Historical review of Lie Theory

1. The theory of Lie groups and their representations is a vast subject (Bourbaki [Bou] has so far written 9 chapters and 1,200 pages) with an extraordinary range of applications. Some of the greatest mathematicians and physicists of our times have created the tools of the subject that we all use. In this review I shall discuss briefly the modern development of the subject from its historical beginnings in the mid nineteenth century.

The origins of Lie theory are geometric and stem from the view of *Felix Klein* (1849–1925) that geometry of space is determined by the group of its symmetries. As the notion of space and its geometry evolved from Euclid, Riemann, and Grothendieck to the supersymmetric world of the physicists, the notions of Lie groups and their representations also expanded correspondingly. The most interesting groups are the semi simple ones, and for them the questions have remained the same throughout this long evolution: what is their structure, where do they act, and what are their representations?

2. The algebraic story: simple Lie algebras and their representations. It was *Sopus Lie* (1842–1899) who started investigating all possible (local)group actions on manifolds. Lie's seminal idea was to look at the action *infinitesimally*. If the local action is by $\mathbf{R}$, it gives rise to a vector field on the manifold which integrates to capture the action of the local group. In the general case we get a *Lie algebra* of vector fields, which enables us to reconstruct the local group action. The simplest example is the one where the local Lie group acts on itself by left(or right) translations and we get the *Lie algebra of the Lie group*. The Lie algebra, being a linear object, is more immediately accessible than the group. It was *Wilhelm Killing* (1847–1923) who insisted that before one could classify all group actions one should begin by classifying all (finite dimensional real) Lie algebras. The gradual evolution of the ideas of Lie, *Friedrich Engel* (1861–1941), and Killing, made it clear that determining all *simple* Lie algebras was fundamental.

What are all the simple Lie algebras (of finite dimension) over $\mathbf{C}$? It was Killing who conceived this problem and worked on it for many years. His researches were published in the *Mathematische Annalen* during 1888–1890 [K]. Although his proofs were incomplete (and sometimes wrong) at crucial places and the overall structure of the theory was confusing, Killing arrived at the astounding conclusion that the only simple Lie algebras were those associated to the linear, orthogonal, and symplectic groups, apart from a small number of isolated ones. The problem was completely solved by *Elie Cartan* (1869–1951), who, reworking the ideas and results of Killing but adding crucial innovations of his own (*Cartan–Killing form*), obtained the rigorous classification of simple Lie algebras in his 1894 thesis, one of the greatest works of nineteenth century algebra [C]. Then in 1914, he classified the simple *real* Lie algebras by determining the real forms of the complex algebras. In particular he noticed that there is exactly one real form (the *compact form*) on which the Cartan–Killing form is negative definite, a fact that would later play a central role in Weyl's transcendental approach to the representation theory of semi simple Lie algebras. For the fascinating account of the story, especially of the trail-blazing work of Killing and Cartan, see [Ha].

The classification. The simple Lie algebras over $\mathbf{C}$ fall into four infinite families $A_n(n \geq 1), B_n(n \geq 2), C_n(n \geq 3), D_n(n \geq 4)$ respectively corresponding to the groups

$\mathrm{SL}(n+1,\mathbf{C}), \mathrm{SO}(2n+1,\mathbf{C}), \mathrm{Sp}(2n,\mathbf{C}), \mathrm{SO}(2n,\mathbf{C})$), and five isolated ones (the *exceptional Lie algebras*) denoted by G_2, F_4, E_6, E_7, E_8, with dimensions $14, 52, 78, 133, 248$ respectively. The key concept for the classification is that of a *Cartan subalgebra* (CSA) $\mathfrak{h}$, which is a special maximal nilpotent subalgebra, unique up to conjugacy as shown by Chevalley much later. In the spectral decomposition of $\mathrm{ad}\,\mathfrak{h}$, the eigenvalues α are certain linear forms on $\mathfrak{h}$ called *roots*, the corresponding (generalized) eigenvectors X_α are *root vectors*, the (generalized) eigenspaces $\mathfrak{g}_\alpha$ are *root spaces*, and the structure of the set of roots captures a great deal of the structure of the Lie algebra itself. For instance, if α, β are roots but $\alpha+\beta$ is non-zero but not a root, then $[X_\alpha, X_\beta] = 0$.

