic aspects of the Berezinian.

In the simplest case when $A = k$ and $T \in \mathbf{End}(M)$ is even, the matrix of T is

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}.$$

If the relation $\det(e^X) = e^{\mathrm{tr}(X)}$ in the commutative situation is to persist in the super commutative situation where the supertrace replaces the trace, one *has to make the definition*

$$\mathrm{Ber}(T) = \det(A)\det(D)^{-1},$$

since the supertrace of the matrix $X = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$ is $\mathrm{tr}(U) - \mathrm{tr}(V)$. Thus already we must have D invertible. In the general case when we are dealing with modules over a general supercommutative super algebra A, we first observe the following lemma.

Lemma 3.6.1. *If*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{End}(M)$$

is even, then T is invertible if and only if A and D are invertible matrices over the commutative ring A_0, i.e., $\det(A)$ *and* $\det(D)$ *are units of A_0.*

Proof. As in a previous situation we do this by going to the case when the odd variables are made 0. Let $J = A_1 + A_1^2$ be the ideal in A generated by A_1, and let $\overline{A} = A/J$. For any matrix L over A let $\overline{L}$ be the matrix over $\overline{A}$ obtained by applying the map $A \longrightarrow \overline{A}$ to each entry of L.

We claim first that L is invertible if and only if $\overline{L}$ is invertible over $\overline{A}$. If L is invertible it is obvious that $\overline{L}$ is invertible. Indeed, if $LM = 1$ then $\overline{LM} = 1$. Suppose that $\overline{L}$ is invertible. This means that we can find a matrix M over A such that $LM = I + X$ where X is a matrix over A such that all its entries are in J. It is enough to prove that $I + X$ is invertible, and for this it is sufficient to show that X is nilpotent, i.e., $X^r = 0$ for some integer $r \geq 1$. There are odd elements $o_1, \ldots, o_N$ such that any entry of X is of the form $\sum_i a_i o_i$ for suitable $a_i \in A$. If $r = N+1$, it is clear that any product $o_{i_1} o_{i_2} \ldots o_{i_r} = 0$ because two of the o''s have to be identical. Hence $X^r = 0$. This proves our claim.

This said, we return to the proof of the lemma. Since T is even, A, D have even entries and B, C have odd entries. Hence

$$\overline{T} = \begin{pmatrix} \overline{A} & 0 \\ 0 & \overline{D} \end{pmatrix}$$

so that T is invertible if and only if $\overline{A}$ and $\overline{D}$ are invertible, which in turn happens if and only if A and D are invertible. The lemma is proved.

For any T as above we have the easily verified decomposition

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix}. \tag{$*$}$$

Since we want the Berezenian to be multiplicative this shows that we have no alternative except to define

$$\mathrm{Ber}(T) = \det(A - BD^{-1}C)\det(D)^{-1} \qquad T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad (A, D \text{ even}).$$

We take this as the definition of $\mathrm{Ber}(T)$. With this definition we have

$$\mathrm{Ber}(T) = 1 \qquad T = \begin{pmatrix} I & 0 \\ C & I \end{pmatrix}, \quad \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}, \quad (B, C, \text{odd }).$$

The roles of A and D appear to be different in the definition of $\mathrm{Ber}(X)$. This is however only an apparent puzzle. If we use the decomposition

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ CA^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix}$$

the we obtain

$$\mathrm{Ber}(X) = \det(D - CA^{-1}B)^{-1}\det(A) \qquad X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

That the two definitions are the same will follow after we have shown that Ber is multiplicative and has the obvious definition on the even (block) diagonal elements. Notice that all the matrices whose determinants are taken have even entries and so the determinants make sense. In particular, $\mathrm{Ber}(T)$ is an element of A_0.

Let $\mathrm{GL}_A(p|q)$ denote the group of all invertible even elements of $\mathbf{End}(\mathbf{R}^{p|q})$. We then have the basic theorem.

Theorem 3.6.2. *Let*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be an even element of $\mathbf{End}(\mathbf{R}^{p|q})$*. Then:*

(a) *T is invertible if and only if A and D are invertible.*

(b) *$\mathrm{Ber}(X)$ is an element of A_0. If $X, Y \in \mathrm{GL}_A(p|q)$, then*

$$\mathrm{Ber}(XY) = \mathrm{Ber}(X)\mathrm{Ber}(Y) \qquad (X, Y \in \mathrm{GL}_A(p|q)).$$

In particular, $\mathrm{Ber}(X)$ is a unit of $A_0^\times$.

