3. The concept of a manifold

3.1. Concepts of space and ringed spaces. For over two thousand years the only geometry for space was the Euclidean geometry, until Non Euclidean geometry was created by the efforts of *Bolyai*, *Lobachevsky*, and *Gauss.* However only after *Riemann* was it possible to think about geometry, and the geometry of the space we live in, in a clear and fundamental manner. In his famous Inaugural lecture of 1854 Riemann advanced certain view points that were revolutionary in his time, and are, even now, remarkable. Riemann made the following points[1].

1. *Space, by itself, has no structure except its topology. The geometry of space arises from matter filling space and the material phenomena that take place in space.*
2. *The geometry of space has to be built from its smallest parts accessible to observation.*
3. *The local geometry is to be decided by local observations.*
4. *In ultra small regions of space, concepts such as a light ray or a solid body lose their meaning and one has to be prepared for the possibility that the structure of space in such regions need not be a manifold; in fact one has to investigate seriously alternative structures.*

The discovery of general relativity by Einstein in which the metric of space time is a dynamic object that varies with the matter filling space time, and the ideas of the high energy physicists leading to the creation of new super symmetric models for space time, show how prophetic Riemann's vision was.

In order to start the process of translating some of these views into mathematics the first step is to axiomatize the idea that one has to build space from its smallest parts. The correct way to do this, at least in the first approximation, is by the concept of a *sheaf of functions.* Let X be a topological space. By a *sheaf of functions on X* we mean an assignment

$$\mathcal{O} = \mathcal{O}_X : U \longmapsto \mathcal{O}(U) \quad (U \subset X \text{ open })$$

[1] Riemann's lecture was published after his death in his Collected Papers. There are English translations, one by Clifford (se his Collected Papers) and by Spivak (Vol II of his books on Differential Geometry, pp. 132–153).

such that

(i) $\mathcal{O}(U)$ is a commutative ring of functions on U with numerical values (say in a field k), and containing the constant function 1

(ii) If $V \subset U$ is open, the restriction map from U to V takes $\mathcal{O}(U)$ into $\mathcal{O}(V)$

(iii) If $U_i \subset X$ are open and $U = \bigcup_i U_i$, and if $f_i \in \mathcal{O}(U_i)$ are given such that f_i and f_j coincide on $U_i \cap U_j$, there is a $f \in \mathcal{O}(U)$ such that f coincides with f_i on U_i.

Elements of $\mathcal{O}(U)$ may be thought of as the results of measurements in U. The pair $(X, \mathcal{O})$ is a *ringed space*; for any open $Y \subset X$ the assignment $\mathcal{O}_Y : U \longmapsto \mathcal{O}(U)$ when U varies over open subsets of Y, gives us a ringed space $(Y, \mathcal{O}_Y)$ which is called the *open subspace* of $(X, \mathcal{O}_X)$. Once we decide what the structure of the ringed spaces $\mathcal{O}_Y$ is for small Y we have the notion of space. We refer to the ringed spaces Y for small Y as *local models.* If $(U, \mathcal{O}_U)$ is a ringed space, a ringed space $(X, \mathcal{O}_X)$ has $(U, \mathcal{O}_U)$ as a local model if we can cover X by open sets U_i such that $(U_i, \mathcal{O}_{U_i})$ is isomorphic to $(U, \mathcal{O}_U)$.

3.2. Manifolds with various types of smoothness. We look at some examples.

Manifolds of class $C^r (0 \leq r \leq \infty)$. Here $k = \mathbf{R}$ and the local model is $(U, \mathcal{C}^r)$ where $U \subset \mathbf{R}^m$ is open and $\mathcal{C}^r$ is the sheaf $V \longmapsto C^r(V)$ where $f \in C^r(V)$ if and only if it has continuous derivatives of order up to and including r. Here r is finite; for $r = \infty$, $C^\infty(V)$ is simply the class of C^∞ functions on V. An equivalent way to define the C^r manifolds is to cover X by an atlas of charts which are linked by C^r maps.

k-analytic manifolds where k is a complete field with absolute value. An *absolute value on a field k* is a function $|\cdot|$ from k into nonnegative reals with the following properties:

(a) $|0| = 0, |1| = 1, |x| > 0$ for $x \neq 0$; moreover there is $a \in k$ such that $0 < |a| < 1$.

