8. The Lie algebra and the exponential map for general Lie groups

8.1. The Lie algebra. We shall show how one can associate to any Lie group G its Lie algebra $\mathrm{Lie}(G)$. We work over $\mathbf{R}$ or $\mathbf{C}$. To begin with let M be any smooth manifold. Then one know that the set of vector fields on M is a Lie algebra, generally infinite dimensional. If D is a set of automorphisms of M, the vector fields that are invariant under the diffeomorphisms of D form a Lie subalgebra of the Lie algebra of vector fields. If we take $M = G$ and D to be the set of left translations $\ell_x (x \in G)$, we get the Lie algebra $\mathrm{Lie}(G)$ of all vector fields on G which are invariant under all left translations–the so-called left invariant vector fields. We call $\mathrm{Lie}(G)$ the *Lie algebra of* G.

If $X \in \mathrm{Lie}(G)$ and X_e is the tangent vector defined by X at the identity element e of G, left invariance requires that $X_x = d\ell_x(X_e)$ so that X is completely determined by X_e. Thus $X \longmapsto X_e$ is an injection of $\mathrm{Lie}(G)$ into the tangent space $T_e(G)$ to G at e. This is actually a bijection. In fact, if $v \in T_e(G)$ and we define $X_x = d\ell_x(v)$ for $x \in G$, we get a left invariant assignment $x \longmapsto X_x$ and it is a vector field. To see this we must show only smoothness and enough to verify smoothness near e. Take coordinates (x_i) near e such that e goes to 0 and let F_i be the smooth functions expressing multiplication: if $u = st$ and (x_i) (resp. $(y_i), (z_i)$) are the coordinates of s (resp. t, u), then

$$z_i = F_i(x_1, \ldots, x_n, y_1, \ldots, y_n) \qquad (1 \le i \le n).$$

We may assume that $v = (\partial/\partial x_i)_0$. It is a question of showing that $t \longmapsto X_t(st)$ is smooth in t, which comes to verifying that

$$\frac{\partial F_i}{\partial x_i}(0, \ldots, 0, t_1, \ldots, t_n)$$

is smooth in the t_i which is obvious. We thus have

Theorem 1. *The map $X \longmapsto X_e$ is a linear isomorphism of* $\mathrm{Lie}(G)$ *with* $T_e(G)$.

Example 1: The case $G = \mathrm{GL}(N, k)$. The ground field is k which is $\mathbf{R}$ or $\mathbf{C}$. Then G is an open subset of k^{N^2} and the matrix entries a_{ij} are the global coordinates on G. Any tangent vector at I is of the form

$\sum_{ij} x_{ij}\partial/\partial a_{ij}$ and so can be identified with the matrix $X = (x_{ij})$. Let $\bar{X}$ be the element of $\mathfrak{g} := \mathrm{Lie}(G)$ such that X is the tangent vector defined by it at the identity element I. Let $\mathfrak{gl}(N, k)$ be the Lie algebra of $N \times N$ matrices with entries from k with the bracket as the matrix commutator. We wish to show that the map

$$X \longmapsto \bar{C}$$

described above is a *Lie algebra isomorphism* of $\mathfrak{gl}(N, k)$ with $\mathfrak{g}$. It is a question of computing $\bar{X}, \bar{Y}, [\bar{X}, \bar{Y}]$ and showing that

$$[\bar{X}, \bar{Y}]_I = [X, Y].$$

To prove this we first compute $\bar{Y}(a_{rs})$. This is just the result of applying the tangent vector Y at I to the function

$$z \longmapsto a_{rs}(gz) = \sum_p a_{rp}(g)a_{ps}(z)$$

leading to the value

$$\sum_{ijp} y_{ij}\delta_{ip}\delta_{js}a_{rp}(g) = \sum_i y_{is}a_{ri}(g)$$

and so

$$\bar{Y}a_{rs} = \sum_i a_{rq}y_{qs}.$$

From this we get

$$(\bar{X}\bar{Y}a_{rs})(I) = \sum_{ijq} x_{ij}\delta_{ir}\delta_{jq}y_{qs} = \sum_j x_{rj}y_{js} = (XY)_{rs}.$$

Thus

$$([\bar{X}, \bar{Y}]a_{rs})(I) = [X, Y]_{rs}$$

proving what we want.