Central to Cartan's work is the *Cartan–Killing form*, the symmetric bilinear form $X, Y \longmapsto \mathrm{Tr}(\mathrm{ad}\,X \mathrm{ad}\,Y)$, invariant under all automorphisms of the Lie algebra. It is non-degenerate if and only if the Lie algebra is *semi simple*. For a semi simple Lie algebra the CSA's are the maximal abelian diagonalizable subalgebras, and they have *one dimensional root spaces*. In this case there is a natural $\mathbf{R}$-form $\mathfrak{h}_\mathbf{R}$ of $\mathfrak{h}$ on which all roots are real and $(\cdot\,,\cdot)$ is positive definite. This allows us to view the set Δ of roots as a *root system*, i.e., a finite subset of the Euclidean space $\mathfrak{h}^*_\mathbf{R} \setminus \{0\}$ with the following key property: it remains invariant under reflection in the hyperplane orthogonal to any root. Thus the reflections generate a *finite* subgroup of the orthogonal group of $\mathfrak{h}_\mathbf{R}$, the *Weyl group*. Root systems thus become special combinatorial objects and their classification leads to the classification of simple Lie algebras. The calculations however remained hard to penetrate till *E. B. Dynkin*, (1924–) discovered the concept of a *simple root* [Dy]. If $\dim(\mathfrak{h}_\mathbf{R}) = n$, then a set of simple roots has n elements α_i, and $a_{ij} := 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ is an integer ≤ 0 for $i \neq j$. The matrix $A = (a_{ij})$ is called a *Cartan matrix*, and it gives rise to a graph, the *Dynkin diagram*, where there are n nodes, with the nodes corresponding to simple roots α_i, α_j linked by $a_{ij}a_{ji}$ lines. Connected Dynkin diagrams, which correspond to simple Lie algebras, fall into 4 infinite families and 5 isolated ones. The integer n, the *rank*, is the one in the Cartan classification. The theory became more accessible when the book of *Nathan Jacobson* (1910–1999) came out in 1962 [J]; till then [Dy] and [L] were the only sources available apart from [C].

Representations. In 1914 Cartan determined the irreducible finite dimensional representations of the simple Lie algebras [C]. In any representation the elements of a CSA $\mathfrak{h}$ are diagonalizable and the simultaneous eigenvalues are elements $\nu \in \mathfrak{h}^*_\mathbf{R}$, the *weights*, which are *integral* in the sense that $\nu_\alpha := 2(\nu, \alpha)/(\alpha, \alpha)$ is an integer for all roots α. Among the weights of an irreducible representation there is a distinguished one λ, the *highest weight*, which has multiplicity 1, determines the irreducible representation, and is *dominant*, i.e., $\lambda_{\alpha_i} \geq 0$ for $1 \leq i \leq n$. The obvious question is whether every dominant integral element of $\mathfrak{h}^*_\mathbf{R}$ is the highest weight of an irreducible representation. It is enough to prove this for the *fundamental weights* μ^i defined by $\mu^i_{\alpha_j} = \delta_{ij}$. For A_n, the actions on the exterior products $\Lambda^i(\mathbf{C}^n)$ are irreducible with highest weights μ^i. Similar calculations show that the fundamental weights are highest weights for the classical Lie algebras. Once again, Cartan showed by explicit calculation that the fundamental weights are highest even for the exceptional Lie algebras. It was in the course of this analysis that Cartan discovered the *spin modules* of the orthogonal Lie algebras which do not occur in the tensor algebra of the defining representation, unlike the case for A_n and C_n. They arise from representations of the Clifford algebras and there is one of them for B_n and two for D_n. They were originally discovered by Dirac in his relativistic treatment of the spinning electron, thus accounting for their name. They act on *spinors*, which, unlike the tensors, *are not functorially attached to the base vector space*, so that one can define the Dirac operators only on Riemannian manifolds with a *spin structure*.

General algebraic methods. In the late 1940's *Claude Chevalley* (1909–1984) and *Harish-Chandra* (1923–1983) (independently) discovered the way to answer, without using classification, the two key questions here: (1) whether every Dynkin diagram comes from a semi simple Lie algebra, and (2) if every dominant integral weight is the highest weight of an irreducible representation [H1] [Ch]. In the mid 1920's, *Hermann Weyl* (1885–1955) had settled (2) as well as the complete reducibility of all representations, by global methods without classification(see below).

