9. The Lie group–Lie algebra correspondence

9.1. The functor Lie. The fundamental theorems of Lie concern the correspondence $G \longmapsto \mathrm{Lie}(G)$. The work of Lie was essentially *local* and led to the following fundamental theorems, usually known as *the fundamental theorems of Lie.*

1. The Lie algebra of a group is a complete invariant of the local group structure. This means that two Lie groups have isomorphic Lie algebras if and only if the groups are locally isomorphic.

2. Every Lie algebra of finite dimension is the Lie algebra of at least a local group.

As the concept of a *global* Lie group became better understood, the goal became one of proving these results with appropriate modifications when we replace local groups by global Lie groups. Since the Lie algebra needs only the connected component of the identity for its construction it is natural to consider only *connected* Lie groups. We shall see presently that the assignment

$$\mathbf{Lie} : G \longmapsto \mathrm{Lie}(G)$$

is *functorial*. The theorems of Lie in their modern incarnation emerge out of the attempt to see how close this functor is to be an *equivalence of categories.*

Categories and functors. The importance of the *categorical view point* was stressed above all by Grothendieck. I shall adopt an informal approach. Categories consist of *objects* and *maps* or *morphisms* between objects. If $\mathcal{C}$ is any category, and O_1, O_2 are two objects in $\mathcal{C}$, the maps or morphisms from O_1 to O_2 form a *set* denoted by $\mathrm{Hom}(O_1, O_2)$. The crucial notion is that of a functor from one category to another. A *covariant functor* from a category $\mathcal{C}_1$ to a category $\mathcal{C}_2$ is a pair of assignments

$$O \longmapsto F(O)$$

from objects of $\mathcal{C}_1$ to objects of $\mathcal{C}_2$ and, for any pair of objects O, O' in $\mathcal{C}_1$ a map

$$F_{O,O'} : \mathrm{Hom}_1(O.O') \longrightarrow \mathrm{Hom}_2(F(O), F(O'))$$

such that compositions correspond.

Functors first became prominent in algebraic topology where they occur, via cohomology or homology, taking topological objects into algebraic objects like abelian groups. Let **LG** be the category of connected Lie groups and **LA** the category of finite dimensional Lie algebras over the base field (**R** or **C**).

Theorem 1. *The assignment*

$$\mathbf{Lie} : G \longmapsto \mathrm{Lie}(G)$$

is a functor from the category **LG** *of connected Lie groups to the category* **LA** *of finite dimensional Lie algebras over the base field* (**R** *or* **C**)*, and* $\dim(G) = \dim(\mathrm{Lie}(G))$*.*

Proof. We must show that if $f(G_1 \longrightarrow G_2)$ is a morphism of Lie groups, there is a morphism, say $df(\mathfrak{g}_1 \longrightarrow \mathfrak{g}_2)$ associated to f and $d(f \circ g) = df \circ dg$; here $\mathfrak{g}_i = \mathrm{Lie}(G_i)$. If $X \in \mathfrak{g}_1$, the tangent map of f applied to X_{e_1} gives a tangent vector v to G_2 at e_2, and hence an element $X' \in \mathfrak{g}_2$ such that $X'_{e_2} = v$. We write $X' = df(X)$. It is clear that df is linear and that $d(f \circ g) = df \circ dg$. It only remains to show that df preserves brackets. Since f preserves multiplication it commutes with left translations:

$$\ell(f(x_1) \circ f = f \circ \ell(x_1) \qquad (x_1 \in G_1).$$

It follows from this that for any function $\psi \in \mathcal{O}_{G_2}(U)$, we have

$$f^* X' \psi = X f^* \psi \quad (\text{on } \mathcal{O}_{G_1}(f^{-1}(U))$$

In other words the map f intertwines the derivations X and X'. It is then immediate that if $Y \in \mathfrak{g}_1, Y' = df(Y)$, then f intertwines $[X, Y]$ and $[X', Y']$.

Corollary. *We have*

$$f \exp tX) = \exp df(X) \qquad (X \in \mathfrak{g}_1).$$

This corollary suggests a method to try to show that any morphism $\alpha(\mathfrak{g}_1 \longrightarrow \mathfrak{g}_2)$ arises as the df of a morphism $f(G_1 \longrightarrow G_2)$. We simply define f near e_1 by

$$f(\exp X) = \exp df(X).$$

The BCH formula would then imply that f is a *local homomorphism*. Since G_1 is connected we can write any element $x \in G_1$ as a product of exponentials $\exp X$ where X is chosen from a neighborhood $\mathfrak{n}_1$ of 0 in $\mathfrak{g}_1$. We then define $f(x)$ as the product of the corresponding exponentials in G_2:

$$x = \exp X_1 \ldots \exp X_n, \qquad f(x) = \exp X_1' \ldots \exp X_n', \quad (X_i' = df(X_i)).$$

Unfortunately the expression for x is not unique and one has to prove that the definition is independent of the choice of the representation of x.