Example 2: *Lie algebra of a closed Lie subgroup.* Let $H \subset G$ be a closed Lie subgroup and let $\mathfrak{h} = \mathrm{Lie}(H)$. The injection $T_e(H) \hookrightarrow T_e(G)$ induces a linear injection $\mathfrak{h} \hookrightarrow \mathfrak{g}$.

Theorem 2. *The map $\mathfrak{h} \longrightarrow \mathfrak{g}$ is a Lie algebra injection.*

Proof. We must verify that the map preserves brackets. This comes down to showing that if $X, Y \in \mathfrak{g}$ are tangent to H at all points of H, the same is true of $[X, Y]$. This is true if G, H are replaced by M, N where M is a manifold and N a submanifold. The result is local and so we may assume that $M = k^a \times k^b, N = k^a \times \{0\}$. Let $x' = (x_r)_{r \leq a}$ be the coordinates on k^a and $x" = (x_s)_{s \geq a+1}$ the coordinates on k^b. If $\sum_i f_j \partial_j$ is a vector field where $\partial_j = \partial/\partial x_j$, then the condition for it to be tangent to N is that $a_r(x', 0) = 0$ for $r \geq a + 1$. If now $X = \sum_j a_j \partial_j, Y = \sum_j b_j \partial_j$, we have $[X, Y] = \sum_s c_s \partial_s$ where

$$c_s = \sum_r (a_r \partial b_s/\partial x_r - b_r \partial a_s/\partial x_r).$$

It is then easy to check that $c_t(x', 0) = 0$ for $t \geq a + 1$; indeed, for $r \geq a + 1$, $a_r(x', 0) = b_r(x', 0) = 0$, while, for $r \leq a$, $(\partial b_t/\partial x_r)(x', 0) = (\partial a_t/\partial x_r)(x', 0) = 0$ because $b_t(x', 0) = a_t(x', 0) = 0$.

Remark. If $X, Y \in \mathrm{Lie}(G)$, the bracket $[X, Y]$ depends only on the values of X and Y in a neighborhood of the identity and so the Lie algebra *is already determined by the connected component of the identity of the Lie group.* So the part of G beyond the component of the identity does not play any role in the Lie algebra. This fact has a rather sharp consequence in physics. Originally it was believed that the laws of physics were invariant under the full Poincaré group which is the semi direct product of the space-time translation group T and the Lorentz group $L := \mathrm{SO}(1, 3)$. Now L is not connected but has 4 connected components respectively containing I, I_s (space reflection $(x_0, x_1, x_2, x_3) \longmapsto (x_0, -x_1, -x_2, -x_3)$), I_t (time reflection $((x_0, x_1, x_2, x_3) \longmapsto (-x_0, x_1, x_2, x_3)$), and I_{st} (space-time reflection $(x_0, x_1, x_2, x_3) \longmapsto (-x_0, -x_1, -x_2, -x_3)$). Space reflection is called *parity* and in the mid 1950's it was discovered in an experiment involving neutrinos performed at Columbia University by Madam Wu and her collaborators at the suggestion of C. N. Yang and T. D. lee (who later won the Nobel Prize for this) that the propagation of neutrinos violates parity. The wave equation for the neutrino which had hitherto been used was then abandoned and replaced by the equation that had been proposed by Hermann Weyl in the 1930's, which had been out of favor till then because it violated parity!