For (2) one works with the *universal enveloping algebra* of $\mathfrak{g}$, say $\mathcal{U}$. For any linear function $\lambda \in \mathfrak{h}^*$ there is a unique irreducible module I_λ with highest weight λ, and one has to show that I_λ is finite dimensional if and only if λ is dominant and integral. For (1) one notes that in a semi simple Lie algebra $\mathfrak{g}$ with a Cartan matrix $A = (a_{ij})$, if $0 \neq X_{\pm i}$ are in the root spaces $\mathfrak{g}_{\pm\alpha_i}$, then we have the commutation rules

$$[H_i, H_j] = 0, \quad [H_i, X_{\pm j}] = \pm a_{ij} X_{\pm j}, \quad [X_i, X_{-j}] = \delta_{ij} H_i \qquad (I).$$

However a deeper study of the adjoint representation yields the higher order commutation rules

$$[X_{\pm i}, [X_{\pm i}, [\ldots [X_{\pm i}, X_{\pm j}]] \ldots] = \mathrm{ad}(X_{\pm i})^{-a_{ij}+1}(X_{\pm j}) = 0 \qquad (II).$$

The universal associative algebra $\mathcal{U}_A$ defined by the relations (I) and (II) bears a close resemblance to the algebra $\mathcal{U}$ mentioned earlier and one can construct a theory of its highest weight representations. One obtains the same criterion for the finite dimensionality of the irreducible representations. Let $\mathfrak{l}$ be the Lie algebra inside $\mathcal{U}_A$ generated by the $H_i, X_{\pm i}$. If the highest weight has a value strictly > 0 at each node of the diagram this representation will be faithful on $\mathfrak{h}$, and the image of $\mathfrak{l}$ under this representation will be the semi simple Lie algebra corresponding to the diagram. Much later Serre discovered the beautiful result that $\mathfrak{l}$ is *already finite dimensional* and hence is the required semi simple Lie algebra with the given Cartan matrix A, thus defining a *presentation* of the semi simple Lie algebra associated to any given diagram [S1] [V1].

Infinite dimensional Lie algebras. Cartan also studied what he called the *infinite simple continuous groups*. Roughly speaking they are the infinite dimensional analogues of the simple Lie groups. The *general* theory of infinite dimensional Lie groups is still very much of a mystery and I cannot say much about these (see [CC]).

In the late 1960's, *Victor Kac* (1943–) and *Robert Moody* (1941–) independently initiated the study of certain infinite dimensional Lie algebras somewhat different from Cartan's. If we relax the properties of a Cartan matrix, especially the one requiring the Weyl group to be finite, (I) and (II) will lead, by the methods of Chevalley-Harish-Chandra, to *new* Lie algebras that *will no longer be finite dimensional*. These are the *Kac-Moody* algebras [Ka1] [Moo]. If we extend the scalars from $\mathbf{C}$ to the ring of finite Laurent series in an indeterminate, the simple Lie algebras give rise to certain Lie algebras, which have *universal central extensions* with one-dimensional center. The latter are the *affine Lie algebras* which are special Kac-Moody algebras, which, along with the *Virasoro algebras*, are important in conformal field theory. Their structure and representation theory resemble closely those of the finite dimensional simple Lie algebras, and their root systems are very beautiful infinite combinatorial objects related to many famous classical formulae.

Classification of restricted simple Lie algebras in characteristic $p > 0$. It is natural to ask what the classification of simple Lie algebras looks like in characteristic $p > 0$. Here one has the concept of a *restricted* Lie algebra which is a Lie algebra together with an automorphism $X \longmapsto X^{[p]}$ that is an infinitesimal version of the Frobenius morphism for algebraic groups. Interestingly there are additional simple Lie algebras, namely those that are finite dimensional analogues of Cartan's infinite simple Lie algebras, the so-called Cartan-type Lie algebras. That the class of restricted simple Lie algebras is exhausted by the classical and Cartan-type Lie algebras (Kostrikin-Shafarevich conjecture) was proved in [BW].

