Trigonometry and the the beginnings of calculus in India

- Calculus in India did not begin with integration, as in Greece and China.
- It began with the perhaps empirical discovery of the 2^{nd} order difference equation for sine, sometime in the first half of the 1^{st} millenium CE.
- Sine, cosine, tan, arctan, etc were important parts of the formulas of astronomy which were expounded in all siddhantas, esp. *tables* were needed.

The *Aryabhatiya*, 499 CE

It contains the following “sine-table”:

makhi-bhakhi-phakhi-dhakhi-ṇakhi-ñakhi-ṅakhi-hasjha-skaki-kiṣga-śghaki-kighva, ghlaki-kigra-hakya-dhaki-kica-sga-jhaśa-ṅva-kla-pta-pha-cha-kalārdhajyāḥ.

These are the 24 numbers 225, 224, 222,219, ….,51, 37, 22, 7, in his unique Sanskrit-gibberish-number system, described by the last word as <differences of> half-chords in arc-minutes. To get the actual sines, you must take cumulative sums and divide by *R*=3438, i.e. the cum. sums:

225, 225+224=449, 449+222=671, 671+219=890, …., 3438

are *R*.sin(*i.3¾°*). *R* ≈ #minutes in a radian, *3¾°* =225 minutes – so for small x, *R*.sin(*x* minutes) ≈ *x.* If you must write sine tables in memorizable verse, it’s clearly better to memorize the differences!

[*3¾°* arose because it is *30°*/8, so the sines come from 48-gons.]

But in the *Paitāmaha-siddhānta*, c. 425 CE

The first RSine is 96th part of 21,600. If one divides the first RSine by the first RSine and subtracts the quotient from the first Rsine, one obtains the difference of the second Rsine; the sum of the first Rsine and the difference of the second Rsine in the second Rsine. If one divides the second Rsine by the first Rsine and subtracts the quotient from the difference of the second Rsine, one obtains the difference of the third Rsine; the sum of this and the second Rsine is the third Rsine …. [and so on up to the 24th and final Rsine.]

The minutes [in the argument of arc] are to be divided by 225; the Rsine corresponding to the serial number is to be put down. One should multiply the remainder by the difference of the next Rsine and divide [the product] by 225. The sum of the quotient and the Rsine which was put down is the desired Rsine.

What's happening?

To compute $S_k = R.\sin(k.\pi / 48)$, use differences $D_k = S_k - S_{k-1}$

$R = \#\text{minutes/radian} = 360.60/2\pi \approx 3438$, $\pi/48$ radians $= 225$ minutes

start with $S_0 = 0$, $S_1 \approx 3438 \cdot \pi/48 \approx 225$, then follow recipe:

$$D_2 = S_1 - \left(S_1 / S_1\right) = 224, \quad S_2 = S_1 + D_2 = 449$$

$$D_3 = D_2 - \left(S_2 / S_1\right) = 222, \quad S_3 = S_2 + D_3 = 771$$

....

$$D_{i+1} = D_i - \left(S_i / S_1\right), \quad S_{i+1} = S_i + D_i$$

....

$S_n = R$ (with any luck)

Nilakantha (15th c.) and Hayashi (1997) claim Aryabhata makes a very crucial change

prathamāc cāpajyārdhād yair ūnaṃ khaṇḍitaṃ dvitīyārdham,
tatprathamajyārdhāṃśais tais tair ūnāni śeṣāṇi.

When the second half-⟨chord⟩[1] partitioned is less than the first half-chord, which is ⟨approximately equated to⟩ the ⟨corresponding⟩ arc, by a certain amount, the remaining ⟨sine-differences⟩ are less ⟨than the previous ones⟩ each by that amount of that (i.e., the corresponding half-chord)[2] divided by the first half-chord. (AB 2.12 [1, 83])

$$D_i - D_{i+1} = \left(D_1 - D_2\right)\cdot\left(\frac{J_i}{J_1}\right) \quad \text{or}$$