Proof. The first statement has been already established. We now prove (b). Let $G = \mathrm{GL}_A(p|q)$ and let G^+, G^0, G^- be the subgroups of G consisting of elements of the respective form g^+, g^0, g^- where

$$g^+ = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}, \qquad g^0 = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \qquad g^- = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}.$$

From $(*)$ we see that any element $g \in G$ can be expressed as a triple product $g = g^+g^0g^-$. We then have $\mathrm{Ber}(g^\pm) = 1$ and $\mathrm{Ber}(g) = \mathrm{Ber}(g^0) = \det(A)\det(D)^{-1}$. The triple product decompositions of g^+g, g^0g, gg^0, gg^- are easy to obtain in terms of the one for g and so it is easily established that $\mathrm{Ber}(XY) = \mathrm{Ber}(X)\mathrm{Ber}(Y)$ for all Y if $X \in G^+, G^0$, and for all X if $Y \in G^-, G^0$. The key step is now to prove that

$$\mathrm{Ber}(XY) = \mathrm{Ber}(X)\mathrm{Ber}(Y) \tag{$*$}$$

for all X if $Y \in G^+$. It is clearly enough to assume that $X \in G^-$. Thus we assume that

$$X = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}.$$

Now

$$B \longmapsto \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$$

maps the additive group of A_1 homomorphically into G^+, and so we may assume in proving $(*)$ that B is elementary, i.e., all but one entry of B is 0, and that one is an odd element β. Thus we have

$$X = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 1 & E \\ 0 & 1 \end{pmatrix} \qquad (E \text{ elementary}).$$

Then

$$XY = \begin{pmatrix} 1 & E \\ C & 1+CE \end{pmatrix} \qquad \mathrm{Ber}(XY) = \det(1 - E(1+CE)^{-1}C)\det(1+CE)^{-1}$$

so that we have to prove that

$$\det(1 - E(1+CE)^{-1}C)\det(1+CE)^{-1} = 1.$$

Since E has a single nonzero entry β, which is odd, all entries of any matrix of the form EX, XE are divisible by β. Hence the product of any two elements of any two of these matrices is 0. This means, in the first place, that $(CE)^2 = 0$, and so

$$(1+CE)^{-1} = 1 - CE$$

and hence

$$1 - E(1+CE)^{-1}C = 1 - E(1-CE)C = 1 - EC.$$

If L is any matrix of even elements such that the product of any two entries of L is 0, then a direct computation shows that

$$\det(1+L) = 1 + \mathrm{tr}(L).$$

Hence

$$\det(1 - E(1+CE)^{-1}C) = \det((1-EC)) = 1 - \mathrm{tr}(EC).$$

Moreover

$$\det((1+CE)^{-1}) = (\det(1+CE))^{-1} = (1+\mathrm{tr}(CE))^{-1}.$$

Hence

$$\det(1 - E(1+CE)^{-1}C)\det(1+CE)^{-1} = (1-\mathrm{tr}(EC))(1+\mathrm{tr}(CE))^{-1}.$$

But, as C, E have only odd entries, $\mathrm{tr}(CE) = -\mathrm{tr}(EC)$ so that

$$\det(1 - E(1+CE)^{-1}C)\det(1+CE)^{-1} = (1+\mathrm{tr}(CE))(1+\mathrm{tr}(CE))^{-1} = 1$$

as we wanted to prove.

The proof of the multiplicativity of Ber can now be completed easily. Let G' be the set of all $Y \in G$ such that $\mathrm{Ber}(XY) = \mathrm{Ber}(X)\mathrm{Ber}(Y)$ for all $X \in G$. We have seen earlier that G' is a subgroup containing G^-, G^0 and we have seen just now that it contains G^+ also. Hence $G' = G$, finishing the proof of the theorem.

Berezinian for odd elements. Odd elements of $\mathbf{End}(A^{p|q})$ are not invertible unless $p = q$. In this case the odd element

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad (A, D \text{ odd}, B, C \text{ even})$$

is invertible if and only if

$$JT = \begin{pmatrix} C & D \\ -A & -B \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

which is even, is invertible. We define

$$\mathrm{Ber}(T) = \mathrm{Ber}(JT).$$

It is then easily verified that the multiplicative property extends to include odd invertible elements as well.