The situation here is reminiscent of what happens in complex function theory where we have multi-valued functions. There the solution to eliminating multi-valuedness lies in simple connectivity. The same is true here also. If G_1 is simply connected we can extend the local homomorphism to a *global homomorphism*:

Theorem 2. *If G_1 is simply connected, any morphism $\alpha(\mathfrak{g}_1 \longrightarrow \mathfrak{g}_2)$ is of the form df for a unique morphism $f(G_1 \longrightarrow G_2)$.*

What happens when G_1 is not simply connected? To understand this we need to give an informal discussion of *covering groups.*

9.2. Covering groups. Although the notions of covering spaces and groups are usually treated in topology in a very wide context, I shall restrict myself to manifolds and Lie groups. A covering manifold of a *connected* manifold X is a pair $(X^\sim, \pi)$ where $X^\sim$ is a connected manifold, π is continuous, and $X^\sim$ is a discrete fiber bundle over X, i.e., for each point $x \in X$ there is an open neighborhood U such that $\pi^{-1}(U) \simeq U \times D$ where D is discrete, and the map $\pi^{-1}(U) \longrightarrow U$ goes over to the projection $U \times D \longrightarrow U$. It is obvious that the smooth structure of X can be transported to $X^\sim$ via π so that $X^\sim$ becomes a smooth manifold and π a smooth map which is a local diffeomorphism. X is *simply connected* if for any covering $(X^\sim, \pi)$, π is a homeomorphism, i.e., $\pi : X^\sim \simeq X$. It is known that X is simply connected if and only if all closed paths in X are homotopic to the constant path and that for any manifold we can construct a simply connected cover which is essentially unique. If X is a Lie group, $(X^\sim, \pi)$ is a simply connected cover, and $1^\sim \in X^\sim$ is above $1 \in X$, we can construct a unique group structure on $X^\sim$ such that π is a homomorphism. We then speak of $X^\sim$ as the covering group of X and π as the covering homomorphism. It is easily proved that $X^\sim$ is a Lie group.

The kernel of π is then a *discrete central* subgroup of $X^{\sim}$. The centrality is seen as follows. Indeed, if D is the kernel, the action of $X^{\sim}$ on itself by inner automorphisms leaves D stable since D is normal, and so gives rise to an action on D; the orbits for this action are connected subsets of a discrete space and so are singletons, i.e., for any $d \in D$, $gdg^{-1} = d$ for all $g \in X^{\sim}$.

Theorem 1. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let $G^{\sim}$ be a simply connected Lie covering group of G. Then the Lie groups G_1 with Lie algebras isomoprhic to $\mathfrak{g}$ are all obtained as $G_1 = G^{\sim}/D$ where D varies over the discrete central subgroups of $G^{\sim}$.*

If $G = \mathbf{R}^n$ and $D = \mathbf{Z}^n$, then $T^n = \mathbf{R}^n/\mathbf{Z}^n$ is covered by $\mathbf{R}^n$; this is the classical example of covering groups. The simplest and most common example of covering groups in the nonabelian world is the covering exact sequence

$$1 \longrightarrow \mathbf{Z}_2 = \{\pm 1\} \longrightarrow \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3) \longrightarrow 1$$

which we have already discussed. The group SU(2) can be realized as the group of *quaternions of unit norm.* One can generalize this aspect of quaternion algebras and introduce the *Clifford algebras.* For any integer $n \geq 2$, the *Clifford algebra* $\mathcal{C}_n$ is the algebra with generators

$$e_1, e_2, \ldots, e_n$$

and relations

$$e_i^2 = -1, \quad e_i e_j = -e_j e_i (i \neq j).$$

One can then find a subgroup Spin(n) sitting inside the group of invertible elements of $\mathcal{C}_n$ and a map $\pi : \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$ such that Spin(n) is simply connected and the kernel of π is $\{\pm 1\}$:

$$1 \longrightarrow \mathbf{Z}_2 = \{\pm 1\} \longrightarrow \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n) \longrightarrow 1.$$

The Spin(n) are the *spin groups* and go back to Elie Cartan; the terminology came after Dirac discovered independently the Clifford algebra in dimension 4 and used it spectacularly to write down the relativistic equation satisfied by the electron which revealed the spin structure of the electron.