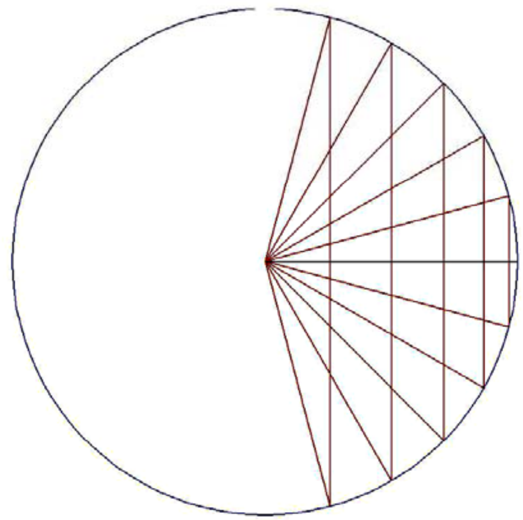
“certain amount”

$$\frac{D_i - D_{i+1}}{J_i} = \frac{D_1 - D_2}{J_1}, \quad \text{or}$$

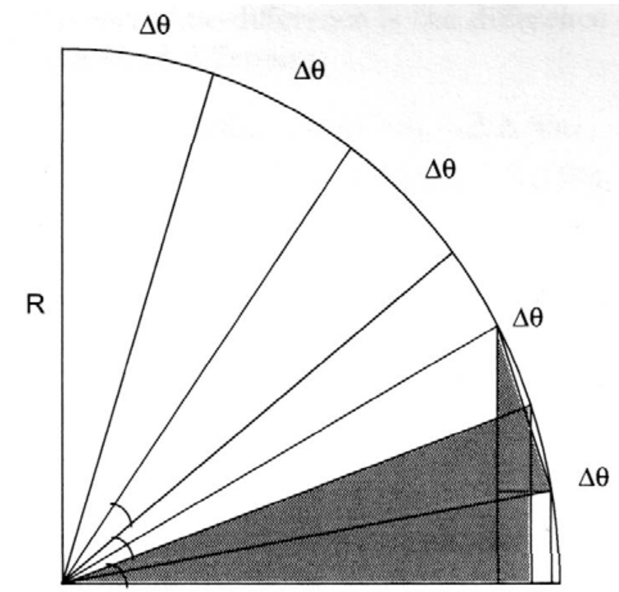
$$J_{i-1} - 2J_i + J_{i+1} = \left(\text{cnst.ind.of } i\right)\cdot J_i$$

This is the finite difference form of $\sin'' = -\sin$!

RADIUS (makes circum. 360*60)	ANGLE in degrees	RAD*SIN (HALF CHORD)	DELTA RAD*SIN	DELTA OF DELTA OF RAD*SIN	Aryabhata's rule
3438	3.75	225	225	1	1
	7.5	449	224	2	2
	11.25	671	222	3	3
	15	890	219	4	4
	18.75	1105	215	4	5
	22.5	1316	211	6	6
	26.25	1521	205	7	6
	30	1719	198	7	7
	33.75	1910	191	8	8
	37.5	2093	183	9	9
	41.25	2267	174	10	10
	45	2431	164	10	10
	48.75	2585	154	11	11
	52.5	2728	143	12	12
	56.25	2859	131	13	12
	60	2977	118	12	13
	63.75	3083	106	13	13
	67.5	3176	93	13	14
	71.25	3256	80	15	14
	75	3321	65	14	14
	78.75	3372	51	14	14
	82.5	3409	37	15	15
	86.25	3431	22	15	15
	90	3438	7		



Madhava (c.1400) – rigorous derivation of the first derivatives of sin and cos



The two shaded triangles are similar. Equating the ratios of their sides to their hypotenuse:

$$\frac{\sin(\theta+\Delta\theta)-\sin(\theta-\Delta\theta)}{(\text{chord of angle } 2\Delta\theta)/R}=\frac{\text{vert.side}}{\text{hypot}}\text{ of small }=\frac{\text{hor.side}}{\text{hypot}}\text{ of large}=\cos(\theta)$$

$$\frac{\cos(\theta+\Delta\theta)-\cos(\theta-\Delta\theta)}{(\text{chord of angle } 2\Delta\theta)/R}=\frac{\text{hor.side}}{\text{hypot}}\text{ of small }=\frac{\text{vert.side}}{\text{hypot}}\text{ of large}=\sin(\theta)$$