Let M be a free module of rank $p|q$ over A. Then $M \simeq A^{p|q}$ and any invertible $\mathbf{End}(M)$ can be represented by a matrix $X^\sim$. If we choose another basis for M, the matrix for X changes to $X'^\sim = CX^\sim C^{-1}$ where C is some invertible *even* matrix. Hence $\mathrm{Ber}(X^\sim) = \mathrm{Ber}(X'^\sim)$. If we define $\mathrm{Ber}(X)$ as $\mathrm{Ber}(X^\sim)$, then $\mathrm{Ber}(X)$ is well defined and gives a homomorphism

$$\mathrm{Ber} : \mathrm{Aut}(M) \longrightarrow A_0^\times = \mathrm{GL}_A(1|0)$$

where $A_0^\times$ is the group of units of A_0. The following properties are now easy to establish:

(a) $\mathrm{Ber}(X^{-1}) = \mathrm{Ber}(X)^{-1}$.

(b) $\mathrm{Ber}(X') = \mathrm{Ber}(X)$.

(c) $\mathrm{Ber}(X \oplus Y) = \mathrm{Ber}(X)\mathrm{Ber}(Y)$.

3.7. The categorical point of view. The category of vector spaces and the category of super vector spaces, as well as the categories of modules over commutative and supercommutative rings are examples of categories where there is a notion of tensor product that is functorial in each variable. Such categories first appeared in the work of Tannaka who proved a duality theorem for compact nonabelian groups that generalized the Pontryagin duality for abelian groups. Now the equivalence classes of irreducible representations of a nonabelian compact group do not form any reasonable algebraic structure, and Tannaka's great originality was that he considered for any compact Lie group G the *category* $\mathrm{Rep}(G)$ of all finite dimensional unitary G-modules where there is an algebraic operation, namely that of $\otimes$, the tensor product of two representations. If $g \in G$, then, for each unitary G-module V we have the element $V(g)$ which gives the action of g on V; $V(g)$ is an element of the unitary group $\mathcal{U}(V)$ of V, and the assignment

$$V \longmapsto V(g)$$

is a functor compatible with tensor products and duals. The celebrated *Tannaka duality theorem*[5] is the statement that *G can be identified with the group of all such functors.* The first systematic study of abstract categories with a tensor product was that of Saavedra[6]. Subsequently tensor categories have been the object of study by Deligne-Milne[7], Deligne[8], and Doplicher and Roberts[9]. In this section we shall give a brief discussion of how the point of view of tensor categories illuminates the theory of super vector spaces and super modules.

The basic structure from the categorical point of view is that of an abstract category $\mathcal{C}$ with a binary operation $\otimes$,

$$\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}, \qquad X, Y \longmapsto X \otimes Y,$$

where $X \otimes Y$ is the "tensor product"of X and Y. We shall not go into the precise details about the axioms but confine ourselves to some remarks. The basic axiom is that the operation $\otimes$ satisfies the following.

Associativity constraint: This means that there is a functorial isomorphism

$$(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$$

satisfying what is called the pentagon axiom involving four objects.

Commutativity constraint: There is a functorial isomorphism

$$X \otimes Y \simeq Y \otimes X$$

satisfying the so-called hexagon axiom involving three objects.

Unit constraint: There is a unit object 1 with an isomorphism $1 \simeq 1 \otimes 1$ such that $X \longmapsto 1 \otimes X$ is an equivalence of categories of $\mathcal{C}$ with itself. In particular, we have unique functorial isomorphisms

$$X \simeq 1 \otimes X, \qquad X \simeq X \otimes 1.$$

To this one adds the general assumption that $\mathcal{C}$ is an abelian category. For any object X we write $\mathrm{End}(X)$ for the ring $\mathrm{Hom}(X, X)$. The category is said to be k-linear, k a field, if $k \subset \mathrm{End}(1)$. Then all $\mathrm{Hom}(X, Y)$ become vector spaces over k and $\mathrm{End}(X)$ become k-algebras.

In the category of vector spaces or modules over a commutative ring with unit, the unit object is the ring itself, and the commutativity isomorphism is just the map

$$u \otimes v \longrightarrow v \otimes u.$$

In the super categories it is the map

$$u \otimes v \longrightarrow (-1)^{p(u)p(v)} v \otimes u.$$

In the general case one can use the associativity and commutativity constraints to define the tensor products of arbitrary finite families of objects in a natural manner and actions of the permutation group on tensor powers of a single object. We have done this in detail in the category of super vector spaces already.