3. Invariant theory. Let us leave the algebraic story here and go to the classical invariant theory which was concerned with computing the *invariants* of the projective varieties under the action of the projective group $\mathrm{PGL}(n, \mathbf{C})$. In the first approximation we may replace the varieties by homogeneous polynomials and study the action of $\mathrm{SL}(n, \mathbf{C})$ on the space $P_{n,d}$ of all homogeneous polynomials of degree d in n variables, and the induced action on the *algebra* $\mathcal{P}_{n,d}$ *of polynomial functions on* $P_{n,d}$. Invariant theory asks for an explicit determination of the subalgebra $\mathcal{I}_{n,d}$ of elements of $\mathcal{P}_{n,d}$ invariant under the group. The work of *Paul Gordan* (1837–1912), had led to the result that $\mathcal{I}_{2,d}$ is finitely generated and to an algorithmic construction of a set of generators for it, when *David Hilbert* (1862–1943) came into the picture and took the entire subject to a new level. In a celebrated paper Hilbert proved the finite generation of $\mathcal{I}_{n,d}$ by very general abstract arguments, but under prodding from Gordan, later examined the question of the finite determination of the invariants.

The finite generation depends on the existence of a *projection operator* R (*Reynold's operator*) from $\mathcal{P}(V)$ to $\mathcal{I}(V)$ that preserves the grading and commutes with multiplication by elements of $\mathcal{I}(V)$; here V is any module for $\mathrm{SL}(n, \mathbf{C})$. Hilbert used what is called the *Cayley* Ω*-process* for this purpose; one can equally well use averaging with respect to $\mathrm{SU}(n)$. However what is essential is the complete reducibility of all finite dimensional representations of $\mathrm{SL}(n, \mathbf{C})$. Weyl, who had proved this for all semi simple groups, was thus able to generalize Hilbert's result to the case where $\mathrm{SL}(n, \mathbf{C})$ is replaced by *any* semi simple Lie group G over $\mathbf{C}$. In his majestic and profound 1939 book *Classical Groups: their Invariants and Representations* [W1] Weyl gave an exposition of the fundamental questions of invariant theory over a field of characteristic 0, emphasizing that they should be studied over any field. For a given G-module V (for classical G, important cases are the direct sum of copies of the defining representation and its dual, as well as the conjugacy action on a number of copies of the matrices) the *first fundamental theorem* (FFT) seeks an explicit description of generators for $\mathcal{I}(V)$, and the *second fundamental theorem* seeks a basis for the ideal of relations among the generators. Of course this process can be continued, and Hilbert's study of the *syzygies* marks the beginning of the homological theory of commutative algebras. For developments since 1939 and a whole lot of other aspects of representations and invariants see the encyclopedia (and encyclopedic) volume [GW]. For a profound study of the action of a semi simple group over the polynomial ring of its Lie algebra, see [Ko].

Semi simple groups in characteristic $p > 0$: Mumford's geometric reductivity. Hilbert's work (see the English translation of his papers on this subject [AH]) lay buried till *David Mumford* (1937–) resurrected it in the 1960's and expanded its scope enormously [M1] [MF]. He showed that the central problems of *moduli* of algebraic geometric objects in *any characteristic* depend upon viewing the orbit space of a projective action of a semi simple (or the slightly more general *reductive*) group as an algebraic variety itself. When the characteristic is 0, the Hilbert-Weyl theory is a perfectly adequate foundation for this. In prime characteristic, it was clear that one should work with the reductive groups that Borel and Chevalley had discovered by then (see below). But, complete reducibility of representations is *not available in characteristic* $p > 0$. Nevertheless Mumford conjectured that semi simple groups in prime characteristic are *geometrically reductive*, a property equivalent to complete reducibility in characteristic 0: given any non-zero vector v fixed by the group, there is a homogeneous invariant polynomial F such that $F(v) \neq 0$. If the characteristic p of the field k divides n, the action of $\mathrm{SL}(n,k)$ on $\mathfrak{gl}(n,k)$ is not completely reducible: $k.I_n$ does not admit an invariant complement since the only invariant linear form is the trace and it vanishes at I_n; but we can take F to be the determinant in Mumford's definition. Mumford's conjecture was proved in 1975 by Haboush [Hab] (independently, for $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$, by Formanek and Processi [FP]). Nagata showed that geometric reductivity implies the finite generation of invariants; he also constructed counterexamples to the question of finite generation of invariants (Hilbert's 14^{th} problem, see [M2]). For simpler counterexamples, see [St2]. For the theory of moduli see [Se].

4. The Weyl character and dimesion formulae. Compact and complex groups. In the mid 1920's Hermann Weyl wrote a series of epoch-making papers ([W2], Band II, 543–647; Band III, 1–33) on representations of semi simple Lie groups and Lie algebras. Weyl found a simple construction for the compact form of a complex semi simple Lie algebra and proved the remarkable fact that the *simply connected group corresponding to the compact form is still compact*. It follows that the category of *continuous* representations of the compact group is equivalent to the category of representations of the complex Lie algebra. The first algebraic proof of the complete reducibility of all representations of a complex semi simple Lie algebra was given by Casimir and Van der Waerden [CW] much later. It is a question of showing that $H^1(\mathfrak{g}) = 0$ for semi simple $\mathfrak{g}$ [V1].