Computing sines in Bhaskara I, c.630 CE

$$\text{On } 0 \le \theta \le \pi, \quad \sin(\theta) \approx \frac{4\left(\theta/\pi\right)\left(1-\theta/\pi\right)}{\frac{5}{4}-\left(\theta/\pi\right)\left(1-\theta/\pi\right)}$$

- Exact at 0, 30, 90, 150, 180 degrees
- Always within 0.0016 of correct value.
- Note germinating idea of the *sine as a function on the full 360 degrees*
- Note the focus on very accurate and workable numerical values – for handy use in astonomical calculations

Full trigonometry in Bhaskara II (c.1150 CE)

"On the construction of the canon of sines"

1. As the Astronomer can acquire the rank of an ACHARYA in the science only by a thorough knowledge of the mode of constructing the canon of sines, BHÁSKARA therefore now proceeds to treat upon this (interesting and manifold) subject in the hope of giving pleasure to accomplished astronomers.

……………………………………………………

Rules for finding the sine of every degree from 1° to 90°.

16. Deduct from the sine of BHUJA its $\frac{1}{6569}$ part and divide the ten-fold sine of KOṬI by 573.

17. The sum of these two results will give the following sine (i. e., the sine of BHUJA one degree more than original BHUJA and the difference between the same results will give the preceding sine, i. e., the sine of BHUJA one degree less than original BHUJA). Here the first sine, i. e., the sine of 1°, will be 60 and the sines of the remaining arcs may be successively found.

21 and 22. If the sines of any two arcs of a quadrant be multiplied by their cosines reciprocally (that is the sine of the first arc by the cosine of the 2d and the sine of the 2d by the cosine of the first arc) and the two products divided by radius, then the quotients will, when added together, be the sine of the sum of the two arcs, and the difference of these quotients will be the sine of their difference.† This excellent rule called JYA-BHÁVANÁ has been prescribed for ascertaining the other sines.

Rules for finding the sines of sum and difference of any two arcs.

<u>Bhaskara II got the volume of the sphere too</u>

In this case, what we would write as the quadratic equation in x, where x is the desired number of monkeys in the troop, that is, $\left(\frac{1}{5}x - 3\right)^2 + 1 = x$, has been solved by Bhāskara as the equation *yā va* 1 *yā* $\dot{5}5$ *rū* 0 [=] *yā va* 0 *yā* 0 *rū* $\dot{2}50$, where *yā va* stands for *yāvattāvat varga*, "the square of the unknown." The two resulting values for x, from the two square roots in the quadratic, are 50 and 5. Only the first of these values, Bhāskara says, is "applicable" to this particular problem, because the second would make the "fifth part of the troop, minus three" equal to a negative number, which makes no sense in this context.

Section 10. Equations in more than one unknown. The remaining three sections build on the basic rules of Brahmagupta in the *Brāhma-sphuṭa-siddhānta*.[37] Linear equations in more than one unknown are solved by finding one unknown in terms of the others. If there are more unknowns than equations, so that the problem is indeterminate, the pulverizer is used.

Section 11. Elimination of the middle with more than one unknown. A single quadratic equation with more than one unknown is to be transformed into a square-nature problem if possible, and solved by indeterminate methods.

Section 12. Products of unknowns. Equations containing products of two or more unknowns are solved with arbitrarily chosen numbers, as directed by Brahmagupta.

Section 13. Conclusion. The last few verses contain information about Bhāskara's background and his work, surveyed at the beginning of this section.

6.2.3 The *Siddhānta-śiromaṇi*

This work (literally "Crest-jewel of *siddhāntas*") was composed when Bhāskara was 36, that is, in 1150. It lived up to the boast in its title by gaining a high place among astronomical treatises, although as an orthodox Brāhma-pakṣa work it could not supplant the canonical texts of the other *pakṣas*. The *Siddhānta-śiromaṇi* is sometimes described as containing the *Līlāvatī* and *Bīja-gaṇita* in addition to its two sections on astronomy proper, and it is evident that Bhāskara considered the subject matter of all these compositions to be very closely linked. But since the arithmetic and algebra texts have individual titles and have usually been copied as individual manuscripts, we will follow the tradition of considering the *Siddhānta-śiromaṇi* a separate work devoted to astronomy.