(b) $|xy| = |x||y|$

(c) $|x + y| \leq |x| + |y|$

The condition (a) ensures that the topology of k is non discrete. The function $d(x, y) = |x - y|$ then converts k into a metric space and we require k to be *complete* in what follows. If $U \subset k^m$ is open we write

$A(U)$ for the space of functions which are analytic on U; here a function f defined on U with values in k is *analytic* if for each $x_0 \in U$ there is a *convergent power series* about the point x_0 that represents the function f in a polydisk around x_0. More precisely, let $r = (r_1, r_2, \ldots, r_m)$ be a polyradius ($r_i > 0$ for all i) and

$$D(r) = \{t = (t_1, t_2, \ldots, t_m) \in k^m, |t_i| < r_i\};$$

then

$$f(x_0 + t) = \sum_{k \geq 0} c_k t^k \qquad (t \in D(r)).$$

here we abbreviate as usual $(k_1, \ldots, k_m), c_{k_1,\ldots,k_m}, t_1^{k_1} \ldots t_m^{k_m}$ into k, c_k, t^k as usual. Multiplication of convergent series tells us that $A(U)$ is a k-algebra. The standard argument of translating a power series shows that if f is given by a convergent power series in $D(r)$, then it is analytic on $D(r)$.

The usual examples are when $k = \mathbf{R}, \mathbf{C}$ with the standard absolute values to give us the real and complex analytic manifolds. The complex analytic manifolds are usually defined differently; the analyticity is defined through the Cauchy-Riemann equations. But Weierstrass showed that the method of defining analytic functions through convergent power series will form a perfectly adequate foundation for the theory of analytic functions and analytic varieties. For arbitrary fields it is the *only* method available.

One of the most interesting examples is obtained when $k = \mathbf{Q}_p$, the field of p-adic numbers. The p-adic absolute value $|\cdot|_p$ on $\mathbf{Q}$ (the rationals) is defined by

$$|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-a} & \text{if } p = p^a \frac{m}{n} \text{ where } a, m, n \in \mathbf{Z}, m, n \text{ are prime to } p. \end{cases}$$

We also write

$$a = v_p(x).$$

It is standard that $|\cdot|_p$ is an absolute value on $\mathbf{Q}$ which satisfies the triangle inequality in a sharp form:

$$|x + y|_p \leq \max(|x|_p, |y|_p).$$

An absolute value with this property (on any field) is called *ultrametric* or *non-archimedean*. The completion of $\mathbf{Q}$ with respect to $|\cdot|_p$ is $\mathbf{Q}_p$.

There are two basic results which one has to keep in mind. The first is that if k is a complete field with absolute value, then either $k = \mathbf{R}, \mathbf{C}$ or else k is non-archimedean. The second is that up to replacing an absolute value by a power of it, any absolute value on $\mathbf{Q}$ is either the standard one or the p-adic for some prime p (Ostrowski's theorem).

Remark 1. The *archimedean axiom* says that if a, b are *any* two non-zero elements of a field with absolute value, we can find an integer $N \geq 1$ such that $|Na| > |b|$. The real and complex fields are thus archimedean. The ultrametric fields are not; if $|a| \leq 1, |b| > 1$, then $|Na| \leq 1$ for all integers N. In physics the non-archimedean field became interesting when Igor Volovich, in a famous paper in the late 1980's, suggested that for studying fundamental phenomena, one should perhaps explore space times based on a non-archimedean geometry. His reasoning (roughly speaking) was that at the Plank scale (10^{-33} cm.), as no observations are possible, length measurements make no sense and so the archimedean principle is invalid. Quantum field theory based on non-archimedean space times (over *finite* or p-adic fields) leads to very interesting and open mathematical problems. On the historical side, Hilbert was the first to construct examples of non-archimedean geometries.