Problems

1. Verify that for $G_1 \times G_2$ the Lie algebra is $\mathfrak{g}_1 \times \mathfrak{g}_2$ where $\mathfrak{g}_i = \mathrm{Lie}(G_i)$ for $i = 1, 2$.

8.2. The exponential map. If M is a manifold and X is a vector field on M, then it is well known that X generates a *local flow.* The flow is obtained by finding the maximal integral curves through the points $m \in M$ which will be defined for $-a(m) < t < b(m)$ where $0 < a(m), b(m) \leq \infty$. The numbers $a(m), b(m)$ depend on m but are locally bounded away from 0 as m varies over compact sets, i.e., for any compact $K \subset M$ there is $\varepsilon = \varepsilon(K) > 0$ such that for all $m \in K$ the integral curve through m is defined for $|t| < \varepsilon$. For the flow to be *global* we should have $a(m) = b(m) = \infty$ for all $m \in M$. In this case the map α_t which takes the point m to the point on the integral curve at time t is a diffeomorphism of M and $t, m \longmapsto \alpha_t(m)$ is a smooth map. In general the local flow is not global, but is so *if the manifold M is compact.* to see this we on=bserve that the integral curves ared efined for $|t| < \varepsilon$ *for all* $m \in M$ and so for all t; one simply keeps going for time $\varepsilon/2$ wherever one is on the manifold. So vector fields generate global flows on a compact manifold. There is another circumstance in which this can be asserted. This is when there is a group D of diffeomorphisms of M that acts *transitively on* M and leve the vector field invariant. Then the $a(m), b(m)$ are independent of m and so by the same argument as in the compact case the integral curve is defined for all t. This case occurs when $M = G$ and $D = \{\ell_x\}_{x \in G}$. Thus:

Theorem 1. *The elements of* $\mathrm{Lie}(G)$ *generate global flows on* G*.*

Write

$$\mathfrak{g} = \mathrm{Lie}(G).$$

Fix $X \in \mathfrak{g}$ and let

$$h(X : \cdot) : t \longmapsto h(X : t), \qquad h(X : 0) = e$$

be the integral curve of X through the identity e. Since X is left invariant, the integral curve through $x \in G$ is

$$t \longmapsto xh(X : t).$$

Furthermore the integral curve through e satisfies the differential equation

$$\frac{dh}{dt} = X_{h(t)}, \qquad h(0) = e$$

for all $t \in \mathbf{R}$. Now the integral curve through $h(s)$ is, on the one hand $t \longmapsto h(s)h(t)$, while it is also $t \longmapsto h(s + t)$. Hence we must have

$$h(X : s + t) = h(X : s)h(X : t) \qquad (s, t \in \mathbf{R}).$$

This shows that $h(X:\cdot)$ is the unique solution to

$$h(s+t) = h(s)h(t), \quad \left(\frac{dh}{dt}\right)_{t=0} = X_e.$$

From this it follows that

$$h(X:st) = h(sX:t) \qquad (s,t \in \mathbf{R}).$$

It also follows that the $h(X:\cdot)$ are precisely all the smooth Lie group morphisms $\mathbf{R} \longrightarrow G$.

If G is a closed subgroup of $\mathrm{GL}(N)$ we can thus identify $h(X:t)$ with $\exp(tX)$. So we define in the general case

$$\exp X = h(X:1).$$

We have

$$\exp tX = h(tX:1) = h(X:t)$$

so that for any $X \in \mathfrak{g}$ the maps

$$t \longmapsto x \exp tX$$

are the integral curves of X through the points $x \in G$. We have thus defined the exponential map in the general case and verified that it coincides with the matrix exponential for the matrix Lie groups.