The treatise is divided into two sections: the first, on planetary calculations, presents standard computational algorithms like discussed in section 4.3 for calculating mean motions, true motions, the Three Questions, lunar and solar eclipses, and so forth. The second, on *gola*, contains chapters on the following subjects: praise of the sphere, the form of the sphere, the sphere of the earth, explanations of various subjects from the *gaṇita* section, instruments, a poetic description of the seasons, and questions to test the student's knowledge. As it is not possible to do justice here to even a substantial portion of this comprehensive work, we will content ourselves with pointing out a few examples of Bhāskara's ingenious manipulations of small quantities and his explanations in his own commentary *Vāsanā-bhāṣya*, or "Commentary of rationales."

[37] They are described more fully in [DatSi1962], vol. 2, pp. 57–59, 181–193, and 199–201, respectively.

The first example, from the first section's chapter on true motions, involves calculating the speed of a planet's motion through the sky. We have seen in section 4.3.2 how to calculate a planet's mean speed R/D, where R is the number of the planet's integer revolutions in a given time period and D is the number of days in that period. But usually a planet's apparent motion will be slower or faster than that mean motion. Roughly, the speed will be least when the planet's anomaly κ (see section 4.3.3) is zero (i.e., when the planet is at apogee), greatest when the anomaly is 180°, and close to the mean speed when the anomaly is about 90°. If we want to know how fast a planet appears to be moving at some given time, how should we compute that? This problem of *tātkālika* or "at-that-time" motion was tackled in various ways by Indian astronomers; one typical strategy involved calculating the difference between the true and mean speeds as approximately proportional to the Sine-difference corresponding to the value of the anomaly. In following this approach, Bhāskara made use of a concept he called an "instantaneous Sine-difference," computed from the Cosine by a Rule of Three Quantities:

> The difference between today's and tomorrow's true [positions of a] planet ... is the true [daily] speed.... [At some point] within that time [or, on average in that time] the planet is required to move with that speed. Yet this is the approximate speed. Now the accurate [speed] for that time [or, instantaneous (*tātkālika*) speed] is described....
>
> If a Sine-difference equal to five-two-two [i.e., 225] is obtained with a Cosine equal to the Radius, then what [is obtained] with a desired [Cosine]? Here, five-two-two is the multiplier and the Radius is the divisor of the Cosine. The result is the accurate Sine-difference at that time.[38]

In other words, the "at-that-time" Sine-difference $\Delta\,\mathrm{Sin}$ for a given arc α is considered simply proportional to the Cosine of α:

$$\Delta\,\mathrm{Sin} = \mathrm{Cos}\,\alpha \cdot \frac{225}{R}.$$

It has been noted[39] that this and related statements reveal similarities between Bhāskara's ideas of motion and concepts in differential calculus. (In fact, perhaps these ratios of small quantities are what he was referring to in

[38] *Vāsanā-bhāṣya* on *Siddhānta-śiromaṇi* Ga.2.36–38, [SasB1989], pp. 52–53. The method is explained in detail in [Ike2004].

[39] For example, in [Rao2004], pp. 162–163.

his commentary on *Līlāvatī* 47 when he spoke of calculations with factors of 0/0 being "useful in astronomy.") This analogy should not be stretched too far: for one thing, Bhāskara is dealing with particular increments of particular trigonometric quantities, not with general functions or rates of change in the abstract. But it does bring out the conceptual boldness of the idea of an instantaneous speed, and of its derivation by means of ratios of small increments.

The *gola* section of the *Siddhānta-śiromaṇi* begins with an exhortation on the importance of understanding astronomy's geometric models that is somewhat reminiscent of Lalla's remarks quoted in section 4.5, but also emphasizes the need for their mathematical demonstration:

> A mathematician [knowing only] the calculation of the planets [stated] here [in the chapters on] mean motions and so forth, without the demonstration of that, will not attain greatness in the assemblies of the eminent, [and] will himself not be free from doubt. In the sphere, that clear [demonstration] is perceived directly like a fruit in the hand. Therefore I am undertaking the subject of the sphere as a means to understanding demonstrations.
>
> Like flavorful food without ghee and a kingdom deprived of [its] king, like an assembly without a good speaker, so is a mathematician ignorant of the sphere.[40]

