Some remarks on a problem of Accardi

G. CASSINELLI and V. S. VARADARAJAN

1. Introduction

Let Q, P be the projection valued measures on the real line $\mathbf{R}$ that are associated to the position and momentum observables of a particle in quantum mechanics moving in one dimension. In his investigations on the foundations of quantum mechanics Accardi has raised the question whether the pair (Q, P) is determined up to unitary equivalence by purely probabilistic i.e., measure theoretic means. For instance, one can show that if E, F are two bounded Borel sets $\subset \mathbf{R}$, then $Q(E)P(F)$ is an operator of trace class and

$$\mathrm{Tr}\ (Q(E)P(F)) = \mu(E)\mu(F) \tag{1}$$

where μ is the Lebesgue measure on $\mathbf{R}$; and following Accardi one can ask if this property, which we call the *trace property,* is enough to determine the pair (Q, P) up to unitary equivalence. In this note we discuss three examples which indicate that the answer is negative for this question as well as for some natural variations of it.

The pair (Q, P) can be defined for any locally compact separable abelian group which is self dual. Let G be such a group and let us choose an isomorphism of G with its character group $\widehat{G}$. Let Q be the canonical projection valued measures on G and let $\widehat{P}$ be the projection valued measure on $\widehat{G}$ obtained by transferring Q from $L^2(G)$ to $L^2(\widehat{G})$ via the Fourier transform. The isomorphism of G with $\widehat{G}$ allows us to transfer $\widehat{P}$ to a projection valued measure P on G. At the same time it allows us to view the duality between G and $\widehat{G}$ as a duality $(\cdot, \cdot)$ of G with itself and so to view the Fourier transform as a unitary operator $\mathbf{F}$ of $L^2(G)$ with itself having the kernel function

$$(x, y) \qquad (x, y \in G)$$

Clearly

$$P = \mathbf{F}Q\mathbf{F}^{-1}$$

Suppose α, β are two measure preserving Borel automorphisms of $\mathbf{R}$ which take bounded Borel sets to bounded Borel sets. We call these *boundedly measure preserving transformations.* Then, defining

$$Q^{\alpha}(E) = Q(\alpha(E)), \qquad P^{\beta}(F) = P(\beta(F))$$

we get projection valued measures Q^{α}, P^{β} on $\mathbf{R}$ which also satisfy (1) for all bounded Borel sets $E, F \subset \mathbf{R}$. In our first example we construct a pair α, β such that (Q^{α}, P^{β}) is *not* unitarily equivalent to (Q, P). This example suggests that we call a pair (Q', P') *weakly equivalent* to (Q, P) if there is a pair α, β of boundedly measure preserving transformations such that (Q', P') is unitarily equivalent to (Q^{α}, P^{β}).

In our second example we give a pair (Q', P') satisfying (1) but not weakly equivalent to (Q, P). The idea behind this example is as follows. The projection valued measure P is $\mathbf{F}Q\mathbf{F}^{-1}$ where $\mathbf{F}$ is the Fourier transform on $\mathbf{R}$. So the real issue is whether (1) determines the Fourier transform up to weak equivalence. Since only the measure theoretic structure of $\mathbf{R}$ is to be used one can obtain an example of this kind by constructing a group which is separable locally compact abelian, with the same underlying set, Borel structure, and Haar measure, but which is different enough from $\mathbf{R}$ so that its Fourier transform operator is very far from that of $\mathbf{R}$.

Our third example deals with finite quantum systems studied by Schwinger [S] and later by Accardi [A], to which the definitions above may be extended without any difficulty. Let X be a finite set with N elements. In a quantum system where any observable has at most N values one can set up a correspondence between the values and the elements of X and associate to any observable a projection valued measure on X. Let us consider two such, say Q, P which have N distinct values. The condition

$$\text{Tr } (Q_a P_b) = \frac{1}{N} \qquad (a, b \in X) \tag{2}$$

is then the condition that these two observables are *maximally incompatible,* namely that if one of them has a sharply defined value in a state, there is no statistical information on the other, in the sense that all its values are equally probable. Notice that in the Hilbert space of dimension N^2 of the endomorphisms of $L^2(X)$ with scalar product $(A, B) = \text{ Tr } (AB^{\dagger})$, we have

$$\begin{aligned}(Q_a P_b, Q_{a'} P_{b'}) &= \text{ Tr } (Q_a P_b P_{b'} Q_{a'}) \\ &= \text{ Tr } (Q_{a'} Q_a P_b P_{b'}) \\ &= \delta(a, a')\delta(b, b')\frac{1}{N}\end{aligned}$$

so that the N^2 elements $N^{1/2}Q_aP_b$ form an orthonormal basis.