As another example we have $\mathrm{Spin}(6) \simeq \mathrm{SU}(4)$, namely,

$$1 \longrightarrow \mathbf{Z}_2 = \{\pm 1\} \longrightarrow \mathrm{SU}(4) \longrightarrow \mathrm{SO}(6) \longrightarrow 1.$$

This comes out of the classical geometry of lines in projective space and the theory of the so-called *Klein quadric.*

9.3. Inverting the functor Lie. The functor **Lie** cannot be inverted because locally isomorphic Lie groups have isomorphic Lie algebras. But *on the subcategory of simply connected Lie groups it can be inverted.* The essential surjectivity of the functor is called the third fundamental theorem of Lie, namely, every Lie algebra of finite dimension over $K (K = \mathbf{R}, \mathbf{C})$ is isomorphic to the Lie algebra of a Lie group. It was first proved in full generality by Elie Cartan. We shall discuss two ways of proving it: by using Ado's theorem, or the Levi-Malcev decomposition. Both routes need a substantial amount of Lie algebra theory.

Theorem 1. *If $\mathfrak{g}$ is a Lie algebra of finite dimension over $k = \mathbf{R}, \mathbf{C}$, there is a connected simply connected Lie group G such that $\mathfrak{g} = \mathrm{Lie}(G)$. G is determined uniquely up to isomoprhism. More precisely,* **Lie** *is an equivalence of categories from the subcategory* **SLG** *of connected simply connected Lie groups of the category* **LG** *to the category* **LA** *of finite dimensional Lie algebras over the base field.*

Elie Cartan was the first to prove this global version of the third fundamental theorem of Lie. There are at least two ways to do this. Both use Lie algebra theory at a very deep level.

Proof based on Ado's theorem. Ado's theorem says that if $\mathfrak{g}$ is a finite dimensional Lie algebra over a field k of characteristic 0, then there is a faithful finite dimensional representation of $\mathfrak{g}$, namely an imbedding $\mathfrak{g} \hookrightarrow \mathfrak{gl}(N, k)$ for some N (Ado's proof had gaps; the first complete proof is due to Harish-Chandra). So we may assume that $\mathfrak{g} \subset \mathfrak{gl}(N, k)$. If $k = \mathbf{R}$ or $\mathbf{C}$, the Lie subgroup $G \subset \mathrm{GL}(N, k)$ defined by $\mathfrak{g}$ is then a Lie group with Lie algebra $\mathfrak{g}$. Its universal cover is then a simply connected Lie group with Lie algebra $\mathfrak{g}$.

Proof based on Levi-Malcev decomposition. First of all the adjoint representation of $\mathfrak{g}$ gives a morphism $\mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ with kernel equal to the center of $\mathfrak{g}$. So, if $\mathfrak{g}$ has trivial center, we have an imbedding $\mathfrak{g} \hookrightarrow \mathfrak{gl}(\mathfrak{g})$ and we can proceed as before. This takes care of the case when $\mathfrak{g}$ is semisimple. The Levi-Malcev theorem says that any Lie algebra is a *semidirect sum* $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{m}$ where $\mathfrak{s}$ is a *solvable* ideal and $\mathfrak{m}$ is a *semisimple subalgebra.* Let M be the simply connected Lie group with Lie algebra $\mathfrak{m}$

and suppose we can find a simply connected Lie group S with Lie algebra $\mathfrak{s}$. The adjoint action of $\mathfrak{m}$ on $\mathfrak{s}$ (remember $\mathfrak{s}$ is a Lie ideal in $\mathfrak{g}$) may be viewed as a morphism $\mathfrak{m} \longrightarrow \mathrm{Der}(\mathfrak{s})$ where $\mathrm{Der}(\mathfrak{s})$ is the Lie algebra of derivations of $\mathfrak{s}$. So we can lift this to a morphism of M into the group of automorphisms of $\mathfrak{s}$. But because S is simply connected, the group of automorphisms of $\mathfrak{s}$ is the same as the group of auotmorphisms of S. We may then form the semidirect product $S \times' M$ which will have as Lie algebra the semidirect sum $\mathfrak{s} \oplus \mathfrak{m} \simeq \mathfrak{g}$.

It thus remains to settle the theorem for $\mathfrak{s}$, namely when $\mathfrak{g}$ is itself solvable. We then have a filtration

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \ldots \supset \mathfrak{g}_m \supset 0$$

where $\mathfrak{g}_i$ is an ideal of codimension 1 in $\mathfrak{g}_{i-1}$. So we can write

$$\mathfrak{g}_{i-1} = \mathfrak{g}_1 \oplus kX_i$$

where X_i is any element of $\mathfrak{g}_{i-1} \backslash \mathfrak{g}_i$, and the sum is semidirect. The preceding argument using semidirect products is applicable if we use induction on i.