An example of what Bhāskara means by a demonstration can be seen in his chapter on the terrestrial globe, where he criticizes the value for the size of the earth given by Lalla in the *Śiṣya-dhī-vṛddhida-tantra*, arguing that it is erroneous because of Lalla's erroneous rule for the surface area of a sphere. This rule, which Lalla allegedly stated in a now lost work on *gaṇita*, says that the surface area is equal to the area of a great circle times its circumference (or $2\pi^2r^3$). Bhāskara remarks, "Because of the error in the computation stated by Lalla, the surface area of the [spherical] earth is wrong too."[41] He justifies his criticism by explaining his own rules from *Līlāvatī* 199–201 for the sphere's surface area and volume, which are equivalent to $4\pi r^2$ and $\frac{4}{3}\pi r^3$, respectively, for a sphere of radius r. He starts out by imagining equidistant great circles like longitude circles on the sphere's surface, and approximating the area of one spherical lune (a portion of the surface cut off between two adjacent great semicircles, like the skin of a segment of an orange):

> The circumference of a sphere is to be considered [as having] measure equal to the amount of Sines, times four [i.e., $24 \times 4 = 96$].

[40] *Siddhānta-śiromaṇi* Go.1.2–3, [SasB1989], pp. 175–176. See also [Srin2005], pp. 228–229.

[41] *Vāsanā-bhāṣya* on *Siddhānta-śiromaṇi* Go.3.54–57, [SasB1989], p. 187. See also [Cha1981] 2, pp. xx–xxi, 250–251, and [Hay1997b], pp. 198–199.

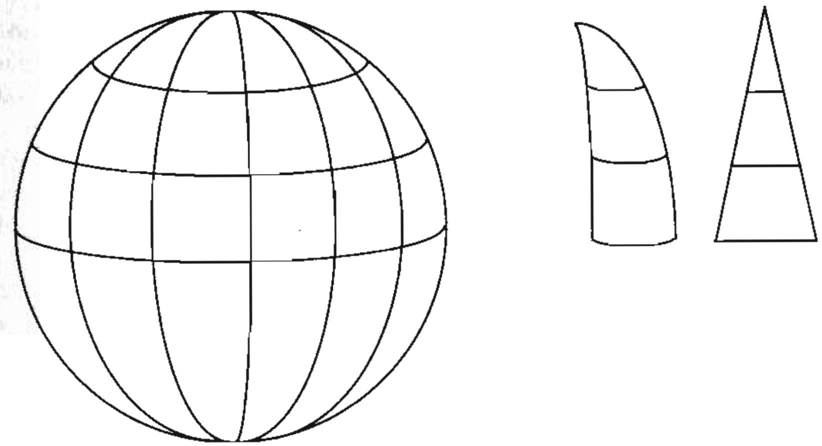


Figure 6.1 Dividing the surface of the sphere to compute its area.

> Spherical lunes [literally "mounds" or "circumferences"] are perceived [when the surface is] divided by multiple lines going from the top to the bottom as on the surface of the ball of an *amla* [Indian gooseberry] fruit. When one has set out spherical lunes [equal in] number to the prescribed [divisions of the] circumference ... the area in one lune is to be determined thus:...
>
> The circumference in the sphere is assumed [to be] equal to ninety-six cubits, and that many lunes are made with vertical lines at each cubit. Then when one has made horizontal lines at each cubit-interval on half of one lune, twenty-four lune-portions are imagined, [equal in] number to the Sines. Then the Sines separately divided by the Radius are the measures of the horizontal lines. In that case the lowest line is equal to [one] cubit, while [the lines] above are successively somewhat less, in accordance with [the sequence of] the Sines. The altitude is always just equal to [one] cubit. When one has found the areas of the portions equal to the sum of the base and the top multiplied by the altitude, [they are all] added. That is the area in half a lune; that times two is the area in one lune.[42]

The following explanation and amplification of Bhāskara's arguments rely on the sphere shown in figure 6.1, with circumference $4n$, where n is the number of Sines in a quadrant. (Bhāskara takes $n = 24$, but our figure for

[42] *Vāsanā-bhāṣya* on *Siddhānta-śiromaṇi* Go.3.58–61, [SasB1989], p. 188. This rationale is preceded in Bhāskara's discussion by a similar one that imagines a hemisphere divided into zones by small circles parallel to the equator, with a spherical cap on top. The areas of the zones spread out into long trapezoidal strips, plus the circular area of the spherical cap, add up to the total area of the hemisphere (*Vāsanā-bhāṣya* on *Siddhānta-śiromaṇi* Go.3.54–57, [SasB1989], pp. 187–188). See the description of both methods in [Sara1979], pp. 211–213, and the translation and explanation in [Hay1997b], pp. 202–217.

simplicity shows $n = 3$.) The sphere's surface is divided vertically into $4n$ equal lunes of unit width at the equator, and the top half of each lune is divided horizontally into n segments by small circles parallel to the equator at unit intervals.