Pairs (Q,P) satisfying (2) can be constructed using finite abelian groups. Let us equip X with the structure of an abelian group and choose an isomorphism of X with its dual group so that we have a map $(\cdot,\cdot)$ of $X\times X$ into the circle group T which expresses the duality of X with itself. The Fourier transform map of $L^2(X)$ with itself is a unitary matrix with entries f_{rs},

$$\mathbf{F}=(f_{rs})_{1\le r,s\le N},\qquad f_{rs}=\frac{(r,s)}{\sqrt{N}}$$

If we define Q,P by

$$(Q_af)(x)=\delta(a,x)f(x),\qquad P_b=\mathbf{F}Q_b\mathbf{F}^{-1}$$

then (Q,P) satisfy (2). The concept of weak equivalence extends in an obvious manner to systems satisfying (2). We shall show by an example that two "sufficiently different" group structures on X lead to pairs (Q,P) which are not weakly equivalent.

2. Proof of the trace property on R and $T\times$ Z.

Let $\mathbf{F}$ be the Fourier transform operator on $\mathbf{R}$ and $P=\mathbf{F}Q\mathbf{F}^{-1}$. We first show that $Q(E)\mathbf{F}Q(F)$ is of trace class; this implies at once that $Q(E)P(F)=Q(E)\mathbf{F}Q(F)\mathbf{F}^{-1}$ is of trace class. We then compute its trace. The argument depends on two lemmas.

Let X be a smooth manifold (second countable) and k a locally bounded Borel function on $X\times X$. Let m be a measure with a smooth *strictly positive* density in any coordinate chart. We assume that k defines a bounded integral operator A_k. This means that

$$(A_k\psi,\varphi)=\int\int k(x,y)\psi(y)\varphi(x)^{\mathrm{conj}}dm(y)dm(x)$$

for any two bounded and compactly supported Borel functions φ,ψ. Let R be the projection valued measure on X defined by

$$R(E)f=\chi_Ef$$

where χ_E is the indicator function of E.

Lemma 1. *Suppose that k is smooth with compact support in $X \times X$. Then for any Borel set $E \subset X$, the operator $R(E)A_k$ is of trace class and its trace is*

$$\int_E k(x,x)dm(x)$$

Proof. This result is a generalization of the one discussed in [V] (p 296). We sketch its proof in a little more detail. By a partition of unity argument we can write k as a finite sum of kernels each of which is one of two types: (i) with support inside $D \times D$ where D is an open set in X diffeomorphic to a cube in $\mathbf{R}^n$ (ii) with support inside $D_1 \times D_2$ where the D_i are open diffeomorphic to a cube in $\mathbf{R}^n$, and have disjoint closures.

In the first case notice that $R(X \setminus D)A_k = 0$ and so we may suppose that $E \subset D$. Thus we may replace X by D and hence assume that X itself is a cube in $\mathbf{R}^n$ and $dm = gd^n x$ where $g > 0$ is a smooth density. The map $\psi \longmapsto \psi\sqrt{g}$ is a unitary isomorphism of $L^2(X, gd^n x)$ with $L^2(X, d^n x)$ that takes the operator A_k to the operator with kernel

$$k^*(x,y) = \sqrt{g(x)}\sqrt{g(y)}k(x,y)$$

Since

$$\int_E k(x,x)g(x)d^n x = \int_E k^*(x,x)d^n x$$

we may suppose that $g = 1$. As k has compact support we may use boundary conditions that do not interfere with k to suppose that X is a torus and m is its Haar measure. The operator A_k is then summable in the usual trigonometric basis and its trace norm is estimated by

$$||A_k||_1 \leq \sum_{r,s \in \widehat{X}} |c_{r,s}(k)|$$

where $c_{r,s}$ are the (rapidly decreasing) Fourier coefficients of k on $X \times X$. We can now approximate A_k arbitrarily closely in the trace norm by A_{k_N} where

$$k_N(x,y) = \sum_{|r|,|s| \leq N} c_{r,s} e_r(x) e_s(y)$$

and so we are reduced to the situation when

$$k(x,y) = e_r(x)e_s(y)$$

where the e_p are the characters of X. The result is trivial in this case.