Remark 2. The topology on $\mathbf{Q}_p$ is *totally disconnected* (t.d); this means that the family of sets which are at the same time open and closed, forms a basis for the topology. In fact the balls $B(t, r) = \{x \in \mathbf{Q}_p, |x - t| \leq r\}$ are compact and open for each $t \in \mathbf{Q}_p, r > 0$ and form a basis for the topology. Th $\mathbf{Q}_p$ are thus locally compact.

The fields $\mathbf{Q}_p$ have the power of the continuum. Let $\mathbf{Z}_p$ be the closure of the ring of integers. the elements of $\mathbf{Z}_p$ are called *p-adic integers* and they form a compact open subring. Let us take a set of residues mod p, say $\{0, , 2, \ldots, p-1\}$. Then the elements of $\mathbf{Z}_p$ have *unique* expansions of the form

$$x = a_0 + a_1 p + a_2 p^2 + \ldots, \qquad (a_i \in \{0, 1, \ldots, p-1\}.$$

So it is obvious that $\mathbf{Z}_p$, hence $\mathbf{Q}_p$, has the power of the continuum. One can solve many equations over $\mathbf{Q}_p$ that cannot be solved over $\mathbf{Q}$; for instance, we can always solve $X^2 = A$ for any integer A if we can solve it mod p. As an example, consider $p \equiv 1 \mod 4$ and $X^2 = -1$, which is solvable mod p (Euler), or more generally, $X^2 = A$ where A is a quadratic residue mod p. This is a consequence of *Hensel's lemma.* Hensel was a great non-archimedean analyst, one of a long line of distinguished non-archimedean analysts from Germany such as Kummer, Hensel, Witt, etc.

Remark 3. Over ultrametric fields the principle of *analytic continuation* fails for analytic functions: the characteristic function of a closed-open set is analytic. In the 1970's John Tate overcame this difficulty by creating a beautiful theory of *rigid analytic spaces.*

Remark 4. Let us return to the function v_p defined on $\mathbf{Q}$ earlier: for a rational number x, $v_p(x) = a$ means that $x = p^a(m/n)$ where m, n are prime to p; we define $v_p(0) = \infty$. Then v_p has the following properties that correspond to the properties of the absolute value:

(a) $v_p(x)$ is a finite integer for $c \in \mathbf{Q}^\times$ and $v_p(p) = 1$

(b) $v_p(xy) = v_p(x) + v_p(y)$

(c) $v_p(x + y) \geq \min(v_p(x), v_p(y))$.

A function such as this on a field is called a *valuation.* Ostrowski's theorem can now be restated as the assertion that the only valuations on $\mathbf{Q}$ are the v_p for various primes p. Thus the absolute values or the valuations determine completely the arithmetic of $\mathbf{Q}$. In the noneteenth century, even great mathematicians like Dirichlet and Gauss were stuck becasue of the failure of unique factorization theorem of number theory in more general number fields. Gauss had studied $\mathbf{Q}[\sqrt{-1}]$ which admits unique factorization, but, defining prime numbers in number fields in imitation of the ordinary numbers leads to unpleasant non uniqueness of prime factorization:

$$6 = 2 \times 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}).$$

It was Kummer who realized that unique factorization can be restored if one identifies primes with valuations. In a great series of papers he constructed *all* the valuations of the cyclotomic fields $\mathbf{Q}[e^{2\pi i/\ell}]$ and laid the founations of modern algebraic number theory. Kummer's theory was later generalized to include *all* algebraic number fields by Kronecker and Dedekind.

The methods of p-adic or more generally, non-archimedean, analysis offers a more transcendental approach to the study of arithmetical questions. The great power of this approach is due to the local compactness of the $\mathbf{Q}_p$ and their finite extensions, so that one has a full-fledged Fourier transform theory on them.