The radius of each ith parallel circle (taking $i = 0$ at the equator) is proportional to the Cosine of its elevation above the equator. (If this is not immediately obvious, think—as Bhāskara certainly would have—of parallel day-circles on the celestial sphere whose radii are equal to the Cosines of their declinations, as explained in section 4.3.4.) If we let A_L be the area of one lune, we can consider that the area $A_L/2$ of the individual half-lune shown on the right side of figure 6.1 is approximately equal to that of the triangular figure corresponding to it. That triangle is a stack of $(n-1)$ trapezoids with a triangle on top, all of which are considered to have unit altitude. The equally spaced horizontal line segments dividing them are the chords of the corresponding unit arcs of the parallel circles in the half-lune. Assuming that the lowest of these line segments s_0 has unit length, and that the unit arc on the sphere contains u degrees, we can express the length s_i of each ith horizontal line segment by

$$s_i = 1 \cdot \frac{\mathrm{Cos}(iu)}{R} = 1 \cdot \frac{\mathrm{Sin}(90 - iu)}{R}.$$

Hence, as Bhāskara says, the measures of the line segments s_i are the Sines separately divided by the Radius. Then the area of the lowest trapezoid in the half-lune will be $\frac{s_0 + s_1}{2} \cdot 1$, and so on up to the top triangle, whose area will be $\frac{s_{n-1} + 0}{2} \cdot 1$. The sum of all of them will be the total area $A_L/2$ of the half-lune, which we may express in terms of the n Sines as follows:

$$\frac{A_L}{2} = \left(\frac{\mathrm{Sin}_n + \mathrm{Sin}_{n-1}}{2} + \frac{\mathrm{Sin}_{n-1} + \mathrm{Sin}_{n-2}}{2} + \ldots + \frac{\mathrm{Sin}_2 + \mathrm{Sin}_1}{2} + \frac{\mathrm{Sin}_1 + 0}{2}\right)\frac{1}{R}$$
$$= \left(\frac{\mathrm{Sin}_n}{2} + \sum_{i=1}^{n-1} \mathrm{Sin}_i\right)\frac{1}{R} = \left(\sum_{i=1}^{n-1} \mathrm{Sin}_i - \frac{\mathrm{Sin}_n}{2}\right)\frac{1}{R}.$$

The area A_L of the whole lune must be twice that amount. Since $\mathrm{Sin}_n = R$, area A_L will indeed be the sum of all the Sines minus half the Radius and divided by half the Radius, just as Bhāskara says.

To find a simpler expression for that sum of all the Sines, he then switches from a geometrical demonstration to a numerical illustration—that is, just adding up their known values:

> For the sake of determining that [lune area], this rule [was stated]: "The sum of all the Sines is decreased by half the Radius [and divided by half the Radius]" [verse Go.3.60cd]. Here the sum of all the sines beginning with 225 is 54,233. [When] that is decreased by half the Radius, the result is 52,514. [When] that is divided by half the Radius, the result is the area of one lune [and] equal to the diameter, 30;33. Because the diameter of a sphere with circumference ninety-six is just that much, 30;33, and the lunes are equal [in number] to [the divisions of] the circumference, therefore the area of the surface of the sphere is equal to the product of the circumference and the diameter; thus it is demonstrated.[43]

The sum of all the Sines in the standard Sine table, from $\mathrm{Sin}_1 = 225$ to $\mathrm{Sin}_{24} = 3438 = R$, is 54232 (Bhāskara says 54,233), which diminished by $R/2$ equals 52,513 (Bhāskara says 52,514). Dividing by $R/2$, we get a little over $30\frac{1}{2}$, or $30;33$ to the nearest sixtieth, which is in fact the diameter of a circle with circumference ninety-six units. From this Bhāskara infers the general result that the area A_L of the lune is equal to the diameter of the sphere. Since there are as many lunes as there are units in the sphere's circumference, the total area A of the sphere's surface therefore is just its circumference times its diameter.

