12. Hilbert's fifth problem for compact groups: Von Neumann's theorem

12.1. The Hilbert problem. In his address entitled *Mathematical Problems* before the International Congress of Mathematicians in Paris in 1900, David Hilbert proposed a list containing 23 problems varying over almost all branches of mathematics with the idea that their solutions would lead to progress in mathematics. To a remarkable extent he was prophetic. Among these problems the 5^{th} concerns us in this section. Hilbert's question was the following:

> *How far Lie's concept of continuous groups of transformations is approachable in our investigations without the assumption of the differentiability of the functions defining the transformations of group.*

More precisely, let G be a topological group acting continuously on a topological space M by the action

$$(g, x) \longmapsto g[x] = F(g, x) \qquad (g \in G, x \in M).$$

The functions

$$F : G \times M \longrightarrow M$$

are continuous and the *functional equations*

$$F(g_1, g_2, x) = F(g_1, F(g_2, x)), \quad F(e, x) = x \tag{$*$}$$

summarize completely the action. Suppose now that both the group and the space have the property that their elements can be described by a finite number of real parameters. In terms of these parameters the functions F become numerical functions of several real variables; Hilbert's question then asks if we can change the parameters in such a way that the functions F become differentiable or analytic when expressed in terms of the new parameters.

Let us call a topological space *locally Euclidean* if each point of it has a neighborhood that is homeomorphic to a cell in a Euclidean space. Here, by an n-cell or a cell in $\mathbf{R}^n$ we mean a subset $I_1 \times I_2 \times \ldots \times I_n$ where each I_j is a nonempty open interval in $\mathbf{R}$. Then in modern terminology, Hilbert's question becomes the following.

> *Let the topological group G and the topological space M be both locally Euclidean. Is it then possible to equip G and M with the structure of*

an analytic group and an analytic manifold, each compatible with its topology, so that the action becomes analytic?

The case $G = M$ and the action is by left (or right) translations is a vry important one. The question then becomes

Is every locally Euclidean topological group a Lie group?

It is in this form that the usual formulation of Hilbert's 5^{th} problem is customarily given.

The first breakthrough came in 1933 when Von Neumann proved that for a *compact* group the answer to Hilbert's question was affirmative:

Theorem (Von Neumann). *A compact locally Euclidean group is a Lie group.*

Partial generalizations were then obtained by Pontryagin (for commutative G) and Chevalley (for solvable G). But the problem remained unsolved till the 1950's when Gleason and Montgomery-Zippin succeeded in proving that the answer to Hilbert's question was affirmative without any restriction. In this section we shall discuss the case when G is compact and give the proof of Von Neumann's theorem that that a compact locally Euclidean group is a Lie group. The proof also yields the case of the transformation group when the action is *transitive.*

12.2. Approximation by Lie groups. Since a locally Euclidean compact group can be covered by a *finite number of cells* it is clear that such a group always satisfies the second countability axiom. Hence we may assume that G is second countable and compact. Fix a fundamental sequence of decreasing compact neighborhoods (U_n) of the identity element e in G. We have seen earlier that we can find a closed normal subgroup $N_n \subset U_n$ such that G/N_n is a Lie group. Now, if H_1, H_2 are closed normal subgroups such that G/H_i is a Lie group for $i = 1, 2$, the imbedding

$$G/H_1 \cap H_2 \hookrightarrow G/H_1 \times G/H_2$$

shows that $G/H_1 \cap H_2$ is also a Lie group. Hence in the above construction of the N_n we may assume that the N_n are decreasing.

We can actually arrange matters so that N_1 is any preassigned closed normal subgroup H such that G/H is a Lie group. In fact, given H, we choose (N_n) as before and note that $G/H \cap N_n$ is a Lie group for all n.

Thus $(H \cap N_n)$ is a sequence which satisfies the requirements; we add H as the first element of the sequence. We thus have

Proposition 1. *Let (U_n) be a fundamental sequence of compact neighborhoods of e in G. Then we can find closed normal subgroups N_n of G such that*

$$N_1 \supset N_2 \supset N_3 \supset \ldots, \cap_n N_n = \{e\}, \quad N_n \subset U_n$$

such that G/N_n is a Lie group for all n. Moreover, we can arrange matters so that $N_1 = H$ where H is any given closed normal subgroup such that G/H is a Lie group.

We say that G is *approximated by Lie groups.*

Let (G_n) be any sequence of compact groups and for each n let us assume that there is a *surjective* morphism of G_{n+1} onto G_n. Let G_∞ be the set of all sequences (x_n) such that $x_n \in G_n$ for all n, and for each n, x_{n+1} lies above x_n. Then G_∞ has a natural imbedding in the product of all the G_n,

$$G_\infty \subset \prod_n G_n$$

and it is immediate that it is a closed subgroup of the product group. Thus G_∞ is a compact group, called the *projective limit of the* (G_n). Let us now write

$$G_n := G/N_n.$$

It is then clear that we have an injection

$$G \hookrightarrow G_\infty.$$

We claim that this is a surjection. Indeed, if (x_n) is a sequence in G_∞, we can find $y_n \in G$ such that y_n lies above x_n for all n. Select a subsequence (y_{n_k}) such that $n_1 < n_2 < \ldots$ and $y_{n_k} \to y$ as $k \to \infty$. It is then easily seen that y lies above x_n for all n. Thus

$$G = \varprojlim G/N_n.$$

We have thus proved

Proposition 2. *With (N_n) as earlier*

$$G = \varprojlim G/N_n.$$

12.3. Lifting of one-parameter groups and cells. By a *compact n-cell* of a topological space X we mean a subset of X homeomorphic to a cube $J_1 \times \ldots J_n$ where the J_i are nonempty compact intervals of $\mathbf{R}$. We want to prove the following.