In order to do anything serious one has to assume that the category $\mathcal{C}$ admits the so-called *internal* Hom, written **Hom**. Before we do this we take time out to describe a general method by which objects are defined in a category. Suppose $\mathcal{T}$ is any category. For any object A the assignment

$$T \longmapsto \mathrm{Hom}(T, A)$$

is then a contravariant functor from $\mathcal{T}$ to the category of sets. If A, B are objects in $\mathcal{T}$ and $f(A \longrightarrow B)$ is an isomorphism, it is immediate that for any object T, there is a functorial bijection

$$\mathrm{Hom}(T, A) \simeq \mathrm{Hom}(T, B), \qquad x \leftrightarrow fx.$$

Conversely, suppose that A, B are two objects in $\mathcal{T}$ with the property that there is a functorial bijection

$$\mathrm{Hom}(T, A) \simeq \mathrm{Hom}(T, B).$$

Then A and B are isomorphic; this is the so-called *Yoneda's lemma.* Indeed, taking $T = A$, let f be the element of $\mathrm{Hom}(A, B)$ that corresponds under the above bijection to id_A; similarly, taking $T = B$ let g be the element of $\mathrm{Hom}(B, A)$ that corresponds to id_B. It is then an easy exercise to show that $fg = \mathrm{id}_B, gf = \mathrm{id}_A$, proving that A and B are uniquely isomorphic given this data. However if we have a contravariant functor F from $\mathcal{T}$ to the category of sets, it is not always true that there is an object A in the category such that we have a functorial identification

$$F(T) \simeq \mathrm{Hom}(T, A).$$

By Yoneda's lemma, we know that A, if it exists, is determined up to a unique isomorphism. Given F, if A exists, we shall say that F is *representable* and is *represented by* A.

This said, let us return to the category $\mathcal{C}$. We now assume that for each pair of objects X, Y the functor

$$T \longmapsto \mathrm{Hom}(T \otimes X, Y)$$

is representable. This means that there is an object $\mathbf{Hom}(X, Y)$ with the property that

$$\mathrm{Hom}(T, \mathbf{Hom}(X, Y)) = \mathrm{Hom}(T \otimes X, Y)$$

for all objects T. This assumption leads to a number of consequences. Using $X \simeq 1 \otimes X$ we have

$$\mathrm{Hom}(X, Y) = \mathrm{Hom}(1, \mathbf{Hom}(X, Y)).$$

In the vector space or module categories Hom is the same as **Hom**. However, in the super categories, Hom is the space of even maps while **Hom** is the space of all maps. If we take T to be $\mathbf{Hom}(X, Y)$ itself, we find that corresponding to the identity map of $\mathbf{Hom}(X, Y)$ into itself there is a map

$$\mathrm{ev}_{X,Y} : \mathbf{Hom}(X, Y) \otimes X \longrightarrow Y.$$

This is the so-called *evaluation map*, so named because in the category of modules it is the map that takes $L \otimes v$ to $L(v)$. It has the property that for any $t \in \mathrm{Hom}(T \otimes X, Y)$, the corresponding element $s \in \mathrm{Hom}(T, \mathbf{Hom}(X, Y))$ is related to t by

$$\mathrm{ev}_{X,Y} \circ (s \otimes \mathrm{id}) = t.$$

Moreover, if X_1, X_2, Y_1, Y_2 are given, there is a natural map

$$\mathbf{Hom}(X_1, Y_1) \otimes \mathbf{Hom}(X_2, Y_2) \simeq \mathbf{Hom}(X_1 \otimes X_2, Y_1 \otimes Y_2).$$

Finally we can define the dual of any object by

$$X^* = \mathbf{Hom}(X, 1), \qquad \mathrm{Hom}(T, X^*) = \mathrm{Hom}(T \otimes X, 1).$$

We have the evaluation map

$$\mathrm{ev}_X := \mathrm{ev}_{X,1} : X^* \otimes X \longrightarrow 1.$$

Using the commutativity isomorphism we then have the map

$$X \otimes X^* \longrightarrow 1$$

which gives a map

$$X \longrightarrow X^{**}.$$

An object is called *reflexive* if

$$X = X^{**}.$$

Already, in the category of vector spaces, only finite dimensional spaces are reflexive. More generally free modules of finite rank are reflexive. In the category of modules over a supercommutative k-algebra A, the free modules $A^{p|q}$ are easily seen to be reflexive. If we assume that *all* objects are reflexive, we obtain a category which is very close in its properties to categories of finite dimensional objects. Such categories are called *rigid*.