Let G be compact and simply connected. G has a *maximal torus* T and all conjugacy classes of G meet T in *Weyl group orbits*. Weyl found a wonderful formula for the integral of a function in terms of its integral on the torus:

$$\int_G f(x)dx = \frac{1}{|\mathfrak{w}|}\int \bar{f}(t)\Delta(t)\overline{\Delta(t)}dt, \qquad \bar{f}(t) = \int_G f(xtx^{-1})dx$$

where $\mathfrak{w}$ is the Weyl group and $|\mathfrak{w}|$ is its order, and dx, dt are the normalized Haar measures on G, T respectively. Here, for $H \in \mathfrak{t} = (-1)^{1/2}\mathfrak{h}_{\mathbf{R}} = \mathrm{Lie}(T)$,

$$\Delta(\exp H) = \prod_{\alpha>0}\left(e^{\alpha(H)/2} - e^{-\alpha(H)/2}\right) = \sum_{s\in\mathfrak{w}} \det(s)e^{(s\rho)(H)} \qquad (H \in \mathfrak{t})$$

where ρ is as usual half the sum of positive roots. Using this formula in conjunction with the orthogonality relations in a stunning fashion, Weyl obtained his famous formula for the characters of the irreducible representations which showed right away that every dominant integral linear form is a highest weight. If λ is the highest weight, then the character Θ_λ, and the dimension of the irreducible representation I_λ with highest weight λ, are given by (for dimension we let $H \to 0$)

$$\Theta_\lambda(\exp H) = \frac{\sum_{s\in\mathfrak{w}} \det(s) e^{(s(\lambda+\rho))(H)}}{\sum_{s\in\mathfrak{w}} \det(s) e^{(s\rho)(H)}}, \qquad \dim(I_\lambda) = \frac{\prod_{\alpha>0}(\lambda+\rho,\alpha)}{\prod_{\alpha>0}(\rho,\alpha)}.$$

The Weyl formulae remained the standard of beauty in the theory till they were joined by the Harish-Chandra formulae for the character and formal dimension of the representations of the discrete series of a real semi simple Lie group ([H2], Vol. III, 537–647).

Real groups. Cartan's theory of *symmetric spaces* [C], the first major advance in the theory of homogeneous spaces after Riemann's discovery of spaces of constant curvature, proved to be of fundamental importance for the real groups [He]. The non-compact symmetric spaces are of the form G/K where G is a real semi simple Lie group and K is a maximal compact subgroup. The existence and uniqueness up to conjugacy of K is a special case of Cartan's theorem that a compact Lie group acting on a space of negative curvature has a fixed point. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, is the setting for *Iwasawa* (1917–1998) who introduced the maximal abelian subspaces $\mathfrak{a}$ of $\mathfrak{p}$, the root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$, and the *Iwasawa decomposition* of G which are fundamental for the structure of real semi simple Lie groups [I]. The roots form a root system which need not always be reduced (twice a root can be a root). The theory of the parabolic subalgebras and subgroups that derive from it, are an essential foundation for the harmonic analysis on real semi simple Lie groups [He] [Kn].

5. Modern developments. Nowadays groups with additional structures are viewed as group objects in categories. One starts with a Lie group G of whatever category one wants to be in, and associates its Lie algebra $\mathrm{Lie}(G)$ to get a *functor* $G \longmapsto \mathrm{Lie}(G)$; the fundamental theorems of Lie amount to studying how close this functor comes to being an equivalence of categories. It was only after the appearance of Chevalley's great 1946 book *The Theory of Lie Groups* (Volume # 8 in the famous Princeton Series with a dedication to Elie Cartan and Hermann Weyl and a blurb on the cover saying that "the reader need no longer be afraid of shrinking neighborhoods of the identity element!"), that the global view became accessible to the general mathematical public.