The exponential map is smooth (in the appropriate sense, namely C^∞ or analytic according as G is a C^∞ or an analytic Lie group). To see this we take a basis $X_1, X_2, \ldots, X_n$ for $\mathfrak{g}$. Then

$$X = \sum_{j=1}^{n} a_j X_j$$

and the differential equations defining $h(X:t)$ are given in local coordinates near e in the form

$$\frac{dh_i(t)}{dt} = \sum_{j=1}^{n} a_j F_{ij}(h_1, \ldots, h_n, t) \qquad (1 \le i \le n).$$

The a_j can now be viewed as *parameters* and we conclude that the h_i are smooth in the parameters. We thus have

Theorem 2. *The exponential map is smooth from* $\mathrm{Lie}(G)$ *to* G.

Problems

1. On any manifold prove that any vector field with compact support generates a global flow,
2. Give the details of the result that for matrix Lie groups the Lie algebra and the exponential map defined here coincide with the earlier definitions.

8.3. The adjoint representation. In defining the Lie algebra we could also have worked with the *right invariant vector fields.* But the theory is equivalent. Moreover, using the exponential map obtained from the left invariant theory we can obtain the right invariant vector fields and their integral curves very simply. If $X \in \mathrm{Lie}(G)$ the integral curves of the right invariant vector field X' with $X'_e = X_e$ are

$$t \longmapsto \exp tXx \qquad (x \in G) = x(x^{-1} \exp tXx).$$

This leads us to consider the action of G on $\mathfrak{g}$ by inner automorphisms. More precisely, for $x \in G$ let

$$\iota_x : y \longmapsto xyx^{-1}$$

be the inner automorphism defined by x. Then, for any $X \in \mathfrak{g}$,

$$t \longmapsto \iota_x(h(X : \cdot)$$

is a Lie group morphism of $\mathbf{R}$ into G and so is of the form $h(X_x : \cdot)$ for some $X_x \in \mathfrak{g}$. Computing differentials at e we get

$$(d\iota_x)_e(X_e) = (X_x)_e$$

showing that $X \longmapsto X_x$ is a linear map of $\mathfrak{g}$ into itself. We write $\mathrm{Ad}(x)$ for this linear map. Thus

$$x \exp tXx^{-1} = \exp t\mathrm{Ad}(x)(X).$$

Since $\iota_{xy} = \iota_x \iota_y$ we see that Ad is a homomorphism of G into $\mathrm{GL}(\mathfrak{g})$. The formula

$$\left(\frac{d}{dt}\right)_{t=0} x \exp tXx^{-1} = \mathrm{Ad}(x)(X)$$

shows that $\mathrm{Ad}(x)(X)$ depends smoothly on x for each X and hence that Ad is a Lie group morphism of G into $\mathrm{GL}(\mathfrak{g})$. It is called the *adjoint representation* of G. We shall see later that it is a morphism of G into $\mathrm{Aut}(\mathfrak{g})$, the group of automorphisms of the*Lie algebra* $\mathfrak{g}$.

There is a very interesting application of the adjoint representation in the theory of complex Lie groups. Notice first that if the adjoint representation is the identity, then the elements of G commute with all exponentials. Since a *connected* G is generated by the exponentials it follows that G is commutative. If now G is a *compact complex* Lie group, we know that it has no non-constant maps into affine space (as a complex manifold) and so the adjoint representation is necessarily trivial. Hence G is abelian. The exponential map is then a *homomorphisn* from $\mathfrak{g}$ to G. It follows from this that

$$G = \mathfrak{g}/L$$

where L is a lattice. However G is not necessarily a *complex torus*; for that L will have to satisfy the so-called *Riemann conditions.*

Problems

1. Prove that G is generated by the elements of the form $\exp X (X \in \mathfrak{g})$.
2. Prove that if G is commutative then exp is a Lie group morphism from the additive group of $\mathfrak{g}$ into G, i.e., $\exp(X+Y) = \exp X \exp Y$ for $X, Y \in \mathfrak{g}$.
3. Prove that for commutative G, the exponential map has bijective differential everywhere and hence derive the representation $G = \mathfrak{g}/L$ for a suitable discrete subgroup L which is a lattice if we know that G is compact.