In the second case we may restrict the operator to $L^2(D_1) \oplus L^2(D_2)$ where it has the form

$$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \qquad B : L^2(D_2) \longrightarrow L^2(D_1)$$

where B is the operator defined by a smooth kernel k with support $\subset D_1 \times D_2$. We may also suppose that $E \subset D_1$. Then $R(E)A_k$ has the form

$$\begin{pmatrix} 0 & R(E)B \\ 0 & 0 \end{pmatrix}$$

Arguing as before we conclude that B, and hence this operator, is of trace class. Its trace is clearly 0 which is also the value of the integral of $k(x,x)$ on E since the integrand is 0 on the diagonal.

Lemma 2. *If k is smooth but is not necessarily compactly supported, the operator $R(E)A_kR(F)$ is of trace class whenever E, F are bounded Borel sets.*

Proof. Choose smooth functions e, f with compact support such that $e = 1$ on E and $f = 1$ on F. Let M_e, M_f be the operators of multiplication in $L^2(\mathbf{R})$ by e, f respectively. Then

$$R(E)A_kR(F) = R(E)M_eA_kM_fR(F)$$

and so it is enough to show that $M_eA_kM_f$ is of trace class. But, for any bounded compactly supported ψ we have

$$(M_eA_kM_f\psi)(x) = e(x)\int k(x,y)f(y)\psi(y)dy = \int k_1(x,y)\psi(y)dy$$

where

$$k_1(x,y) = e(x)f(y)k(x,y)$$

Since k_1 is compactly supported and smooth on $X \times X$ the conclusion follows from Lemma 1.

Proposition 3. *The operator $Q(E)P(F)$ is of trace class with trace $\mu(E)\mu(F)$ whenever E, F are bounded Borel subsets of $\mathbf{R}$.*

Proof. We work with the notation of Lemma 1. The operator $K_1 = M_e \mathbf{F} Q(F) \mathbf{F}^{-1} M_e$ has the kernel function

$$k_1 = e(x)(\mathbf{F}\chi_F)(x-y)e(y)$$

which is smooth with compact support, and

$$\text{Tr } (Q(E)P(F)) = \text{ Tr } (Q(E)\mathbf{F}Q(F)\mathbf{F}^{-1}Q(E)) = \text{ Tr } (Q(E)K_1Q(E))$$

But

$$\begin{aligned} \text{Tr } (Q(E)K_1Q(E)) &= \text{ Tr } (Q(E)K_1) \\ &= \int_E k_1(x,x)dx \\ &= (\mathbf{F}\chi_F)(0)\int_E e(x)^2 dx \\ &= \mu(F)\mu(E) \end{aligned}$$

since $e(x) = 1$ on E.

Proposition 4. *The trace property (1) is valid for $G = T \times \mathbf{Z}$.*

Proof. The proof is the same as that for Proposition 3 except that we use the Fourier transform operator for the group G in place of $\mathbf{F}$.

3. The examples.

Example 1. We first write down the condition that (Q^α, P^β) is unitarily equivalent to (Q, P). Let R^α be the unitary map defined by $(R^\alpha f)(x) = f(\alpha^{-1}(x))$. Then $Q^\alpha = R^\alpha Q R^{\alpha^{-1}}$. We want to know when there is a unitary operator U such that

$$UR^\alpha Q R^{\alpha^{-1}} U^{-1} = Q, \qquad U\mathbf{F}R^\beta Q R^{\beta^{-1}}\mathbf{F}^{-1}U^{-1} = \mathbf{F}Q\mathbf{F}^{-1}$$

The first condition implies that $UR^\alpha = M_a$ where a is a Borel function of absolute value 1, i.e., whose values have absolute value 1. The second condition can be rewritten as the statement that $\mathbf{F}^{-1}M_a R^{\alpha^{-1}}\mathbf{F}R^\beta$ commutes with Q and so is of the form M_b where b is a Borel function of absolute value 1. This gives the condition

$$\mathbf{F} = R^\alpha M_{a^{-1}} \mathbf{F} M_b R^{\beta^{-1}}$$

The operator on the right side is easily checked to have the kernel function

$$a^{-1}(\alpha^{-1}(x))e^{i\alpha^{-1}(x)\beta^{-1}(y)}b(\beta^{-1}(y))$$

and so the condition is that

$$e^{ixy} = a^{-1}(\alpha^{-1}(x))e^{i\alpha^{-1}(x)\beta^{-1}(y)}b(\beta^{-1}(y))$$