Proposition 1. *Let G be a compact group and H a closed normal subgroup such that G/H is a Lie group. If $a(t)$ is an one-parameter subgroup of G/H we can find an one-parameter subgroup $b(t)$ in G such that $b(t)$ lies above $a(t)$ for all t.*

Proof. We use approximation. Let (N_n) be as above with $N_1 = H$. Write $G_n = G/N_n$. Suppose we have found an one-parameter subgroup $(a_n(t))$ in G_n. Since G_{n+1} is a Lie group and $G_{n+1} \longrightarrow G_n$ is surjective, the map $\mathrm{Lie}(G_{n+1}) \longrightarrow \mathrm{Lie}(G_n)$ is surjective, and so we an find an one-parameter subgroup $(a_{n+1}(t))$ in G_{n+1} such that $a_{n+1}(t)$ lies above $a_n(t)$ for all t. So by induction we have one-parameter subgroups $(a_n(t))$ in G_n for all n such that $a_{n+1}(t)$ lies above $a_n(t)$ for all n. Since $G = \lim_{\longleftarrow} G_n$ we have unique $b(t) \in G$ for each t such that for each n, $b(t)$ lies above $a_n(t)$. It is immediate that $(b(t))$ is an one-parameter subgroup in G and lies above $(a_n(t))$ for all n.

Proposiiton 2. *Let $m = \dim(G/H)$. Then we can find a compact neighborhood C of the identity in G/H and a compact subset D of G such that C nd D are compact m-cells and the natural map $G \longrightarrow G/H$ is a homeomorphism of D onto C.*

Proof. By using canonical coordinates of the second kind for G/H we can find one-parameter groups $a_1(t), a_2(t), \ldots, a_m(t)$ in G/H such that the map

$$\varphi : (t_1, t_2, \ldots, t_m) \longmapsto a_1(t_1)a_2(t_2)\ldots a_m(t_m)$$

is a homeomorphism of the compact unit cube $J \subset \mathbf{R}^m$ onto a compact neighborhood C of the identity e in G/H. Let $(b_j(t))$ be an one-parameter group in G above $(a_j(t))$ and let us cnsider the map

$$\psi : (t_1, t_2, \ldots, t_m) \longmapsto b_1(t_1)\ldots b_m(t_m)$$

of J into G; let D be the image of J under ψ. If π is the natural map $G \longrightarrow G/H$, we have $\pi \circ \psi = \varphi$. Since φ is a homeomorphism, it follows that ψ is also a homeomorphism, hense also the restriction of π to D.

12.4. Proof of Von Neumann's theorem. We need the following Lemma which is a consequence of dimension theory of compact spaces.

Lemma. *If A (resp. B) is a compact m-cell (resp. n-cell), and $n > m$, B cannot be homeomorphic to a subset of A.*

We start with the approximation

$$G = \varprojlim G_n, \qquad G_n = G/N_n.$$

The proof depends on the following facts.

Proposition 1. *If G has a neighborhood of the identity that is an m-cell, then $\dim(G_n) \leq m$ for all n, and we have $\dim(G_n)$ is constant for all sufficiently large n. In particular N_n/N_{n+1} is finite for all sufficiently large n.*

Proof. Otherwise we can find n such that $\dim(G_n) = k > m$. We can then find a compact k-cell inside G, hence a compact k-cell D containing the identity element of G. If E is a compact m-cell in G which is a neighborhood of the identity, then $E \cap D$ is a compact neighborhood of the identity in D and so there is a compact k-cell F such that $F \subset E \cap D \subset E$. This contradicts the Lemma.

We shall henceforth assume that the conditions of the Proposition are satisfied for all n. In particular $\dim(G_n) = k$ for all n. It will turn out that $k = m$ but this is not needed at this time.

Proposition 2. *Let C be a compact k-cell which is a neighborhood of the identity in G_1 and let $D \subset G$ be a compact set such that the natural map $\pi : G \longrightarrow G/N_1$ is a homeomorphism of D onto C. Then the map*

$$f : D \times N_1 \times N_1 \longrightarrow DN_1, \qquad f(x, y) = xy$$

is a homeomorphism and DN_1 is a compact neighborhood of the identity in G.

Proof. The first statement follows at once from the fact that π is a homeomorphism on D. Since $DN_1 = \pi^{-1}\pi(D)$, it is immediate that DN_1 is a neighborhood of the identity in G.

Proposition 3. *N_1 is totally disconnected.*

Proof. We have noted already that the N_1/N_n are all finite. Hence

$$N_1 = \varprojlim (N_1/N_n)$$

is a closed subgroup of a product of finite groups and so is totally disconnected.

Proposition 4. *We have $k = m$, N_1 is finite, and G itself is a Lie group.*

Proof. Let E be a compact m-cell which is a neighborhood of the identity element e in G. By shrinking this cell we may assume that $E \subset DN_1$. Since $DN_1 \simeq D \times N_1$ we may speak of the *projection* of E on N_1. Let E' be this projection. ThenE' is a *connected* compact neighborhood of the identity in N_1. Since N_1 is totally disconnected, it follows that $E' = \{e\}$. Hence $E \subset D$. But then $E \cap N_1 = \{e\}$, showing that H is *discrete*. Hence N_1 is finite and so $G \longrightarrow G/N_1$ has finite kernel. This proves that G is a Lie group and $m = \dim(G) = \dim(G/N_1) = k$.