Chevalley's Princeton book. In his book [Ch1] Chevalley developed all the major results: the construction of the Lie algebra of a Lie group, the exponential map, the subgroup–subalgebra correspondence, Von Neumann's theorem that a closed subgroup of a real Lie group is a Lie group, and the fact that every C^∞ (in fact, every C^2) Lie group is a real analytic Lie group; the analytic structure underlying the topology is unique because any continuous homomorphism between Lie groups is analytic. In addition he treated compact Lie groups in depth: complete reducibility of all representations, Peter-Weyl completeness theorem, Tannaka-Krein duality, existence of a faithful finite dimensional representation σ, and the theorem that every irreducible representation is contained in the tensor product of a number of copies of σ and its contragredient. This list does not indicate the originality of his treatment of these topics. For instance he had to extend the notion of Lie subgroups to include the cases when the subgroup is not closed and its topology and smooth structures are not induced by the ambient group. He constructed the subgroup and its cosets as the maximal global integral manifolds of the involutive distribution on the group defined by the subalgebra, giving in the process the first global treatment of the Frobenius theorem of integrability of involutive distributions. In the Tannaka duality he proved that there is a unique *complex Lie group* of which the given compact Lie group is a real form, thereby giving an entirely new perspective on the Weyl correspondence between compact and complex groups. Chevalley's theorem is the beginning of the *Tannakian point of view* that reconstructs an algebraic group from the tensor category of its finite dimensional modules [D]. For Chevalley, the ring of matrix elements of a compact Lie group is a reduced finitely generated algebra with a Hopf algebra structure, and its spectrum is the complex semi simple group enveloped by the compact group, thus foreshadowing the point of view of quantum groups which arose almost forty years later.

Perhaps some remarks on the *fifth problem of Hilbert* are in order here. Hilbert, motivated by his insights into foundations of geometry, felt that the condition of differentiability in the definition of a Lie group was a deficiency, and proposed the problem of proving that any topological group which is locally homeomorphic to a manifold, must be a Lie group. The problem was eventually solved in the affirmative by the efforts of Gleason, Iwasawa, Montgomery-Zippin, Yamabe, and Lazard (in the p-adic case) (see [MZ] [Laz]) after partial solutions by Von Neumann (compact groups), and Chevalley (solvable).

Linear algebraic groups and the classification of simple groups over an algebraically closed field of arbitrary characteristic. Chevalley himself, along with *Armand Borel* (1923–2003), was a central player in the next great development of Lie theory, the theory of *linear algebraic groups in arbitrary characteristic*. Chevalley's initial attempts (in tomes II and III of [Ch 1]) did not go very far because they were tied to the exponential map. But the work [B1] of Borel, which used only global methods based on algebraic geometry, changed the picture dramatically. Starting from Borel's work Chevalley went forward (by "analytic continuation"in his own words) to the classification of semi simple algebraic groups and their representations [Ch2] [Ch3]. He discovered the remarkable fact that complex semi simple groups form group schemes over $\mathbf{Z}$, so that one can tensor them with any field to produce algebraic semi simple groups over that field. If the field is algebraically closed this procedure will yield essentially *all* semi simple algebraic groups. If the field is finite one will get *new finite simple groups* beyond those first studied by Dickson [Di]. For algebraic groups [B2] and [Sp1] are good sources; The book of Borel was profoundly influential in the development of the subject. For the theory of the *Chevalley groups* see [St1]. Chevalley's original papers and articles are available in [Ch 2] Ch3]. For a simpler proof that isomorphic root data determine isomorphic groups see [St3].

Reductive groups over arbitrary fields. The Chevalley groups are *split*, i.e., they have a maximal torus split over the ground field. The theory of roots of reductive groups which are not split was carried out by Borel and Tits [BT] and is fundamental for rationality questions. The subgroups P that contain the Borel subgroups are the *parabolic subgroups*. The associated homogeneous spaces G/P are the *flag manifolds* which are the only projective homogeneous spaces for the semi simple groups. The representation theory of semi simple groups is thus tied up intimately with the geometry and analysis of these flag spaces. The terminology derives from the fact that for $G = \mathrm{SL}(n)$ they are the spaces of actual flags. In this case the maximal parabolic subgroups are the ones that leave a fixed subspace invariant, and so we get the grassmannians. The geometry of the parabolic subgroups in the general case is thus a far-reaching generalization of classical projective geometry (Tits geometries) [F dV].

The group of K-points of a semi simple group defined over K, a p-adic field, is locally compact and second countable, and its structure is important for its infinite dimensional representation theory. Maximal compact subgroups (for example, $\mathrm{GL}(n, \mathbf{Z}_p) \subset \mathrm{GL}(n, \mathbf{Q}_p)$) exist, but they are not always conjugate. The structures have a strong combinatorial component ("buildings") [Br T]. For the basics of the *general* theory of Lie groups over all local fields see [S2].