for almost all x, y. Take now

$$\alpha(x) = \begin{cases} -x, & |x| \leq 1 \\ x & |x| > 1 \end{cases}$$

and

$$\beta(x) = x$$

Then the condition becomes

$$e^{ixy} = e^{i\alpha(x)y}u(x)v(y)$$

where u, v are Borel functions of absolute value 1, the identity being satisfied for almost all (x, y). Taking $|x| \leq 1$ we see that

$$e^{2ixy} = u(x)v(y)$$

for almost all (x, y) with $|x| \leq 1$. For some x_0 this is true for almost all y and so we have

$$v(y) = \gamma e^{iry}$$

for almost all y where γ, r are constants with $|\gamma| = 1$ and r real. Similarly

$$u(x) = \delta e^{isx}$$

for almost all $|x| \leq 1$ where δ is a constant of absolute value 1 and s is a real constant. Hence

$$e^{2ixy} = \gamma\delta e^{i(sx+ry)}$$

for almost all (x, y) with $|x| \leq 1$, hence for all (x, y) with $|x| \leq 1$, and hence, by analyticity, for all (x, y). Taking $x = y = 0$ we get $\gamma\delta = 1$. But then this means that

$$2xy = sx + ry$$

for all (x, y) which is absurd. Thus, for this choice of α we have the system (Q^α, P) which is not unitarily equivalent to (Q, P) although the criterion (1) of §1 is satisfied.

Example 2. Here we wish to construct another separable locally compact abelian group structure on $\mathbf{R}$ whose Haar measure is still μ. Let $G = T \times \mathbf{Z}$ where T is the circle group. We identify G with $\mathbf{R}$ by the *Borel isomorphism* of $\mathbf{R}$ with G defined by

$$n + t \longmapsto (e^{2i\pi t}, n) \qquad (n \in \mathbf{Z}, 0 \leq t < 1)$$

It is clear that this preserves bounded sets and Haar measures. We transfer the group structure from G to $\mathbf{R}$ by this map and denote the separable locally compact abelian group thus obtained on $\mathbf{R}$ by $\mathbf{R}'$. It is clear that $\mathbf{R}'$ has the same Borel structure and Haar measure as $\mathbf{R}$. Let $\mathbf{F}'$ be the Fourier transform map of $\mathbf{R}'$. The group G is self dual and so we have a standard pair (Q', P') on it. By Proposition 2.4 we have two projection valued measures on G which satisfy the criterion (1) of §1. Transferring these to $\mathbf{R}$ we obtain two projection valued measures on Q', P' on $\mathbf{R}$ which satisfy (1) of §1. Note that

$$Q' = Q, \qquad P' = \mathbf{F}' Q \mathbf{F}'^{-1}$$

Our claim is that this pair is not weakly equivalent to (Q, P).

The argument is similar to the one given in the previous example. The condition for weak equivalence is that for some unitary operator U we have

$$UQ^\alpha U^{-1} = Q, \quad U\mathbf{F}R^\beta Q R^{\beta^{-1}} \mathbf{F}^{-1} U^{-1} = \mathbf{F}' Q \mathbf{F}'^{-1}$$

This leads to

$$\mathbf{F}' = M_a R^{\alpha^{-1}} \mathbf{F} R^\beta M_{b^{-1}}$$

Let us apply both sides to the function $\chi_{[0,1)}$. The left side reproduces this function. But the right side becomes

$$a(x) \int_{\beta([0,1))} b^{-1}(\beta^{-1}(y)) e^{i\alpha(x)y} dy$$

If these two are to be equal, the function

$$\int_{\beta([0,1))} b^{-1}(\beta^{-1}(y)) e^{i\alpha(x)y} dy$$

must vanish if x is outside $[0,1)$. If

$$g(t) = \int_{\beta([0,1))} e^{ity} b^{-1}(\beta^{-1}(y))dy \qquad (t \in \mathbf{C})$$

then the boundedness of $\beta([0,1))$ and of b^{-1} imply that g is an *entire* function vanishing on the complement of the bounded set $\alpha([0,1))$. So it must be identically 0. But this is impossible as it is the Fourier transform of a nonzero function.