Remark 5. The proof that a convergent power series defines an analytic function inside the polydisk of its convergence goes as follows. I give it in one variable, but the proof needs no change in several variables. Consider a series

$$f(z) = \sum_{n \geq 0} c_n z^n$$

convergent for $|z| < r$ and fix z_0 with $|z_0| < r$. Choose r_1, r_2 with $|z_0| < r_1 < r_2 < r$, and write $z = z_0 + h$ where $|h| < r_1 - |z_0|$. Then

$$f(z_0 + h) = \sum_{m \geq 0} b_m h^m, \qquad b_m = \sum_{n \geq m} c_n \binom{n}{m} z_o^{n-m}.$$

The main argument is that everything converges if we replace all the quantities by their absolute values, and use $|r.1| \leq r$ for $r = 0, 1, 2, \ldots$. If $|c_n| r_2^n \leq M$ then

$$\begin{aligned} \sum_{n \geq m \geq 0} |h|^m |c_n| \binom{n}{m} |z_0|^{n-m} &\leq M \sum_{n \geq m \geq 0} r_2^{-n} |z_0|^{n-m} \binom{n}{m} (r_1 - |z_0|)^m \\ &\leq M \sum_{n \geq 0} (r_1/r_2)^n < \infty. \end{aligned}$$

For the basic aterial on k-analytic manifolds see Serre's Harvard lecture Notes *Lie Algebras and Lie groups.*

Problems

1. Prove the analyticity of a function defined by a power series when the number of variables is > 1.
2. Prove from first principles that if we can solve the equation $X^2 \equiv A \bmod p$ where p is an odd prime and A is a fixed integer, then we can find a solution in $\mathbf{Q}_p$. (*Hint*: If there is $x_n \in \mathbf{Z}$ such that $x_n^2 \equiv A \bmod p^n$, show that we can find $x_{n+1} \in \mathbf{Z}$ such that $x_{n+1} \equiv x_n \bmod p^n$ and $x_{n=1}^2 \equiv A \bmod p^{n+1}$; then $x = \lim x_n$ exists in $\mathbf{Q}_p$ and satisfies $x^2 = A$.)
3. Verify that the expansion in powers of p of p-adic integers does *not* give an isomorphism of $\mathbf{Z}_p$ with a formal power series ring (there is the "carrying over"phenomenon characteristic of multiplying or adding numbers in a fixed scale). But if F is a finite field prove that the formal Laurent series in an indeterminate T with coefficients in F is a complete field $F((T))$ with an absolute value such that $|T| < 1$.

4. Show how to construct products in the category of k-analytic manifolds.
5. Prove that the $\mathbf{Q}_p$ as well as $F((T))$ are locally compact.

Algebraic (*Serre*) *varieties.* The modern foundations of algebraic geometry were laid by Zariski and Weil. But in the procesxs a large part of the geometric intuition was lost, and it was Serre who restored it substantially. In a famous paper *Faisceaux algbriques cohrents* in the Annals in 1955 he showed how an algebraic variety could be defined in a manner very similar to a differentiable manifold, using the Zariski topology instead of the usual one. We call these the *Serre varieties.* They are adequate to study almost all parts of the theory of linear algebraic groups as well as large parts of classical algebraic geometry. A little while after Serre's paper appeared Grothendieck introduced the notion of a *scheme* that finally became the definitive foundation for algebraic geometry.

A Serre variety is a ringed space of functions $(X, \mathcal{O})$; the only point to decide is the choice of the local model. The main departure from the theory of differentiable or analytic manifolds is the fact that the local models are *not* open (even Zariski open) subsets of k^m but affine varieites. I shall elucidate this point now.

Let k be an algebraically closed field of arbitrary characteristic. WE define an *algebraic set* to be a subset of some k^m defined as the set of zeros of a set of polynomial functions on k^m. The functions on the algebraic sets are the restrictions to the set of the polynomials on the ambient space. On k^m we have the *Zariski topology* a basis of which consists of the sets of the form $\{x \in k^m \mid P(x) \neq 0\}$ where P is a polynomial. An algebraic set in k^m inherits the relative Zariski topology. One can use Hilbert's Nullstellensatz to defiine the algebraic sets with theor Zariski topology in a coordinate-independent manner as follows. We start with an *affine algebra* A, namely a finitely generated k-algebra which is *reduced* in the sense that it has no nonzero nilpotents. Then $\mathrm{Spec}(A)$ as the set of its homomorphisms into k,