Given a map $f(X \longrightarrow Y)$ we can define naturally its *transpose* $f^*(Y^* \longrightarrow X^*)$. For any X we have a map

$$X^* \otimes Y \longrightarrow \mathbf{Hom}(X, Y);$$

in fact, it is the map that corresponds to the composite map

$$X^* \otimes Y \otimes X \simeq X^* \otimes X \otimes Y \overset{\mathrm{ev}_X \otimes \mathrm{id}}{\longrightarrow} 1 \otimes Y \simeq Y.$$

In case X is reflexive this map is an isomorphism. The inverse map can be defined as the transpose of

$$X \otimes Y^* \longrightarrow \mathbf{Hom}(\mathbf{Hom}(X, Y), 1).$$

if we can define this map. To do this one needs to define a natural map

$$X \otimes Y^* \otimes \mathbf{Hom}(X, Y) \longrightarrow 1$$

and this is just the composite map

$$X \otimes Y^* \otimes \mathbf{Hom}(X, Y) \simeq \mathbf{Hom}(X, Y) \otimes X \otimes Y^* \stackrel{\mathrm{ev}_{X,Y} \otimes \mathrm{id}}{\longrightarrow} Y \otimes Y^* \simeq Y^* \otimes Y \longrightarrow 1.$$

For reflexive X we have, in addition to the evaluation map ev_X, its transpose, the *coevaluation map*, namely,

$$\delta : 1 \longrightarrow X \otimes X^*.$$

For X reflexive, we also have

$$\mathrm{Hom}(X, Y) = \mathrm{Hom}(1 \otimes X, Y) = \mathrm{Hom}(1, \mathbf{Hom}(X, Y)) = \mathrm{Hom}(1, X^* \otimes Y).$$

Thus for any $f(X \longrightarrow Y)$ we have the map

$$\delta(f) : 1 \longrightarrow X^* \otimes Y.$$

If $Y = X$ we then have the composite

$$\mathrm{Tr}(f) \stackrel{\mathrm{def}}{=} \mathrm{ev}_X \circ \delta(f) \in \mathrm{End}(1).$$

We have thus a categorical way to define the trace of any element of $\mathrm{End}(X)$ of any reflexive X.

Let us see how Tr reduces to the supertrace in the category of modules over a supercommutative k-algebra A with unit. We take

$$X = A^{p|q}.$$

In this case we can explicitly write the isomorphism

$$\mathbf{Hom}(A^{p|q}, Y) \simeq (A^{p|q})^* \otimes Y.$$

Let (e_i) be a homogeneous basis for $A^{p|q}$ and let $p(i) = p(e_i)$. Let (ξ^j) be the dual basis for $(A^{p|q})^*$ so that $\xi^j(e_i) = \delta_{ij}$. The map

$$(A^{p|q})^* \otimes Y \simeq \mathbf{Hom}(A^{p|q}, Y)$$

is then given by

$$\xi \otimes y \longmapsto t_{\xi \otimes y}, \qquad t_{\xi \otimes y}(x) = (-1)^{p(x)p(y)} \xi(x) y.$$

A simple calculation shows that any homogeneous $f \in \mathbf{Hom}(A^{p|q}, Y)$ can be expressed as

$$f = \sum_j (-1)^{p(j)+p(f)p(j)} \xi^j \otimes f(e_j).$$

Take $X = Y$ let $f \in \mathrm{Hom}(X, Y)$. Then $p(f) = 0$ and so

$$\delta(f) = \sum_j (-1)^{p(j)} \xi^j \otimes f(e_j).$$

Suppose now f is represented by the matrix (M^i_j) so that

$$f(e_j) = \sum_i e_i M^i_j.$$

Then

$$\delta(f) = \sum_{ij} (-1)^{p(j)} \xi^j \otimes e_i M^i_j$$

so that, as $p(M^i_j) = p(e_j) = p(j)$,

$$\mathrm{Tr}(f) = \mathrm{ev}_X(\delta(f)) = \sum_{ij} (-1)^{p(j)} \delta_{ij} M^i_j = \sum_{a \text{ even}} M^a_a - \sum_{b \text{ odd}} M^b_b.$$

We have thus recovered our ad hoc definition. This derivation shows also that the supertrace is independent of the basis used to compute it.