Example 3. We work in the context of finite systems. Let X be a finite set with N elements. Let G_1, G_2 be two finite abelian groups whose underlying set is X. We choose an isomorphism of G_i with its dual $\widehat{G}_i$, and write $(\cdot,\cdot)_i$ for the duality between G_i and itself thus obtained. Let $\mathbf{F}_1, \mathbf{F}_2$ be the Fourier transform unitary matrices corresponding to G_1, G_2. We have

$$\mathbf{F}_i(x,y) = \frac{(x,y)_i}{\sqrt{N}}$$

Our aim is to show that if G_1 and G_2 are "sufficiently different" then the corresponding pairs (Q_i, P_i) are not weakly equivalent.

We make some remarks on the Fourier transform matrix. Let G be a finite abelian group whose underlying set is X. As usual we choose a duality of G with itself. Then G is a direct sum of groups of the form $\mathbf{Z}/p^r\mathbf{Z}$ where the p's are prime divisors of N and $r \le s$ where p^s is the largest power of p dividing N. If s^* is the maximum of the integers r such that $\mathbf{Z}/p^r\mathbf{Z}$ occurs in the standard decomposition of G, then $s^* \le s$ and $s^* = s$ if and only if the p–component of G is the cyclic group $\mathbf{Z}/p^s\mathbf{Z}$. Let $R(G)$ be the subgroup of the circle group generated by $N^{1/2}$–times the entries of the Fourier transform matrix of G. Then it is clear from a look at the character table of G that $R(G)$ is the group $R(N^*)$ of all $N^{*\text{th}}$ roots of unity where $N^* = \prod_{p|N} p^{s^*}$:

$$R(G) = R(N^*)$$

We write

$$N^* = N^*(G)$$

For instance if $N = p^4$ (p a prime), then $s = 4$ and $s^* = 1, 2, 2, 3, 4$ for $G = \mathbf{Z}_p^4, \mathbf{Z}_p^2 \oplus \mathbf{Z}_{p^2}, \mathbf{Z}_{p^2}^2, \mathbf{Z}_p \oplus \mathbf{Z}_{p^3}, \mathbf{Z}_{p^4}$.

The condition for the two pairs associated to G_1 and G_2 to be weakly equivalent is easily derived to be

$$(x,y)_2 = a(x)(\alpha(x),\beta(y))_1 b(y) \qquad (x,y \in X)$$

where $|a(x)| = |b(y)| = 1$ and α, β are permutations of X. If y_0 is such that $\beta(y_0) = e_1$ where e_1 is the identity of G_1, then we have $(x,y_0)_2 = a(x)b(y_0)$. Similarly $(x_0,y)_2 = a(x_0)b(y)$ where $\alpha(x_0) = e_1$. This gives

$$\frac{(x,y)_2}{(x,y_0)_2(x_0,y)_2} = \frac{(\alpha(x),\beta(y))_1}{b(y_0)a(x_0)}$$

So there is a constant θ with $|\theta| = 1$ such that

$$\theta(x,y)_1 \in R(G_2) \qquad (x,y \in X)$$

Taking $x = e_1$ we see that $\theta \in R(G_2)$. So

$$(x,y)_1 \in R(G_2) \qquad (x,y \in X)$$

This implies that

$$R(G_1) \subset R(G_2)$$

By symmetry we have the reverse inclusion also and so

$$R(G_1) = R(G_2)$$

So

$$N^*(G_1) = N^*(G_2)$$

Thus if the group structures are different enough so that

$$N^*(G_1) \neq N^*(G_2)$$

then the pairs (Q_i, P_i) are not weakly equivalent.

If G_1 is the cyclic group of order N, then $N^*(G_1) = N$ and then $N^*(G_2) = N$ if and only if G_2 is isomorphic to G_1. Thus, if G_1 is the cyclic group of order N and G_2 is not isomorphic to it, the pairs (Q_i, P_i) are not weakly equivalent.

If N is a prime, or more generally, if it is a product of *distinct* primes, then there is only one abelian group of order N. In this case it is an interesting question whether any two pairs (Q, P) are weakly equivalent. We can take $Q' = Q$ and $P' = WQW^{-1}$ for some unitary matrix W. The condition for weak equivalence is that

$$|w(a,b)| = 1 \qquad (a, b \in X)$$

Let us fix $x_0 \in X$ and use left and right multiplication by diagonal unitary matrices to make

$$w(x_0, y) = w(x, x_0) = 1 \qquad (x, y \in X)$$

Our question reduces to showing that W is then up to permutations of rows and columns the character table of the group $\mathbf{Z}/N\mathbf{Z}$. One can show that in the set of all such W's, the ones that are permutations of the character table are isolated points. So it appears plausible that these are the only such matrices W but we do not have a proof.

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