$$X_A := \mathrm{Spec}(A) := \mathrm{Hom}(A, k).$$

For each $a \in A$ we define

$$\hat{a}(\chi) = \chi(a) \qquad (\chi \in \mathrm{Spec}(A)),$$

thus allowing us to view each a as a function on $\mathrm{Spec}(A)$, namely the function $\chi \longmapsto \chi(a)$. The map $a \longmapsto \hat{a}$ is a homomorphism of A into the k-algebra of all functions on $\mathrm{Spec}(A)$; the fact that A is reduced implies that this map is *injective.* On $\mathrm{Spec}(A)$ we have the *weak* or *Zariski topology*, namely the topology whose open sets are unions of sets of the form $\Omega_a := \{\chi \mid \hat{a}(\chi) \neq 0\}$. For any open $U \subset X_A$ the elements f of $\mathcal{O}(U)$ are the functions $f \longmapsto k$ such that for each $\chi \in U$ there is some $a \in A$ with the property that $\Omega_a \subset U$ and $f(\chi) = \hat{b}(\chi/\hat{a}^n(chi)$ for a suitable $b \in A$ and integer $n \geq 0$. It can be shown that this is a sheaf and converts X_A into a ringed space, called the *affine Serre variety.* . A Serre variety is then a ringed space such that for some *finite covering* (U_i), each of the open subspaces defined by the U_i is isomoprhic to an affine Serre variety (plus a separation axiom)[2]

The notion of a ringed space of functions needed to be generalized before the definitive treatment of algebraic geometry was possible. This was achieved by Grothendieck who replaced the sheaf of rings of functions by an *abstract sheaf of rings*. The ringed space is now a pair $(X, \mathcal{O})$ where for each open U, $\mathcal{O}(U)$ is a commutative ring with unit; the restriction maps $\mathcal{O}(U) \longrightarrow \mathcal{O}(V)$ have now to be specified as part of the data defining the ringed space. In defining the morphisms the pull-back maps have also to be specified as part of the data defining the morphisms.

Super manifolds. In the 1970's the physicists, propelled by a desire to build divergence-free quantum field theories, began the exploration of new models for space time based on the newly discovered concept of *super symmetry.* A super symmetric manifold, or a *super manifold* for short, ias a new type of manifold whose local cooedinates include, ina ddition to the usual ones, a set of *Grassmann* variables. The Grassmann variables were intended to encode the *fermionic structure* of matter. The point here is that matter is composed of fermionic particles like the electrons, protons, neutrons and so on, which obey the so-called *exclusion principle of Pauli*: *no two fermions occupy the same quantum state*. Mathematically, a super manifold is a ringed space $(X, \mathcal{O})$ where X has the structure of a smooth manifold in the classical sense and the rings $|oo(U)$ are mildly non-commutative, of the form $C^\infty(U) \otimes \Lambda[\xi_1, \xi_2, \ldots, \xi_n]$, the ξ_j being indeterminates satisfying

$$\xi_r \xi_s + \xi_s \xi_r = 0 \qquad (r, s = 1, 2, \ldots, n).$$

[2] For a very nice treatment of Serre varieties see Volume 2 of J. Dieudonne's beautiful little book *Cours de géometrie algébrique*; the first volume of this book gives a nice historical perspective on algebraic geometry.

Notice that the local rings $\mathcal{O}(U) = C^\infty(U) \otimes \Lambda[\xi_1, \xi_2, \ldots, \xi_n]$ are non-commutative. Moreover, the Grassmann variables become 0 when evaluated so that the ringed space is *not* a ringed space of functions. So the techniques of working with super manifolds have to borrow quite a bit from the Grothendieck theory of schemes.

Problems

1. Explicitly construct projective space $\mathbf{P}^m(k)$ as a Serre variety.
2. Explicitly describe the curve $y^2 = x^3 + x$ as an affine variety in k^2 and as a projective variety in $\mathbf{P}^2(k)$.
3. If k is a field with absolute value, show that a rational function is analytic. Can you construct analytic functions which are not rational?
4. Show that the ringed space $\mathbf{R}^{m|n} := (\mathbf{R}^m, \mathcal{O}^{m|n})$ is a super manifold if $\mathcal{O}^{m|n}(U) = C^\infty(U) \otimes \Lambda[\xi_1, \ldots, \xi_n]$.