Now that the formula for the surface area is demonstrated, Bhāskara uses it to explain the formula for the volume:

> And in the same way, that area produced from the surface of a sphere, multiplied by the diameter [and] divided by six, is called the accurate solid [volume] within the sphere.... Here is the demonstration: Square pyramidal holes [literally "needle-excavations"] with unit [base]-sides [and] depth equal to the half-diameter, [equal in] number to [the divisions of] the area, are imagined in the surface of the sphere. The meeting-point of the tips of the pyramids is inside the sphere. Thus the sum of the pyramid amounts is the solid amount; thus it is demonstrated.[44]

Using the same imagined unit grid, this time covering the whole of the sphere's surface, Bhāskara now considers it as made up of unit squares which are the bases of square pyramidal holes bored into the sphere, with depth equal to the sphere's radius r. The sum of the volumes of all the pyramidal holes is the total volume of the sphere. Bhāskara leaves it to the reader to recall that the volume of each pyramid will be one-third the product of its depth and the area of its base. So the sum of the volumes will be one-third the product of the total surface area times the depth, or $\frac{1}{3}A \cdot r = \frac{1}{6}A \cdot 2r$.

Elsewhere in the *Siddhānta-śiromaṇi*, in his chapters on eclipses, Bhāskara again criticizes Lalla's mathematics, this time concerning the geometry underlying computations for eclipse diagrams. As we saw in section 4.3.5, these computations involve a quantity called the "deflection" or deviation of the path of the ecliptic away from the east-west direction on the disk of the

[43] *Vāsanā-bhāṣya* on *Siddhānta-śiromaṇi* Go.3.58–61, [SasB1989], pp. 188–189.

[44] *Vāsanā-bhāṣya* on *Siddhānta-śiromaṇi* Go.3.58–61, [SasB1989], p. 189. See also [Sara1979], p. 213.

Now, in order to teach the beginner, one should show 〈the derivation of the correct formula〉 on a sphere. Having made an earthern or wooden globe of the earth, having supposed that it has a circumference equal to the 〈number of〉 minutes, 21600, of a disc, and having put a dot at its summit, one should produce, from that dot 〈as a center〉, a circular line 〈on the surface〉 by means of 〈a thread〉, which, corresponding to a ninety-sixth part of the globe, has a length of 《the arrows, two, a twin》, 225, and a form of an arc. Again, from the same dot, 〈one should produce〉 another 〈circular line〉 by means of a thread twice as long as that, and another by means of 〈a thread〉 thrice as long as that, and so on, up to twenty-four times as long. Twenty-four circles are produced 〈in this way〉.

The radii of these circles will be the half chords (i. e., Sines) beginning with '《the arrows, eyes, arms》', 225, 〈which have been versified in the Sine table of Lalla.[28]〉

From them by proportion the sizes of 〈the circumferences of〉 the circles 〈are obtained〉. Among them, first of all, the size of the last circle is the minutes in a disc, 21600. Its radius is the Sine of Three 〈Houses〉 (i. e., of ninety degrees), 3438. The half chords (i. e., Sines), when multiplied by the minutes of a disc (21600) and divided by the Sine of Three (3438), become the sizes of the circles.

There is one geometric figure having the shape of a belt (*valaya*) between every two consecutive circles. They are twenty-four in number. In a case where many 〈more〉 Sines are supposed, there would be many 〈more belt-like figures〉.

There, when one has supposed the lower, greater circle to be the base, the above, smaller one to be the face, and 《the arrows, two, a twin》 (225) to be the perpendicular, the area of each 〈belt-like figure is calculated〉 severally by means of 〈the rule〉:

Half the sum of the base and the face is multiplied by the perpendicular. (Tr 42d)

The sum of those areas is the surface area of the hemisphere. That multiplied by two is the surface area of the whole sphere. It shall be equal to the product of the diameter and the circumference. 〈The end of the commentary on Stanzas〉 54—57.

Now, 〈the derivation of the rule〉 is explained in a different way.