The irreducible representations. For the geometer they arise from the Borel-Weil-Bott picture of the cohomology of line bundles over the flag manifold. Over $\mathbf{C}$ the setting is that of the flag manifold $F = G/B = U/T$. Here G is a simply connected complex semi simple group, B is a Borel subgroup of G, U is a compact form of G, and T is a maximal torus of U with $T = U \cap B$. Then the characters of T, which can be identified with algebraic characters of B, give rise to line bundles on F. The resulting action of the groups G or U on the cohomologies of the line bundles gives rise to the irreducible representations [Bo].

Super Lie groups. The notion of a super manifold was created by the physicists in the 1970's. Confronted with the failure to erect divergence-free quantum field theories they suggested that this was partly due to the failure of conventional pictures of space time in ultra-small regions. In particular they conceived of the idea that the local algebras of space time must be $\mathbf{Z}_2$-graded (=super) algebras that reflect the fermionic structure of matter (isomorphic to $C^\infty(x_1, \ldots, x_p, \xi_1, \ldots, \xi_q)$ where the x_i are the usual commutative local coordinates and the ξ_j are grassmann variables). The *super Lie groups* are the group objects in the category of super manifolds. In the theory of super Lie groups one is forced to use the view points of the theory of *group schemes* systematically [DM] [V4] [Wat]. For *unitary representations* of super Lie groups from this point of view, with applications to super particle classification, see [CCTV].

Almost immediately after the discovery of super symmetry some special super Lie algebras were also discovered by the physicists (super Poincaré, $\mathfrak{sl}(4|1)$, see [V4]). Kac [Ka2] then obtained a classification of the simple super Lie algebras.

Quantum groups. The notion of a *quantum group* arose from the idea that quantum mechanics is a *deformation* of classical mechanics, namely, there is an essentially unique deformation of the Lie algebra of smooth functions on phase space with the *Poisson Bracket* [Mo] [BFFLS]. Given this point of view it is natural to ask whether the symmetry groups of classical geometry can also be deformed into interesting objects. In the 1980's such a theory of deformations emerged, under the impulses of several groups of people. Since classical semi simple Lie algebras are classified by *discrete data*, they are *rigid*. So, in order to deform them one must enlarge the category. The idea is to work in the wider category of general Hopf algebras [Dr] [Wo]. For thorough accounts with full references see [CP] [Kas] [Lu].

Infinite dimensional representations of semisimple Lie groups and Lie algebras. In order to complete this bird's eye view of the subject I would like to add a few remarks on infinite dimensional representation theory. The beginnings of this theory go back to the work of *Bargmann* (1908–1989), *Gelfand* (1913–) and *Naimark* (1909–1978) (see [V2]). In the early 1950's Harish-Chandra began his monumental study of the representations of all *real* semi simple Lie groups. His work led to a categorical equivalence between unitary irreducible representations of G and certain modules of the Lie algebra, and to the existence of a *character*, nowadays called the *Harish-Chandra character*, for the irreducible unitary representations. The character is a *distribution* on the group; it is the sum, in the weak topology of distributions, of the diagonal matrix coefficients, determines the representation, and is an eigen distribution for the algebra of bi-invariant differential operators on the group. By a deep study of these distributions Harish-Chandra constructed the representations of the *discrete series* (the building blocks of infinite dimensional representation theory) by *explicitly constructing their characters*. The Harish-Chandra formulae for the character and formal degree of the discrete series represntations reduce to Weyl's when the group is compact.

There are many expositions of Harish-Chandra's work and other aspects of the theory beside the original papers [H2], for instance [V5] [Wa1] [Wa2] and the reviews by Wallach and by Howe in [H2], Vol. 1. For algebraic aspects see [EV] [E] [EW] [Z]; for geometric methods see [AS] [Sch] [HS]. For the p-adic groups the theory is still incomplete because the discrete series has not been completely constructed. If the ground field is *finite*, the groups are finite and their *complex* representations become interesting. Their theory is deeply influenced by the theory over reals and p-adics. In particlular one can speak of the discrete series [Ha3] [Sp2] and the Whittaker series of Gel'fand-Graev (see [St1]). The general theory needs a deep use of algebraic geometry [DL].

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