Even rules. In the early days of the discovery of supersymmetry the physicists used the method of introduction of auxiliary odd variables as a guide to make correct definitions. As an illustration let us suppose we want to define the correct symmetry law for the super bracket. If X, Y are odd elements, we introduce auxiliary odd variables ξ, η which supercommute. Since ξX and ηY are both even we have

$$[\xi X, \eta Y] = -[\eta Y, \xi X].$$

But, using the sign rule, we get

$$[\xi X, \eta Y] = -\xi\eta[X, Y], \qquad [\eta Y, \xi X] = -\eta\xi[Y, X]$$

so that, as $\xi\eta = -\eta\xi$, we have

$$[X, Y] = [Y, X].$$

A similar argument can be given for the definition of the super Jacobi identity. These examples can be generalized into a far-reaching principle from the categorical point of view.

The even rules principle. For any vector space V over k and any supercommutative k-algebra B we write

$$V(B) = (V \otimes B)_0 = \text{the even part of } V \otimes B.$$

Clearly $B \longmapsto V(B)$ is functorial in B. If

$$f : V_1 \times \ldots \times V_N \longrightarrow V$$

is multilinear, then, for any B, we have a natural extension

$$f_B : V_1(B) \times \ldots \times V_N(B) \longrightarrow V(B)$$

which is B_0-multilinear and functorial in B. The definition of f_B is simply

$$f_B(b_1 v_1, \ldots, b_N v_N) = (-1)^{m(m-1)/2} b_1 \ldots b_N f(v_1, \ldots, v_N)$$

where the $b_i \in B, v_i \in V_i$ are homogeneous and m is the number of b_i (or v_i) which are odd. The system (f_B) is functorial in B. The *principle of even rules* states that any functorial system (f_B) of B_0-multilinear maps

$$f_B : V_1(B) \times \ldots \times V_N(B) \longrightarrow V(B)$$

arises from a unique k-multilinear map

$$f : V_1 \times \ldots \times V_N \longrightarrow V.$$

The proof is quite simple; see[10]. The proof just formalizes the examples discussed above. It is even enough to restrict the B's to the exterior algebras. These are just the auxiliary odd variables used heuristically.

The categorical view is of course hardly needed while making calculations in specific problems. However it is essential for an understanding of super linear algebra at a fundamental level. One can go far with this point of view. As we have seen earlier, one can introduce Lie objects in a tensor category and one can even prove the Poincaré-Birkhoff-Witt theorem in the categorical context. For this and other aspects see[10].

Everything discussed so far is based on the assumption that k has characteristic 0. In positive characteristic the main results on the Tannakian categories require interesting modifications[11]

REFERENCES

[1] Yu. A. Gol'fand and A. P. Likhtman, *Extension of the algebra of the Poincaré group generators and violation of P invariance*, JETP Lett., **13** (1971), 323–326.

[2] J. Wess and B. Zumino, *Supergauge transformations in four dimensions*, Nucl. Phys., **B70** (1974), 39–50.

[3] V. G. Kac, *Lie superalgebras*, Adv. Math., **26** (1977), 8–96.

[4] S. Helgason, *Differential geometry, Lie Groups, and Symmetric Spaces*, Academic press, 1978.

[5] K. Yosida, *Functional Analysis*, Springer 1968.

[6] N. Saavedra Rivano, *Catégories Tannakiennes*, Lecture Notes in Mathematics, **265**, Springer, 1972.

[7] P. Deligne and J. S. Milne, *Tannakian Categories*, Lecture Notes in Mathematics **900**, Springer, 1982.

[8] P. Deligne, *Catégories Tannakiennes*, Grothendieck Festschrift, vol 2, Birkhüser, 1991, 111–195.

[9] S. Doplicher and J. E. Roberts, *Endomorphisms of C^*-algebras, cross products and duality for compact groups*, Ann. Math., **130** (1989), 75–119; *A new duality theory for compact groups*, Inv. Math., **98** (1989), 157–218.

[10] P. Deligne and J. W. Morgan, *Notes on Supersymmetry* (*following Joseph Bernstein*). *Quantum Fields and Strings*: *A Course for Mathematicians*, 41. Amer. math. Soc., 1999.

[11] S. Gelfand and D. Kazhdan, *Examples of tensor categories*, Inv. Math., **109** (1992), 595–617.