58. The circumference of a sphere should be supposed to be measured by

the number of the Sines multiplied by 《the Vedas》 (4). Just as segments (*vaprakas*) on an *āmalaka*[29] fruit are observed to be 〈separated〉 by means of lines passing through the top and the bottom,
59. just so one should suppose segments on a sphere, 〈separated〉 by means of lines made vertically, as many as 〈the units in〉 the circumference told above.
60. There, the area of one segment is obtained by means of parts (*khaṇḍas*); that is, the sum of all the Sines is decreased by half the Sine of Three Houses and divided by half the Sine of Three.
61. Thus is 〈obtained〉 the area of a segment. Since it must be equal to the diameter of the sphere, the area of the surface of a sphere is remembered to be the product of the circumference and the diameter.

Here, the number of the Sines 〈tabulated〉 in any optional book is multiplied by four. On a sphere, the circumference should be regarded as being measured by it. Just as, on the surface of the globe of an *āmalaka* fruit, segments divided by natural lines passing through the top and the bottom are observed, just so, on the surface of any optional sphere, when one has supposed segments 〈divided〉 by lines going from the top to the bottom as many as the supposed units in the circumference, the area on one segment should be obtained. It is as follows.

In this *Dhīvṛddhida*[30] twenty-four Sines 〈have been tabulated〉. The circumference is, therefore, supposed to be ninety-six *hastas* (i. e., units). The same number of segments are made with vertical lines 〈drawn〉 at every *hasta*. On the 〈upper or lower〉 half of one of those segments, when one has made horizontal lines at intervals of one *hasta*, as many parts as the number of the 〈tabulated〉 Sines, twenty-four 〈in the present case〉, are supposed. The Sines there, severally divided by the Sine of Three, become the sizes of the horizontal lines. The low〈est〉 line among them is measured by one *hasta*, while the upper ones decrease little by little due to 〈the diminution of〉 the Sines. The perpendicular is measured by one *hasta* everywhere. When one has calculated the areas of the parts by means of 〈the rule〉:

Half the sum of the base and the face is multiplied by the perpendicular, (Tr 42d)

they are made into one (i. e., summed up). That is the area on the half of a segment. That multiplied by two becomes the area on one segment. What exists here in order to obtain it is this rule:

The sum of all the Sines is decreased by half the Sine of Three Houses, etc. (Stanza 60cd)

Here (in our case), 'the sum' of 'all the Sines', that is, '《the arrows, eyes, arms》 (225)', etc.,[31] is equal to 《the gods, a twin, Kṛta, arrows》, 54233. This, decreased by half the

Sine of Three, becomes what is measured by 《Manu, the principles, five》, 52514. This, divided by half the Sine of Three, becomes the area on one segment, 30 ; 33, which is equal to the diameter, since this much will be the diameter, 30 ; 33, of a sphere whose circumference is ninety-six. The segments are equal 〈in number〉 to 〈the number of units in〉 the circumference. It has been proved (or derived), therefore, that 'the product of the circumference and the diameter is the surface area of a sphere'. The same has also been stated in our mathematics of algorithms (*pāṭīgaṇita*):

> In a circular figure, a quarter of the diameter multiplied by the circumference is the area, which, multiplied (lit. pounded) by 《the Vedas》 (4), is the surface area (lit. the fruit produced from the surface) of the sphere like a net all around the surface of a hand ball. That, too, when multiplied by the diameter and divided by six, becomes the exact thing called a cube (or solid) 〈contained〉 inside the sphere. (L 201)

A sixth part of the surface area of a sphere multiplied by the diameter shall be the volume. Proof of this: Needle-like ditches, as many as 〈the units in〉 the surface area, whose arms (sides) are unity, and whose depth is equal to the radius, should be supposed on 〈the inner side of〉 the surface of a sphere. The tips of the needles fall together to the center of the sphere. In this way, the sum of the volumes of the needles is the volume 〈of the sphere〉. Thus has been proved 〈the formula for the volume of a sphere〉.

'The area multiplied by the square root of the area shall be the volume': the teacher Caturveda 〈Pṛthūdakasvāmin〉 stated this[32] perhaps as an opinion of others.[33] 〈The end of the commentary on Stanzas〉 58–61.